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Wave kinetic description of quantum pair plasmas

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Abstract. The dispersion relation for a quantum pair plasma is derived, by using a wave kinetic description. A general form of the kinetic dispersion relation for electrostatic waves in a two-component quantum plasma is established. The particular case of an electron–positron pair plasma is considered in detail. Exact expressions for Landau damping are derived, and the quasi-classical limit is discussed.

1. Introduction

Collective interactions in quantum plasmas have been considered in the past by Pines [1] who showed that electron density correlations at quantum scales modify the electron plasma wave spectrum. Recently, there has been a growing interest [2–9] in the investigation of longitudinal waves and their instabilities [2–5], as well as the formation of dark solitons and vortices [8] in unmagnetized quantum plasmas. This area of research is pertinent to laser plasmas [10, 11], to astrophysical plasmas [12–14], as well as to electronic devices [15].

Here we apply the wave kinetic formulation [16–18] to quantum pair plasmas, thus extending previous results [2, 5] where only the electron motion was retained. We will derive a general dispersion relation that is valid for quantum plasmas with two different particle species. Hence, our results are applicable to an electron–ion plasma, as well as to an electron–positron/hole (pair) plasma. The particular case of a pair plasma has been examined in detail. Exact expressions for Landau damping, and their quasi-classical approximation, have been derived.

2. Basic equations

We start from the Schrödinger–Poisson system of equations for a low-temperature quantum pair plasma, which describes the evolution of the collective wave function, $\psi_j(\mathbf{r}, t)$, in a self-consistent mean field potential $V(\mathbf{r}, t)$. In a normalized form, we have

$$i\partial_t \psi_j + A_j \nabla^2 \psi_j + (s_j V - |\psi_j|^\alpha)\psi_j = 0, \tag{1}$$

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where the electrostatic potential V between the particle species is determined from the Poisson equation

$$\nabla^2 V = |\psi_{\rm e}|^2 - |\psi_{\rm i}|^2. \tag{2}$$

Here j equals e for the electrons, and i for positively charged species (ions, holes or positrons). We have also used the charge sign s_j, defined in such way that $s_{\rm e} = +1$ and $s_{\rm i} = -1$. Furthermore, we have denoted $A_j = m_j e^2/2\epsilon_0 \hbar^2 \sqrt{n_0}$, where m_j is the mass of the different species and n_0 is the mean plasma density. The wave functions ψ_j are normalized by $\sqrt{n_0}$, the potential V by $T_{\rm F}/e$, where $T_{\rm F}$ is the Fermi temperature, time t is normalized by $\hbar/T_{\rm F}$, and space $\mathbf{r}$ by the Debye–Fermi radius $\lambda_{\rm D} = \sqrt{\epsilon_0 T_{\rm F}/e^2 n_0}$. We have also used the exponent $\alpha = 4/d$, where d is the dimension of the system. These two equations result from a straightforward generalization of the model equations used in [2, 4, 8].

Our wave kinetic formulation is based on the use of a Wigner function for each particle species, defined as [18]

$$F_j(\mathbf{r}, \mathbf{k}, t) = \int \psi_j(\mathbf{r} + \mathbf{s}/2, t)\psi_j^*(\mathbf{r} - \mathbf{s}/2, t)e^{i\mathbf{k}\cdot\mathbf{s}}\, d\mathbf{s}. \tag{3}$$

Following the usual procedure [17], evolution equations for these quantities can be derived. They take the form

$$i(\partial_t + A_j \mathbf{k}\cdot\nabla)F_j = -\int V_j(\mathbf{q}, t)[F_{j-} - F_{j+}]e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d\mathbf{q}}{(2\pi)^3}, \tag{4}$$

where we have used the Fourier components of the nonlinear potentials

$$V_j(\mathbf{r}, t) = s_j V - \rho_j^{\alpha/2}. \tag{5}$$

Here ρ_j is the probability density, defined by

$$\rho_j(\mathbf{r}, t) = |\psi_j(\mathbf{r}, t)|^2 = \int F_j(\mathbf{r}, \mathbf{k}, t) \frac{d\mathbf{k}}{(2\pi)^3}. \tag{6}$$

In (4) we have also made use of the following quantities:

$$F_{j\pm} \equiv F_j(\mathbf{k} \pm \mathbf{q}/2). \tag{7}$$

By using the definition of the particle density ρ_j, we can rewrite the Poisson equation (2) as

$$\nabla^2 V = \rho_{\rm e} - \rho_{\rm i}. \tag{8}$$

The wave kinetic equation (3) and the new form of the Poisson's equation will be used as our basic set of equations for studying the eigen-oscillations of a quantum pair plasma.

3. Perturbative analysis

We first assume an equilibrium state of our quantum plasmas, where we have $V = 0$, and the quasi-probability functions have values F_{j0}, such that

$$\rho_{j0}(\mathbf{r}, t) = \int F_{j0}(\mathbf{r}, \mathbf{k}, t) \frac{d\mathbf{k}}{(2\pi)^3}, \tag{9}$$

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with $\rho_{e0} = \rho_{i0}$. We then assume a sinusoidal perturbation such that

$$V(\mathbf{r},t) = \tilde{V}\exp(i\mathbf{q}\cdot\mathbf{r} - i\Omega t),$$

$$F_j(\mathbf{r},\mathbf{k},t) = F_{j0} + \tilde{F}_j\exp(i\mathbf{q}\cdot\mathbf{r} - i\Omega t), \tag{10}$$

$$\rho_j(\mathbf{r},t) = \rho_{j0} + \tilde{\rho}_j\exp(i\mathbf{q}\cdot\mathbf{r} - i\Omega t).$$

From (8) we conclude that $\tilde{V} = (\tilde{\rho}_i - \tilde{\rho}_e)/q^2$. After linearization of (4), we then obtain

$$(\Omega - A_j\mathbf{k}\cdot\mathbf{q})\tilde{F}_j = -\tilde{V}_j(F_{j0-} - F_{j0+}). \tag{11}$$

Using $\mathbf{v}_j = A_j\mathbf{k}$ and replacing $\tilde{V}_j$, we have

$$\tilde{F}_j = -\frac{s_j}{q^2}\,\frac{(F_{j0-} - F_{j0+})}{(\Omega - \mathbf{v}_j\cdot\mathbf{q} - \rho_{j0}^{\alpha/2})}(\tilde{\rho}_i - \tilde{\rho}_e). \tag{12}$$

After integration over the momentum space $\mathbf{k}$, we obtain

$$q^2\tilde{\rho}_j = -s_j(\tilde{\rho}_i - \tilde{\rho}_e)I_j(\Omega,\mathbf{q}), \tag{13}$$

where we have denoted the integral

$$I_j(\Omega,\mathbf{q}) = \int \frac{(F_{j0-} - F_{j0+})}{(\Omega - \mathbf{v}_j\cdot\mathbf{q} - \rho_{j0}^{\alpha/2})}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{14}$$

From these equations we can derive the dispersion relation for electrostatic oscillations in a quantum pair plasma.

Let us first rewrite (13) in a matrix form as

$$\begin{bmatrix}(q^2 - I_e) & I_e\\ I_i & (q^2 - I_i)\end{bmatrix}\begin{bmatrix}\tilde{\rho}_e\\ \tilde{\rho}_i\end{bmatrix} = 0. \tag{15}$$

The existence of a non-trivial solution for the perturbations $\tilde{\rho}_j$ implies that the determinant of their coefficients has to be zero, which means that

$$(q^2 - I_e)(q^2 - I_i) - I_eI_i = 0. \tag{16}$$

This is the dispersion relation for the eigen-oscillations in the quantum pair plasma. Its explicit contents will be discussed below.

4. The dispersion relation

In order to simplify our discussion, let us first consider that the mass of the positively charged particles tends (the ions) to infinity $M_i \to \infty$, and therefore their oscillations in the electrostatic field can be neglected, $\tilde{\rho}_i = 0$. In this case, only the electrons will oscillate, and the ions only provide a neutralizing positive background. Equation (16) is now reduced to $q^2 = I_e$, or, more explicitly (after dropping the subscript 'e' for simplicity),

$$q^2 = \int \frac{(F_0(\mathbf{k} - \mathbf{q}/2) - F_0(\mathbf{k} + \mathbf{q}/2)}{(\Omega - \mathbf{v}\cdot\mathbf{q} - \rho_0^{\alpha/2})}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{17}$$

This can also be written as

$$q^2 = \int F_0(\mathbf{k})\left[\frac{1}{(\Omega' - \mathbf{v}_+\cdot\mathbf{q})} - \frac{1}{(\Omega' - \mathbf{v}_-\cdot\mathbf{q})}\right]\frac{d\mathbf{k}}{(2\pi)^3}, \tag{18}$$

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where we have used $\Omega' = \Omega - \rho_0^{\alpha/2}$ and defined

$$\mathbf{v}_{\pm}(\mathbf{k}) \equiv \mathbf{v}(\mathbf{k} \pm \mathbf{q}/2) = A_{\mathrm{e}}(\mathbf{k} \pm \mathbf{q}/2). \tag{19}$$

In order to illustrate the meaning of such a dispersion relation, let us consider the simple but physically relevant case of a mono-energetic beam of electrons moving across the medium with velocity $\mathbf{v}_0 = A_{\mathrm{e}}\mathbf{k}_0$. We can describe this electron population by using the following equilibrium quasi-distribution:

$$F_0(\mathbf{k}) = (2\pi)^3 F_0 \delta(\mathbf{k} - \mathbf{k}_0). \tag{20}$$

The integral in (18) is easily solved and the dispersion relation for the electron oscillations takes the form

$$q^2(\Omega' - \mathbf{v}_{0+} \cdot \mathbf{q})(\Omega' - \mathbf{v}_{0-} \cdot \mathbf{q}) = F_0 \mathbf{q} \cdot (\mathbf{v}_{0+} - \mathbf{v}_{0-}). \tag{21}$$

For the electrons at rest, $\mathbf{k}_0 \to 0$, this reduced to

$$q^2 \left(\Omega'^2 - \frac{A_{\mathrm{e}}^2}{4} q^4 \right) - F_0 A_{\mathrm{e}} q^2 = 0, \tag{22}$$

which has a non-trivial solution

$$\Omega'^2 = F_0 A_{\mathrm{e}} + \tfrac{1}{4} A_{\mathrm{e}}^2 q^4. \tag{23}$$

Noting that for the distribution (20) we have $\rho_0 = F_0$, we can finally write

$$(\Omega - \rho_0^{\alpha/2})^2 = \rho A_{\mathrm{e}} + \tfrac{1}{4} A_{\mathrm{e}}^2 q^4, \tag{24}$$

which has to be compared with the classical result

$$\Omega^2 = \omega_{\mathrm{pe}}^2 (1 + 3q^2 \lambda_{\mathrm{D}}^2). \tag{25}$$

Equation (24) is only valid in the zero temperature limit, which would correspond to $\lambda_{\mathrm{De}} \to 0$, where the thermal dispersion is not present. We can then identify (using the appropriate units) the electron plasma frequency as $\omega_{\mathrm{pe}} = \sqrt{\rho_0 A_{\mathrm{e}}}$. This means that two purely quantum corrections can be identified in (24): first, the frequency shift $\rho_0^{\alpha/2}$ appearing in the left-hand side; second, the q^4 dispersion term associated with the spread of the wave packet. Quantum dispersion terms of this kind have already been obtained in [5].

Instead of an electron–ion plasma (with infinitely heavy ions), let us now consider the opposite case of an electron–positron quantum plasma, such that $A_{\mathrm{e}} = A_{\mathrm{i}} = A$. The dispersion relation for a cold pair plasma can be established along the same lines. We can use $F_{j0} = (2\pi)^3 \rho_0 \delta(\mathbf{k})$, where the quasi-neutrality implies that $\rho_{\mathrm{e}0} = \rho_{\mathrm{i}0} = \rho_0$, and from (14) we have $I_{\mathrm{e}} = I_{\mathrm{i}} = I$, with

$$I(\Omega, q) = \frac{\rho_0 A}{2} \frac{q^2}{(\Omega'^2 - A^2 q^4/4)}. \tag{26}$$

The cold quantum pair plasma oscillations will be determined by the dispersion relation $q^2 = 2I(\Omega, q)$. This is similar to (24), but with an additional factor of two multiplying the first term on the right-hand side. The effective plasma frequency will be increased by a factor of $\sqrt{2}$, as expected.

Finally, for more general conditions of electron–ion quantum plasmas, with arbitrary drift velocities $\mathbf{v}_{j0} = A_j \mathbf{k}_{j0}$, the dispersion relation will be given by the

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general expression (16), with the integrals I_j determined from

$$I_j(\Omega,\mathbf{q}) = \rho_{j0}\,\frac{\mathbf{q}\cdot(\mathbf{v}_{j0+}-\mathbf{v}_{j0-})}{(\Omega'-\mathbf{q}\cdot\mathbf{v}_{j0+})(\Omega'-\mathbf{q}\cdot\mathbf{v}_{j0-})}, \tag{27}$$

and $\mathbf{v}_{j0\pm} = \mathbf{v}_{j0} \pm A_j\mathbf{q}/2$.

5. Kinetic effects

Let us now introduce finite-temperature and other kinetic effects. As in the previous section, we start by considering the case of immobile ions. We go back to (18) and make the distinction between the parallel and perpendicular motions with respect to the wavevector $\mathbf{q}$, by using the decomposition

$$\mathbf{k} = p\mathbf{e}_q + \mathbf{k}_\perp, \quad \mathbf{v} = u\mathbf{e}_q + \mathbf{v}_\perp, \tag{28}$$

with the unit vector $\mathbf{e}_q = \mathbf{q}/q$, and the parallel electron velocity $u = A_e p$. A straightforward analysis then leads to the following kinetic dispersion relation:

$$q^2 = \frac{1}{qA_e}\int \frac{[G_0(p-q/2)-G_0(p+q/2)]}{(p-p_0)}\,\frac{dp}{2\pi}, \tag{29}$$

where $p_0 = \Omega'/qA_e$ is the resonant value associated with the parallel electron motion, such that its parallel velocity equals the wave phase speed $u_0 = A_e p_0 = \Omega'/q$. Note that, in the definition of this resonance condition, the quantum frequency shift $\rho_0^{\alpha/2}$ has been retained. In (29) we have also used the parallel quasi-distribution $G_0(p)$, defined by the integral of $F_0(\mathbf{k})$ over the perpendicular velocities

$$G_0(p) = \int F_0(p,\mathbf{k}_\perp)\frac{d\mathbf{k}_\perp}{(2\pi)^2}. \tag{30}$$

In order to evaluate the integral in the dispersion relation (29), we follow the usual plasma kinetics procedure, by first considering the principal part of the integral, and then adding the contribution from the pole $p = p_0$. Define

$$\langle p^n\rangle = \mathbf{P}\int G_0(p)p^n\,\frac{dp}{2\pi} \tag{31}$$

for n an integer, where $\mathbf{P}$ denotes the principal part. Noting that $\langle p\rangle = 0$ if $G_0(p)$ is even, and assuming that the main contribution for this principal part comes from the regions of low parallel momentum, $p < p_0$, we obtain

$$\mathbf{P}\int \frac{G_0(p)}{(p-p_\pm)}\frac{dp}{2\pi} \simeq -\frac{1}{p_\pm}\left(1+\frac{1}{p_\pm^2}\langle p^2\rangle\right), \tag{32}$$

with $p_\pm = p_0 \pm q/2$. We can then write the dispersion relation for our quantum plasmas in the general form

$$\epsilon_r(\Omega,q) + i\epsilon_i(\Omega,q) = 0, \tag{33}$$

where the real part of the electron susceptibility is determined by the above principal part of the integral, obtaining

$$\epsilon_r(\Omega,q) = q^2 - \frac{\rho_0}{A_e}\left[\frac{1}{(p_0^2-q^2/4)} + \frac{2p_0\langle p^2\rangle}{(p_0^2-q^2/4)^2}\right], \tag{34}$$

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and the imaginary part of the susceptibility contains the resonant contributions, and reads

$$\epsilon_{\rm i} = -\frac{\pi}{4qA_{\rm e}}[G_0(p_0 - q/2) - G_0(p_0 + q/2)]. \tag{35}$$

If we split the wave frequency into its real and imaginary parts, according to $\Omega = \Omega_{\rm r} + i\gamma$, then in the limit $|\gamma| \ll \Omega_{\rm r}$ the real part will be determined by $\epsilon_{\rm r}(\Omega_{\rm r}, q) = 0$, and the wave damping follows from

$$\gamma = -\epsilon_{\rm i}(\Omega_{\rm r}, q)/(\partial\epsilon_{\rm r}/\partial\Omega)_{\rm r}. \tag{36}$$

The explicit expression for the real dispersion relation can be written as

$$\left(\Omega'^2 - q^4\frac{A_{\rm e}^2}{4}\right) - \rho_0 A_{\rm e} q^2 - \frac{2\rho_0 p_{\rm th}^2 A_{\rm e}^2 \Omega'}{(\Omega'^2 - q^4 A_{\rm e}^2/4)} = 0, \tag{37}$$

where we have used $\langle p^2\rangle \equiv p_{\rm th}^2$. Comparing this new dispersion relation with the cold plasma result of the previous section, we see the appearance of an additional term, depending on the temperature through the quantity $p_{\rm th}^2$. This is equivalent, in the present quantum description, to the well-known classical term $3q^2\lambda_{\rm D}^2$ of (25).

On the other hand, by using (36) we can obtain the following approximate expression for the kinetic (or Landau) damping of waves by nearly resonant electrons in the quantum description:

$$\gamma = -\frac{\pi}{8}\frac{\rho_0^{1/2}}{qA_{\rm e}^{1/2}}[G_0(p_0 - q/2) - G_0(p_0 + q/2)]. \tag{38}$$

For a thermal distribution, this will always lead to a negative value of γ, which means that the wave is damped in the medium. However, for a non-equilibrium distribution of electrons, for instance in the presence of electron beams, we will eventually have an inversion of population leading to $\gamma > 0$, which means instability of plasma waves.

It is interesting to consider this expression in the quasi-classical limit, such that $q \ll p_0$, when the parallel momentum of the particles is much larger than the wave momentum. In this case, we can expand in (38) the functions $G_0(p_0 \pm q/2)$ around the resonant momentum $p = p_0$, and obtain

$$\gamma = \frac{\pi}{2^4}\left(\frac{\rho_0}{A_{\rm e}}\right)^{1/2}\left(\frac{dG_0}{dp}\right)_{p=p_0}. \tag{39}$$

This approximate result, with the derivative of the dispersion relation at the resonant momentum, is familiar to plasma physicists. It is clear from here that it can only be used in the quasi-classical limit. Inversion of population corresponds here to a positive derivative of the parallel distribution at the resonance $p = p_0 = \Omega'/qA_{\rm e}$. It is interesting to note that, even in this limit, the quantum frequency shift of order $\rho^{\alpha/2}$ still persists in the definition of p_0.

Extension to the case of a pair plasma can easily be made. In the dispersion relation (37) we replace ρ_0 by $2\rho_0$, except in the definition of Ω' which remains unchanged. In the wave damping, we now have to include Landau damping contributions due to both species, and write

$$\gamma = \sum_j \gamma_j, \tag{40}$$

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with

$$\gamma_j = -\frac{\pi}{8}\frac{(2\rho_0)^{1/2}}{qA_j^{1/2}}[G_{j0}(p_0 - q/2) - G_{j0}(p_0 + q/2)]. \tag{41}$$

6. Conclusions

In this paper, we have used a wave kinetic description to study the properties of electrostatic waves in quantum pair plasmas. A general dispersion for electrostatic waves in a two-component quantum plasma has been derived. The particular case of an electron–positron plasma has been analysed in detail, but the present results are also relevant to an electron–ion or an electron–hole plasma. Explicit expressions for the wave dispersion relation and damping or growth rates have been presented. The cold plasma limit was considered, and the kinetic effects were also taken into account. Exact expressions for the Landau wave damping have been derived, and the quasi-classical limit has been briefly discussed.

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