Combinatorics, Probability and Computing (2003) 12, 359–364. © 2003 Cambridge University Press
DOI: 10.1017/S0963548303005674 Printed in the United Kingdom

Schütte's Tournament Problem and Intersecting Families of Sets

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Received 12 May 2002; revised 18 November 2002

In this paper we consider a bipartite version of Schütte's well-known *tournament problem*. A bipartite tournament $T = (A, B, E)$ with teams A and B, and set of arcs E, has the property $S_{k,l}$ if for any subsets $K \subseteq A$ and $L \subseteq B$, with $|K| = k$ and $|L| = l$, there exist *conquerors* of K and L in opposite teams. The task is to estimate, for fixed k and l, the minimum number $f(k,l) = |A| + |B|$ of players in a tournament satisfying property $S_{k,l}$. We achieve this goal by reformulating the problem in terms of *intersecting set families* and applying probabilistic as well as constructive methods. Intriguing connections with some famous problems of this area have emerged in this way, leading to new open questions.

1. Introduction

In this paper we consider some variations on the following well-known problem of Schütte (see [2]). A tournament $T = (V, E)$ has property S_k ($k \geqslant 1$) if, for any subset $K \subseteq V$ of size k, there exists a player $v \in V \setminus K$ that beats all players of K. Such a v is called a *conqueror* of K. The problem is to determine, for any given $k \geqslant 1$, the smallest possible number $f(k)$ of players in a tournament satisfying S_k. The existence of $f(k)$ for every $k \geqslant 1$ is not at all obvious. However, it was proved by Erdős [3], applying a simple probabilistic

† Research supported in part by the Hungarian Scientific Research Fund under grant OTKA T-32969.

https://doi.org/10.1017/S0963548303005674 Published online by Cambridge University Press

argument, that

$$f(k) \leqslant k^2 2^k (\ln 2)(1 + o(1)).$$

This was one of the first and the most transparent examples of using the probabilistic method in a combinatorial problem (*cf.* [2]).

Since Erdős's proof is probabilistic, the question of an explicit construction of a tournament satisfying S_k arises naturally. One such example, based on finite fields, was found by Graham and Spencer [7]. Let q be a prime power of the form $4m + 3$ and let $\mathrm{GF}(q)$ denote the field of order q. The *Paley tournament* T_q is defined on $\mathrm{GF}(q)$ by the following rule: (x, y) is an arc of T_q whenever $x - y$ is a nonzero square in $\mathrm{GF}(q)$. Then, as was proved in [7], T_q satisfies S_k, provided $q > k^2 2^{2k-2}$. The argument uses sharp estimates for character sums implied by Weil's famous theorem, known as the Riemann hypothesis for curves over finite fields (see [1]).

On the other hand, it was shown by Szekeres and Szekeres [11] that

$$f(k) \geqslant 2^{k-1}(k + 2) - 1,$$

for all $k \geqslant 1$. The exact values of this function are known only for $k = 1, 2, 3$: namely we have $f(1) = 3$, $f(2) = 7$, and $f(3) = 19$. It seems that determining precise estimates of $f(k)$ is as hard as for the classical Ramsey numbers $R(k)$ (though the lower and upper bounds on the latter are much farther from each other).

Our aim in this paper is to investigate the following *bipartite version* of Schütte's problem. Let $T = (A, B, E)$ be a bipartite tournament with the set of arcs E between teams A and B of sizes a and b, respectively. A vertex v dominating all members of some set X of vertices is called a *conqueror* of X. A bipartite tournament $T = (A, B, E)$ satisfies property $S_{k,l}$ if, for any subsets $K \subseteq A$ and $L \subseteq B$, with $|K| = k$ and $|L| = l$, there exist conquerors of K and L in opposite teams. In analogy to the original problem of Schütte's, we ask for the minimum order $f(k, l) = a + b$ of a bipartite tournament with property $S_{k,l}$. The existence of such tournaments for all pairs (k, l) can be established along the same lines as in the probabilistic proof of Erdős. This gives an upper bound on the function $f(k, l)$. However, we also present an explicit construction, based on finite projective spaces, which provides much better bounds if one of the parameters is significantly smaller than the other.

Perhaps the most interesting feature of this topic lies in its connections with some famous hypergraph problems. One of them is the Erdős–Hanani conjecture [4] on covering and packing numbers, confirmed by the probabilistic proof of Rödl [10] (see [2], [6]). Another related problem comes from Erdős's famous 'list of three' and concerns transversal numbers of r-uniform intersecting hypergraphs. It was posed in a seminal paper [5] by Erdős and Lovász. The question was to decide whether $n(r) = O(r)$, where $n(r)$ is the minimum number of edges in such a hypergraph $\mathscr{H}$, which additionally satisfies $\tau(\mathscr{H}) = r$. The unexpected positive answer was given by Kahn in [8] (see [6], [9]).

A relationship with the bipartite version of Schütte's problem is obtained as follows. Let $X = \{1, 2, \ldots, n\}$ be a finite set and let $\mathscr{F} = \{F_1, F_2, \ldots, F_m\}$ denote a family of subsets of X. The family $\mathscr{F}$ is *k-wise intersecting* if any k sets from $\mathscr{F}$ have a nonempty intersection.

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Let $\tau(\mathscr{F})$ denote the *transversal number* of $\mathscr{F}$, that is, the minimum size of a subset P of X, which has a nonempty intersection with every member of $\mathscr{F}$.

Now, consider a bipartite tournament $T = (\mathscr{F}, X, E)$, with teams $\mathscr{F}$ and X, and the set of arcs E defined as follows: a vertex $x \in X$ dominates a vertex $F_i \in \mathscr{F}$ if and only if $x \in F_i$. It is easy to see that property $S_{k,l}$ means simply that $\mathscr{F}$ is k-wise intersecting and, at the same time, $\tau(\mathscr{F}) > l$. Also conversely, from each tournament with property $S_{k,l}$, the corresponding family $\mathscr{F}$ can be derived. Thus, $f(k,l)$ is the minimum number $a+b$ such that we can pick a subsets of a b-element set, constituting a k-wise intersecting family with transversal number at least $l+1$.

Example 1. *Let* $X = \{1,2,3,4,5,6\}$ *and let* $\mathscr{F}$ *consist of the subsets* $\{1,2,6\}$, $\{1,3,4\}$, $\{1,5,6\}$, $\{2,3,5\}$, $\{2,4,5\}$ *and* $\{3,4,6\}$. *Since* $\mathscr{F}$ *is 2-wise intersecting and* $\tau(\mathscr{F}) = 3$, *the associated bipartite tournament* $(\mathscr{F}, X, E)$ *has property* $S_{2,2}$. *This shows that* $f(2,2) = 12$, *since, as is easily seen,* $f(2,2)$ *cannot be smaller than* 12.

A slight change in the definition of $(\mathscr{F}, X, E)$ leads to the following dual setting. Let $(x, F_i) \in E$ if and only if $x \notin F_i$. This results in reversing directions of arcs in the previously defined tournament. Now, $S_{k,l}$ is equivalent to the following two properties of $\mathscr{F}$: it is an *l-covering* of X (that is, every l-element subset of X is contained in some member of $\mathscr{F}$), but no k members of $\mathscr{F}$ cover the entire X.

2. Probabilistic bound on $f(k,l)$

First, we establish the existence of bipartite tournaments satisfying $S_{k,l}$ by applying the probabilistic argument of Erdős. This will provide an upper bound on the function $f(k,l)$ defined in the Introduction.

Theorem 2.1. *If* $\binom{a}{k}(1-2^{-k})^b + \binom{b}{l}(1-2^{-l})^a < 1$, *then there exists a bipartite tournament of order* (a,b) *that has property* $S_{k,l}$.

Proof. Consider a random bipartite tournament with teams A and B of sizes a and b, respectively, where the orientation of each edge is taken with probability $1/2$ in either direction, independently of all the other edges. Let $K \subseteq A$ and $L \subseteq B$ be fixed subsets of sizes k and l, respectively. Let A_K denote the event that there is no conqueror of K in the team B, and, similarly, let B_L be the event that no vertex from A is a conqueror of L. Clearly, we have $\Pr(A_K) = (1-2^{-k})^b$ and $\Pr(B_L) = (1-2^{-l})^a$. Hence

$$\Pr\left(\bigcup_{\substack{K\subset A, L\subset B\\ |K|=k, |L|=l}} (A_K \cup B_L)\right) \leqslant \sum_{\substack{K\subset A, L\subset B\\ |K|=k, |L|=l}} (\Pr(A_K) + \Pr(B_L))$$
$$= \binom{a}{k}(1-2^{-k})^b + \binom{b}{l}(1-2^{-l})^a < 1.$$

This completes the proof. □

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For instance, assume in Theorem 2.1 that $k = l$ and $a = b$. Then, applying standard inequalities $\binom{a}{k} < (\frac{ea}{k})^k$ and $1 - x < e^{-x}$, for $x > 0$, leads to the estimate $f(k,k) \leqslant 2k^2 2^k \ln 2(1 + o(1))$. However, the same asymptotic bound in this special case may be deduced directly from the existence of tournaments satisfying S_k.

Theorem 2.2. *Let $T = (V, E)$ be a tournament of order n satisfying S_k. Then there exists a bipartite tournament of order $2n$ that has property $S_{k,k}$. In consequence, assuming $k \geqslant l$, we have*

$$f(k,l) \leqslant 2f(k) \leqslant k^2 2^{k+1} (\ln 2)(1 + o(1)).$$

Proof. Denote the set of players of T by $V = \{1, 2, \ldots, n\}$, and consider a bipartite tournament with bipartition $V \cup V'$, where $V' = \{1', 2', \ldots, n'\}$, with the set of arcs D defined as follows. If $i \neq j$ and $(i,j) \in E$, then both (i, j') and (i', j) are in D. If $i = j$ then choose any of the two possible orientations. Clearly, such a bipartite tournament satisfies $S_{k,k}$, provided the initial tournament T had property S_k. □

3. The case of small *l*

Below, we consider the case where l is relatively small with respect to k.

3.1. Upper bound

The constructive upper bound given below beats the probabilistic one significantly when $l = o(\frac{k}{\ln k})$.

Theorem 3.1. *If k is a prime power, then*

$$f(k,l) \leqslant 2k^l + O(k^{l-1}).$$

Proof. This is shown by the hypergraph $(X, \mathscr{F})$, where

$$|X| = |\mathscr{F}| = \sum_{i=0}^{l} k^i,$$

and $\mathscr{F}$ is the collection of hyperplanes in the l-dimensional projective space $PG(k,l)$ of order k over the set X of points. Since each point is incident to precisely $\sum_{i=0}^{l-1} k^i$ hyperplanes only, the transversal number of $(X, \mathscr{F})$ is greater than k. Moreover, it is well known that any l hyperplanes have a non-empty intersection. Thus, the bipartite tournament associated with $(X, \mathscr{F})$ satisfies property $S_{k,l}$. □

3.2. Lower bound

The following lower bound shows that the construction applying projective spaces is not very far from optimal if l is significantly smaller than k. In particular, if l is fixed and k gets large, then the quotient of the two bounds is just polylogarithmic in k.

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Theorem 3.2. *For any $k, l \geqslant 1$, with $k/l > e^2$,*

$$f(k,l) > k\left(\frac{k}{l \ln \frac{k}{l}}\right)^{l-1}.$$

Proof. Suppose that $T = (A, B, E)$ is a bipartite tournament such that $|A| + |B| = f(k,l)$, and that property $S_{k,l}$ holds for T. We first select a sequence of distinct vertices $b_1, b_2, \ldots, b_s \in B$ and subsets $A \supset A_1 \supset A_2 \supset \cdots \supset A_s$ as follows. Set $A_0 = A$, and let $p = \frac{l \ln \frac{k}{l}}{k}$. If A_j has been defined, and some vertex of B dominates all but at most $p|A_j|$ vertices of A_j, then choose such a vertex as b_{j+1}, and let A_{j+1} be the set of those vertices in A_j which dominate b_{j+1}. If no such vertex exists, or $j = l-1$, then we stop with $s = j$. Suppose that we could successfully define A_{l-1}. Then $|A_{l-1}| > k$, for otherwise some vertex b_l would dominate the entire A_{l-1} (by property $S_{k,l}$), and then the l-tuple $\{b_1, \ldots, b_l\}$ would not have a conqueror, a contradiction to $S_{k,l}$. Thus, in this situation,

$$|A| \geqslant \frac{|A_{l-1}|}{p^{l-1}} > k\left(\frac{k}{l \ln \frac{k}{l}}\right)^{l-1}$$

and the assertion is proved. Hence, suppose that the selection procedure terminates with $s < l-1$. It means that every vertex of the set

$$B_0 = B \setminus \{b_1, \ldots, b_s\}$$

is dominated by at least $p|A_s|$ vertices of A_s. Therefore, the average number of vertices in B_0 dominated by a vertex of A_s is at least $p|B_0|$. Let $a_1 \in A_s$ be a vertex dominating the largest number of vertices in B_0, and define $B_1 \subset B_0$ as the set of vertices not dominated by a_1. By the choice of a_1, we have $|B_1| \leqslant (1-p)|B_0|$. Since each vertex of B_1 is dominated by at least $p|A_s|$ vertices of A_s, and none of them is dominated by a_1, we can again select an $a_2 \in A_s$ dominating all but at most $(1-p)|B_1|$ vertices of B_1. The non-dominated subset of B_1, denoted by B_2, satisfies $|B_2| \leqslant (1-p)|B_1|$. In general, having b_j at hand for some $j < k-1$, we can select an $a_{j+1} \in A_s$ such that the set $B_{j+1} \subset B_j$ non-dominated by $a_1, \ldots, a_{j+1}$ has cardinality $|B_{j+1}| \leqslant (1-p)|B_j|$. The property $S_{k,l}$ implies that B_{k-1} is non-empty and has more than l vertices. Thus, applying $e^{-p} > 1-p$ for $p > 0$, we obtain

$$|B| \geqslant \frac{|B_{k-1}|}{(1-p)^{k-1}} > le^{p(k-1)} = l\left(\frac{k}{l}\right)^{\frac{l(k-1)}{k}} = k\left(\frac{k}{l}\right)^{l-1}\left(\frac{k}{l}\right)^{-l/k}.$$

In order to show that this is larger than the lower bound claimed, it suffices to observe that $(k/l)^{-l/k}$ always exceeds $e^{-1/e} > \frac{1}{2}$, while

$$\left(\ln \frac{k}{l}\right)^{l-1} > 2$$

holds whenever $l \geqslant 2$ and $k/l > e^2$. Finally, the assertion is obvious for $l = 1$. □

4. Edge colourings of $\binom{X}{k}$

We would like to conclude the paper with another reformulation of the bipartite problem of Schütte. Let $\binom{X}{k}$ denote the k-uniform complete hypergraph on the set X of n

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vertices. Consider edge colourings of $\binom{X}{k}$ such that no l colour classes form a spanning subhypergraph. Assuming $n \geqslant kl + 1$, we may define $\mu(n, k, l)$ as the minimum number of colours in such a colouring.

It is not hard to see that

$$f(k, l) = \min_{n \geqslant kl+1} n + \mu(n, k, l).$$

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