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Bounds For The Real Zeros of Chromatic Polynomials

F. M. DONG[1] and K. M. KOH[2†]

[1] Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616
(e-mail: fengming.dong@nie.edu.sg)
[2] Department of Mathematics, National University of Singapore, Singapore 117543

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Sokal in 2001 proved that the complex zeros of the chromatic polynomial $P_G(q)$ of any graph G lie in the disc $|q| < 7.963907\Delta$, where Δ is the maximum degree of G. This result answered a question posed by Brenti, Royle and Wagner in 1994 and hence proved a conjecture proposed by Biggs, Damerell and Sands in 1972. Borgs gave a short proof of Sokal's result. Fernández and Procacci recently improved Sokal's result to $|q| < 6.91\Delta$. In this paper, we shall show that all real zeros of $P_G(q)$ are in the interval $[0, 5.664\Delta)$. For the special case that $\Delta = 3$, all real zeros of $P_G(q)$ are in the interval $[0, 4.765\Delta)$.

1. Introduction

Throughout this paper, a graph is considered to be simple, finite and undirected if no specification is given. Let G be any graph and let $P_G(q)$ be its chromatic polynomial. The study of the real and complex zeros of $P_G(q)$ has attracted many mathematicians since the chromatic polynomial was introduced by Birkhoff [2] in 1912. Many results and conjectures on the real and complex zeros of chromatic polynomials can be found in the book [9] and in the review article [14].

One important class of results concerns bounds on the zeros of $P_G(q)$ in terms of the graph's maximum degree. A few years ago, Sokal [19] proved the following fundamental result, which answers in the affirmative a three-decade-old conjecture of Biggs, Damerell and Sands [1] and Brenti, Royle and Wagner [4].

Theorem 1.1 ([19]). *For any graph G of maximum degree Δ, all zeros (real or complex) of $P_G(q)$ lie in the disc $|q| < 7.963907\Delta$.*

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See also Borgs [3] for a slight simplification of the proof.

Of course, the constant 7.963907 is an artifact of the proof and is presumably far from sharp. Indeed, Fernández and Procacci [11] have recently improved the constant to 6.907652. The linear dependence on Δ is, however, best possible, since the complete graph $K_{\Delta+1}$ has chromatic roots $0, 1, 2, \ldots, \Delta$. Furthermore (and perhaps surprisingly), the complete graph $K_{\Delta+1}$ is *not* the extremal graph for this problem, and a bound $|q| \leqslant \Delta$ is *not* possible. In fact, a non-rigorous (but probably rigorizable) asymptotic analysis, confirmed by numerical calculations [17], shows that the complete bipartite graph $K_{\Delta,\Delta}$ has a chromatic root $\alpha\Delta + o(\Delta)$, where $\alpha = -2/W(-2/e) \approx 0.678345 + 1.447937i$ and W denotes the principal branch of the Lambert W function (the inverse function of $w \mapsto we^w$) [5]. So the constant in Theorem 1.1 presumably cannot be better than $|\alpha| \approx 1.598960$.

The real roots of chromatic polynomials are much better understood than the complex roots [9, 14]. For instance, it has long been known that chromatic polynomials never have roots in $(-\infty, 0)$ or $(0, 1)$, and Jackson [13] proved over a decade ago that they have no roots in $(1, 32/27]$. It is therefore quite astonishing that the only currently available *upper* bound on the real chromatic roots in terms of maximum degree is Theorem 1.1 itself (or its improvement by Fernández and Procacci)! Nevertheless, Sokal has made the following appealing conjecture.

Conjecture 1.2 ([8, 20]). *For any graph G of maximum degree Δ, we have $P_G(q) > 0$ whenever $q > \Delta$.*

(The example of the complete graph $K_{\Delta+1}$ shows that this bound, if true, is best possible.) At present we are very far from having a proof of Conjecture 1.2, though the authors [8] have proved it for graphs satisfying $\Delta(G) \geqslant \lfloor n/3 \rfloor - 1$, where $n = |V(G)|$ is the number of vertices. It is worth noting that an extension of Conjecture 1.2 to complex roots could conceivably be true.

Conjecture 1.3 ([20]). *For any graph G of maximum degree Δ, we have $P_G(q) \neq 0$ whenever $\operatorname{Re} q > \Delta$.*

The purpose of the present paper is to make some small progress towards proving Conjecture 1.2. We shall prove the following.

Theorem 1.4. *For any graph G of maximum degree Δ, we have $P_G(q) > 0$ whenever $q \geqslant 5.664\Delta$.*

Our method of proof will be a variant on the polymer expansion employed by Sokal [19] in proving Theorem 1.1. We will show that when q is real, his inequalities can be sharpened.

The plan of this paper is as follows. In Section 2 we recall the polymer expansion of the multivariate Tutte polynomial, which includes the chromatic polynomial as a special case. In Section 3 we recall a recent result on the complex zeros of the independent-set polynomial (or polymer-model partition function) by Fernández and Procacci [10], and we extend it to a bound on the real zeros. This extended bound will be our main tool. In Section 4 we apply this bound to prove Theorem 1.4.

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2. Potts model, multivariate Tutte polynomial, and polymer representation

Let $G = (V, E)$ be a graph where each edge e is assigned a real or complex weight v_e. Let $\mathbf{v} = \{v_e\}_{e \in E}$. For any positive integer q, define

$$Z_G^{\text{Potts}}(q, \mathbf{v}) = \sum_{\sigma} \prod_{e \in E} [1 + v_e \delta(\sigma_{x_1(e)}, \sigma_{x_2(e)})], \tag{2.1}$$

where the sum runs over all maps $\sigma : V \to \{1, 2, \ldots, q\}$, $x_1(e)$ and $x_2(e)$ are the two ends of e, and $\delta(q_1, q_2)$ is the function defined by

$$\delta(q_1, q_2) = \begin{cases} 1, & \text{if } q_1 = q_2, \\ 0, & \text{otherwise.} \end{cases}$$

In statistical mechanics, when q is a positive integer, the notion $Z_G^{\text{Potts}}(q, \mathbf{v})$ given in (2.1) is known as the *partition function of the q-state Potts model*. For positive integer q, it can be shown that

$$Z_G^{\text{Potts}}(q, \mathbf{v}) = \sum_{E' \subseteq E} q^{c(E')} \prod_{e \in E'} v_e, \tag{2.2}$$

where $c(E')$ is the number of components of the subgraph (V, E'). We then define $Z_G(q, \mathbf{v})$ for general (non-integer) q by the right-hand side of (2.2), and call it *the multivariate Tutte polynomial* [20]. Thus the Potts-model partition function is simply a specialization to positive integer q of the multivariate Tutte polynomial.

If $v_e = v$ for all $e \in E$, then $Z_G(q, \mathbf{v})$ is known as the *dichromatic polynomial* of G. If $v_e = -1$ for all $e \in E$, then $Z_G(q, \mathbf{v})$ is the *chromatic polynomial* of G.

For any graph $H = (V', E')$, define the connected-spanning-subgraph polynomial

$$C_H(\mathbf{v}) = \sum_{\substack{A \subseteq E' \\ (V', A) \text{ connected}}} \prod_{e \in A} v_e. \tag{2.3}$$

The multivariate Tutte polynomial $Z_G(q, \mathbf{v})$ can be transformed into a particular *polymer-model partition function*.

Theorem 2.1 ([19]). *Let $G = (V, E)$ be a graph equipped with edge weights $\{v_e\}_{e \in E}$. Then*

$$Z_G(q, \mathbf{v}) = q^{|V|} Z_{\text{polymer},G}(q, \mathbf{v}),$$

where

$$Z_{\text{polymer},G}(q, \mathbf{v}) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{S_1, \ldots, S_N \\ \text{disjoint}}} \prod_{i=1}^{N} w(S_i) \tag{2.4}$$

and

$$w(S) = \begin{cases} q^{-(|S|-1)} C_{G[S]}(\mathbf{v}), & \text{if } |S| \geqslant 2, \\ 0, & \text{if } |S| = 1, \end{cases} \tag{2.5}$$

where $G[S]$ denotes the subgraph of G induced by S and the sum in (2.4) *runs over pairwise disjoint subsets $S_1, \ldots, S_N$ of V, and the term $N = 0$ in* (2.4) *is understood to contribute* 1.

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3. Real zero-free region of independent-set polynomials

As stressed by Sokal [19], the polymer representation (2.4) of the multivariate Tutte polynomial is none other than the independent-set polynomial of a special graph $\mathcal{G}$, namely the intersection graph associated to the set V, with fugacities defined by (2.5). It is thus appropriate to begin (as Sokal does) by discussing the (multivariate) independent-set polynomial for an *arbitrary* graph $\mathcal{G}$ (Section 3.1). Then we specialize to the intersection graph with weights (2.5) (Section 3.2).

3.1. Independent-set polynomials

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and let $\mathbf{w} = \{w_x\}_{x\in\mathcal{V}}$ be a collection of real or complex vertex weights. For any subset $\Lambda \subseteq \mathcal{V}$, we define the (multivariate) *independent-set polynomial* of $\mathcal{G}[\Lambda]$ [18] by

$$Z_\Lambda(\mathbf{w}) = \sum_{\substack{I\subseteq\Lambda \\ I\in\mathcal{I}(\mathcal{G})}} \prod_{x\in I} w_x,$$

where $\mathcal{I}(\mathcal{G})$ denotes the family of independent (or stable) sets of vertices in $\mathcal{G}$, and $\prod_{x\in I} w_x = 1$ if $I = \emptyset$. Note, in particular, that $Z_\Lambda(\mathbf{0}) = 1$.

For any $x \in \mathcal{V}$, we denote by $\Gamma(x)$ the set of vertices in $\mathcal{G}$ that are adjacent to x and write $\Gamma^*(x) = \Gamma(x) \cup \{x\}$. More generally, for any $S \subseteq \mathcal{V}$, we write $\Gamma(S) = \bigcup_{x\in S} \Gamma(x)$.

Fernández and Procacci [10] recently proved the following fundamental result concerning the complex zeros of independent-set polynomials, which improves Dobrushin's result [6, 7].

Theorem 3.1 ([10]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex weight $\mathbf{w} = \{w_x\}_{x\in\mathcal{V}}$. If there exist positive constants $\mu = \{\mu_x\}_{x\in\mathcal{V}}$ such that*

$$|w_x| Z_{\Gamma^*(x)}(\mu) \leqslant \mu_x \tag{3.1}$$

holds for all $x \in \mathcal{V}$, then $Z_\Lambda(\mathbf{w}) \neq 0$ for all $\Lambda \subseteq \mathcal{V}$.

It follows immediately from Theorem 3.1 that for *real* $\mathbf{w}$ satisfying (3.1) for all $x \in \mathcal{V}$, we have $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}$, by continuity from $Z_\Lambda(\mathbf{0}) = 1$.

Our main tool in this paper is the following easy lemma that allows extending this latter result – indeed, on a slightly *weakened* hypothesis – to $-R_x \leqslant w_x < +\infty$.

Lemma 3.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with real vertex weight $\mathbf{w} = \{w_x\}_{x\in\mathcal{V}}$ and $R_x \geqslant 0$ for all $x \in \mathcal{V}$. Suppose that $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}$ whenever $-R_x \leqslant w_x \leqslant 0$ for all x. Then $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}$ whenever $-R_x \leqslant w_x < +\infty$ for all x.*

Proof. For $\Lambda \subseteq \mathcal{V}$, decompose $\Lambda = \Lambda_+ \cup \Lambda_-$, where

$$\Lambda_+ = \{x \in \Lambda : \ w_x \geqslant 0\},$$
$$\Lambda_- = \{x \in \Lambda : \ w_x < 0\}.$$

We then have

$$Z_\Lambda(\mathbf{w}) = \sum_{\substack{I\subseteq\Lambda_+ \\ I\in\mathcal{I}(\mathcal{G})}} \left(\prod_{x\in I} w_x\right) Z_{\Lambda_-\setminus\Gamma(I)}(\mathbf{w}).$$

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All quantities $Z_{\Lambda_- \setminus \Gamma(I)}(\mathbf{w})$ are strictly positive by hypothesis – note that restricting to the subset $\Lambda_- \setminus \Gamma(I) \subseteq \Lambda$ is equivalent to setting $w_x = 0$ for $x \in \Lambda \setminus (\Lambda_- \setminus \Gamma(I))$ – and $w_x \geqslant 0$ for all $x \in I$. Therefore, the contribution of $I = \emptyset$ is strictly positive and all other contributions are non-negative. □

By Theorem 3.1 and Lemma 3.2, we immediately obtain the following result.

Lemma 3.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with real vertex weight $\mathbf{w} = \{w_x\}_{x \in \mathcal{V}}$. Let $\mathcal{V}_- = \{x \in \mathcal{V} : w_x < 0\}$. If there exist positive constants $\mu = \{\mu_x\}_{x \in \mathcal{V}_-}$ such that*

$$|w_x| Z_{\Gamma^*(x) \cap \mathcal{V}_-}(\mu) \leqslant \mu_x \tag{3.2}$$

holds for all $x \in \mathcal{V}_-$, then $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}$.

Proof. For any $x \in \mathcal{V}$ and $\Lambda \subseteq \mathcal{V}_-$, as $Z_{\Gamma^*(x) \cap \Lambda}(\mu) \leqslant Z_{\Gamma^*(x) \cap \mathcal{V}_-}(\mu)$, we have

$$|w_x| Z_{\Gamma^*(x) \cap \Lambda}(\mu) \leqslant \mu_x.$$

By Theorem 3.1, we have $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}_-$. Then, by Lemma 3.2, we have $Z_\Lambda(\mathbf{w}) > 0$ for all $\Lambda \subseteq \mathcal{V}$. □

3.2. The intersection graph

The moral of Lemma 3.2 is that in proving $Z_G(\mathbf{w}) \neq 0$ for some set of *real* fugacities $\mathbf{w}$, we need only worry about the induced subgraph on the set of vertices where the fugacity is *negative*, *i.e.*, $\mathcal{V}_- = \{x \in \mathcal{V} : \ w_x < 0\}$: if we can prove $Z_{\mathcal{G}[\mathcal{V}']}(\mathbf{w}) \neq 0$ for all $\mathcal{V}' \subseteq \mathcal{V}_-$, then $Z_\mathcal{G}(\mathbf{w}) > 0$ follows. This simple idea is, in fact, the main new technical tool underlying the results of the present paper.

We will apply the following result, by Fernández and Procacci [10], which gives a sufficient condition for (3.1).

Theorem 3.4 ([10]). *Let $G = (V, E)$ be a graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph defined by*

$$\mathcal{V} = \{S \subseteq V : |S| \geqslant 2\} \quad \text{and} \quad \mathcal{E} = \{S_1 S_2 : S_1, S_2 \in \mathcal{V}, S_1 \cap S_2 \neq \emptyset\}$$

with vertex weight $w(S)$. Let $\mathcal{V}' \subseteq \mathcal{V}$. Then $Z_\Lambda(\mathbf{w}) \neq 0$ for all $\Lambda \subseteq \mathcal{V}'$ if the following inequality holds for some constant a:

$$\sup_{x \in V} \sum_{\substack{S \in \mathcal{V}' \\ x \in S}} |w(S)| e^{a|S|} \leqslant e^a - 1. \tag{3.3}$$

This is an improvement over the Kotecký–Preiss [16] criterion used by Sokal [19], in which the $e^a - 1$ on the right-hand side of (3.3) is replaced by a.

Thus, by Lemma 3.3 and Theorem 3.4, we have the following result.

Corollary 3.5. *Let $G = (V, E)$ be a graph and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the graph defined in Theorem 3.4 with real vertex weight $\mathbf{w} = \{w(S)\}$. Then $Z_\mathcal{V}(\mathbf{w}) > 0$ if the following inequality holds for some*

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constant a:

$$\sup_{x\in V}\sum_{\substack{S\in\mathcal{V}_-\\ x\in S}}|w(S)|e^{a|S|}\leqslant e^a-1. \tag{3.4}$$

4. Bounds

Let $G=(V,E)$ be a graph with edge weights $\mathbf{v}=\{v_e\}_{e\in E}$. Define the graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where

$$\mathcal{V}=\{S\subseteq V : S\neq\emptyset\}\quad\text{and}\quad\mathcal{E}=\{S_1S_2 : S_1\cap S_2\neq\emptyset\}$$

with vertex weight $w(S)$ given by (2.5) for each $S\subseteq\mathcal{V}$. Let $\mathbf{w}=\{w(S)\}_{S\in\mathcal{V}}$. By Theorem 2.1,

$$Z_G(q,\mathbf{v})=q^{|V|}Z_{\mathcal{V}}(\mathbf{w}). \tag{4.1}$$

Sokal [19, Remark 2 after Proposition 4.1] (see also [18, Propositions 2.5 and 2.6]) has proved the following general result concerning the connected-spanning-subgraph polynomial of a graph.

Proposition 4.1. *Let $H=(V',E')$ be a graph equipped with real edge weights $\mathbf{v}=\{v_e\}_{e\in E'}$ satisfying $-1\leqslant v_e\leqslant 0$ for all $e\in E(G)$. Then*

$$(-1)^{|V'|-1}C_H(\mathbf{v})\geqslant 0.$$

For our polymer fugacities $w(S)$ defined by (2.5), this gives the following immediate corollary.

Corollary 4.2. *Suppose that q is* real *and that $-1\leqslant v_e\leqslant 0$ for all $e\in E(G)$.*
(a) *If $q<0$, then $w(S)\geqslant 0$ for all S.*
(b) *If $q>0$, then $(-1)^{|S|-1}w(S)\geqslant 0$ for all S.*

The main goal of this paper is to obtain an improved bound on the *positive real* chromatic roots by exploiting Corollary 4.2(b) together with Corollary 3.5. Indeed, Corollary 3.5 tells us that we need only worry about the subsystem consisting of polymers with *negative* fugacity; and Corollary 4.2(b) tells us that these are the polymers with $|S|$ *even*. By discarding all the polymers with $|S|$ odd, we are able to improve on the bounds obtained in [19] and [11].

Theorem 4.3. *Assume that $\mathbf{w}=\{w(S)\}$ is given by (2.5) for each $S\subseteq\mathcal{V}$, where $v_e=-1$ for all $e\in E(G)$. Then $P_G(q)=q^{|V|}Z_{\mathcal{V}}(\mathbf{w})>0$ if the following inequality holds for some constant a:*

$$\sup_{x\in V}\sum_{\substack{x\in S\subseteq V\\ |S|\text{ even}}}|w(S)|e^{a|S|}\leqslant e^a-1. \tag{4.2}$$

Now we need some results to get an upper bound for the left-hand side of (4.2).

Lemma 4.4 ([19]). *Let $H=(V',E')$ be a graph equipped with complex edge weights $\mathbf{v}=\{v_e\}_{e\in E'}$ satisfying $|1+v_e|\leqslant 1$ for all e. Then*

$$|C_H(\mathbf{v})|\leqslant T_H(|\mathbf{v}|)\equiv\sum_{\substack{A\subseteq E'\\ (V',A)\text{ tree}}}\prod_{e\in A}|v_e|.$$

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Lemma 4.5 ([19]). *Let $G = (V, E)$ be a graph of maximum degree Δ, and let $x \in V$. Then the number of n-vertex trees in G containing x is bounded above by*

$$t_n^{(\Delta)} \equiv \Delta \frac{[(\Delta-1)n]!}{(n-1)!\,[(\Delta-2)n+2]!}.$$

Furthermore, for all $\Delta, n \geqslant 1$ we have

$$t_n^{(\Delta)} \leqslant \frac{(\Delta n)^{n-1}}{n!}.$$

By Lemmas 4.4 and 4.5, we have the following result.

Lemma 4.6. *Let $G = (V, E)$ be a graph equipped with complex edge weights $\mathbf{v} = \{v_e\}_{e\in E}$ satisfying $|1 + v_e| \leqslant 1$ for all $e \in E$. Then, for any $x \in V(G)$ and any $n \geqslant 2$,*

$$\sum_{\substack{x\in S\subseteq V\\ |S|=n}} |w(S)| \leqslant q^{1-n} M^{n-1} t_n^{(\Delta)},$$

where $M = \max\{|v_e| : e \in E\}$.

Proof. By Lemmas 4.4 and 4.5,

$$\begin{aligned}
\sum_{\substack{x\in S\subseteq V\\ |S|=n}} w(S) &= \sum_{\substack{x\in S\subseteq V\\ |S|=n}} q^{1-n} |C_{G[S]}(\mathbf{v})| \\
&\leqslant q^{1-n} \sum_{\substack{x\in S\subseteq V\\ |S|=n}} \sum_{\substack{A\subseteq E(G[S])\\ (S,A)\,\text{tree}}} \prod_{e\in A} |v_e| \\
&\leqslant q^{1-n} \sum_{\substack{x\in S\subseteq V\\ |S|=n}} \sum_{\substack{A\subseteq E(G[S])\\ (S,A)\,\text{tree}}} M^{n-1} \\
&= q^{1-n} M^{n-1} \sum_{\substack{x\in S\subseteq V\\ |S|=n}} \sum_{\substack{A\subseteq E(G[S])\\ (S,A)\,\text{tree}}} 1 \\
&\leqslant q^{1-n} M^{n-1} t_n^{(\Delta)}.
\end{aligned}$$

□

By Theorem 4.3 and Lemma 4.6, the next result follows immediately.

Lemma 4.7. *Let $q > 0$ and G be a graph with maximum degree Δ and equipped with real edge weights $\mathbf{v} = \{v_e\}_{e\in E}$. Then $Z_{\text{polymer},G}(q, \mathbf{v}) > 0$ if $v_e < 0$ for all $e \in E$ and there exists $\alpha > 0$ such that*

$$\sum_{k=1}^{\infty} e^{2\alpha k} (q/M)^{1-2k} t_{2k}^{(\Delta)} \leqslant e^{\alpha} - 1, \tag{4.3}$$

where $M = \max\{|v_e| : e \in E\}$. In particular, $P_G(q) > 0$ if there exists $\alpha > 0$ such that (4.3) holds for $M = 1$.

Finally, we need the following result.

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Lemma 4.8. *Let η be the smallest positive real number such that*

$$\inf_{\alpha>0}\left\{1-e^{\alpha}+\sum_{k=1}^{\infty}e^{2\alpha k}\frac{(2k)^{2k-1}}{\eta^{2k-1}(2k)!}\right\}<0. \tag{4.4}$$

Then $\eta \leqslant 5.664$.

Proof. Let $\alpha = 0.452$ and $\eta = 5.664$. Using the software Maple, we can verify that

$$1-e^{\alpha}+\sum_{k=1}^{100}e^{2\alpha k}\frac{(2k)^{2k-1}}{\eta^{2k-1}(2k)!}<-0.000102. \tag{4.5}$$

Note that $(2k)^{2k-1} \leqslant (2k)! \times e^{2k-1}$. Thus

$$\sum_{k=101}^{\infty}e^{2\alpha k}\frac{(2k)^{2k-1}}{\eta^{2k-1}(2k)!}\leqslant\frac{\eta}{e}\sum_{k=101}^{\infty}\left(\frac{e^{2\alpha+2}}{\eta^2}\right)^k=\frac{\eta}{e}\times\frac{x^{101}}{1-x}, \tag{4.6}$$

where $x = \frac{e^{2\alpha+2}}{\eta^2} < 0.57$. It is easy to find that

$$\frac{\eta}{e}\times\frac{x^{101}}{1-x}\leqslant 1.1\times 10^{-24}.$$

Hence the result holds. □

We are now ready to prove our main result.

Theorem 4.9. *Let G be any graph with maximum degree Δ. Then $P_G(q) > 0$ for all real $q \geqslant \eta\Delta$, where η is the number defined in Lemma 4.8. In particular, $P_G(q) > 0$ for all real $q \geqslant 5.664\Delta$.*

Proof. By Lemma 4.5, we have

$$q^{1-n}t_n^{(\Delta)}\leqslant q^{1-n}\frac{(\Delta n)^{n-1}}{n!}=\left(\frac{\Delta}{q}\right)^{n-1}\frac{n^{n-1}}{n!}.$$

Let η be the number defined in Lemma 4.8. Then the result follows from Lemmas 4.7 and 4.8. □

Finally, we consider the cases when $\Delta \leqslant 6$, in which we can improve the result of Theorem 4.9 significantly.

Lemma 4.10. *For any integer $3 \leqslant \Delta \leqslant 10$ and real numbers $\eta \geqslant 4.76$ and $0 < \alpha \leqslant 0.557$,*

$$\sum_{k=101}^{\infty}e^{2\alpha k}\frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}}<0.2\times 10^{3-169/\Delta}. \tag{4.7}$$

Proof. We can first show that, for any integers $3 \leqslant \Delta \leqslant 10$ and $n > 200$,

$$t_n^{(\Delta)}<\frac{\Delta^{n-1}e^{n-1}}{10^{169/\Delta}}. \tag{4.8}$$

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Let

$$f_n^{(\Delta)} = \frac{10^{169/\Delta} t_n^{(\Delta)}}{\Delta^{n-1} e^{n-1}} = \frac{10^{169/\Delta}}{\Delta^{n-1} e^{n-1}} \times \frac{\Delta[(\Delta-1)n]!}{(n-1)!\,[(\Delta-2)n+2]!}.$$

It is easy to verify by Maple that $f_{201}^{(\Delta)} < 1$ for all $3 \leqslant \Delta \leqslant 10$. So the inequality (4.8) holds as $f_{n+1}^{(\Delta)} < f_n^{(\Delta)}$ for any $\Delta \geqslant 3$ and $n \geqslant 2$ as shown below:

$$\begin{aligned}
\frac{f_{n+1}^{(\Delta)}}{f_n^{(\Delta)}} &= \frac{1}{\Delta e n} \frac{[(\Delta-1)(n+1)]!}{[(\Delta-2)(n+1)+2]!} \times \frac{[(\Delta-2)n+2]!}{[(\Delta-1)n]!} \\
&= \frac{(\Delta-1)(n+1)}{\Delta e n} \prod_{i=1}^{n-1} \frac{(\Delta-1)(n+1)-i}{(\Delta-1)n-i+1} \\
&\leqslant \frac{n+1}{en} \left(\frac{(\Delta-2)(n+1)+2}{(\Delta-2)n+2} \right)^{n-1} \\
&< \frac{1}{e} \left(\frac{n+1}{n} \right)^{n} \\
&< 1.
\end{aligned}$$

By (4.8),

$$\sum_{k=101}^{\infty} e^{2\alpha k} \frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}} < \frac{1}{10^{169/\Delta}} \sum_{k=101}^{\infty} \frac{e^{2\alpha k+2k-1}}{\eta^{2k-1}} = \frac{e^{2\alpha+1}}{\eta \times 10^{169/\Delta}} \times \frac{x^{100}}{1-x},$$

where $x = \frac{e^{2\alpha+2}}{\eta^2} < 0.994$ as $\eta \geqslant 4.76$ and $0 < \alpha \leqslant 0.557$. The result thus follows immediately. □

Theorem 4.11. *Let G be any graph with maximum degree Δ.*

(a) *If $\Delta = 3$, then $P_G(q) > 0$ for all real $q \geqslant 4.765\Delta$.*
(b) *If $\Delta = 4$, then $P_G(q) > 0$ for all real $q \geqslant 4.97\Delta$.*
(c) *If $\Delta = 5$, then $P_G(q) > 0$ for all real $q \geqslant 5.0995\Delta$.*
(d) *If $\Delta = 6$, then $P_G(q) > 0$ for all real $q \geqslant 5.189\Delta$.*

Proof. Note that

$$t_n^{(\Delta)} = \frac{\Delta((\Delta-1)n)!}{(n-1)!(n(\Delta-2)+2)!}.$$

Let $q = \Delta\eta$. It can be verified by Maple that for $\Delta = 3$, $\eta \geqslant 4.765$ and $\alpha = 0.557$,

$$1 - e^{\alpha} + \sum_{k=1}^{100} e^{2\alpha k} \frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}} \leqslant -0.00007;$$

for $\Delta = 4$, $\eta \geqslant 4.97$ and $\alpha = 0.53$,

$$1 - e^{\alpha} + \sum_{k=1}^{100} e^{2\alpha k} \frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}} \leqslant -0.00009;$$

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for $\Delta = 5$, $\eta \geqslant 5.0995$ and $\alpha = 0.513$,

$$1 - e^{\alpha} + \sum_{k=1}^{100} e^{2\alpha k} \frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}} \leqslant -0.00001;$$

and for $\Delta = 6$, $\eta \geqslant 5.189$ and $\alpha = 0.5035$,

$$1 - e^{\alpha} + \sum_{k=1}^{100} e^{2\alpha k} \frac{t_{2k}^{(\Delta)}}{(\Delta\eta)^{2k-1}} \leqslant -0.00005.$$

Thus the results hold by Lemmas 4.10 and 4.7. □

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