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Constant slope, entropy, and horseshoes for a map on a tame graph

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Abstract. We study continuous countably (strictly) monotone maps defined on a tame graph, i.e. a special Peano continuum for which the set containing branch points and end points has countable closure. In our investigation we confine ourselves to the countable Markov case. We show a necessary and sufficient condition under which a locally eventually onto, countably Markov map f of a tame graph G is conjugate to a map g of constant slope. In particular, we show that in the case of a Markov map f that corresponds to a recurrent transition matrix, the condition is satisfied for a constant slope $e^{h_{\mathrm{top}}(f)}$, where $h_{\mathrm{top}}(f)$ is the topological entropy of f. Moreover, we show that in our class the topological entropy $h_{\mathrm{top}}(f)$ is achievable through horseshoes of the map f.

Key words: Markov map, tame graph, constant slope, conjugacy, entropy
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1. *Introduction*

In this paper we explore when a *countably (strictly) monotone* map on a tame graph is conjugate to a map of constant slope. We show a necessary and sufficient condition under which a countably Markov and locally eventually onto map f of a tame graph G is conjugate to a constant-slope map g of a metric tame graph. In particular, we show that in the case of a Markov map f that corresponds to a recurrent transition matrix, the condition is satisfied for a constant slope $e^{h_{\text{top}}(f)}$, where $h_{\text{top}}(f)$ is the topological entropy of f. We give also a partial solution to the constant-slope problem when the locally eventually onto hypothesis is weakened to topological mixing. Moreover, we show that in this broader class of maps the topological entropy $h_{\text{top}}(f)$ is achievable through horseshoes of the map f.

Let us consider continuous maps $f : X \to X$, $g : Y \to Y$, and $\phi : X \to Y$, where X, Y are compact Hausdorff spaces, such that

$$\phi \circ f = g \circ \phi. \tag{1}$$

If ϕ is surjective/a homeomorphism, we say that f is *semiconjugate/conjugate* to g via the map ϕ. We call the map ϕ a *semiconjugacy/conjugacy*.

Recall that a *finite graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points, and that a *finite tree* is a finite graph not containing a simple closed curve.

Let G be a finite graph. A continuous map $f : G \to G$ is said to be *piecewise strictly monotone* if there is a finite subset $P \subset G$ such that for each connected component i of $G \setminus P$, $f|_i$ is a homeomorphism of i onto its image.

The classical results on conjugacy of a piecewise-monotone interval map to a map of constant slope come from Parry [**15**] and later on Milnor and Thurston [**10**] (see also [**4**]). A tree version concerning piecewise strictly monotone tree maps was proved by Baillif and de Carvalho in [**3**]. Recently, Alsedà and Misiurewicz [**2**] proved, among other results, its graph version.

THEOREM 1.1. [**2**] *Let $f : G \to G$ be a piecewise-monotone map of a finite graph G with positive topological entropy $h_{\text{top}}(f)$. Then there is a semiconjugacy ϕ between f and a piecewise-monotone graph map $g : G' \to G'$ with constant slope $e^{h_{\text{top}}(f)}$. The map ϕ is a homeomorphism if f is transitive.*

Throughout the paper we will use topological properties of special continua—as a general reference with an extensive exposition concerning Peano continua (i.e. locally connected continua, equivalently described as Hausdorff continuous images of the unit interval)—see [**14**].

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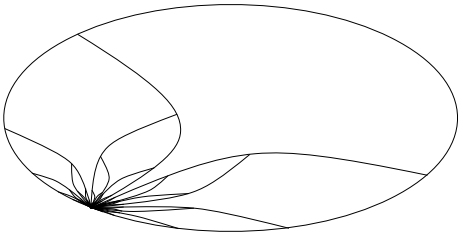


FIGURE 1. A planar tame graph that is a quotient of the Gehman dendrite.

1.1. *Main goal and structure of exposition.* The question we want to address is: *when is a continuous countably Markov and mixing map f of a tame graph G conjugate to a map $g : (G', d) \to (G', d)$ of constant slope $e^{h_{\mathrm{top}}(f)}$?* Our general strategy will follow that of [**5**] with some modification to adapt it to more general tame graphs.

In §2, we define our class of spaces—tame graphs; we generalize the notion of slope to maps between metric tame graphs; we review the basic properties of Markov partitions; and we define our class of maps—countably Markov mixing maps on tame graphs. Section 3 develops our main results, giving conditions for the existence of a conjugate map of constant slope. Section 4 makes a stronger connection between a Markov tame graph map and its symbolic dynamics, allowing us to show that entropy in our class of maps is given by horseshoes. Finally, §5 relates the entropy of the maps we consider to the Lipschitz constants of compatible metrics.

2. *Preliminaries*

2.1. *Tame graphs.* We will consider the class of so-called *tame graphs*, a particular class of 'countable graphs' generalizing finite graphs.

Following [**14**, Definition 9.3], by $\mathrm{ord}(x, X)$ we denote the *order of a point* x in a space X, i.e. the smallest cardinal κ such that x has a neighborhood basis consisting of sets with boundary of cardinality $\leq \kappa$. We denote by $E(X)$, respectively $B(X)$, the set of all points $x \in X$ such that $\mathrm{ord}(x, X) = 1$ (*end point*), respectively $\mathrm{ord}(x, X) \geq 3$ (*branch point*). An arc α in a continuum X is called a *free arc* if its *interior* $\alpha^\circ = \alpha \setminus E(\alpha)$ is open in X.

Definition 2.1. A continuum G will be called a *tame graph* if the set $E(G) \cup B(G)$ has countable closure. A *tame partition* for G is a countable family $\mathcal{P}$ of free arcs with pairwise-disjoint interiors covering G up to a countable set of points. See Figure 1 for an example of a tame graph.

THEOREM 2.2. *The following conditions are equivalent for a continuum G.*

(i) *G is a tame graph.*
(ii) *G is the union of a countable sequence of free arcs and a countable set.*
(iii) *G admits a tame partition.*
(iv) *Each point of G but countably many has a neighborhood that is a finite graph.*

Proof. If G is degenerate or a simple closed curve, the theorem follows easily.

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(i) $\Longrightarrow$ (ii): By [**14**, Theorem 10.4], one obtains that every tame graph is *hereditarily locally connected* (otherwise there would be a so-called *continuum of convergence* and hence uncountable many points of infinite order).

Let F be the countable closure of $E(G) \cup B(G)$. For every point $x \in G \setminus F$, there is a closed and connected neighborhood A of x which does not intersect F. Clearly, every point in A is of order at most two. It follows by [**14**, Proposition 9.5] that A is either an arc or a simple closed curve. Since we assume that G is neither degenerate nor a simple closed curve, A has to be an arc. It follows that x is an element of a free arc. Now, we are able to find countably many free arcs covering $G \setminus F$.

(ii) $\Longrightarrow$ (iii): This follows from the fact that every free arc of G is contained in a maximal free arc or in a free loop (i.e. a simple closed curve $\alpha \subset G$ such that $\alpha \setminus \{x\}$ is open in G for some point $x \in \alpha$). These maximal free arcs and free loops have pairwise-disjoint interiors.

(iii) $\Longrightarrow$ (iv): Clearly, every point in the interior of a free arc has this free arc as a neighborhood.

(iv) $\Longrightarrow$ (i): No point of G that has a finite graph neighborhood can be a limit point of $E(G) \cup B(G)$. □

Remark 2.3. We have seen in the proof that every tame graph is hereditarily locally connected. In particular, it is a one-dimensional Peano continuum. It is an easy exercise to verify that every subcontinuum of a tame graph is a tame graph as well.

Remark 2.4. Every finite graph is a tame graph. Also, being a finite graph is equivalent to any of the following conditions: (i) $E(G) \cup B(G)$ is finite; (ii) G is the finite union of free arcs; (iii) G admits a finite tame partition; (iv) each point of G has a finite graph neighborhood.

2.2. *Slope, arc length, and inner metric.* In this section we consider metric properties of tame graphs and of continuous maps on them. By a *metric tame graph* we mean a pair (G, d), where G is a tame graph and d is a metric compatible with the topology of G.

Definition 2.5. Let $f : (G, d) \to (G', d')$ be a continuous map between metric tame graphs. We say that f has *slope* $\lambda \geq 0$ on an arc $\alpha \subset G$ if:

(i) for every $x \leq y \leq z \in \alpha$, we have $d(x, z) = d(x, y) + d(y, z)$, i.e. α is a *line segment*;

(ii) for every $x, y \in \alpha$, we have $d'(f(x), f(y)) = \lambda\, d(x, y)$.

We say that f has *constant (respectively bounded) slope* λ if all but countably many points $x \in G$ have a free arc neighborhood $\alpha_x \subset G$ such that f has slope λ (respectively $\lambda_x \leq \lambda$) on α_x.

Notation 2.6. Let (G, d) be a metric tame graph.

- Let $f : \alpha \to G$ be a continuous map from an arc α, i.e. a *path* in G. By $\ell(f)$ we denote its *length*, i.e.

$$\sup\left\{\sum_{i<n} d(f(x_i), f(x_{i+1})) : (x_i)_{i\leq n} \text{ a monotone sequence in } \alpha\right\}.$$

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The length may be finite or infinite. Clearly, $\ell(f) = \ell(f \circ h)$ for any homeomorphism $h : \beta \to \alpha$.

- $\mathcal{A}(G)$ denotes the collection of all arcs in G together with all singletons in G. For $\alpha \in \mathcal{A}(G)$, we put $\ell(\alpha) := \ell(\mathrm{id}_\alpha)$ if α is an arc, and $\ell(\alpha) := 0$ if α is a singleton.
- By the *total length* of G we mean $\sum_{i\in\mathcal{P}} \ell(i)$ for any tame partition $\mathcal{P}$. This does not depend on the choice of tame partition and may be equivalently expressed as $\sup_{\mathcal{F}} \sum_{\alpha\in\mathcal{F}} \ell(\alpha)$, where the supremum is taken over all finite sets $\mathcal{F} \subset \mathcal{A}(G)$ whose members intersect at most in end points. Equivalently, the total length of G is its one-dimensional Hausdorff measure.
- $\mathcal{A}_{xy}(G)$ denotes the set of all arcs with end points x and y in G if $x \neq y \in G$, and $\mathcal{A}_{xx}(G) := \{\{x\}\}$.
- For $x, y \in \alpha \in \mathcal{A}(G)$, the symbol $\alpha(xy)$ denotes the subarc of α from x to y or the singleton $\{x = y\}$. Clearly, $\alpha(xy) \in \mathcal{A}_{xy}(G)$.

Definition 2.7. Let (G, ϱ) be a metric tame graph. Recall that ϱ is called an *inner metric* if

$$\varrho(x, y) = \inf\{\ell(\alpha) : \alpha \in \mathcal{A}_{xy}(G)\} \quad \text{for every } x, y \in G.$$

Since our space G is compact, the infimum is in fact the minimum.

Example 2.8. Let S denote a unit circle in $(\mathbb{R}^2, d)$ endowed with the induced inner metric ϱ, i.e. $\varrho(x, y)$ is the length of the shorter of the two arcs in S joining x and y. Note that the length with respect to d is the same as the length with respect to ϱ.

Every arc $\alpha \subset S$ such that $\ell(\alpha) \leq \pi$ is a line segment, i.e. satisfies 2.5(i), and hence id_S has slope 1 on α. It follows that id_S has constant slope 1.

On the other hand, if $\alpha \subset S$ is an arc of length $> \pi$, then α is not a line segment since the distance of its end points is realized via the complementary shorter arc and 2.5(i) does not hold. This shows that the global property 'f has slope λ on α' is strictly stronger than the local property '$f|_\alpha$ has constant slope λ'.

It turns out that the half-length observation from the previous example holds for any free arc in an inner metric space.

LEMMA 2.9. *Let (G, ϱ) be an inner metric space and let $\alpha \subset G$ be a free arc.*

(i) *For every $x \in \alpha$ and $\varepsilon > 0$, there is a subarc $\beta \subset \alpha$ such that β is a neighborhood of x in α and $\ell(\beta) < \varepsilon$. It follows that $\ell(\alpha) < \infty$.*

(ii) *For every subarc $\beta \subset \alpha$ such that $\ell(\beta) \leq \ell(\alpha)/2$, we have $\varrho(x, y) = \ell(\beta(xy))$ for every $x, y \in \beta$, so β is a line segment satisfying Definition 2.5(i).*

Proof. (i) First suppose that x is an end point of α. Let y be the other end point, let $z \in \alpha^\circ$, and let $w \in \alpha(xz)^\circ$ be such that $\varrho(x, w) < \min\{\varepsilon, \ell(\alpha(zy))\}$. Consider any arc $\gamma \in \mathcal{A}_{xw}(G)$. Since α is a free arc, either $\gamma = \alpha(xw)$ or $\gamma \supset \alpha(wy)$. Since ϱ is an inner metric, we have

$$\min\{\ell(\alpha(xw)), \ell(\alpha(wy))\} \leq \varrho(x, w) < \ell(\alpha(zy)) \leq \ell(\alpha(wy)).$$

It follows that $\ell(\alpha(xw)) = \varrho(x, w) < \varepsilon$, so $\alpha(xw)$ is our desired β.

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If x is an inner point of α, we split α into two subarcs meeting at x, use the previous construction on these subarcs with $\varepsilon/2$, and join the resulting half-neighborhoods together.

To obtain the last claim, for every $x \in \alpha$ we have an arc neighborhood β_x of finite length. By compactness, α is covered by finitely many of these and so its length is also finite.

(ii) Let 0, 1 be the end points of α and let p, q be the end points of β such that $0 \le p \le q \le 1$. Again, every arc $\gamma \in \mathcal{A}_{p,q}(G)$ is either β or contains $\alpha(0p) \cup \alpha(q1)$. Since by (i) $\ell(\alpha) < \infty$, we have $\ell(\alpha(0p)) + \ell(\alpha(q1)) \ge \ell(\alpha)/2 \ge \ell(\beta)$, so $\varrho(p, q) = \ell(\beta)$. Clearly, the same holds for every subarc of β and so for every $x \le y \le z$ in β, we have

$$\varrho(x, z) = \ell(\beta(xz)) = \ell(\beta(xy)) + \ell(\beta(yz)) = \varrho(x, y) + \varrho(y, z). \qquad \square$$

PROPOSITION 2.10. *Let $f : (G, d) \to (G', d')$ be a continuous map between metric tame graphs and let $\lambda \ge 0$.*

(i) *If f has constant slope λ, then $\ell(f|_\alpha) = \lambda\, \ell(\alpha)$ for every arc $\alpha \subset G$.*

(ii) *If f has bounded slope λ, then $\ell(f|_\alpha) \le \lambda\, \ell(\alpha)$ for every arc $\alpha \subset G$.*

Moreover, suppose that d is an inner metric.

(iii) *If f has bounded slope λ, then it is λ-Lipschitz, i.e. $d'(f(x), f(y)) \le \lambda\, d(x, y)$ for every $x, y \in G$.*

(iv) *If $\alpha \subset G$ is an arc such that $f|_\alpha$ is monotone and $\ell(f|_\beta) = \lambda\, \ell(\beta)$ for every subarc $\beta \subset \alpha$, then $\lambda < \infty$ and $f|_\alpha$ has constant slope λ.*

Proof. To prove (i) and (ii), observe that there is a tame partition $\mathcal{P}$ of G such that f has slope λ_i on every $i \in \mathcal{P}$. (There is a collection $\mathcal{A}$ of free arcs in G such that $G \setminus \bigcup \mathcal{A}$ is countable and f has slope λ_α on each arc $\alpha \in \mathcal{A}$. It can be reduced to a countable subfamily $\{\alpha_n : n \in \omega\} \subset \mathcal{A}$ such that $\bigcup_{n\in\omega} \alpha_n^\circ = \bigcup_{\alpha\in\mathcal{A}} \alpha^\circ$. To obtain a tame partition, we replace each α_n by the closures of the finitely many components of $\alpha_n \setminus \bigcup_{i<n} \alpha_i$.) It follows that $\ell(f|_j) = \lambda_i\, \ell(j)$ for every arc $j \subset i$ and $i \in \mathcal{P}$.

Let $P := G \setminus \bigcup_{i\in\mathcal{P}} i^\circ$ and $\alpha \subset G$ be an arc. The closures of the components of $\alpha \setminus P$ form a tame partition $\mathcal{P}_\alpha$ of α induced by $\mathcal{P}$. For every $j \in \mathcal{P}_\alpha$, there is a unique $i_j \in \mathcal{P}$ such that $j \subset i_j$. Since the set P is countable, we have $\ell(\alpha) = \sum_{j\in\mathcal{P}_\alpha} \ell(j)$ and $\ell(f|_\alpha) = \sum_{j\in\mathcal{P}_\alpha} \ell(f|_j) = \sum_{j\in\mathcal{P}_\alpha} \lambda_{i_j}\, \ell(j)$. In (i) we have $\lambda_i = \lambda$ and in (ii) we have $\lambda_i \le \lambda$ for every $i \in \mathcal{P}$. That concludes the proof.

To prove (iii), let $x \ne y \in G$. Since d is an inner metric, there is an arc $\alpha \in \mathcal{A}_{xy}(G)$ such that $d(x, y) = \ell(\alpha)$. By (ii) we have $d'(f(x), f(y)) \le \ell(f|_\alpha) \le \lambda\, \ell(\alpha) = \lambda\, d(x, y)$.

(iv) Let $\mathcal{P}$ be any tame partition for G and $P := G \setminus \bigcup_{i\in\mathcal{P}} i^\circ$. For every $x \in \alpha \setminus P$, we have $x \in i_x^\circ$ for some $i_x \in \mathcal{P}$. By Lemma 2.9(i), there is a subarc $\alpha_x \subset i_x$ such that $x \in \alpha_x^\circ$ and $\ell(\alpha_x) \le \ell(i_x)/2 < \infty$. By Lemma 2.9(ii), the arc α_x satisfies 2.5(i).

If $\lambda = 0$, then $f|_\alpha$ is constant and f has slope 0 on α_x. On the other hand, if $f|_\alpha$ is constant on any subarc of α, then $\lambda = 0$. Hence, by the monotonicity assumption we may suppose that $0 < \lambda \le \infty$ and that f maps α homeomorphically onto its image.

Let $\mathcal{P}'$ be a tame partition of G' and let $P' := G' \setminus \bigcup_{j\in\mathcal{P}'} j^\circ$. For every $x \in \alpha$ not in the countable set $P \cup (f|_\alpha)^{-1}(P')$, we have $f(x) \in j_x^\circ$ for some $j_x \in \mathcal{P}'$ and, as before, there is a smaller arc neighborhood $\beta_x \subset \alpha_x$ of x such that $f(\beta_x) \subset j_x$ and $\ell(f(\beta_x)) \le \ell(j_x)/2 < \infty$. By Lemma 2.9(ii), for every $y, z \in \beta_x$ we have

$$d'(f(y), f(z)) = \ell(f(\beta_x(yz))) = \lambda\, \ell(\beta_x(yz)) = \lambda\, d(y, z).$$

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It follows that $\lambda < \infty$ and that f has slope λ on β_x. Since $x \in \alpha$ was arbitrary up to countably many exceptions, $f|_\alpha$ has constant slope λ. □

2.3. *Markov maps on tame graphs.*

Definition 2.11. Let $f : G \to G$ be a continuous map on a tame graph. A family $\mathcal{P}$ is called a *countable Markov partition* for f if:

- $\mathcal{P}$ is a tame partition for G. In particular, for every $i \in \mathcal{P}$, $i^\circ = i \setminus E(i)$ is open in G and so any point in $i \cap B(G)$ is an end point of i;
- for every $i \in \mathcal{P}$, $f|_i$ is *monotone*, i.e. f maps i homeomorphically onto its image $f(i)$. So, the partition $\mathcal{P}$ witnesses that the map f is *countably monotone*;
- for every $i, j \in \mathcal{P}$, if $f(i) \cap j^\circ \neq \emptyset$, then $f(i) \supset j$.

The map f is said to belong to the class $\mathcal{CMM}(G)$ (*countably Markov and mixing*) if:

- f admits a countably infinite Markov partition;
- f is *topologically mixing*, i.e. for every pair of non-empty open sets U, V, there is an n such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq n$.

Remark 2.12. We will argue in Corollary 4.2 that for each $f \in \mathcal{CMM}(G)$, the set $\mathrm{Per}(f)$ of periodic points is dense in G and $h_{\mathrm{top}}(f) > 0$.

Remark 2.13. There are two possibilities for a map f from $\mathcal{CMM}(G)$—either there exists a non-empty open set U such that for every n, $f^n(U) \subsetneq G$; or for every non-empty open set U there is an n for which $f^n(U) = G$. In the latter case, f will be called *leo* (*locally eventually onto*). Our attention will be paid to both leo/non-leo types of maps.

Remark 2.14. Let G be a finite graph and let $f : G \to G$ be a continuous piecewise-monotone Markov graph map, i.e. such that f admits a finite Markov partition. If f is topologically mixing, then f also admits countably infinite Markov partitions and we will consider f as an element of $\mathcal{CMM}(G)$.

The basic properties of Markov partitions as regards iteration are summarized in the next lemma.

LEMMA 2.15. *Suppose that $f \in \mathcal{CMM}(G)$ with a countable Markov partition $\mathcal{P}$ and put $P = G \setminus \bigcup_{i \in \mathcal{P}} i^\circ$. Let $n \geq 0$. Then:*

(i) *$P_n := f^{-n}(P)$ defines a countable Markov partition $\mathcal{P}_n$ for f^j whose elements are the closures of connected components of $G \setminus P_n$, provided $j \leq n + 1$;*

(ii) *the partition arcs of $\mathcal{P}_n$ are the arcs of the form*

$$[i_0 i_1 \cdots i_n] := i_0 \cap f^{-1} i_1 \cap \cdots \cap f^{-n} i_n, \quad \text{where } i_0, \ldots, i_n \in \mathcal{P};$$

(iii) *for each $[i_0 \cdots i_n] \in \mathcal{P}_n$, the restricted map $f^n|_{[i_0 \cdots i_n]} : [i_0 \cdots i_n] \to [i_n]$ is a homeomorphism;*

(iv) *$Q := \bigcup_{n=0}^\infty P_n = \bigcup_{i=0}^\infty f^{-i}(P)$ is dense in G;*

(v) *P is forward invariant, i.e. $f(P) \subset P$. Consequently, $P = P_0 \subset P_1 \subset P_2 \subset \cdots$.*

We omit the proof of Lemma 2.15. Parts (i), (ii), (iii), and (iv) are essentially the same as in the theory of finite Markov partitions; see [**1**]. Part (v) follows from the global continuity of f and the fact that every element of P is a limit of end points of partition arcs

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(this uses the fact that a tame graph contains no continuum of convergence and so diameters in any sequence of disjoint arcs go to zero).

For a given $f \in \mathcal{CMM}(G)$ with a Markov partition $\mathcal{P}$, we associate to f and $\mathcal{P}$ the *transition matrix* $M = M(f) = (m_{ij})_{i,j\in\mathcal{P}}$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } f(i) \supset j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This matrix in turn determines a countable-state Markov shift (Σ_M, σ), where

$$\Sigma_M = \{i_0 i_1 \cdots \in \mathcal{P}^{\mathbb{N}_0} : m_{i_n i_{n+1}} = 1,\ n = 0, 1, \ldots\}$$

and $\sigma(i_0 i_1 \cdots) = i_1 i_2 \cdots$ is the shift map. The topology of Σ_M is inherited from the product space $\mathcal{P}^{\mathbb{N}_0}$, where $\mathcal{P}$ has the discrete topology. The partition arcs of $\mathcal{P}_n$ described in Lemma 2.15(ii) correspond to all the words $i_0 \cdots i_n$ which are allowable in this shift space, i.e. such that $m_{i_0 i_1} = \cdots = m_{i_{n-1} i_n} = 1$.

3. *Conjugacy to maps of constant or bounded slope*

The conjugacy of a countably Markov mixing map on a tame graph to a map of constant or bounded slope is related to the existence of positive eigenvectors of the corresponding transition matrix.

Definition 3.1. Let G be a tame graph, let $f \in \mathcal{CMM}(G)$ with transition matrix $M = (m_{ij})_{i,j\in\mathcal{P}}$, and let $v = (v_i)_{i\in\mathcal{P}}$ be a sequence of positive numbers. We say that v is a $(\lambda_i)_{i\in\mathcal{P}}$*-eigenvector* if it satisfies the equalities

$$\sum_{j\in\mathcal{P}} m_{ij} v_j = \lambda_i\, v_i \quad \text{for } i \in \mathcal{P}. \tag{3}$$

Since the entries v_i are positive, the numbers λ_i lie in $(0, \infty]$ and are uniquely determined by $(v_i)_{i\in\mathcal{P}}$. Let $\lambda \in (0, \infty)$.

- v is a λ*-eigenvector* if $\lambda_i = \lambda$ for every $i \in \mathcal{P}$.
- v is a λ*-subeigenvector* if $\lambda_i \le \lambda$ for every $i \in \mathcal{P}$. We say that v is *deficient in coordinate* i if $\lambda_i < \lambda$. Hence, being a λ-eigenvector means no deficiency.
- v is *summable* if $\sum_{i\in\mathcal{P}} v_i < \infty$.
- v *expansive* if $\liminf_{i\in\mathcal{P}} \lambda_i > 1$, i.e. $\inf_{i\in\mathcal{P}\setminus\mathcal{F}} \lambda_i > 1$ for some finite $\mathcal{F} \subset \mathcal{P}$.

Remark 3.2. Let us consider the situation from the previous definition.

(i) We could have considered all non-negative sequences $(v_i)_{i\in\mathcal{P}}$, but for the constant-zero sequence the situation is trivial, and it follows from the mixing property that M is *irreducible* (for all $i, j \in \mathcal{P}$, there is $n \in \mathbb{N}$ such that $m_{ij}(n) > 0$, where $M^n = (m_{ij}(n))_{i,j\in\mathcal{P}}$) and consequently if one entry v_i is strictly positive, all of them are.

(ii) If the map f is leo, then every λ-subeigenvector is summable. This is because for every $i \in \mathcal{P}$, there is $n > 0$ such that $f^n(i) = G$ and so $m_{ij}(n) \ge 1$ for every $j \in \mathcal{P}$. Hence, $\sum_{j\in\mathcal{P}} v_j \le \sum_{j\in\mathcal{P}} m_{ij}(n) v_j \le \lambda^n\, v_i < \infty$.

(iii) If the sequence $(v_i)_{i\in\mathcal{P}}$ is summable, then every λ_i is finite.

(iv) We introduce the expansivity condition because we need it in our construction. Clearly, a λ-eigenvector is expansive if and only if $\lambda > 1$. More generally, if a λ-subeigenvector is expansive, then $\lambda > 1$, and if it is deficient in only finitely many coordinates, then the other implication holds as well.

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3.1. *Statement of main results.* We now state our main results. We give a partial solution to the constant-slope problem for mixing maps and a complete solution for leo maps. Since we will work with maps from $\mathcal{CMM}(G)$, which are topologically mixing, we formulate our statement for topological conjugacies only—compare with [**1**, Proposition 4.6.9].

Our proof techniques apply not only to constant-slope maps, but also to bounded-slope maps, leading us to the following theorem whose consequences we explore in §5. The interval version of the following theorem for bounded slope was recently proved in [**5**, Theorem 3.7].

THEOREM 3.3. *Let G be a tame graph and let $f \in \mathcal{CMM}(G)$ with transition matrix M. In order for f to be conjugate to a map of constant (respectively bounded) slope λ:*
(i) *it is necessary that M admits a λ-(sub)eigenvector; and*
(ii) *it is sufficient that M admits a summable expansive λ-(sub)eigenvector.*

By Remark 3.2(ii), this gives us our solution for leo maps.

COROLLARY 3.4. *Let $f \in \mathcal{CMM}(G)$ be leo with transition matrix M. Then f is conjugate to a map of constant slope $\lambda > 1$ if and only if M admits a λ-eigenvector.*

Remark 3.5. The condition of Corollary 3.4 with $\lambda = e^{h_{\mathrm{top}}(f)}$ is fulfilled when the map f is leo and its transition matrix M is recurrent. We shall see in §4 that $\lambda_M = e^{h_{\mathrm{top}}(f)} > 1$ and, by [**18**, Theorem 4.1], there is a λ_M-eigenvector when M is recurrent, which is summable since f is leo. Moreover, even in the non-leo case, existence of a λ-(sub)eigenvector is possible only when $\lambda \geq \lambda_M = e^{h_{\mathrm{top}}(f)}$; see [**18**, Corollary 1].

The necessity of the (sub)eigenvector condition is more or less trivial to prove given the following proposition.

PROPOSITION 3.6. *Let (G, d) be a metric tame graph and let $f \in \mathcal{CMM}(G)$ with partition $\mathcal{P}$. If f has bounded slope λ, then $\ell(i) < \infty$ for every $i \in \mathcal{P}$.*

Proof. It follows from the definition of slope that there is an arc $\alpha \subset G$ of finite length. By Lemma 2.15(iv) and (v), there is n such that at least two points of P_n fall in α. Therefore, α contains a whole component of $G \setminus P_n$. By Lemma 2.15(i) and (ii), we can identify this component as having the form $[i_0 i_1 \cdots i_n]$. Applying Lemma 2.15(iii), we get $f^n(\alpha) \supset i_n$. Therefore, $\ell(i_n) \leq \ell(f^n|_\alpha) \leq \lambda^n \ell(\alpha) < \infty$. Since f is Markov and topologically mixing, there is k such that $f^k(i_n) \supset i$. Reasoning as before, we get $\ell(i) \leq \ell(f^k|_{i_n}) \leq \lambda^k \ell(i_n) < \infty$. □

Proof of Theorem 3.3(i). Without loss of generality, suppose that f has constant (respectively bounded) slope λ with respect to some compatible metric d on G. Let $\mathcal{P}$ denote the countable Markov partition of f and let $(m_{ij})_{i,j \in \mathcal{P}}$ denote the coefficients of M. We put $v_i := \ell(i)$ for $i \in \mathcal{P}$. By Proposition 3.6, we have $0 < v_i < \infty$ for every $i \in \mathcal{P}$. For every $i \in \mathcal{P}$, we have

$$\sum_{j \in \mathcal{P}} m_{ij} v_j = \sum_{f(i) \supset j \in \mathcal{P}} \ell(j) = \ell(f(i)) = \lambda_i \, v_i \quad \text{where } \lambda_i = \ell(f(i))/\ell(i).$$

By Proposition 2.10, $\lambda_i = \lambda$ (respectively $\lambda_i \leq \lambda$) for every $i \in \mathcal{P}$ if f has constant (respectively bounded) slope λ. □

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The sufficiency of the (sub)eigenvector condition is much harder to prove. It requires us to construct the conjugate map of constant (bounded) slope or, equivalently, to find a compatible metric ϱ on G such that f will have constant (or bounded) slope with respect to ϱ. To do it, we first construct an *arc map* A for G with the same properties as we want the length function ℓ to have with respect to ϱ. This is the goal of the next subsection, which may be summarized in the following theorem.

THEOREM 3.7. *Let G be a tame graph and let $f \in \mathcal{CMM}(G)$ with partition $\mathcal{P}$ and transition matrix $(m_{ij})_{i,j\in\mathcal{P}}$. If there is a summable expansive $(\lambda_i)_{i\in\mathcal{P}}$-eigenvector, then G admits an inner metric ϱ such that (G, ϱ) has finite total length and for every $i \in \mathcal{P}$, the map $f|_i$ has constant slope λ_i.*

Proof of Theorem 3.3(ii). It is enough to directly apply Theorem 3.7. By the assumption, $\lambda_i = \lambda$ (respectively $\lambda_i \le \lambda$) for every $i \in \mathcal{P}$ and, since the arcs $i \in \mathcal{P}$ are free, it follows that f has constant (respectively bounded) slope λ with respect to ϱ. □

3.2. *Arc maps.* In what follows, we assume a tame graph G, a map $f \in \mathcal{CMM}(G)$ with countably infinite Markov partition $\mathcal{P}$ and transition matrix $M = (m_{ij})_{i,j\in\mathcal{P}}$, and a summable expansive $(\lambda_i)_{i\in\mathcal{P}}$-eigenvector $v = (v_i)_{i\in\mathcal{P}}$ that is normalized so that $\sum_{i\in\mathcal{P}} v_i = 1$. We shall also use the notation introduced in 2.6 and Lemma 2.15. Denote for $[i_0 \cdots i_n] \in \mathcal{P}_n$,

$$\Delta([i_0 \cdots i_n]) := \frac{v_{i_n}}{\prod_{k=0}^{n-1} \lambda_{i_k}}, \tag{4}$$

where the empty product (when $n = 0$) is taken to be 1. For every $i \in P$, we have $v_i > 0$ by the definition and $\lambda_i < \infty$ by Remark 3.2(iii). Hence,

$$\Delta([i_0 \cdots i_n]) > 0 \quad \text{for each } n \ge 0.$$

For a Peano continuum X, any map $F : \mathcal{A}(X) \to \mathbb{R}$ is called an arc map defined on X. Using (4), we define arc maps A, $A_n : \mathcal{A}(G) \to [0, \infty)$ by

$$A_n(\gamma) = \sum_{\gamma \supset [i_0 \cdots i_n] \in \mathcal{P}_n} \Delta([i_0 \cdots i_n]) \tag{5}$$

and $A_n(\gamma) = 0$ when $[i_0 \cdots i_n] \not\subset \gamma$ for any $[i_0 \cdots i_n] \in \mathcal{P}_n$. Finally, we put

$$A(\gamma) = \sup_{n\in\mathbb{N}} \quad A_n(\gamma) \tag{6}$$

for every $\gamma \in \mathcal{A}(G)$. For an arc $\gamma \in \mathcal{A}(G)$ and a point $t \in \gamma$, we denote by γ_t, γ_{1-t} the elements from $\mathcal{A}(G)$ satisfying $\gamma = \gamma_t \cup \gamma_{1-t}$ and $\{t\} = \gamma_t \cap \gamma_{1-t}$.

PROPOSITION 3.8. *Let A_n, A be the arc maps defined in (5), (6), and $\gamma \in \mathcal{A}(G)$. Then:*

(i) *$A_n(\gamma) \le A_{n+1}(\gamma)$ and hence $A(\gamma) = \lim_{n\to\infty} A_n(\gamma)$;*
(ii) *if γ is not a point, then $A(\gamma) > 0$;*
(iii) *the arc map A is additive (on γ), i.e. for every $t \in \gamma$, $A(\gamma) = A(\gamma_t) + A(\gamma_{1-t})$;*
(iv) *for every sequence $(\gamma^m)_m \subset \mathcal{A}(G)$, if $\gamma^m \supset \gamma^{m+1}$ and $\bigcap_m \gamma^m = \{x\}$, then $\lim_m A(\gamma^m) = 0$;*

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(v) *for each pair of distinct points $x, y \in G$, there exists an arc $\alpha \in \mathcal{A}_{xy}$ such that*

$$A(\alpha) = \inf\{A(\beta) : \beta \in \mathcal{A}_{xy}\}. \tag{7}$$

Proof. In the proof we repeatedly use some topological properties of Peano continua. All of them can be found in [**14**]. We will also often apply the conclusions of Lemma 2.15.

As before, let us denote $P = G \setminus \bigcup_{i\in\mathcal{P}} i^\circ$, $P_n = f^{-n}(P)$, $Q = \bigcup_{n\geq 0} P_n$. Clearly, $f^{-1}(Q) = Q$ and by our choice of $\mathcal{P}$ the set Q is countable and dense in G. The set $\mathcal{P}_n$ contains the closures of connected components of $G \setminus P_n$. In particular, $\mathcal{P}_0 = \mathcal{P}$. Also (see [**5**, (3) and (4)] for the details),

$$\sum_{j\subseteq f(i_n)} \Delta([i_0 \cdots i_n j]) = \Delta([i_0 \cdots i_n]) \tag{8}$$

and

$$\Delta([i_1 \cdots i_n]) = \lambda_{i_0} \cdot \Delta([i_0 \cdots i_n]). \tag{9}$$

Notice that $P_n \subset P_{n+1}$ and that each $[i_0 \cdots i_n] \in \mathcal{P}_n$ is subdivided by the points of P_{n+1} into the arcs $[i_0 \cdots i_n j]$, where j ranges over all members of $\mathcal{P}$ contained in $f(i_n)$. This fact together with (4) implies that for each n,

$$\sum_{[i_0\cdots i_n]\in\mathcal{P}_n} \Delta([i_0 \cdots i_n]) = \sum_{i\in\mathcal{P}} \Delta([i]) = \sum_{i\in\mathcal{P}} v_i = 1. \tag{10}$$

It follows from (5) and (8) that for every $\gamma \in \mathcal{P}_n$, $A_m(\gamma) = A_n(\gamma)$ whenever $m \geq n$. Moreover, again from (8), we obtain for any arc $\gamma \in \mathcal{A}(G)$,

$$A_n(\gamma) \leq A_{n+1}(\gamma) \quad \text{and hence } A(\gamma) = \lim_{n\to\infty} A_n(\gamma).$$

It proves the property (i).

Let us verify (ii). Assume that $\gamma \in \mathcal{A}(G)$ is an arc. Since G is a tame graph, γ is not nowhere dense and then, because Q is dense in G, there are $n \in \mathbb{N}$ and $[i_0 \cdots i_n] \in \mathcal{P}_n$ such that $[i_0 \cdots i_n] \subseteq \gamma$. Since $v_{i_n} > 0$, we obtain that

$$A_n(\gamma) > 0 \quad \text{and hence also } A(\gamma) > 0.$$

(iii) We want to verify that the arc map A is additive on γ. Clearly, $A(\{x\}) = 0$. Suppose now that $\gamma \in \mathcal{A}(G)$ is non-degenerate and $t \in \gamma$ is given. Let us distinguish two cases for t.

If $t \in Q$, then there is $n \in \mathbb{N}$ such that $t \in P_n$. Now, for $m \geq n$,

$$A_m(\gamma) = A_m(\gamma_t) + A_m(\gamma_{1-t}).$$

Hence, in the limit for $m \to \infty$ we get the required equality

$$A(\gamma) = A(\gamma_t) + A(\gamma_{1-t}).$$

If $t \notin Q$, then there is for every $n \in \mathbb{N}$ exactly one arc $[i_0 \cdots i_n] \in \mathcal{P}_n$ containing t in the interior. It follows that

$$A_n(\gamma) = A_n(\gamma_t) + A_n(\gamma_{1-t}) + \Delta([i_0 \cdots i_n])$$

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and clearly it is sufficient to show that

$$\lim_{n\to\infty} \Delta([i_0 \cdots i_n]) = 0. \tag{11}$$

Let us call the sequence $i_0 i_1 \cdots$ the *itinerary* of t. From (8), it follows that

$$\Delta([i_0 \cdots i_n]) \geq \Delta([i_0 \cdots i_{n+1}]) \quad \text{for each } n.$$

Suppose first that some symbol j occurs infinitely often in the itinerary of t. If $f(j)$ is a single-partition arc, then it also occurs infinitely often in the itinerary of t and we may replace j by $f(j)$. Since the mixing hypothesis does not allow for a cycle of partition arcs, we may conclude (after making finitely many such replacements) that $f(j)$ contains more than one partition arc. Among all the partition arcs contained in $f(j)$ there must exist some k with $v_k/(\lambda_j v_j)$ maximal; call this ratio c and notice that $c < 1$. We have

$$\Delta([i_0 \cdots jk]) = \frac{v_k}{\lambda_j v_j} \Delta([i_0 \cdots j]) = c \cdot \Delta([i_0 \cdots j]).$$

Therefore, our decreasing sequence shrinks by the factor c or better infinitely many times and hence converges to zero.

Suppose now that each symbol in the itinerary of t occurs only finitely often. By the hypothesis, there are $\lambda > 1$ and a finite set $\mathcal{F} \subset \mathcal{P}$ such that $\lambda_i \geq \lambda$ for every $i \in \mathcal{P} \setminus \mathcal{F}$. Since the arcs from $\mathcal{F}$ can occur only finitely many times in the itinerary, we find that the denominator in the expression for $\Delta([i_0 \cdots i_n])$ in (4) is eventually monotone increasing with respect to n, growing by a factor $\geq \lambda$ at each step. Moreover, the numerator in this expression is always bounded by 1. Therefore, the limit (11) is zero, as desired.

Let us prove the property (iv). Since by (iii) the arc map A is additive, we can assume that x is an end point of every γ^m. Let $\mathcal{P}_n(\gamma^m)$ denote the set of all elements of $\mathcal{P}_n$ non-intersecting $(\gamma^m)^\circ$. There are two cases. Either there is n such that $\mathcal{P}_n = \bigcup_m \mathcal{P}_n(\gamma^m)$, or for every n there is an arc $\alpha^n \in \mathcal{P}_n \setminus \bigcup_m \mathcal{P}_n(\gamma^m)$. In the first case, we can miss any finitely many members of $\mathcal{P}_n$ by an arc γ^m and so for every $\varepsilon > 0$ there are a finite family $\mathcal{F} \subset \mathcal{P}_n$ such that

$$\sum_{[i_0 \cdots i_n] \in \mathcal{F}} \Delta([i_0 \cdots i_n]) \geq 1 - \varepsilon$$

by (10) and an arc γ^m such that $\mathcal{F} \subset \mathcal{P}_n(\gamma^m)$ and so $A_{n'}(\gamma^m) \leq \varepsilon$ for every $n' \geq n$ by (8). In the second case, $\alpha^n \cap (\gamma^m)^\circ \neq \emptyset$ for every m and hence there is m_n such that $\gamma^m \subset \alpha^n$ for every $m \geq m_n$. It follows that there is an itinerary $i_0 i_1 \cdots$ of x such that $\alpha^n = [i_0 \cdots i_n]$. Therefore, $A(\gamma^{m_n}) \leq \Delta([i_0 \cdots i_n])$ and we use (11). In both cases we have $\lim_m A(\gamma^m) = 0$.

The property (v) is clear when there are finitely many arcs in $\mathcal{A}_{xy}(G)$. Assume to the contrary that for some distinct points $x, y \in G$ and a sequence $(\alpha^m)_m \subset \mathcal{A}_{xy}(G)$,

$$\text{for all } m : A(\alpha^m) > A(\alpha^{m+1}) \quad \text{and} \quad \lim_{m\to\infty} A(\alpha^m) = \inf\{A(\beta) : \beta \in \mathcal{A}_{xy}(G)\}.$$

It is known that the hyperspace $C(G)$ of all subcontinua of G is compact [**14**, Theorem 4.17] and metrizable (by the Hausdorff metric induced by any compatible metric on G). So, we can assume that

$$\lim_{m\to\infty} \alpha^m = C \tag{12}$$

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for some subcontinuum C of G. Clearly, $x, y \in C$ and C contains some $\alpha \in \mathcal{A}_{xy}(G)$ since G is hereditarily locally connected (Remark 2.3). In what follows, we show that the arc α satisfies (7).

From (ii), we have $A(\alpha) > 0$. Fix an $\varepsilon > 0$. For an arc δ, let $\delta(uv)$ denote the unique subarc of δ with end points $u, v \in \delta$ and, for $n > 0$ put

$$S(\delta, u, v, n) = \{[i_0 \cdots i_n] \in \mathcal{P}_n : [i_0 \cdots i_n] \subseteq \delta(uv), [i_0 \cdots i_n] \cap P_0 = \emptyset\}.$$

By the definition of the arc map A, there is $n \in \mathbb{N}$ such that $A_n(\alpha) > A(\alpha) - \varepsilon$, so using (5) and a larger n (if necessary) we can consider points $u, v \in \alpha^\circ$ such that for some finite subset $S' \subset S(\alpha, u, v, n)$,

$$\sum_{[i_0 \cdots i_n] \in S'} \Delta([i_0 \cdots i_n]) > A(\alpha) - \varepsilon. \tag{13}$$

Since the set $\bigcup S'$ is a compact subset of α° non-intersecting P_0, for each $x \in \bigcup S'$ there exists an open neighborhood $U(x)$ for which

$$\overline{U(x)} \cap P_0 = \emptyset, \quad U(x) \cap \alpha = U(x) \cap G \subset \alpha^\circ;$$

then it follows from (12) that in fact for each $U(x)$ there exists m_x such that for each $m > m_x$,

$$U(x) \cap \alpha = U(x) \cap \alpha^m = U(x) \cap G.$$

Since $\{U(x) : x \in \bigcup S'\}$ is an open cover of the compact set $\bigcup S'$, we can consider its finite subcover $\{U(x_1), \ldots, U(x_k)\}$ and put $m_1 = \max_{1 \le i \le k} m_{x_i}$. By the previous work, for each $m > m_1$ we obtain $\alpha^m \supset \bigcup S'$ and hence by (6) and (13),

$$A(\alpha^m) \ge A_n(\alpha^m) > A(\alpha) - \varepsilon.$$

Since ε was arbitrary, it proves the conclusion of the claim. □

Now that we have our arc map A and its basic properties, we are ready to define the desired metric. Using Proposition 3.8(v), let us define the map $\varrho : G \times G \to [0, 1]$ by

$$\varrho(x, y) := \min\{A(\alpha) : \alpha \in \mathcal{A}_{xy}(G)\}. \tag{14}$$

The following proposition concludes the proof of Theorem 3.7.

PROPOSITION 3.9.

(i) *The map ϱ is a metric compatible with the topology of G, so we may consider the corresponding length ℓ.*

(ii) *We have $\ell(\alpha) = A(\alpha)$ for every $\alpha \in \mathcal{A}(G)$. It follows that ϱ is an inner metric.*

(iii) *The metric tame graph (G, ϱ) has finite total length.*

(iv) *For every arc $\alpha \subset i \in \mathcal{P}$, we have $\ell(f(\alpha)) = \lambda_i \ell(\alpha)$. It follows that $f|_i$ has constant slope λ_i.*

Proof. (i) Clearly, $\varrho(x, x) = 0$, and $\varrho(x, y) = \varrho(y, x)$ for each distinct $x, y \in G$. By Proposition 3.8(ii) and (v), $\varrho(x, y) > 0$ if $x \neq y$. In order to show that ϱ satisfies the triangle inequality, let $x, y, z \in G$ be arbitrary. There are arcs $\alpha, \beta, \gamma \in \mathcal{A}(G)$ with end points x, y; y, z; x, z respectively and such that

$$\varrho(x, y) = A(\alpha), \quad \varrho(y, z) = A(\beta), \quad \varrho(x, z) = A(\gamma).$$

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If $z \in \alpha$, then since $\alpha(xz) \subset \alpha$ we can write

$$\varrho(x, z) \le A(\alpha(xz)) \le A(\alpha) = \varrho(x, y) \le \varrho(x, y) + \varrho(y, z).$$

Assume that $z \notin \alpha$. There exists the unique point $t \in \alpha$ such that $\alpha \cap \beta(tz) = \{t\}$. Since $\alpha(xt) \cup \beta(tz)$ is an arc from $\mathcal{A}_{xz}$, using the additivity of A we can write

$$\begin{aligned}\varrho(x, z) = A(\gamma) &\le A(\alpha(xt) \cup \beta(tz)) = A(\alpha(xt)) + A(\beta(zt)) \\ &\le A(\alpha) + A(\beta) = \varrho(x, y) + \varrho(y, z).\end{aligned}$$

Thus, ϱ is a metric.

Let us show that

$$\lim_{n\to\infty} \max\{A(\alpha) : \alpha \in \mathcal{P}_n\} = 0. \tag{15}$$

We know that for each $\alpha \in \mathcal{P}_n$, $\alpha = [i_0 \cdots i_n]$ and by (8) $A(\alpha) = A_n(\alpha) = \Delta([i_0 \cdots i_n])$. So, by (10) every maximum in (15) is less than 1. Moreover, every element of $\mathcal{P}_{n+1}$ is contained in some element of $\mathcal{P}_n$, so

$$\max\{A(\alpha) : \alpha \in \mathcal{P}_n\} \ge \max\{A(\alpha) : \alpha \in \mathcal{P}_{n+1}\}$$

and hence the limit in (15) exists. Assume to the contrary that it equals some $\varepsilon > 0$. Let $\mathcal{T}$ be the tree $\{\alpha \in \bigcup_n \mathcal{P}_n : \ A(\alpha) \ge \varepsilon\}$ ordered by inclusion. By our assumption, $\mathcal{T}$ is infinite, but by (10) every level $\mathcal{T} \cap \mathcal{P}_n$ is finite. Hence, $\mathcal{T}$ has an infinite branch $[i_0] \supset [i_0 i_1] \supset \cdots$, so $A([i_0 \cdots i_n]) = \Delta([i_0 \cdots i_n]) \ge \varepsilon$ for each n. This contradicts (11) and hence the equality (15) is proved.

We need to verify that the metric ϱ is compatible with the original topology τ on G. This means verifying that the identity $(G, \tau) \to (G, \varrho)$ is a homeomorphism. Since (G, τ) is compact and (G, ϱ) is clearly Hausdorff, it is enough to show that the identity is continuous. Thus, suppose that we have a sequence x_n converging to x in the topology of G. We need to show that it converges also in the metric ϱ. Fix $\varepsilon > 0$. We want to find n_0 such that $\varrho(x, x_n) < \varepsilon$ for $n \ge n_0$.

By (15), there is $n_1 \in \mathbb{N}$ such that $A(\alpha) < \varepsilon$ for each $\alpha \in \mathcal{P}_{n_1}$. Since

$$\sum_{\alpha\in\mathcal{P}_{n_1}} A(\alpha) = \sum_{[i_0\cdots i_{n_1}]\in\mathcal{P}_{n_1}} \Delta([i_0 \cdots i_{n_1}]) = 1,$$

there is a finite set $\mathcal{F} \subset \mathcal{P}_{n_1}$ such that $\sum_{\alpha\in\mathcal{P}_{n_1}\setminus\mathcal{F}} A(\alpha) < \varepsilon$. Let $\mathcal{G} := \{\alpha \in \mathcal{F} : x \in \alpha\}$ and let $\mathcal{H} := \mathcal{P}_{n_1}\setminus\mathcal{F}$. The set $U := \bigcup(\mathcal{G} \cup \mathcal{H})$ is a τ-neighborhood of x since it omits only finitely many arcs from $\mathcal{P}_{n_1}$ not containing x. There is a smaller τ-neighborhood $x \in V \subset U$ that is arcwise connected and disjoint from the finite set $E := \{e : e$ an end point of some $\alpha \in \mathcal{G}$ distinct from $x\}$. It follows that every point $y \in V$ is connected to x by an arc from $\mathcal{G}$ or by an arc covered by arcs from $\mathcal{H}$ up to a countable closed subset. Since V is a τ-neighborhood of x, there is n_0 such that $x_n \in V$ for every $n \ge n_0$. For such n, we have

$$\varrho(x, x_n) \le \begin{cases} \max\{A(\alpha) : \alpha \in \mathcal{G}\} & \text{if } x_n \in \bigcup \mathcal{G}, \\ \displaystyle\sum_{\alpha\in\mathcal{H}} A(\alpha) & \text{otherwise.} \end{cases}$$

In both cases $\varrho(x, x_n) < \varepsilon$. Thus, the metric ϱ is compatible with the original topology τ on G.

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(ii) For every $x_0 \le \cdots \le x_n \subset \alpha$, we have

$$\sum_{k<n} \varrho(x_k, x_{k+1}) \le \sum_{k<n} A(\alpha(x_k x_{k+1})) = A(\alpha(x_0 x_n)) \le A(\alpha).$$

Hence, $\ell(\alpha) \le A(\alpha)$.

Let us prove the other inequality. For every subarc $\beta \subset \alpha$, let x_β, y_β denote the end points of β. First suppose that the arc α is free. Similarly to Lemma 2.9(ii), for every subarc $\beta \subset \alpha$ such that $A(\beta) \le A(\alpha)/2$, we have $\varrho(x_\beta, y_\beta) = A(\beta)$. The map $\alpha \ni t \mapsto A(\alpha(x_\alpha t))$ is continuous by Proposition 3.8(iii) and (iv). Hence, there is a mid point $m \in \alpha$ such that $A(\alpha(x_\alpha m)) = A(\alpha(m y_\alpha)) = A(\alpha)/2$. Therefore,

$$A(\alpha) = A(\alpha(x_\alpha m)) + A(\alpha(m y_\alpha)) = \varrho(x_\alpha, m) + \varrho(m, y_\alpha) \le \ell(\alpha).$$

Now let α be arbitrary, let $\varepsilon > 0$, and let us enumerate $P \cap \alpha$ as $\{p_n : n \in \mathbb{N}\}$. By Proposition 3.8(iv), for every n there is a subarc $\alpha_n \subset \alpha$ such that α_n is a neighborhood of p_n in α and $A(\alpha_n) \le \varepsilon \cdot 2^{-n}$. Since $P \cap \alpha$ is compact, there is a finite family of arcs $\mathcal{B}^- \subset \mathcal{A}(\alpha)$ with disjoint interiors such that $P \cap \alpha \subset \bigcup \mathcal{B}^-$ and every $\beta \in \mathcal{B}^-$ is contained in a different α_n. It follows that $\sum_{\beta\in\mathcal{B}^-} A(\beta) \le \sum_{n=1}^{\infty} A(\alpha_n) \le \varepsilon$. Let $\mathcal{B}^+$ be the complementary finite family of subarcs of α, i.e. $\mathcal{B}^+$ consists of the closures of the components of $\alpha \setminus \bigcup \mathcal{B}^-$. Clearly, every arc $\beta \in \mathcal{B}^+$ is free. Let $\{x_k\}_{k=0}^n$ be a strictly monotone enumeration of all end points of $\beta \in \mathcal{B}^+ \cup \mathcal{B}^-$. Altogether, we have

$$A(\alpha) - \varepsilon \le \sum_{\beta\in\mathcal{B}^+} A(\beta) = \sum_{\beta\in\mathcal{B}^+} \varrho(x_\beta, y_\beta) \le \sum_{k<n} \varrho(x_k, x_{k+1}) \le \ell(\alpha).$$

Since $\varepsilon > 0$ was arbitrary, we have $A(\alpha) \le \ell(\alpha)$.

(iii) For every $i \in \mathcal{P}$, we have $\ell(i) = A(i) = \Delta([i]) = v_i$, so the finite total length follows directly from the summability of $(v_i)_{i\in\mathcal{P}}$.

(iv) Suppose first that the end points of α are two points $x, y \in Q$. Take n minimal so that $x, y \in P_n$. Notice that f induces a bijective correspondence between the set of arcs $[i_0 \cdots i_n] \in \mathcal{P}_n$ contained in the arc α (necessarily $i_0 = i$) and the set of arcs $[i_1 \cdots i_n] \in \mathcal{P}_{n-1}$ contained in the arc $f(\alpha)$. Using (9) and taking sums, we find that

$$A(f(\alpha)) = \sum_{[i_1\cdots i_n]\subseteq f(\alpha)} \Delta([i_1 \cdots i_n]) = \sum_{[i_0\cdots i_n]\subseteq\alpha} \lambda_{i_0} \Delta([i_0 \cdots i_n]) = \lambda_i A(\alpha).$$

Since Q is dense, the conclusion is true by Proposition 3.8(iii) and (iv) and by the equality $\ell(\alpha) = A(\alpha)$ (ii). By Proposition 2.10(iv), $f|_i$ has constant slope λ_i for every $i \in \mathcal{P}$. □

3.3. *Realization in* $\mathbb{R}^3$. It turns out that the inner metric structure with constant slope obtained in the previous section may be realized in $\mathbb{R}^3$ in such a way that most of its nice properties are preserved—the lengths of all arcs are preserved, the original inner metric may be recovered from the metric induced from $\mathbb{R}^3$, and the conjugate map has the same slope. Moreover, the halves of the partition arcs become line segments in $\mathbb{R}^3$ and the whole tame graph in $\mathbb{R}^3$ may be easily visualized.

LEMMA 3.10. *Let (X, d) be a countable compact metric space. There is a $\frac{1}{2}$-Lipschitz embedding $\pi : X \to \mathbb{R}$.*

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Proof. To avoid trivial cases and to simplify the notation, assume that $X = \{x_n : n \in \mathbb{N}\}$ is infinite and $\operatorname{diam}(X) \le 1$. Consider the Tychonoff product $R := \prod_{k\in\mathbb{N}}[0,\ 1/2^{k+1}]$. Clearly, R is a compact metrizable space. Using the metric d and an element $r = (r_k)_{k\in\mathbb{N}} \in R$, we can define the map

$$\pi_r : X \to \mathbb{R}, \quad \pi_r(x_n) = \sum_{k\in\mathbb{N}} r_k\, d(x_n, x_k).$$

We have

$$|\pi_r(x_n) - \pi_r(x_m)| \le \sum_{k\in\mathbb{N}} r_k |d(x_n, x_k) - d(x_m, x_k)| \le \sum_{k\in\mathbb{N}} r_k\, d(x_n, x_m) \le \frac{d(x_n, x_m)}{2},$$

so the map $\pi_r : (X, d) \to (\mathbb{R}, |\cdot|)$ is $\frac{1}{2}$-Lipschitz. For every $i < j \in \mathbb{N}$, let $G_{i,j} \subset R$ denote the set $\{r \in R : \pi_r(x_i) \ne \pi_r(x_j)\}$. It is easy to verify that $G_{i,j}$ is open and dense in R. For every $r \in G := \bigcap_{i<j\in\mathbb{N}} G_n$, the map π_r is injective and, since the space X is compact, π_r is an embedding. □

THEOREM 3.11. *Let (G, ϱ) be a metric tame graph of finite total length with a tame partition $\mathcal{P}$. Let $\mathcal{P}'$ be the tame partition where each arc $i \in \mathcal{P}$ is split into two half-length arcs. There is an embedding $\phi : G \to (\mathbb{R}^3, d)$ such that:*

(i) *every arc $\phi(i)$, $i \in \mathcal{P}'$, is a line segment;*

(ii) *the lengths of the arcs are preserved, i.e. $\ell(\phi(\alpha)) = \ell(\alpha)$ for every arc $\alpha \subset G$.*

It follows that:

(iii) *if ϱ is an inner metric, then ϕ is 1-Lipschitz, $\phi|_i$ is an isometry for every $i \in \mathcal{P}'$, and $\varrho(x,\ y) = \min_{\alpha\in\mathcal{A}_{xy}(G)} \ell(\phi(\alpha)) = \min_{\beta\in\mathcal{A}_{\phi(x)\phi(y)}(\phi(G))} \ell(\beta)$ for every $x,\ y \in G$, i.e. ϱ may be recovered from $(\phi(G), d)$;*

(iv) *if $f : G \to G$ is a map of constant (respectively bounded) slope λ, then the conjugate map $\phi \circ f \circ \phi^{-1} : (\phi(G), d) \to (\phi(G), d)$ has constant (respectively bounded) slope λ as well.*

Proof. First, since every partition arc $i \in \mathcal{P}$ has finite length, the map $t \mapsto \ell(i(0t))$ (where 0 denotes an end point of i) is a homeomorphism $i \to [0, \ell(i)]$. Hence, the partition $\mathcal{P}'$ is well defined.

Due to Remark 2.14, we can assume that $\mathcal{P}$ is countably infinite. In order to construct the embedding ϕ, we will fix a real line $\mathbb{R} \subset \mathbb{R}^3$ and a sheaf of planes $\Sigma = \{\sigma_i : i \in \mathcal{P}\}$ in $\mathbb{R}^3$ containing $\mathbb{R}$. To make our construction as simple as possible, we will assume that Σ is convergent, i.e. $\lim_{i\in\mathcal{P}} n_i = n$, where each n_i is a normal vector of σ_i and n is a non-zero vector in $\mathbb{R}^3$.

The set $P = G \setminus \bigcup_{i\in\mathcal{P}} i^\circ$ is a countable compact subspace of (G, ϱ). We define $\phi|_P$ as a $\frac{1}{2}$-Lipschitz embedding $P \to \mathbb{R} \subset \mathbb{R}^3$, which exists by Lemma 3.10. Let $i \in \mathcal{P}$ and let $x,\ y \in P$ be its end points. Since $d(\phi(x), \phi(y)) < \varrho(x,\ y) \le \ell(i)$, there is an isosceles triangle in σ_i with base $[\phi(x), \phi(y)] \subset \mathbb{R}$ and legs $L_{i,x},\ L_{i,y} \subset \sigma_i$ such that $\ell(L_{i,x}) = \ell(L_{i,y}) = \ell(i)/2$. We define $\phi|_i$ as the unique homeomorphism $i \to L_{i,x} \cup L_{i,y}$ consistent with $\phi|_P$ such that $\ell(\phi|_i(\alpha)) = \ell(\alpha)$ for every subarc $\alpha \subset i$.

This defines an injective map $\phi : G \to \mathbb{R}^3$. We need to show that it is continuous. By the definition, the restrictions $\phi|_P$ and $\phi|_i$ for $i \in \mathcal{P}$ are continuous. Let $x_n \to x$ be a

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convergent sequence in G. The only case that has to be covered is when $x \in P$ and every x_n is a point of i_n° for a distinct partition arc $i_n \in \mathcal{P}$. For every n, let y_n be an end point of i_n. We have $\varrho(x_n, y_n) \le \ell(i_n) \to 0$ since $\sum_{i\in\mathcal{P}} \ell(i) < \infty$. Similarly, $d(\phi(x_n), \phi(y_n)) \le \ell(\phi(i_n)) = \ell(i_n) \to 0$. It follows that $y_n \to x$; by the continuity of $\phi|_P$, also $\phi(y_n) \to \phi(x)$, and finally $\phi(x_n) \to \phi(x)$. Thus, ϕ is continuous and, since G is compact, it is an embedding.

We have (i) directly from the construction. The property (ii) is true for subarcs of the partition arcs, but extends to every arc $\alpha \subset G$ since $\alpha \cap P$ is countable.

(iii) Suppose that ϱ is an inner metric, so for every $x, y \in G$ there is $\alpha \in \mathcal{A}_{xy}(G)$ such that $\varrho(x, y) = \ell(\alpha)$. By (ii), $\ell(\alpha) = \ell(\phi(\alpha)) \ge d(\phi(x), \phi(y))$, so the map ϕ is 1-Lipschitz. Moreover, if $x, y \in i$ for some $i \in \mathcal{P}'$, then by Lemma 2.9(ii) the arc α above may be taken in i and hence $\phi(\alpha)$ is a line segment in a leg of the corresponding triangle from the construction. In this case we have $\varrho(x, y) = \ell(\alpha) = \ell(\phi(\alpha)) = d(\phi(x), \phi(y))$. The last claim follows directly from (ii):

$$\varrho(x, y) = \min_{\alpha\in\mathcal{A}_{xy}(G)} \ell(\alpha) = \min_{\alpha\in\mathcal{A}_{xy}(G)} \ell(\phi(\alpha)) = \min_{\beta\in\mathcal{A}_{\phi(x)\phi(y)}(\phi(G))} \ell(\beta).$$

(iv) Let $f : G \to G$ be a map of constant (respectively bounded) slope λ and let $g : \phi(G) \to \phi(G)$ be the conjugate map $\phi \circ f \circ \phi^{-1}$. Let $x \in G$ be such that there is a free arc neighborhood $\alpha \subset G$ of x such that f has slope $\lambda_x = \lambda$ (respectively $\le \lambda$) on α. Moreover, suppose that x is not in the countable set $P' := G \setminus \bigcup_{i\in\mathcal{P}'} i^\circ$, which is P together with mid points of the partition arcs from $\mathcal{P}$. It follows that $\phi(x)$ lies in the interior of a line segment in $\phi(G)$ and so there is a smaller arc neighborhood $\beta \subset \alpha$ of x such that whole $\phi(\beta)$ lies in the line segment in $\phi(G)$. Hence, $\phi(\beta)$ satisfies the slope condition 2.5(i).

If $\lambda_x = 0$, then f and g are constant, and g has constant slope 0 on $\phi(\beta)$. If $\lambda_x > 0$, then f is injective on α and so $f(x) \notin P'$ (unless x is one of the countably many exceptions). This means that $g(\phi(x))$ also lies in the interior of a line segment in $\phi(G)$ and there is a smaller arc neighborhood $\gamma \subset \beta$ of x such that the whole arc $\phi(f(\gamma)) = g(\phi(\gamma))$ lies in the line segment. Altogether, for every $y, z \in \gamma$ we have

$$\begin{aligned} d(g(\phi(y)), g(\phi(z))) &= \ell(\phi(f(\gamma(yz)))) = \ell(f(\gamma(yz))) \\ &= \lambda_x \ell(\gamma(yz)) = \lambda_x\, \ell(\phi(\gamma(yz))) = \lambda_x\, d(\phi(y), \phi(z)) \end{aligned}$$

and so the map g has slope λ_x on $\phi(\gamma)$. □

3.4. *Discussion—an open question and examples.* The main question left open by our work is the following.

Question 3.12. What conditions on the transition matrix are both necessary and sufficient for a map $f \in \mathcal{CMM}(G)$ on a tame graph G to be conjugate to a map of constant slope?

It is easy to see that condition (ii) in Theorem 3.3 is not necessary. It is enough to construct a metric tame graph G of infinite total length and a constant-slope map $f \in \mathcal{CMM}(G)$ whose transition matrix M has no summable eigenvectors. This was done in [**13**] on the extended real line $[-\infty, \infty]$. Note that the constant slope is with respect to the standard metric on $\mathbb{R}$, which cannot be extended to $[-\infty, \infty]$, so to fit the example to

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our setting we formally realize $[-\infty, \infty]$ as a double-infinite countable zig-zag in $\mathbb{R}^2$ of infinite length. The induced metric is not an inner metric; it is only locally inner with the exception of the breakpoints and end points.

Now we shall give two examples on a locally connected fan—an inner metric example showing that the condition 3.3(ii) is not necessary, and an example showing that the condition 3.3(i) is not sufficient.

In both examples it is convenient to specify the Markov dynamics using *slack Markov partitions* [**4**], where the map is allowed to be piecewise monotone on the partition arcs, and where the corresponding transition matrix may contain any non-negative integers as coefficients. The point is that to every slack Markov partition $\mathcal{P}$ with transition matrix M there corresponds a Markov partition $\mathcal{P}'$ with transition matrix M' such that all important properties are translated back and forth: for every $\lambda \geq 0$, there is a one-to-one correspondence between λ-eigenvectors of M and M'; moreover, the (non-)summability is preserved. For details, see the proof of [**4**, Theorem 3].

Also, for both examples we consider $\mathbb{R}^2$ with polar coordinates, writing (r, θ) in place of $(r\cos\theta, r\sin\theta)$.

Example 3.13. For every $n \in \mathbb{N}$, we put $I_n = \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ and we consider the positive sequence

$$\begin{aligned} v &= (v_n)_{n\in\mathbb{Z}} \\ &= (\ldots, \underbrace{\tfrac{1}{2}, \tfrac{1}{2}}_{-I_1}, \underbrace{1}_{-1}, \underbrace{1}_{0}, \underbrace{1}_{1}, \underbrace{\tfrac{1}{2}, \tfrac{1}{2}}_{I_1}, \underbrace{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}}_{I_2}, \underbrace{\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}}_{I_3}, \ldots). \end{aligned}$$

For every $n \in \mathbb{N}_0$, we consider the angle $\theta_n = \pi/(n+1)$ and the arcs in $\mathbb{R}^2$

$$F_n = \{(r, \theta_n) : r \in [0, v_n]\} \quad \text{and} \quad F_{-n} = \{(r, -\theta_n) : r \in [0, v_n]\}.$$

Our space is the locally connected fan $F = \bigcup_{n\in\mathbb{Z}} F_n$. We will refer to the arcs F_n as the 'blades' of the fan. Let d be the standard metric on $\mathbb{R}^2$. On F, we consider the induced inner metric ϱ, i.e. $\varrho = d$ on every blade F_n and $\varrho(x, y) = d(x, 0) + d(0, y)$ for every $x \in F_n \neq F_m \ni y$. Altogether, (F, ϱ) is an inner metric tame graph.

Using v, let us define the matrix $M = (m_{ij})_{i,j\in\mathbb{Z}}$: for $n \in \mathbb{N}_0$, we put

$$(m_{n(n-1)}, m_{nn}, m_{n(n+1)}) = \begin{cases} (2, 2, 4) & \text{if } v_{n-1} = v_n = v_{n+1}, \\ (2, 2, 8) & \text{if } v_{n-1} = v_n = 2v_{n+1}, \\ (2, 2, 2) & \text{if } v_{n-1} = 2v_n = 2v_{n+1}; \end{cases}$$

all the other elements m_{ij} for $i \geq 0$ are zero and, for $(i, j) \in \mathbb{N} \times \mathbb{Z}$, we put $m_{(-i)(-j)} = m_{ij}$. Then $Mv = 8v$ and the matrix M can be considered as a transition matrix of a countably piecewise affine map $f : F \to F$ leaving the point $(0, 0)$ fixed and such that the blade F_n, $n \in \mathbb{Z}$, covers $m_{n(n-1)}$ times, m_{nn} times, and $m_{n(n+1)}$ times the blades F_{n-1}, F_n, and F_{n+1}, respectively. One can see that f is a topologically mixing map with constant slope 8 on F.

We claim that no positive 8-eigenvector $u = (u_n)_{n\in\mathbb{Z}}$ of M is summable.

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Proof sketch. It can be shown that every solution $u = (u_k)_{k\in\mathbb{Z}}$ of the equation $Mu = 8u$ is of the form

$$u_k = \alpha_n + \frac{\beta_n}{2^k}, \quad u_{-k} = \alpha'_n + \frac{\beta'_n}{2^k} \quad \text{for } k \in I_n,$$

where $n \in \mathbb{N}_0$, $I_0 := \{0, 1\}$, and the constants α_n, β_n, $n \in \mathbb{N}_0$, are bounded by

$$\alpha_n = u_{2^{n+1}} + u_{2^{n+1}+1} = 2\alpha_{n+1} + \frac{3\beta_{n+1}}{2^{2^{n+1}+1}},$$
$$\beta_n = 2^{2^{n+1}}(u_{2^{n+1}} - u_{2^{n+1}+1}) = \frac{\beta_{n+1}}{2}.$$

It follows that

$$\alpha_n = \frac{\alpha_0}{2^n} - \frac{3}{4}\cdot\frac{\beta_0}{2^n}\cdot\sum_{j=1}^{n}\frac{(2^j)^2}{2^{(2^j)}} \quad \text{and} \quad \beta_n = 2^n\beta_0.$$

The same holds with I_n, α_n, β_n replaced by $-I_n$, α'_n, β'_n. The positive and the negative parts are bounded by $\alpha'_0 = \alpha_0 + 3\beta_0$ and $\beta'_0 = -2\beta_0$. Moreover, we can show that

$$\lim_{n\to\infty}\sum_{k\in I_n} u_k = \alpha_0 - C\beta_0 \quad \text{and} \quad \lim_{n\to\infty}\sum_{k\in I_n} u_{-k} = \alpha'_0 - C\beta'_0$$

for some $C > 0$. We have $\alpha_0 = u_2 + u_3 > 0$ and similarly $\alpha'_0 = u_{-2} + u_{-3} > 0$. Since $\beta'_0 = -2\beta'_0$, either $\beta_0 \le 0$ and so $0 < \alpha_0 \le \alpha_0 - C\beta_0$, or $\beta'_0 \le 0$ and so $0 < \alpha'_0 \le \alpha'_0 - C\beta'_0$. In both cases $\sum_{k\in\mathbb{Z}} u_k = \infty$. □

Finally, we construct an example to show that condition (i) in Theorem 3.3 is not sufficient. It is a modification of a piecewise-continuous interval map from [**13**], made continuous by working on a dendrite.

Example 3.14. Fix two decreasing sequences $r_0 > r_1 > \cdots$ and $\theta_0 > \theta_1 > \cdots$ with $\theta_0 < \pi$ and with limits $\lim_n r_n = 0$, $\lim_n \theta_n = 0$. Consider the tame graph $G \subset \mathbb{R}^2$ formed as the union of the arcs $A_n = \{(r, \theta_n);\ 0 \le r \le r_n\}$, $n \ge 0$; $C_n = \{(r, -\theta_n);\ 0 \le r \le r_n\}$, $n \ge 1$; and $B = \{(r, -\theta_0);\ 0 \le r \le r_0\}$. The graph G is a dendrite with a single branch point. Moreover, G is a locally connected fan with blades A_n, B, C_n. We define the map $f : G \to G$ so that the branch point is fixed, C_1 maps affinely onto B, B maps affinely onto A_0, and on each of the remaining blades f is the piecewise affine 'connect-the-dots' map with the following dots:

$C_{n+1} \xrightarrow{2:1} C_n$	$C_{2^n} \xleftarrow{2:1} A_n \longrightarrow A_{n+1}$
$f(0, 0) = (0, 0)$	$f(0, 0) = (0, 0)$
$f(r_{n+1}/2, -\theta_{n+1}) = (r_n, -\theta_n)$	$f(r_n/3, \theta_n) = (r_{2^n}, -\theta_{2^n})$
$f(r_{n+1}, -\theta_{n+1}) = (0, 0)$	$f(2r_n/3, \theta_n) = (0, 0)$
	$f(r_n, \theta_n) = (r_{n+1}, \theta_{n+1})$.

Figure 2 illustrates the map f. It is straightforward to verify that if U is an open set, then there is $n_0 \ge 0$ such that $f^{n_0}(U)$ contains a whole blade, and then $\bigcup_{n=0}^{\infty} f^n(U) = G$ is the whole dendrite. This shows that f is topologically mixing.

We will use the partition into blades as a *slack* Markov partition. The corresponding transition matrix admits (up to scaling) exactly one non-negative eigenvector with

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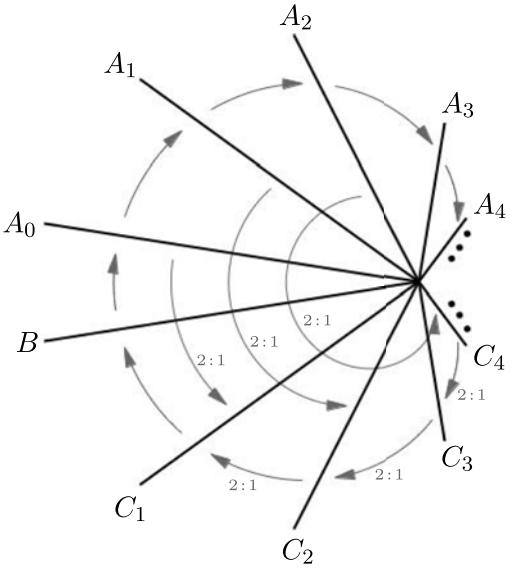


FIGURE 2. A dendrite map with no conjugate map of constant slope.

eigenvalue $\lambda = 2$ (there are other non-negative eigenvectors for $\lambda > 2$, but they do not interest us). Its entries are $v_{A_n} = 2^n + 1$, $v_B = 1$, and $v_{C_n} = \frac{1}{2}$. The choice to work with the eigenvalue $\lambda = 2$ is intentional, as it can be shown (cf. [**4**]) that $h_{\text{top}}(f) = \log 2$.

Now suppose that there is a compatible metric d on G such that f has constant slope 2 with respect to d. Applying the proof of Theorem 3.3(i), we see that the lengths of the blades of G with respect to d must form an eigenvector with eigenvalue 2. But there is only one such eigenvector, namely, the vector v given above. It follows that the set $\bigcap_{n=0}^{\infty} f^{-n}(A_n)$ is an arc of length

$$2 \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{2^n + 1}{2^n + 2} \times \cdots = 1$$

(cf. [**13**]), which means that the map f is not topologically mixing.

4. *Entropy and horseshoes for maps from $\mathcal{CMM}(G)$*

For a matrix $M = (m_{ij})_{i,j\in\mathcal{P}}$, we can consider the powers $M^n = (m_{ij}(n))_{i,j\in\mathcal{P}}$ of M:

$$M^0 = I = (\delta_{ij})_{i,j\in\mathcal{P}}, \quad M^n = \left(\sum_{k\in\mathcal{P}} m_{ik}m_{kj}(n-1)\right)_{i,j\in\mathcal{P}}, \quad n \in \mathbb{N}. \tag{16}$$

PROPOSITION 4.1. *Let $f \in \mathcal{CMM}(G)$ with $M(f) = (m_{ij})_{i,j\in\mathcal{P}}$.*

(i) *For each $n \in \mathbb{N}$ and $i, j \in \mathcal{P}$, the entry $m_{ij}(n)$ of M^n is finite.*

(ii) *The entry $m_{ij}(n) = m$ if and only if there are exactly m arcs $[ii_1^k \cdots i_{n-1}^k j] \in \mathcal{P}_n$, $k \in \{1, \ldots, m\}$, for which $f^n([ii_1^k \cdots i_{n-1}^k j]) = j$, $k = 1, \ldots, m$.*

Proof. (i) From the continuity of f and the definition of $M(f)$, it follows that the sum $\sum_{i\in\mathcal{P}} m_{ij}$ is finite for each $j \in \mathcal{P}$, which directly implies (i). (ii) For $n = 1$, this is given

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by the relation (2) defining the matrix $M = M(f)$. The induction step follows immediately from the definition (16) of the product of the non-negative matrices M and M^{n-1} and Lemma 2.15(iii). □

COROLLARY 4.2. *With the help of Proposition 4.1, one can show that for each* $f \in \mathcal{CMM}(G)$*, the set* $\mathrm{Per}(f)$ *of periodic points is dense in* G *and* $h_{\mathrm{top}}(f) > 0$.

Let us recall that a matrix $M = (m_{ij})_{i,j\in\mathcal{A}}$, where the index set $\mathcal{A}$ is finite or countably infinite, is:

- *irreducible* if for each pair of indices i, j, there exists a positive integer n such that $m_{ij}(n) > 0$;
- *aperiodic* if for each index $i \in \mathcal{A}$, the value $p(i) = \gcd\{\ell : m_{ii}(\ell) > 0\}$ is equal to 1.

Remark 4.3. For $f \in \mathcal{CMM}(G)$ with Markov partition $\mathcal{P}$, its transition matrix $M = M(f) = (m_{ij})_{i,j\in\mathcal{P}}$ is irreducible and aperiodic.

In the following we use the approach suggested by Vere-Jones [**17**].

PROPOSITION 4.4. *Let* $M = (m_{ij})_{i,j\in\mathcal{A}}$ *be a non-negative irreducible aperiodic matrix indexed by a countable index set* $\mathcal{A}$*. There exists a common value* λ_M *such that for each* i, j,

$$\lim_{n\to\infty} [m_{ij}(n)]^{1/n} = \lambda_M.$$

The *Gurevič entropy* of a non-negative irreducible aperiodic matrix M is defined as

$$h(M) = \sup\{\log r(M') : M' \text{ is a finite square submatrix of } M\},$$

where $r(M')$ is the largest eigenvalue of the finite matrix M'.

We can also ask about the entropy of the corresponding Markov shift (Σ_M, σ). When the index set is infinite, this system is non-compact, so there are many possible notions of entropy. We will write $h(\Sigma_M)$ to denote the supremum of entropies of ergodic shift-invariant Borel probability measures.

In [**7**], Gurevič proved the following proposition. An accessible proof in English can be found in [**9**].

PROPOSITION 4.5. $h(\Sigma_M) = h(M) = \log \lambda_M$.

We wish to interpret the Gurevič entropy for the transition matrix $M = M(f)$ of a graph map $f \in \mathcal{CMM}(G)$. To do it, we need to understand more closely the connection between the dynamical system (G, f) and its symbolic dynamics (Σ_M, σ). From the point of view of topological dynamics, the relationship is given by the map

$$\psi : \Sigma_M \to G, \quad \psi(i_0 i_1 \cdots) = x, \quad \text{where } \{x\} = \bigcap_{n=0}^{\infty} [i_0 i_1 \cdots i_n]. \tag{17}$$

Using Lemma 2.15, it is easy to verify that ψ is well defined, continuous, and that $\psi \circ \sigma = f \circ \psi$. Unfortunately, ψ in general is neither injective nor surjective, so we cannot speak of a conjugacy or semiconjugacy.

Nevertheless, the relationship between our graph map and its symbolic dynamics is useful from the point of view of ergodic theory. The following definition is due to Hofbauer [**8**].

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Definition 4.6. In a measurable dynamical system (X, T), a measurable set $N \subset X$ is called *small* if it is backward invariant ($T^{-1}(N) \subset N$) and if every invariant probability measure μ concentrated on N has entropy $h_\mu = 0$. Two measurable dynamical systems (X, T), (X', T') are *isomorphic modulo small sets* if there exist small sets $N \subset X$, $N' \subset X'$ and a bimeasurable bijection $\psi : X\backslash N \to X'\backslash N'$ such that $T' \circ \psi = \psi \circ T$.

LEMMA 4.7. *A small set is a measure-zero set with respect to every ergodic invariant probability measure of positive entropy.*

Proof. Because a small set is invariant, each ergodic measure assigns to it either full or zero measure. If it is full, then the entropy must be zero. □

Our dynamical systems (G, f), (Σ_M, σ) become measurable dynamical systems as soon as we equip them with their Borel σ-algebras. Then we have the following result.

THEOREM 4.8. *Let $f \in \mathcal{CMM}(G)$ with Markov partition $\mathcal{P}$ and transition matrix M. Then (G, f) and (Σ_M, σ) are isomorphic modulo small sets.*

Proof. Recall the definitions of ψ in (17) and Q in Lemma 2.15. We will show that:

(i) $Q \subset G$ and $N = \psi^{-1}(Q) \subset \Sigma_M$ are small sets;
(ii) the restricted map $\psi : \Sigma_M\backslash N \to G\backslash Q$ is a bijection; and
(iii) with respect to the Borel σ-algebras, both $\psi : \Sigma_M\backslash N \to G\backslash Q$ and its inverse are measurable.

We start with (i). Backward invariance of Q is clear from the formula $Q = \bigcup_{n=0}^{\infty} f^{-n}(P)$. The set N inherits backward invariance from Q because of the relation $f \circ \psi = \psi \circ \sigma$. By Lemma 2.15(i), Q is countable and therefore every invariant measure concentrated on Q has entropy zero. Finally, we argue that N is also countable. Let $x \in P_n$. It is enough to show that any two points $i = i_0 i_1 \cdots$, $i' = i'_0 i'_1 \cdots$ in $\psi^{-1}(x)$ with $i_0 \cdots i_n = i'_0 \cdots i'_n$ must be equal. We do it by induction. Suppose that $i_0 \cdots i_t = i'_0 \cdots i'_t$ with $t \geq n$. Since $x \in P_n \subset P_t$, we see that x is an end point of the $\mathcal{P}_t$ partition arc $[i_0 \cdots i_t]$. Both of the subarcs $[i_0 \cdots i_{t+1}]$ and $[i'_0 \cdots i'_{t+1}]$ also contain x and therefore their interiors are not disjoint. Since these are $\mathcal{P}_{t+1}$ partition arcs, we get $i_{t+1} = i'_{t+1}$. This concludes the induction step and the proof that N is countable.

We prove (ii) by giving the formula for the inverse map. It is the so-called itinerary map $\phi : G\backslash Q \to \Sigma_M\backslash N$ given by setting $\phi(x) = i_0 i_1 \cdots$ if $f^n(x) \in i_n \in \mathcal{P}$ for all $n \geq 0$. It is just a matter of checking the definitions to see that $\phi(x) = i_0 i_1 \cdots$ if and only if $x \in [i_0 \cdots i_n]$ for all n, which happens if and only if $\psi(i_0 i_1 \cdots) = x$. Thus, ϕ, ψ are inverses to each other.

Next we prove (iii). We already know that $\psi : \Sigma_M \to G$ is continuous. It follows that the restricted map $\psi : \Sigma_M\backslash N \to G\backslash Q$ is also continuous and therefore measurable. To prove measurability of the inverse, we will show that ϕ is continuous at each point where it is defined. Fix a point $x \in G\backslash Q$ and write $\phi(x) = i_0 i_1 \cdots$. Fix $n \geq 0$. We will show that x has a neighborhood U in the graph G such that $U \subset [i_0 \cdots i_n]$. Since $x \notin P_n$ and P_n is closed, we know that x has a neighborhood V in G with $V \cap P_n = \emptyset$. By local connectedness of G, there is a connected neighborhood U of x contained in V.

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Since U is connected and does not intersect P_n, it must be contained in a single arc of $\mathcal{P}_n$ and therefore $U \subset [i_0 \cdots i_n]$, as desired. □

COROLLARY 4.9. *If $f \in \mathcal{CMM}(G)$ has transition matrix M, then $h_{\text{top}}(f) = \log \lambda_M$.*

Proof. Consider the sets $\mathcal{E}_+(\Sigma_M, \sigma)$, $\mathcal{E}_+(G, f)$ of positive-entropy ergodic invariant Borel probability measures on our two systems. If we take the supremum of entropies of measures over these two sets, we get $\log \lambda_M$ and $h_{\text{top}}(f)$, respectively; this uses Proposition 4.5, the variational principle for the continuous map f on the compact space G, and the fact that both Σ_M and f have positive entropy. By Lemma 4.7, our isomorphism modulo small sets ψ induces a bijection $\psi_* : \mathcal{E}_+(\Sigma_M, \sigma) \to \mathcal{E}_+(G, f)$, $\psi_*(\mu) = \mu \circ \psi^{-1}$ which preserves entropy: $h_{\psi_*\mu}(f) = h_\mu(\sigma)$. □

THEOREM 4.10. *Let $f \in \mathcal{CMM}(G)$. Then there is a sequence $(s_n)_n$ of positive integers such that f^n has an s_n-horseshoe and*

$$\lim_n \frac{1}{n} \log s_n = h_{\text{top}}(f).$$

Proof. Choose a partition $\mathcal{P}$ and let M be the associated transition matrix. Fix a partition arc $j \in \mathcal{P}$. We use Propositions 4.4, 4.1(ii), and Corollary 4.9. By those statements, $h_{\text{top}}(f) = \log(\lim_n [m_{jj}(n)]^{1/n})$ and, for each n, the arc j contains $m_{jj}(n)$ arcs $j_1, \ldots, j_{m_{jj}(n)}$ with pairwise-disjoint interiors such that $f^n(j_i) \supset j_\ell$ for all $1 \le i, \ell \le m_{jj}(n)$. Clearly, the map f^n has an $m_{jj}(n)$-horseshoe [**11**]. Let $s_n = m_{jj}(n)$ and the proof is finished. □

COROLLARY 4.11. *Similarly as for the interval case* [**1**]*, one can deduce from Theorem 4.10 the following:*

(i) $h_{\text{top}}(f) = \sup\{h_{\text{top}}(f|_M) : M \subset G \text{ is minimal}\}$;

(ii) $\limsup_n (1/n) \log \text{card}\{x \in G : f^n(x) = x\} \ge h_{\text{top}}(f)$.

5. *Entropy, Hausdorff dimension, and Lipschitz constants*

Let (X, d) be a non-empty compact metric space with Hausdorff dimension $\text{HD}_d(X)$ and let $f : X \to X$ be a Lipschitz continuous map with Lipschitz constant $\text{Lip}_d(f) = \sup_{x \ne y} d(f(x), f(y))/d(x, y)$. We write $\log^+(x)$ to denote the maximum of $\log(x)$ and 0. The following inequality is well known [**6**, **12**].

PROPOSITION 5.1. *An upper bound for the topological entropy is given by*

$$h_{\text{top}}(f) \le \text{HD}_d(X) \cdot \log^+ \text{Lip}_d(f).$$

Replacing the metric on X with another compatible metric can change both the Hausdorff dimension and the Lipschitz constant. Thus, a natural question arises: by varying the metric d, can we make the product $\text{HD}_d(X) \cdot \log^+ \text{Lip}_d(f)$ as close to $h_{\text{top}}(f)$ as we like? Our construction of conjugate maps of bounded slope gives us a way to address this question for tame graph maps.

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Since by Proposition 4.4 the value λ_M^{-1} is a common radius of convergence of the power series $M_{ij}(z) = \sum_{n\geq 0} m_{ij}(n)z^n$, we immediately obtain for each pair $i,\ j \in \mathcal{A}$,

$$M_{ij}(\lambda^{-1}) \begin{cases} \in \mathbb{R}, & \lambda > \lambda_M, \\ = \infty, & \lambda < \lambda_M. \end{cases}$$

The following result first used in [**16**] will be useful when proving our theorem.

PROPOSITION 5.2. *Let $f \in \mathcal{CMM}(G)$ with a partition $\mathcal{P}$ and consider its transition matrix $M = (m_{ij})_{i,j\in\mathcal{P}}$. For each $\lambda > \lambda_M$ and $j \in \mathcal{P}$,*

$$\text{for all } i \in \mathcal{P} : \sum_{k\in\mathcal{P}} m_{ik}M_{kj}(\lambda^{-1}) = \lambda M_{ij}(\lambda^{-1})\left(1 - \frac{\delta_{ij}}{M_{ij}(\lambda^{-1})}\right) \leq \lambda M_{ij}(\lambda^{-1}). \quad (18)$$

Proof. See [**16**, Theorem 1]. The right-hand inequality follows from the fact that for irreducible M, $M_{jj}(\lambda^{-1}) > 1$ for each $j \in \mathcal{P}$. □

PROPOSITION 5.3. *Let $f \in \mathcal{CMM}(G)$ with a partition $\mathcal{P}$ and consider its transition matrix $M = (m_{ij})_{i,j\in\mathcal{P}}$. For $\lambda > \lambda_M$ and $j \in \mathcal{P}$, put $v = (v_i = M_{ij}(\lambda^{-1}))_{i\in\mathcal{P}}$. Then v is an expansive λ-subeigenvector which is deficient in the coordinate j only.*

Proof. Since the matrix M is irreducible and Proposition 4.1 is true, our definition of λ_M gives $1 < \lambda_M < \lambda$. The Kronecker symbol $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{jj} = 1$, so the inequality in (18) is in fact an equality except when $i = j$. Since $\lambda > 1$ and there is deficiency in only one coordinate, the subeigenvector is expansive. □

If a map $f \in \mathcal{CMM}(G)$ is leo, then by Remark 3.2(ii) every λ-subeigenvector is summable. So, by Proposition 5.3 for such a map, for each $\lambda > \lambda_M$ we have a summable expansive λ-subeigenvector and Theorem 3.3(ii) applies.

THEOREM 5.4. *Let $f \in \mathcal{CMM}(G)$ be leo. Then for each $\varepsilon > 0$ there is a metric ϱ on G compatible with the topology τ such that*

$$\mathrm{HD}_\varrho(G) \cdot \log^+ \mathrm{Lip}_\varrho(f) < h_{\mathrm{top}}(f) + \varepsilon.$$

Proof. Fix a partition $\mathcal{P}$ and let M be the corresponding transition matrix. In Corollary 4.9, we saw that $h_{\mathrm{top}}(f) = \log \lambda_M$ and from Corollary 4.2 we know that $\lambda_M > 1$. Given $\varepsilon > 0$, choose $\lambda > \lambda_M$ with $\log \lambda < h_{\mathrm{top}}(f) + \varepsilon$. In Proposition 5.3, we identified an expansive λ-subeigenvector v. Summability of v follows from the leo property; see Remark 3.2(ii). By Theorem 3.7, there is an inner metric ϱ on G with respect to which f has bounded slope λ. By Proposition 2.10(iii), $\mathrm{Lip}_\varrho(f) \leq \lambda$. Moreover, G is the union of countably many subsets which are either singletons or segments with respect to ϱ, so by the countable stability of Hausdorff dimension we get $\mathrm{HD}_\varrho(G) = 1$. Thus,

$$\mathrm{HD}_\varrho(G) \cdot \log^+ \mathrm{Lip}_\varrho(f) \leq 1 \cdot \log^+ \lambda < h_{\mathrm{top}}(f) + \varepsilon.$$ □

Question 5.5. Does Theorem 5.4 apply also to non-leo maps $f \in \mathcal{CMM}(G)$?

This seems to be a difficult question. Even in the case $G = [0, 1]$, we do not know the answer. This is a different issue than the infimum of Lipschitz constants addressed in [**5**].

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REFERENCES

[**1**] L. Alsedà, J. Llibre and M. Misiurewicz. *Combinatorial Dynamics and Entropy in Dimension One (Advanced Series in Nonlinear Dynamics, 5)*, 2nd edn. World Scientific, Singapore, 2000.
[**2**] L. Alsedà and M. Misiurewicz. Semiconjugacy to a map of a constant slope. *Discrete Contin. Dyn. Syst. Ser. B* **20** (2015), 3403–3413.
[**3**] M. Baillif and A. de Carvalho. Piecewise linear model for tree maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **11**(12) (2001), 3163–3169.
[**4**] J. Bobok and H. Bruin. Constant slope maps and the Vere-Jones classification. *Entropy* **18**(6) (2016), paper No. 234, 27 pp.
[**5**] J. Bobok and S. Roth. The infimum of Lipschitz constants in the conjugacy class of an interval map. *Proc. Amer. Math. Soc.* **147**(1) (2019), 255–269.
[**6**] X. Dai, Z. Zhou and X. Geng. Some relations between Hausdorff-dimensions and entropies. *Sci. China Ser. A* **41** (1998), 1068–1075.
[**7**] B. M. Gurevič. Topological entropy for denumerable Markov chains. *Dokl. Akad. Nauk SSSR* **10** (1969), 911–915.
[**8**] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Israel J. Math.* **34**(3) (1979), 213–237.
[**9**] B. P. Kitchens. *Symbolic Dynamics: One-sided, Two-sided, and Countable State Markov Shifts (Universitext)*. Springer, New York, 1998.
[**10**] J. Milnor and W. Thurston. On iterated maps of the interval. *Dynamical Systems (Lecture Notes in Mathematics, 1342)*. Springer, Berlin, 1988, pp. 465–563.
[**11**] M. Misiurewicz. Horseshoes for mappings of an interval. *Bull. Acad. Pol. Sci. Sér. Sci. Math.* **27** (1979), 167–169.
[**12**] M. Misiurewicz. On Bowen's definition of topological entropy. *Discrete Contin. Dyn. Syst. Ser. A* **10** (2004), 827–833.
[**13**] M. Misiurewicz and S. Roth. Constant slope maps on the extended real line. *Ergod. Th. & Dynam. Sys.* **38** (2018), 3145–3169.
[**14**] S. B. Nadler. *Continuum Theory: An Introduction*. Marcel Dekker, New York, 1992.
[**15**] W. Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.* **122** (1966), 368–378.
[**16**] W. Pruitt. Eigenvalues of non-negative matrices. *Ann. Math. Statist.* **35**(4) (1964), 1797–1800.
[**17**] D. Vere-Jones. Geometric ergodicity in denumerable Markov chains. *Q. J. Math. Oxford Ser.* **13** (1962), 7–28.
[**18**] D. Vere-Jones. Ergodic properties of nonnegative matrices I. *Pacific J. Math.* **22** (1967), 361–386.

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