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WEAK DISHARMONY: SOME LESSONS FOR PROOF-THEORETIC SEMANTICS

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Abstract. A logical constant is weakly disharmonious if its elimination rules are weaker than its introduction rules. Substructural weak disharmony is the weak disharmony generated by structural restrictions on the eliminations. I argue that substructural weak disharmony is not a defect of the constants which exhibit it. To the extent that it is problematic, it calls into question the structural properties of the derivability relation. This prompts us to rethink the issue of controlling the structural properties of a logic by means of harmony. I argue that such a control is possible and desirable. Moreover, it is best achieved by global tests of harmony.

§1. Preliminaries. Within the proof-theoretic tradition, the meaning of the logical constants is thought of as determined by the rules that govern their introduction and elimination in proofs. Famously, Michael Dummett argued that a logical constant is endowed with a coherent meaning if and only if it is *harmonious*: if, that is, the inferential strength of the rules which govern the constant's introduction matches that of the rules governing its elimination. This is the *desideratum of harmony*. Given the introduction rules (I-rules) for a logical constant there are two ways in which its elimination rules (E-rules) may be defective: they may be too strong or they may be too weak. Conversely, given the E-rules, the I-rules are defective if they are either too strong or too weak. Too weak E-rules (or too strong I-rules) generate *weak disharmony* (cf. Steinberger 2011b). (Henceforth, I shall refer to I/E pairs as 'intelims'.)

While somehow neglected in the traditional debate, weak disharmony recently became a prominent topic in proof-theoretic semantics, where much effort is devoted to finding suitable ways of dealing with it. Particularly challenging is *substructural* weak disharmony. This is the type of intelim mismatch that occurs when the weakness of the E-rules ensues from structural restrictions affecting them.

This paper investigates weak disharmony and the lessons that it teaches us about the proof-theoretic project. I argue that substructural weak disharmony and common (non-structural) weak disharmony are different phenomena. The latter is a defect of the constants which instantiate it. The former, while potentially troublesome, testifies to problems in the structure of the derivations. Thus it is not necessarily a defect of the logical constant. This brings about the question of whether and in what form it is possible to control the structural properties of a logical system via harmony. To answer this, I start with the observation that the rules which govern a logical constant's behaviour also codify structural information. I show that some of this information is a *sine qua non* prerequisite for the constant in question. For it, structural control via harmony is unavoidable. When the structural information

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is more complex, the control is both desirable and achievable via Belnap's dual criterion of conservativeness and uniqueness.

Section 2 is an interlude on a somehow preliminary problem: whether the rules for *qor* even qualify as the kind of rules that should be tested for harmony. As it is easy to guess, my answer is affirmative. In Section 3, I briefly discuss Dummett's account of harmony, tracking the split of the aspirational notion, i.e., the *desideratum* of harmony, into harmony and *stability*. Lack of stability is weak disharmony. In Section 4, I substantiate the claim that substructural weak disharmony is distinct from nonsubstructural weak disharmony. I show, for different accounts of stable harmony, that while they successfully rule out nonsubstructural weakly disharmonious constants, they fail when it comes to substructural weak ones. My main test cases are two weakly disharmonious connectives: *knot* (dual to Prior's *tonk*) and *qor* (a substructural disjunction). On this basis, in Section 5, I reassess the intuitive diagnosis of *qor* as weakly disharmonious. If the substructural nature of *qor* is taken into account, this diagnosis is misguided. The addition of *qor* to a logical system is also a modification of the given 'context of deducibility'. This is a salient parameter when judging harmony and stability. I argue that while *qor* can generate problems because of the type of structural context of deducibility that it engenders, it is not disharmonious. In Section 6, I wrap all this together by discussing the possibility of structural control by means of harmony. I conclude in Section 7.

Before moving on, I pause to settle some technical details. Throughout this paper, symbols are their own names; I rely on context to disambiguate between use and mention. Notation is generally standard and assumed to be familiar. Less usual options are explained as they occur. Familiar abbreviations are used without explanation (e.g., 'iff' for 'if and only if'). Natural deduction rules are presented in sequent format (Dummett 1977), with assumptions given with maximal generality. Some terminology borrowed from Humberstone (2011), will recur often; I explain it here. A structure $\Gamma : \Delta$ is a sequent; the left-hand side of a sequent is its antecedent, the right-hand side its succedent. Antecedents and succedents are (possibly empty) finite multisets of formulae. The set-theoretic structure of a sequent constitutes its *framework*; by extension, a framework is the set-theoretic structure underlying a calculus. In the case of single conclusion natural deduction the framework is MSET:FMLA, with 'mset' and 'fmla' contractions for, respectively, 'multiset' and 'formula' (i.e., 'singleton multiset'). Multiple-conclusion natural deduction is based on the framework MSET:MSET, etc.

§2. Interlude. Substructural connectives of the type I consider could be ruled out by the *context generality* condition, CGC, *mentioned* in Humberstone (2011) and *defended* in Francez (2015).[1] According to CGC an I/E rule must not be subject to structural restrictions. But *qor* is given by such a restriction: it is a disjunction in whose E-rule no assumptions apart from those of the disjuncts can be used in the subordinate derivations contained in the rule (see below, Section 4). This solves the problem before it is born—and leaves one to wonder why so much effort was put into developing an account of harmony that rules out *qor*. It also confirms my subsequent diagnosis, that *qor* should not be dealt with by way of local harmony. It does not confirm my belief that there is nothing wrong with *qor simpliciter*. Be that as it may, I reject CGC as too costly and claim that a rule being subject to structural restrictions is not, *eo ipso*, a reason to think that it is unfit to act

[1] I am grateful to an anonymous reviewer for this journal for having raised this issue; this section benefited greatly from the input of the reviewer.

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as a defining rule. As far as I know, Francez's is the only extant defence of the CGC. This defence concerns a version of the CGC best understood as applying to single-conclusion natural deduction which reads as follows:

> Context variables should be present in every premise and conclusion, ranging (...) over arbitrary object language contexts. (p. 47)

Notice that the restriction to natural deduction in MSET:FMLA is significant and chips down the generality of the condition. Natural deduction in MSET:MSET forces parametric formulae in the succedent. In such a calculus, disjunction could be eliminated by the rule $\frac{\Gamma : A \vee B}{\Gamma : A, B}$, provided that in the subsequent rule application at most one of A and B are used. But in such rule applications the unused conclusion becomes a context. So it looks like a fully general CGC forces us to pick MSET:MSET natural deduction formulations over MSET:FMLA ones. Apart from that, the status of the additive connectives (i.e., those in which the multi-premise rules require identical contexts in the premises) is unclear (in the absence of Weakening and Contraction, cf. Humberstone (2011)). Is the CGC meant to rule them out? I assume that this is an unwanted result—thus it seems to me that the CGC overgenerates. Questions of scope and genuine generality aside, I also see an infelicitous disparity between the reasons for defending the CGC and its effects in relation to *qor*. Francez (2015) claims that the CGC is required in order to ensure the noncircularity (and sometimes the purity) of the rules. It is true that it has this effect. There are some logical constants the behaviour of which is characterised by rules subject to side restrictions which themselves mention logical constants. Such is, e.g., the rule for necessity introduction in S4: $\frac{\Gamma : A}{\Gamma : \Box A}$, which can only be applied if all the formulae in Γ are of the form $\Box B$ or $\neg\Diamond B$. Clearly, CGC rules it out. This rule has an air of circularity about it, as it seems that grasping the 'box' is required for introducing it. So blocking it may be a good, if risky, idea. But *this* justification of the CGC does not apply to *qor*. The condition which affects it induces no threat of circularity, as it mentions no connectives and is thus indifferent of the languages to which *qor* may be added. So, given the reasons for adopting the CGC it seems that *qor* is a collateral victim.[2]

§3. Harmonies. In Dummett's discussion of harmony in (1991), the term no longer stands for the desiderative notion mentioned in the previous section. Instead, it receives a more precise meaning, which, however, captures only *half* of the aspirational notion. Harmony is connected with an introductions-first (or verificationist) approach to meaning. Thus, it captures the match between a *given* set of I-rules and the E-rules that can be justified on their basis. It is deployed to rule out strong disharmony.

Even this notion receives different elucidations. There is a *global* form of harmony, identified with conservativeness. A constant $\natural$ is conservative over (a system of constants) $\mathcal{L}$ if the addition of its rules to $\mathcal{L}$ does not increase the stock of provable sentences that are expressed solely in the original ($\natural$-free) vocabulary. A second criterion of harmony applies only to the logical constants; this is *intrinsic* harmony. An E-rule is justified (and thus forms, together with the corresponding I-rule, an intrinsically harmonious intelim)

[2] The reader will discover that I will, in fact, suggest that we use a dual condition, of *minimal* generality with respect to side conditions. I provide a more comprehensive discussion of the context generality condition in Dicher (2016).

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iff its conclusion can be obtained without applying it, provided that the last step in the derivation of its major premise is an I-rule (Prawitz 1965, Gentzen 1969). To show this, it is enough to show that applications of $\natural$E which immediately follow the application of $\natural$I are redundant. We do this by way of the familiar reduction procedures which lie at the core of the proof of normalisation. Neither conservativeness nor reducibility guard against weak disharmony. Hence neither global nor intrinsic harmony correspond to the aspirational notion of harmony.

To ward off weak disharmony, Dummett introduces a requirement of *stability*. This is based on what Dummett calls the 'downward justification procedure' or 'complementation', i.e., a justification of the I-rules on the basis of E-rules. The idea is, roughly, that an arbitrary inference rule is justified iff the consequences of its conclusion are derivable, it being assumed that the E-rules are justified. Since the range of consequences of a given sentence is quite vast, it is more convenient to consider only *canonical* consequences of a given sentence, where a consequence is canonical iff it is obtained by an application of the E-rules for the sentence's main operator. Suppose that there is a derivation δ of a conditional $A \rightarrow B$. Since we are interested in justifying I-rules, we can assume, without (much) loss of generality, that δ ends in an application of $\rightarrow$I. Thus there is a derivation δ' of B from A, which we assume to be valid. To show that $\rightarrow$I is valid we need to show that, given the conditions for its application, we can get the canonical consequences of $A \rightarrow B$. There is only one such consequence, namely B, which also depends on there being a valid derivation π of A. Extending δ with $\rightarrow$E, we obtain:

$$\dfrac{\dfrac{\begin{matrix}\vdots\delta'\\ \Gamma : A \rightarrow B\end{matrix}}{}\rightarrow\mathrm{I} \qquad \begin{matrix}\vdots\pi\\ \Gamma' : A\end{matrix}}{\Gamma, \Gamma' : B}\rightarrow\mathrm{E}$$

Now π extended by δ' is a valid derivation which establishes B. So, an application of $\rightarrow$I is justified: whenever we can apply $\rightarrow$I, we can get the canonical consequences of $A \rightarrow B$.[3]

Dualising *tonk* (Prior 1960) gives us a straightforward example of an unstable (weakly disharmonious) connective *knot* ($\triangledown$), defined by the rules:[4]

$$\dfrac{\Gamma : A \qquad \Gamma' : B}{\Gamma, \Gamma' : A \triangledown B}\triangledown\mathrm{I} \qquad \dfrac{\Gamma : A \triangledown B \qquad \Gamma', A : C \qquad \Gamma'', B : C}{\Gamma, \Gamma', \Gamma'' : C}\triangledown\mathrm{E}$$

It is trivial to check that *knot* cannot generate nonconservativeness and that it generates reducible maximal formulae. Yet *knot* is unstable, as $\triangledown$E is clearly weaker than $\triangledown$I. How will complementation show this?

E-rules such as those for *knot*—and thus disjunction—have the particularity that the conclusion of their application need not be a subformula of their major premise.[5] To deal with this, we modify the definition of canonical consequence so that only atomary sentences obtained by means of E-rules count as such. With that out of the way, suppose that δ is a derivation of $A \triangledown B$. Again, without loss of generality, suppose also that δ ends in an application of $\triangledown$I; so before this last rule, δ has two branches, δ_1 and δ_2, which we assume to be valid. The canonical consequences of $A \triangledown B$ are the atomary sentences that follow from both *knot*-juncts. Suppose that p is such a consequence.

[3] The precise description of the procedure, including a recursive definition of validity based on E-rules, can be found in Prawitz (2007), which corrects the account in Dummett (1991).

[4] Unnamed, the connective is mentioned in Došen & Schroeder-Heister (1985). It is called *knot* in Humberstone (2010), *tunk* in Hjortland (2009), and *plonk* in Steinberger (2011a).

[5] Such rules are known as *generalised* E-rules (Schroeder-Heister 1984, Read 2000).

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Then δ expanded by $\triangledown$E is:

$$\dfrac{\dfrac{\begin{matrix}\vdots\delta_1 \\ \Gamma : A\end{matrix} \qquad \begin{matrix}\vdots\delta_2 \\ \Gamma' : B\end{matrix}}{\Gamma, \Gamma' : A \triangledown B}\triangledown\mathrm{I} \qquad \Gamma'', A : p \qquad \Gamma''', B : p}{\Gamma, \Gamma', \Gamma'', \Gamma''' : p}\triangledown\mathrm{E}$$

The open assumptions A and B on the last two branches can be closed with, respectively, δ_1 and δ_2. In each case, we obtain a derivation of p. Thus $\triangledown$I is justified: whenever we can apply it, we can also apply $\triangledown$E and get the canonical consequences of a *knot*-sentence.

Therefore, complementation on its own fails to detect any flaw in the rules for *knot* and it does not *directly* deal with *knot*'s lack of stability. For all we know, $\triangledown$E may be the best E-rule we can justify.

If we want to expose *knot* as unstable, we cannot rely solely on this procedure. We need some principled way to determine which I-rule to pair with the given E-rule. Dummett describes the situation as follows:

> Suppose that we adopt [complementation], and start with a set [E] of elimination rules. By our procedure, we can determine which introduction rules are valid: say these form a set [I]. Now, with respect to this set [I] of introduction rules, [intrinsic harmony] is well-defined: so we can use it to determine which elimination rules are valid [. . .]. If we get back by this means to the set [E], or to some set interderivable with [E], in the ordinary sense, in the presence of [I], stability prevails; otherwise not. (Dummett 1991: 287–288)

The required principle is not explicitly stated, but it must be that one should choose the *strongest* available rules which are intrinsically harmonious and complementable.

This, then, is Dummett's story of stable harmony. Many a philosopher has found reasons for dissatisfaction with this account (Prawitz 1994, Read 2000, Steinberger 2011b; 2013) and many of these reasons have to do with the treatment of too weak E-rules. The received wisdom is that Dummett's account fails to deal with weak disharmony. Strictly speaking, this is true only because of the split of 'aspirational harmony' into harmony *per se* and stability. Nevertheless, Dummett's attitude towards stability is, to say the least, ambivalent. Whatever merits his notion of stability has, it ends up playing a very modest part in the discussion of intelim balance. He recognises the danger of weak disharmony, proposes some means to deal with it, and goes on to dismiss it as having 'less deleterious' effects. All in all, Dummett's solution to weak disharmony is sub-optimal. The test for weak disharmony is ineffective on its own and the combination of reducibility and complementation sabotaged by Dummett's insistence that conservativeness takes precedence over reducibility.

§4. *Qor* **and** *knot*. In this section I examine alternative accounts of stable harmony, looking at how they deal with substructural weak disharmony. I retain *knot* as a test case, instantiating nonsubstructural weak disharmony. Quantum disjunction—*qor* for short and $\curlyvee$ in symbols—will serve to illustrate substructural weak disharmony.[6] *Qor* has the

[6] This is Dummett's own example (1991: 288 sq.), and much of the discussion in the literature revolves around it. Quantum logic *per se* plays no part in what follows.

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same I-rules as 'standard' disjunction, but its E-rule is subject to the restriction that the subderivations originating in the disjuncts cannot use supplementary assumptions:

$$\frac{\Gamma : A \curlyvee B \qquad A : C \qquad B : C}{\Gamma : C}\ \curlyvee\text{-E}$$

Qor appears to be weakly disharmonious: $\curlyvee$E validates the $\vee$I rules, which in turn validate $\vee$E. It is plain that the pathologies of *qor* and *knot* have different underlying causes. *Knot*'s problem is the arbitrary character of the intelim pairing: $\triangledown$E *blatantly* disregards the meaning determined by $\triangledown$I and conversely. *Qor*, on the other hand, is a substructural variant of standard disjunction. As we shall see, this makes all the difference in the world.

The accounts to be discussed are all united by the aspiration to provide *local* criteria of harmony. That is, they rely on tests which purport to employ only the information conveyed by the tested intelim. The main finding is that while internal tests disqualify *knot*, they fail to expose substructural connectives like *qor* as unstable. We will see what to make of this in Section 5.

Deductive equilibrium harmony. Tennant proposes an account of harmony on which both I- and E-rules have a meaning conferring role, as both impose a determinate inferential behaviour of the constants. A logical constant is harmonious if these behaviours agree; hence the name *deductive equilibrium* harmony (de-harmony). So the intelim rules for a logical constant are determined simultaneously and harmony and stability are joined together. The proposal, refined in a series of papers (Tennant 1997; 2005), finds its ultimate form in Tennant (forthcoming). The intelim for a constant $\natural$ is harmonious iff the following two conditions are satisfied:

(1) The result of applying $\natural$I is the strongest proposition inferrable from the premises. To show that this condition is satisfied: (a) *all* the conditions imposed by $\natural$I must be employed; (b) full use must be made of the available $\natural$E rules; (c) one cannot use $\natural$I.

(2) The major premise of $\natural$E is the weakest proposition that can so feature. To show that this condition is satisfied: (a) *all* the conditions imposed by $\natural$E must be employed; (b) full use must be made of the $\natural$I rules available; (c) one cannot use $\natural$E.

Take, for instance, disjunction. Condition (1) obtains if $A \vee B$ is the strongest possible conclusion of $\vee$I. Suppose then that C follows from both $\vee$I rules, as per clause (a). We must show that C follows from $A \vee B$ by using $\vee$E, as per clause (b) and none of the $\vee$I rules, as per clause (c). This is shown by the derivation:

$$\frac{\Gamma : A \vee B \qquad \Gamma', A : C \qquad \Gamma'', B : C}{\Gamma, \Gamma', \Gamma'' : C}\ \vee\text{E}$$

According to condition (2), $A \vee B$ must be the weakest proposition which can be the major premise of $\vee$E. For conformity with clause (a), it must be that whenever $A \vee B$ follows from A as well as from B, it is entailed by any C that can be the major premise of $\vee$E. By using both I-rules for $\vee$, we can show this by the derivation:

$$\frac{\Gamma : C \qquad \dfrac{\Gamma', A : A}{\Gamma', A : A \vee B}\ \vee\text{I} \qquad \dfrac{\Gamma'', B : B}{\Gamma'', B : A \vee B}\ \vee\text{I}}{\Gamma, \Gamma', \Gamma'' : A \vee B}$$

which clearly complies with clauses (a)–(c) of (2). (Clause (c) is satisfied because $\vee$E is not *used* in the derivation, whose last step is *given*.)

Consider now *knot* and *qor*. In order to show that some proposition C, which can be the major premise of $\triangledown$E, entails $A \triangledown B$ in the prescribed fashion, we must show that

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the schema:

$$\frac{\Gamma : C \qquad \Gamma', A : D \qquad \Gamma'', B : D}{\Gamma, \Gamma', \Gamma'' : D}$$

can be employed, alongside $\triangledown$I, to derive $A \triangledown B$. This is possible only if $A \triangledown B$ could be substituted for D while preserving derivability, which, obviously, is not always the case.

By contrast, *qor* turns out to be stable: the derivations which showed $\vee$ to be harmonious go through even if the assumptions Γ' and Γ'' are removed. Tennant (forthcoming: 22) mends this *à la* Dummett, by postulating an external maximality condition: Counts as harmonious the *strongest* intelim which complies with the stipulated conditions. With this extension of the criterion for de-harmony, *qor* is exposed as unstable. Still, someone who finds Dummett's external solution dissatisfactory, is unlikely to be satisfied with Tennant's proposal.[7]

Local intrinsic harmony. Francez & Dyckhoff (2011) expand Dummett's intrinsic harmony with the further requirement of 'local completeness', which is designed to rule out unstable constants. (The resulting conception of harmony is known as *local intrinsic harmony*.) A derivation δ of a formula A is locally complete iff it can be expanded by an application of the E-rule for A's main operator and applications of the I-rules for it. This is clearly possible in the case of disjunction:

$$\frac{\begin{array}{ccc}\vdots\delta & \dfrac{\Gamma', A : A}{\Gamma', A : A \vee B}\vee\text{I} & \dfrac{\Gamma'', B : B}{\Gamma'', B : A \vee B}\vee\text{I} \\ \Gamma : A \vee B & & \end{array}}{\Gamma, \Gamma', \Gamma'' : A \vee B}\vee\text{E}$$

which is, therefore, stable. Once again, *qor* satisfies the local completeness condition: the derivation above, with $\curlyvee$ instead of $\vee$ and without the contexts Γ', Γ'' is valid. On the other hand, *knot* fails the test. In a derivation of the form:

$$\frac{\begin{array}{ccc}\vdots\delta & \dfrac{\Gamma', A : A}{\Gamma', A : ?}\triangledown\text{I} & \dfrac{\Gamma'', B : B}{\Gamma'', B : ?}\triangledown\text{I} \\ \Gamma : A \triangledown B & & \end{array}}{\Gamma, \Gamma', \Gamma'' : ?}\triangledown\text{E}$$

it is not in general possible to re-create the conditions for the application of $\triangledown$I. Faced with *qor*'s asinine determination to survive, the authors concede defeat. Local completeness, they admit, is merely a necessary condition for stability (Francez & Dyckhoff 2011: Section 3).

General-elimination harmony. Read (2000) proposes a test of harmony which relies on reading the appropriate E-rules from the given I-rules. The algorithm for determining harmonious E-rules is as follows. The conclusion of the I-rules figures as the major premise of the E-rule. Alongside the major premise, the E-rule features as many subderivations as there are I-rules. Each subderivation originates in assumptions of the premises of an I-rule and ends in some formula C. The E-rule concludes C and discharges the assumptions.

[7] Furthermore, Steinberger (2009) points out a potentially disastrous effect of the maximality condition. The quantifier rules are constrained by side conditions which control the variables. A rule which is not subject to restrictions is stronger than a rule so constrained. Tennant's co-determination procedure works for, e.g., $\exists$E even if one drops the usual variable restriction (that the parametric variable in the derivation of the minor premise is not free in the conclusion of the rule, in its major premise, or in the hypotheses on which the minor premise depends). Hence, the strengthened de-harmony criterion picks the unrestricted rule. This generates quantifiers that allow inferring $A(b)$ from $A(a)$ for arbitrary a and b, effectively acting as first-order *tonk*-ish operators.

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For instance, in the case of $\vee$, given the I-rules $\dfrac{\Gamma : A}{\Gamma : A \vee B}$ and $\dfrac{\Gamma' : B}{\Gamma' : A \vee B}$ we obtain the E-rule:

$$\frac{\Gamma : A \vee B \qquad \Gamma', A : C \qquad \Gamma'', B : C}{\Gamma, \Gamma', \Gamma'' : C}$$

i.e., the familiar $\vee$-E rule.[8]

The proposal for a stability test comes later, in Read (2015), and is a strengthened version of local completeness.[9] It requires not merely obtaining the major premise of the E-rule, but obtaining its *grounds*, i.e., the premises which licensed its application. This is very much in the spirit of local completeness as normally the recreation of a complexity maximum goes through the derivation of its grounds.[10] Disjunction, however, is rather different. The strengthened stability test requires each disjunct to be obtainable below a maximal formula. If multiple conclusions are allowed, then this is possible, since $\vee$E is equivalent to $\dfrac{\Gamma : A \vee B}{\Gamma : A, B}\ \vee E_m$. (The sense in which each disjunct is obtainable in the course of the derivation is given by the usual restriction operating upon $\vee E_m$: that its conclusions are not to be used simultaneously.) If, however, one insists on single conclusions, the requirement is unsatisfiable. So Read softens it: in single-conclusion frameworks, the best one can do is to show that if a disjunction holds, then if a disjunct does not hold, the other one does (2015: 300). This can be shown via a derivation of the form:[11]

$$\dfrac{A \vee B : A \vee B \qquad \dfrac{\dfrac{A : A}{A, \neg A : A}\,\mathrm{W} \qquad \dfrac{\neg A : \neg A}{A, \neg A : \neg A}\,\mathrm{W}}{A, \neg A : B}\,\neg\mathrm{E} \qquad B : B}{A \vee B, \neg A : B}\,\vee\mathrm{E}$$

Clearly, both *knot* and *qor* fail to satisfy this requirement.

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[8] In general, if a constant has k introductions, then its E-rule has $k + 1$ premises. If the I-rule features m concurrent premises, then there will be m E-rules, each employing one premise as the root of the pertinent subderivation (Read 2000; 2010). Complications arise in the case of I-rules which discharge premises; here, (higher-order) discharges of entire derivations may be required (Schroeder-Heister 1984). However, we need not worry about this here.

[9] The obvious choice would have been to appeal to an inverse determination procedure, which determines I-rules on the basis of given E-rules. For instance, $\vee$E indicates that disjunction is introduced by two rules (it has two minor premises), none of which discharges assumptions (the E-rule contains no higher order discharges); $\rightarrow$E generates a single I-rule, which discharges the assumption of the conditional's antecedent, etc. This purported test of stability disqualifies *knot*, as $\triangledown$E determines I-rules isomorphic to $\vee$I. On the other hand, there is nothing inherent in the structure of $\curlyvee$I that could constrain one way or another the use of supplementary assumptions in the minor premises of the $\curlyvee$E. Thus *qor* emerges unscathed. A fully inverse procedure, relying on *general* I-rules, has been developed by Milne (2012; 2015). The reader can check that it offers no more control over the assumptions employed in a rule than the procedures discussed here do (cf. Milne 2015: 195 sq., and also 198, esp. note 9).

[10] For instance, conditional local maxima are recreated as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma : A \rightarrow B \qquad \Gamma' : A}{\Gamma, \Gamma' : B}\,\rightarrow\mathrm{E}}{\Gamma, \Gamma', A : B}\,\text{Weakening}}{\Gamma, \Gamma' : A \rightarrow B}\,\rightarrow\mathrm{I}$$

and this derivation recovers the deduction of B from A which grounds $A \rightarrow B$. (For simplicity, I use the standard *modus ponens* as $\rightarrow$E. This is of no consequence, as *modus ponens* is a particular case of the ge-rule).

[11] For simplicity, I leave the contractions implicit.

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But is the test satisfactory? I set aside the fact that its yield, 'the best' we can do, still falls short of establishing the desired result. Taken to its letter, the test would render disjunction unstable in anything but multiple-conclusion frameworks. There is also the delicate matter of the involvement of negation in the test. Much of the appeal of ge-harmony lies in its nonrelational character. An intelim's justification depends only on parameters expressed in its component rules, with no extraneous factors to consider. One would expect that what goes for strong disharmony goes for weak disharmony too. But the presence of negation in the test shows that stability is assessed on the basis of *more* than just the form of the rules—indeed, on the basis of more than just the properties of the tested constant. Furthermore, the involvement of negation is hardly innocent. In languages which lack negation, no disjunction could qualify as stable. Yet there are disjunctive connectives which, while failing this test, are governed by perfectly balanced rules. This is the case with lattice disjunction, *lor*, which is like *qor* except that its I-rules are subject to the same restriction on the use of supplementary assumptions (Restall & Paoli 2005). (So *lor* is a disjunction in FMLA:FMLA.) Undoubtedly, this is not all there is to say about Read's stable ge-harmony. Nevertheless, it does seem that the test is not entirely adequate.

The not-so-subtle strategy. The previous accounts of stable harmony are fairly gentle in their approach. A bolder strategy is proposed by Kürbis (2013): it takes the presence of lateral restrictions as a mark of weak disharmony. Any rule affected by such restrictions is unstable. Alas, the results are rather disconcerting.

The proposal is a variation of Read's 'read off' approach, except that it doesn't regiment the rules into a specific form. The logical constants are classified into two types, based on their rules. Type 1 comprises constants with strictly one I-rule, which may or may not discharge assumptions, can have any form whatsoever, and whose conclusion joins all (and only) the premises and the discharged hypotheses of the rule. Type 2 consists of constants with strictly one E-rule, which is in general form.[12] The recipes for reading E-rules from I-rules and conversely are as expected. For instance, for type 2 constants ($\vee$, $\curlyvee$, $\triangledown$) each subderivation of the E-rule induces an I-rule whose premise is the discharged hypothesis of the subderivation and which concludes the major premise of the E-rule (2013: 6). An intelim is harmonious iff it consists of rules 'of type 1 or 2, with E/I-rules determined from an I/E rule in the way specified, and there are [*sic*] conditions on the application of the rules'.[13] Stability obtains when there are *no* restrictions on the application of the rules.

Both *knot* and *qor* are ruled out by the account—although for different reasons: the first as a straightforward result of the 'read off' approach, and the second because $\curlyvee$E is affected by lateral restrictions.

Unfortunately, the solution prompts more head scratches than *eureka* moments. The examples used in the presentation are either propositional connectives or modal operators. What about the quantifiers? While mentioned nowhere in the paper, there is no indication that they are excepted from the scope of the analysis. If the account covers them, then it runs into troubles. Since the quantifiers are subject to the well-known variable restrictions, it follows that they are weakly disharmonious. Should this be something that requires fixing,

[12] Officially, Kürbis allows the rule to be 'of any form' but then goes on to describe, somehow imprecisely, the ge-form.

[13] Presumably, the last clause misformulates the intended condition and the restrictions on the application of the rules are to be *allowed*, not *required*.

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we would have to drop the restrictions. Yet as already noticed in footnote 7, the obliteration of the restrictions generates disturbingly 'tonkish' quantifiers.[14]

§5. What is wrong with *qor*? The upshot is that none of the accounts of stability investigated in the previous section deliver a satisfactory solution to substructural weak disharmony. Even if they rule out *qor*, they generate collateral damages that give this victory a rather Pyrrhic flavour.

This failure shouldn't be surprising. On the contrary, if *qor*'s peculiarity is taken into account, it is precisely the result to expect. Quite simply, *qor* generates (or originates in) a modification of the deducibility framework, albeit one that is *ad hoc* and, so to speak, posterior to the official specification of the framework. Suppose that one adds *qor* to some standard system of natural deduction. It would be a mistake to describe this as *only* an enrichment of the vocabulary of the system. On top of that, it is also a modification of the framework: when *qor* is present, it is no longer the case that antecedents can uniformly be multisets of arbitrary finite cardinality. The addition of *qor* is *eo ipso* a modification of the structure of the deductions in that system.

This surreptitious framework modification is the source of our troubles with *qor*; it also explains why internal tests of stability fail to detect *qor*'s (purported) deficiency. Its E-rule merely adds supplementary restrictions upon derivations. A disjunction can be introduced on the basis of (as little as) either of its disjuncts. This means that whenever a disjunct is available, one can recover the entire disjunction. But the hypothetical availability of the disjuncts is inbuilt into the E-rule for disjunctive connectives. Thus the restriction cannot affect the connective considered on its own. Nevertheless, since it affects the structure of the derivations, the restriction on $\curlyvee$E affects *qor*'s ability to interact with other connectives. This is why *qor* fails Read's test. That test actually checks how *qor* interacts with negation and this is not a matter that can be assessed independently of the structure of the derivations. By contrast, *knot*, whose defects are otherwise promptly detected by every internal test for stability, impinges via $\triangledown$E upon the 'natural' grounds of its introduction.

The same facts explain the success of the tests which impose a maximality condition. Essentially, these require each I/E-rule to receive its strongest match (while, at the same time, not generating troubles). However, the strength of a rule is measured by its 'yield'. In general, even if a constant is assessed on its own, separately of any possible interactions, it still is the case that the 'yield' is a factor of both the constant's rules and the structure of the derivations, including the available structural transformations (weakening, contraction, etc.).

In itself and with some hindsight, this is not a particularly insightful observation. After all, it records a completely transparent fact about *qor*. What is surprising is that the observation played virtually no role in the debate about harmony. Somehow, the visible lack of balance between $\curlyvee$I and $\curlyvee$E ended up being more important than the exact nature of this unbalance. I would speculate that this is explainable given the history of *qor* and the *primus inter pares* status of standard natural deduction in proof-theoretic semantics.

[14] One might think that a more careful formulation of the criterion for stability could save the day for Kürbis. Since the quantifier restrictions do not appear to concern the framework, one could define a constant as stable if its rules are not affected by *framework restrictions*. This seems to be the strategy in Kürbis (2008), where he speaks of 'list restrictions'. But *lor*, although subject to structural restrictions (if MSET:FMLA is the default framework), seems perfectly balanced. Thus the formal test for stability seems to misfire.

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In relation to harmony, qor appears in Dummett (1991), in the wake of a heroic struggle to establish MSET:FMLA as *the* framework for proof-theoretic semantics. Officially, framework variations were ruled out and the adopted ideology prevented the recognition of *ad hoc* variations of the sort induced by qor.[15] Moreover, the exclusivity of MSET:FMLA lead to qor being introduced as a *clearly distinct* connective. It is, however, possible to regard qor in a different light, which emphasises both its substructural character and its relatedness to standard (indeed any) disjunction. I shall say more about this after an interlude in which I examine in more detail the precise manner in which qor generates troubles.

The example to follow, familiar from Dummett (1991), shows structural weak disharmony at its worst. The entire point of qor is that it does not obey some distributivity laws—for instance, the entailment $A \wedge (B \curlyvee C) \vdash (A \wedge B) \curlyvee (A \wedge C)$ is not valid. But an unsettling thing happens in $\{\vee, \curlyvee, \wedge\}$: the sequent $A \wedge (B \curlyvee C) : (A \wedge B) \curlyvee (A \wedge C)$ is derivable.

In order to grasp what really goes on here, it is convenient to derive the offending sequent in the sequent calculus. This is the derivation δ in Figure 1.[16] The derivation shows that $\vee$ and $\curlyvee$ cannot survive alongside (in the presence of $\wedge$), as the weaker connective collapses into the stronger. Moreover, their co-existence generates nonconservativeness since the above sequent, while not featuring $\vee$, can only be proved by using its rules. There is some irony in this spectacular mess: if nonconservativeness is the sign of trouble, then the offending constant is standard disjunction. So, standard disjunction is strongly disharmonious, despite having an intrinsically harmonious and stable intelim.

The conspicuous feature of the derivation in Figure 1 is the noneliminable presence of Cut, which occurs at a crucial junction in the derivation. Everything above the sequent $B \curlyvee C : B \vee C$ on its branch is obtained in a deducibility context which is compatible with qor. In a sense, up to the application of Cut, $\vee$ is a mere notational variant of $\curlyvee$. Cut forces qor into a richer structural context, which can accommodate $\vee$. Here, one exploits the rules for $\wedge$ and then contracts back to a context suitable for qor. Despite appearances, the problem of the derivation lies neither with $\vee$, nor with $\curlyvee$. It is a purely structural problem which resides in cutting across incompatible derivability contexts. To say that the contexts are incompatible means just that their interaction is destructive. Unchecked, the more generous context simply engulfs the other. But if Cut is dropped, then the two contexts of deducibility can survive unproblematically within the same system. Actually, there is no need to drop Cut: banning its cross-framework application suffices.[17]

[15] One may object that although the restriction which makes the existence of qor possible is structural, it is so intimately connected with the constant that emphasis on structure is pointless if not misleading. After all, a structural modification that affects only a constant is, trivially, constant-related. Moreover, the objection continues, the connection between the constant and the restriction is unavoidable. The obvious reply is to deny the objectors claims and maintain that there is a way of severing this connection. A general clause like 'no dischargeable assumption can be used alongside supplementary assumptions' when deployed so that it only applies to disjunctive connectives is all we need. Now this appears as a clumsy cheat. What the objector wants is a constant-independent formulation that can be used in any case whatsoever (within reason). But this is a misguided demand. A structural restriction is a structural restriction, irrespective of whether it applies to all or only to some derivations in a system.

[16] For typographical reasons δ is broken into the subderivations α, β, and γ. The rules used in the derivation are presented in Figure 2.

[17] For more on nontransitive or restrictedly transitive systems see Ripley (2013; 2015), Tennant (1994; 1997), Weir (2015).

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$$\alpha:\ \dfrac{\dfrac{\dfrac{B:B}{B:B\vee C}\vee\mathrm{R}\qquad\dfrac{C:C}{C:B\vee C}\vee\mathrm{R}}{B\curlyvee C:B\vee C}\curlyvee\mathrm{L}}{}$$

$$\beta:\ \dfrac{\dfrac{\dfrac{\dfrac{A:A}{A,B:A}\mathrm{W}}{A,B:A,C}\mathrm{W}\qquad\dfrac{\dfrac{A:A}{A,B:A}\mathrm{W}}{A,B:A,A}\mathrm{W}}{A,B:A,A\wedge C}\wedge\mathrm{R}\qquad\dfrac{\dfrac{B:B}{A,B:B}\mathrm{W}}{A,B:B,A\wedge C}\mathrm{W}}{A,B:A\wedge B,A\wedge C}\wedge\mathrm{R}$$

$$\gamma:\ \dfrac{\dfrac{\dfrac{\dfrac{A:A}{A,C:A}\mathrm{W}}{A,C:A,B}\mathrm{W}\qquad\dfrac{\dfrac{A:A}{A,C:A}\mathrm{W}}{A,C:A,A}\mathrm{W}}{A,C:A,A\wedge B}\wedge\mathrm{R}\qquad\dfrac{\dfrac{C:C}{A,C:C}\mathrm{W}}{A,C:C,A\wedge B}\mathrm{W}}{A,C:A\wedge B,A\wedge C}\wedge\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\beta}{A,B:A\wedge B,A\wedge C}\qquad\dfrac{\vdots\gamma}{A,C:A\wedge B,A\wedge C}}{A,B\vee C:A\wedge B,A\wedge C}\vee\mathrm{L}\qquad\dfrac{\vdots\alpha}{B\curlyvee C:B\vee C}}{A,B\curlyvee C:A\wedge C,A\wedge B}\mathrm{Cut}}{A\wedge(B\curlyvee C),A\wedge(B\curlyvee C):A\wedge C,A\wedge B}\wedge\mathrm{L\ twice}}{A\wedge(B\curlyvee C):A\wedge C,A\wedge B}\mathrm{C}}{A\wedge(B\curlyvee C):(A\wedge B)\curlyvee(A\wedge C),(A\wedge B)\curlyvee(A\wedge C)}\curlyvee\mathrm{R\ twice}}{A\wedge(B\curlyvee C):(A\wedge B)\curlyvee(A\wedge C)}\mathrm{C}$$

Fig. 1. SC derivation of $\curlyvee$-$\vee$ collapse.

$$\mathrm{Cut:}\ \frac{\Gamma:A,\Gamma'\qquad\Gamma'',A:\Gamma'''}{\Gamma,\Gamma'':\Gamma',\Gamma'''}\qquad\mathrm{Id:}\ \frac{}{A:A}$$

$$\mathrm{RW:}\ \frac{\Gamma:\Gamma'}{\Gamma:\Gamma',A}\qquad\mathrm{LW:}\ \frac{\Gamma:\Gamma'}{\Gamma,A:\Gamma'}$$

$$\mathrm{RC:}\ \frac{\Gamma:\Gamma',A,A}{\Gamma:\Gamma',A}\qquad\mathrm{LC:}\ \frac{\Gamma,A,A:\Gamma'}{\Gamma,A:\Gamma'}$$

$$\wedge L\mathrm{:}\ \frac{\Gamma,A\Rightarrow\Gamma'}{\Gamma,A\wedge B\Rightarrow\Gamma'}\qquad\wedge L\mathrm{:}\ \frac{\Gamma,B\Rightarrow\Gamma'}{\Gamma,A\wedge B\Rightarrow\Gamma'}\qquad\wedge R\mathrm{:}\ \frac{\Gamma\Rightarrow A,\Gamma'\qquad\Gamma\Rightarrow B,\Gamma'}{\Gamma\Rightarrow A\wedge B,\Gamma'}$$

$$\vee L\mathrm{:}\ \frac{\Gamma,A\Rightarrow\Gamma'\qquad\Gamma,B\Rightarrow\Gamma'}{\Gamma,A\vee B\Rightarrow\Gamma'}\qquad\vee R\mathrm{:}\ \frac{\Gamma\Rightarrow A,\Gamma'}{\Gamma\Rightarrow A\vee B,\Gamma'}\qquad\vee R\mathrm{:}\ \frac{\Gamma\Rightarrow B,\Gamma'}{\Gamma\Rightarrow A\vee B,\Gamma'}$$

Fig. 2. Sequent calculus rules.

What of the structural connection between *qor* and standard disjunction mentioned above, on p. 593? On a proof-theoretic account, the constants are identified by the (I/E) rules which they obey. So far, I treated $\curlyvee$ and $\vee$ as different *stricto sensu*. That is, I have treated their rules as different enough to make them different constants. Restall (2014) has a different take on constants which differ like $\curlyvee$ and $\vee$ do. With reference to the sequent calculus rules for negation, Restall states the gist of his position as follows:

> One of Gentzen's great insights in proof theory was that these rules [the sequent calculus rules for classical negation] also provide us with rules for negation if we are to reason using the strictures of intuitionistic logic. If we take the same rules but be careful to use only proofs in which there is at most one formula on the right hand side of a sequent, the instances

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> of the rules we use are [...] the *intuitionist* instances of the rules for negation. (Restall 2014: 283)

The idea is that rules which differ only structurally are *identical* and thus define identical logical constants. This is the *sameness thesis*. If the sameness thesis is true, $\curlyvee$ and $\vee$ have the same meaning and are the *same* constant.

While I do not believe that structural information can always be disregarded when assessing the identity of a logical constant, I do believe that the sameness thesis is true of constants like standard disjunction and qor—and, for that matter, lor. Since I do no plan to engage here in a full defence of (my favourite version of) the thesis, I will confine myself to making the following observation, which has the merit of both providing support for the sameness thesis and helping the case I am making in this paper.

The type of interaction between qor and standard disjunction documented above is an instance of connective-collapse. That is, in a logical system like that which accommodated the derivation from Figure 1, there is no distinction in terms of provable sequents between $\curlyvee$ and $\vee$. The former collapses into the latter and becomes a mere notational variant for it. I submit that this is best explained by positing that $\curlyvee$ and $\vee$ have the same meaning and thus are identical. The clues pointing in this direction are quite obvious. The collapse is triggered by cross-framework cuts that 'suspend' the substructural parsimony peculiar to qor. These cuts void the limitation modulating qor's deductive output. Consequently, the structurally restricted constant 'expands' to cover its full deductive potential as determined by the novel structural environment. It grows up into a standard disjunction.

So, what is wrong with qor? Not much, it would seem. Plenty can go wrong in its presence, but this has less to do with it, and more with the management of the deducibility context. The entire case against qor is, to put it metaphorically, a case of proof-theoretic entrapment. For the big picture of stable harmony, the moral is that (intelim-based) internal stability tests provide almost no means of structural control and cannot be used to regulate frameworks.

§6. Structural control via harmony. Let us see what this entire story teaches us about proof-theoretic semantics. An account of harmony meant to disqualify substructural connectives like qor professes to exert *some* form of control over structural properties. The question I want to address here is whether structural control via harmony is feasible. The answer is a definite 'Yes and no!', though fortunately not in the same sense and not at the same time. I will begin with the 'No!' part of the answer.

The correct answer to our question is negative insofar as the attempt at structural control is set up within the parameters of the accounts of stable harmony discussed in Section 4. These accounts aim to provide internal tests of stable harmony. For the reasons explored previously, neither account succeeds—insofar as the mark of success is rulling out qor. The paradigm case is that of the 'maximality' condition put forward by both Dummett and Tennant: that an intelim should contain the strongest acceptable rules. The need to regulate the rule match by *fiat* shows that neither test is true to its internal(ist) aspirations. Despite (slim) appearances to the contrary, the true scope of the maximality condition extends well beyond the rules in an intelim. Although it acts only upon them, it affects the entire structural framework which supports the deployment of the logical constants.

Nevertheless, the stubborn survival of disjunction (as qor and even lor) when its structural environment shrinks radically should prompt us to rethink the situation. It suggests a change of direction: instead of upward structural control, why not attempt a downward

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form? With this we reach the two-stage 'Yes!' part of the answer.[18] As we shall see in what follows, downward structural control via harmony is possible even by means of local tests. More importantly, seeing how it can be achieved will prompt us to rethink the desirability and value of insisting upon locality.

The previous discussion makes it plain that *every* operational rule carries *some* structural information.[19] This is easy to identify, especially in a sequent calculus setting, where the passive formulae of a rule do just that: encode structural information. Within the structural information expressed by a rule one can distinguish between *cardinal* and *ancillary* information. The former is information without which the rule is not formulable. The latter is structural information that is not essential for the survival of the constant. One needs no sophisticated test to distinguish in a rough fashion cardinal from ancillary information; it is enough to look at the rules. There is no way to define, say, multiplicative disjunction if the succedents cannot feature at least two formulae. Likewise, there is no way to define intuitionistic negation if antecedents cannot accommodate at least two formulae.[20]

There is a slightly less uncouth marker for discerning the two kinds of structural information. Think of intelims stripped of everything except (*i*) the active formulae in their rules and (*ii*) whatever passive formulae are required as a matter of syntax. There is no guarantee that these inference patterns define coherent constants. Whether this is or not the case can be determined by applying one's favourite (purely) local test for stable harmony to this most parsimonious formulation of the rules. If the rules pass the test, then they carry all the required structural information; moreover, all the information carried is cardinal. If they fail it, the constant requires more structural information. Just what information is needed can be determined by enriching the formulation of the rules and repeating the test. In general, for each logical constant it is possible to determine the lowest structural threshold at which the constant is viable. For instance, if standard (extensional) disjunction is stripped of unnecessary parameters, one obtains the rules for *lor* (+):

$$\frac{C : A}{C : A + B} +\mathrm{R} \quad \frac{C : B}{C : A + B} +\mathrm{R} \quad \frac{A : C \qquad B : C}{A + B : C} +\mathrm{L}$$

which also encode the lowest structural threshold that makes disjunctive connectives viable. These rules pass muster no matter what local test of stable harmony is applied: they pass Tennant's de-tests, they generate reducible local maxima and are locally complete, relative to FMLA:FMLA they even satisfy the maximality condition, etc.[21] Is more structural parsimony compatible with *lor*? No: taken together, these two rules show that *lor* needs two positions, one to the left and one to the right of the sequent sign. The most

[18] In effect, this amounts to endorsing a condition dual to the CGC discussed in section 2, i.e., a minimum generality condition for contexts.

[19] As an anonymous reviewer reminds me, this idea is not new. In various forms, the idea has been discussed in Paoli (2002), Humberstone (2007), Hjortland (2012); see also the earlier papers refered to therein. However, my proposal, developed at length in Dicher (forthcoming) is significantly different from these and more in line with that of Došen (1989).

[20] Remember that we are working in a sequent setting and that we take things at face value. If a rule 'looks like' it needs *n* positions on a side, then it does. There may be (subtle) ways to avoid this, but that need not bother us. They merely encode structural information in less conspicuous ways.

[21] They fail Read's test, but that test doesn't qualify as local to the extent that it involves negation.

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parsimonious rules for the conditional are:

$$\frac{A : B}{: A \to B} \to\text{R} \qquad \frac{C : A \qquad B : D}{C, A \to B : D} \to\text{L}$$

Again, the rules are in good order. Note though that the structural threshold is higher in this case. The rule $\to$L, which corresponds to $\to$E, shows that a conditional must have two positions available on the left of the sequent symbol. Is more parsimony possible? The answer is negative. Suppose that one imposes the visibility condition of Sambin et al. (2000). In this case, every active formula in a rule must appear alone. Thus $\to$L becomes:

$$\frac{: A \qquad B : D}{A \to B : D}$$

which is not enough to stabilise $\to$. For instance, it cannot be expanded in the manner prescribed by Francez & Dyckhoff (2011). In a sequent setting, this amounts to obtaining the sequent $A \to B : A \to B$ by applying $\to$R and $\to$L, which cannot be done without having an active formula accompanied by a parametric formula.

This is a trivial form of structural control via harmony which is unavoidable. One of the functions of harmony is to determine if the intelims carry enough structural information to render them viable inferential definitions. At this level, the locality condition makes perfect sense. Note, however, that here the connection between a logical constant and the structure of the derivability relation is so tight that the distinction itself becomes at best blurry, if not altogether spurious.

Still, this is an excessively modest form of structural control. There is little use for logical constants viable (only) in such parsimonious environments. Moreover, not all logical constants are bound together by the same minimal threshold. What is enough for disjunctions is not enough for conditionals, etc. So how do they come together? The minimal structure of a system of constants that have different viability thresholds must be that of the more demanding constant. Only this allows the different constants to interact fully within the system. (For this is the purpose of the ancillary structural information carried by a constant: to make room for its interaction with other constants.) Does this mean that any constant must be endowed with the maximally admissible degree of structural information? Not at all. As long as one does not harm the constant, one is free to control the constant's interaction in whichever fashion one desires. Whether anything good comes out of this is an entirely different question. What matters is that the *meaning* of the defined constant is coherent even if more structural information is squeezed into its rules. Besides, if the aim is achieving full interactiveness, relative maximality with respect to ancillary information is not enough. The static information alone is not enough to ensure it. For instance, even MSET:MSET is insufficient for the distributivity of conjunction over disjunction. This requires the use of weakening and contraction. So full interactiveness is not achieved by maximising the structural endowment of every constant in the system. One also needs structural inference principles for the manipulation of structural information.

What is the upshot of all this? On the face of it, one may accept most of what I have said in the previous paragraphs and still believe that harmony should be tested locally or at least as locally as possible. This would prevent us from endorsing a fully global test for harmony such as conservativeness. The idea is that while every bit of information available with respect to a given intelim should be part of the test's parameters, the test itself must, nonetheless, remain local. This strategy is as global as it needs to be (in the sense that it considers globally relevant information), while really keeping as close as possible to

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the constant (since the assessment concerns only the impact of this information over its intelim). One may, thus, side with Steinberger when he says that:

> conservativeness simply is not a relational property of the right kind: general harmony [. . .] is a property of pairs of inference rules; conservativeness is a property of systems and pairs of inference rules. Unlike the notion of conservativeness, the notion of general harmony we are after is a local, not a global property. Consequently, systematic conservativeness could not possibly offer a suitable formalization of the intuitive notion of general harmony. (Steinberger 2011b: 625)

Unfortunately, this is a very ineffective treatment of harmony. First, the setup whose victims were *qor* and standard disjunction shows that although the structural information which is encoded in each constant is very modest, it can have far reaching consequences in appropriate circumstances. Moreover, these are not locally detectable. A notion of harmony that fails to deliver an answer to the question of whether a logical constant is usable *hic et nunc* is sub-optimal in comparison with one that does deliver this information. Second, as long as we live under the impression that intelims convey only information pertaining to the constant which features in them, we may find some rationale for preferring locality. Nevertheless, if the previous analysis hits the mark, this impression is false. Most of the time most intelims are packed with information that in no way originates with or is required for the viability of the constant they govern. It only makes sense to go global because the rules themselves are, in a sense, global.

All in all, I submit that *local* structural control, while certainly possible (recte: unavoidable), is unfeasible when it comes to anything but the most structurally restrictive logical systems. There are no benefits associated with such tests for the simple reason that in most cases the logical constants are presented together with a great deal of structural information that does not characterise them directly. (Although it is crucial for characterising them given the potential presence of other constants.) Conservativeness is uniquely capable to expose and convey the requisite information about the behaviour of the logical constants in concrete instances of use.

Nevertheless, conservativeness alone is not enough: *knot* illustrates this aplenty. Still, *knot* can be dealt with satisfactorily by conservativeness' forgotten twin: *uniqueness* (Belnap 1962).[22] A logical constant is *uniquely* characterised by its itelim iff it is interderivable with any constant characterised by a notational variant of the given intelim. Consider now the sequent calculus rules for *knot* in MSET:MSET:

$$\frac{\Gamma : A, \Gamma' \qquad \Gamma'' : B, \Gamma'''}{\Gamma, \Gamma'' : A \triangledown B, \Gamma', \Gamma'''} \triangledown\mathrm{R} \qquad \frac{\Gamma, A : \Gamma' \qquad \Gamma'', B : \Gamma'''}{\Gamma, \Gamma'' A \triangledown B : \Gamma', \Gamma'''} \triangledown\mathrm{L}$$

[22] The fate of uniqueness is another semi-mystery in the history of harmony. Somehow, only conservativeness made its way into the 'public conscience' as a criterion of harmony. Uniqueness ended up playing almost no role at all. Again, it all boils down to Dummett's revisionist agenda. While perfectly aware of this second criterion's existence, none of Dummett's considered examples were are risk of being exposed as nonunique. From here on, the ball just rolled almost without a hitch: as far as I know, only Restall (2007; Restall; 2014) endorses Belnap's *dual* criterion of harmony. Recently, uniqueness as a criterion of definitional success has come under criticism in Naibo & Petrolo (2015). Space limitations prevent from going into a discussion of their criticism here, but the reader should be warned that a separate defence of uniqueness might be called for.

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as well as a notational variant of them formulated with the help of $\triangle$. A simple inverted proof search will reveal that neither of the sequents $A \triangledown B \ : \ A \triangle B$; $A \triangle B \ : \ A \triangledown B$ can be obtained by applying these rules. Thus *knot* is not uniquely defined by its rules. This ends the second stage of the 'Yes' answer to the question regarding structural control via harmony.

I shall wrap up this discussion by considering an objection—truly, the most straightfoward objection—to the idea of structural control via harmony. The established tradition of the harmony debate provides some *prima facie* support for the opinion that the negative answer is appropriate come what may. The traditional picture is that assessing the harmonious character of a logical constant is a matter posterior to the determination of the framework and of the valid structural transformations which characterise a logic. In some cases, this is simply presupposed—indeed, most of the accounts of harmony which employ a standard natural deduction formulation simply take its structural commitments *tale quale*. In other cases, the structural commitments are explicitly recognised. For instance, Belnap's solution to *tonk*, which inspired Dummett's criterion of total harmony, is lifted of the ground by the observation that:

> we are not defining our connectives *ab initio*, but rather in terms of an antecedently given context of deducibility, concerning which we have some definite notions. [...] [B]efore arriving at the problem of characterizing connectives, we have already made some assumptions about the nature of deducibility.[23] (Belnap 1962: 131)

Nevertheless, turning this into a full fledged argument against structural control via harmony is too quick a move. In particular, it is ignoring an important lesson contained, though not exploited, in Belnap's solution to *tonk*. If the definability of a logical constant goes hand in hand with the properties of the deducibility context, then their interaction speaks of both the constant and the context of deducibility. So, if a constant under-performs in a context, there are, in principle, two options for rectifying this: fix the constant, or fix the context. As Cook points out:

> [The] discussion of the legitimacy of logical connectives should focus not only on which ones are and are not acceptable *simpliciter*, but should also take into account which connectives are acceptable in which contexts (i.e., particular formal approaches to logical consequence). A mismatch between our account of the general acceptability of a particular connective [...] and its acceptability within a particular formal logic [...] should then lead us to rethink both the acceptability of the connective itself and the legitimacy of the particular conception of consequence in question. (Cook 2005: 223-224)

Mono-directionality is not an inbuilt feature of the test; it is merely a convenient solution to the problem at hand. All things considered, our *ratiocinative* practice is better served by banning *tonk* than by dropping Cut—or so Belnap thought. On top of this, since I have just showed that every logical constant carries some *sine qua non* structural information, we have good reasons to believe that the structure of the derivations in a logical system

[23] At the very least, the context of deducibility consists of framework properties and structural rules. More generous interpretations (like Dummett's) add the rules governing the behaviour of the other logical constants.

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is, in part, determined by the logical constants themselves. So there can be no absolute priority of the context of deducibility over the intelims which act within it. *A fortiori*, there can be no framework immutability.

§7. Conclusion. *Qor* is a stumbling block for the extant accounts of proof-theoretic stability. Insomuch as the relevant tests rely on intelim internal information, they fail to detect it. Mended by the addition of external criteria the tests generate undesirable consequences. I have argued that the precise source of the lack of balance exhibited by its rules must be taken into account. With that, we had to go back to the drawing board and show that *qor* is one of the determinants of the structure of the derivations. This subverts the intuition that *qor* is disharmonious; consequently, there is no need to mend the internal stability tests. That leaves open the possibility of criticising *qor*'s effects over the structural properties of the context of deducibility. However, this matter is best disconnected from considerations of harmony.

The most important lessons of this story emerge if we pay attention to the aspiration of the local accounts of stable harmony to exert control over the structural properties of a logic. I argue that this is unavoidable insofar as we use it to determine the minimal degree of structural involvement of the logical constants. This tells us what structural pressures are exerted by the constants upon the deducibility context. As we allow more structural information in the formulation of the rules (i.e., ancillary information), the control is possible and desirable. But since the structural information with which we are dealing here regards the interaction between the constants, it makes little sense to insist on local forms of control. It is best to go global. Conservativeness and uniqueness will tell us if a constant is harmonious.[24]

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