Rethinking Newton's *Principia*

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It is widely accepted that the notion of an inertial frame is central to Newtonian mechanics and that the correct space-time structure underlying Newton's methods in *Principia* is neo-Newtonian or Galilean space-time. I argue to the contrary that inertial frames are not needed in Newton's theory of motion, and that the right space-time structure for Newton's *Principia* requires the notion of parallelism of spatial directions at different times and nothing more.

1. Introduction. Few writings in physics have occasioned as much philosophical interest as Newton's *Principia*, but still puzzles remain. One is an alleged inconsistency in Newton's stated principles in application to a universe that is, quite possibly like the actual universe, infinite and approximately homogeneous in space.[1] Another is that the motions identified by Newton's methods as privileged ("true," according to Newton) could not, realistically, be the motions relative to an inertial frame—or not if the universe is sufficiently large.[2]

The two problems are evidently related. Indeed, if the material universe is in fact infinite and if Newton's theory really is inconsistent in that application, we should expect that no true motions in Newton's sense could have been identified. But that raises a further puzzle in its own right. For Newton surely did pick out definite motions as somehow preferred, his theory as laid

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1. See, e.g., Norton (1993) and Vickers (2009). For the early history of this problem, see Norton (1999).

2. This question was first raised by Kant. I shall come back to Kant's views shortly.

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down in the *Principia* (what I shall call Newton's theory of motion [NTM]) was empirically successful, and it was genuinely informative: how can an inconsistent theory be empirically informative?

Of course there are other cases of alleged inconsistencies in empirically successful theories. Relativistic quantum electrodynamics, with its famous divergences, is another example. It is tempting to place NTM in the same category and to view the inconsistency problem as an instance of a more general puzzle (of how inconsistencies can in practice be "contained"). That has been the tenor of several recent discussions of the subject.[3]

That is not the strategy taken here. My objective is to give a consistent reading of Newton's *Principia*—but largely discarding the Scholium to the Definitions—under which it applies equally to an infinite as to a finite mass distribution and, hence, (if only in the nonrelativistic limit) to the actual universe. On my reading, the nearest thing to an inertial frame is a nonrotating frame with arbitrary (and possibly time-dependent) linear acceleration. The privileged frame (the center of mass of the solar system), wrongly identified by Newton as inertial, indeed as at rest, is exemplary: it is a local freely falling nonrotating frame. The latter, moreover, is just the sort of frame that is privileged in Einstein's general theory of relativity (GTR) and in Cartan's nonrelativistic theory of gravity (so-called Newton-Cartan theory [NCT]).[4] These theories have more in common with NTM than is normally thought.

There is more. As John Norton has recently explained (Norton 1995), acceleration in NCT in an infinite universe is *relative*, not absolute. On the reading of *Principia* I am proposing the same is true but in a rather more traditional sense. It is that accelerations, as all other motions, are relational, in roughly the sense of the absolute-relational debate in Newton's time.[5] This reading extends to any theory satisfying Newton's laws, yielding a form of relationalism that applies to the entire scope of classical particle mechanics and Newtonian gravity.[6] In this respect it is unlike relationalism in the Machian sense, as developed by Julian Barbour.[7] The latter has the following feature: only systems that (in NTM terms) have total energy and angular mo-

3. An exception is Malament (1995), which showed how the problem can be solved (effectively by going over to the Newton-Cartan theory; see below).

4. See Knox (2009) for a defense of the claim that such frames play the same functional role in GTR that inertial frames in practice played in NTG.

5. I say "roughly," as its motivation is somewhat different: relationalists typically insisted that only directly observable quantities are physically real, abjuring indirect, dynamical determinations of other quantities, as I, following Newton, use here.

6. I also believe that it extends to nonrelativistic field theory, but I will not try to defend that claim here. Certain nonrelativistic field theories (including quantum mechanics) appear to already satisfy the relevant covariance requirements: see Hood (1970) and Rosen (1972).

7. See Barbour (1999) for a self-contained introduction.

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mentum zero can be defined. No such constraint—and no such prediction—follows from relationalism in our sense.

What about relationalism in Norton's sense, in NCT? To explain this we will need to use the modern apparatus of differentiable geometry, and the concept of a connection Γ, a rule for defining differentiation of vectors by continuous transport along smooth curves. Thus, suppose there is a manifold M equipped with global rectilinear coordinates $x^\alpha = \langle x^0, x^j \rangle$, $j = 1, 2, 3$ (so M has the topology of a Euclidean space). The coordinates x^α are eventually to be identified as inertial coordinates. In terms of these, let the connection Γ have as its only nonzero components the quantities

$$\Gamma^j_{00} = \frac{\partial \phi}{\partial x^j}, \tag{1}$$

where ϕ (the gravitational potential) satisfies

$$\nabla^2 \phi = -4\pi G \rho \tag{2}$$

with G the universal gravitational constant $G = 6.67 \times 10^{-11}$ $\mathrm{m^3 sec^{-2} kg^{-1}}$ and ρ the mass density. The equation of motion for a point particle with curve $x : \mathbb{R} \to M$, parameterized by real numbers λ, is then the same as in GTR. It is the geodesic equation:

$$\frac{d^2 x^\alpha}{d\lambda^2} = \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}. \tag{3}$$

Affine geodesics are curves whose tangent vector $dx^\beta / d\lambda$ at each point is the parallel transport of the tangent vector at any other point. Curves that depart from these represent the motions of accelerated bodies.

One can free the theory from dependence on a special coordinate system by introducing spatial and temporal metrics, satisfying suitable compatability conditions, and introducing a Riemann tensor as defined by the connection. The theory can then be written in a generally covariant form as in GTR. The result is NCT. It is more general than the theory based on equations (1)–(3). When the total mass is finite (an 'island universe'), the connection has a unique decomposition into a gravitational and inertial part:

$$\Gamma = \Gamma_{\text{grav}} + \Gamma_{\text{inertial}} \tag{4}$$

such that there exist global rectilinear coordinates in which Γ_{grav} satisfies equation (1) and Γ_{inertial} vanishes. Acceleration, as defined by departures from straight-line motions with respect to Γ_{inertial}, is then uniquely defined. More prosaically, in the coordinates in which Γ_{inertial} vanishes, nonaccelerating motions are given by linear equations in x^0 and x^j. In this context (and in this sense) acceleration is absolute. But Γ can still be defined even in the case of an infinite and homogeneous mass distribution. In that case the decomposition (4) is no longer unique, there are no privileged inertial frames, and there is no absolute meaning to acceleration. Acceleration is only defined rel-

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ative to a decomposition of the connection into an inertial and a gravitational part; in the case of an infinite mass distribution, no such decomposition is preferred. It is in this sense that Norton says acceleration is relativized in NCT.

When I say NTM can be read as a relational theory, I mean something rather different. In the first instance we are speaking of a particle theory, not a field theory. The mathematics is also elementary—it is a reading of *Principia*. And my claim is that motions are relative whether or not the matter distribution is infinite, in roughly the way relationalists like Huygens and Leibniz thought it was.[8] In the case of a finite number of particles, it is true, the system as a whole can be taken to define a frame of reference that is inertial as in the usual reading of *Principia*. Kant indeed took this as the only true inertial frame obtainable by Newton's methods, with the solar system only the first step (yielding only a first approximation) of an iterative procedure that, in principle, can only be completed when the entire universe is taken into account. Mach, and arguably Leibniz, thought the same. I am taking the opposite strategy of Kant's. I am suggesting a reading in which this limit is not necessary to the structure of the theory and plays no role in its application, and in which the concept of inertial motion in the usual sense may not even be defined.

As emphasized by Michael Friedman (1992) and Robert DiSalle (2006), the key to applying Newton's theory of gravity to only a part of the actual matter distribution—to beginning the iterative procedure envisaged by Kant as concluding only with the entire universe—is Corollary VI to the Laws. On my reading of *Principia* it is put to wider use: Corollary VI frees the theory from the need to give operational significance to the notion of inertial frame in the conventional sense. And it poses a further question. Newton thought absolute space and time and absolute velocities were needed to make sense of his Definitions and Laws, but the latter admitted more symmetries; they included boosts among inertial frames as proved in Corollary V (the relativity principle). As a result, Newton's absolute space and time and absolute velocities were dispensed with and replaced by Galilean space-time and relative velocities, with accelerations as the only absolute quantities. But the Definitions and Laws, as applied to the motions of bodies "among themselves," have still another symmetry, namely, the group of time-dependent boosts among nonrotating frames, as demonstrated by Corollary VI. What should then replace Galilean space-time and absolute accelerations? What is the real space-time structure implicit in *Principia*?

8. As we shall see, the reading of *Principia* I shall present meets Leibniz's objections based on the principle of sufficient reason. However Leibniz, erroneously, believed that rotations could be defined by sequences of linear accelerations, for reasons that had little or nothing to do with his general principles. For this reason, if no other, he would have had difficulty in accepting my reading. Huygens, by contrast, in his late writings, gave a very similar analysis of rotation to mine.

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I begin with the passage from Newtonian to Galilean space-time. This was consistent with all the propositions of Books 1 and 2 of *Principia*, but it did violence to the Scholium to the Definitions; it is a conservative extension of NTM all the same. It is our template for what follows. In section 3 I give an argument from cosmology—a near neighbor to the one I began with above—to show we cannot in practice by Newton's principles determine any inertial motions, neither in an infinite nor in a finite universe (so long as it is sufficiently large), so knowing those motions cannot possibly have played any role in the actual application of NTM to planetary motions or to any other motions that have actually been observed. Section 4 is on Corollary VI, and the explanation of the irrelevance of inertial motions to Newton's model of the solar system even in the favorable case of an island universe. In sections 5–7 I kick the ladder away, giving up Galilean space-time. These last sections are on relationalism and the definition of the "right" space-time for *Principia*.

2. Newton's Absolute Space. Newton worked hard, in the Scholium to the Definitions, to explain why a theory of motion required an absolute, immaterial backdrop—absolute space—that was not subject to motions of its parts or to any change. He saw plainly the inadequacy of the appeal, at the foundational level, to the stellar background or any other observable body as an absolute standard of rest: that could define only a "relative space," a "movable dimension or measure" of motion—quantities that may have nothing to do with true motions. No observable body was immovable. No idealization was safe.

The avowed aim of *Principia*, as stated in the Scholium to the Definitions, was to show how these true motions were to be determined, including absolute velocities. The Laws that immediately followed spoke of straight-line motions, and motions deviating from these—absolute accelerations. Nonaccelerating motions were "straight" by reference to the sequence of positions of bodies in absolute space at equal times, with equal distances traversed in equal (absolute) times (in four-dimensional terms, straight lines in Newtonian space-time). These locations in space, and space itself, were part of the furniture of the universe, even if they were insensible. The relative motions of bodies, subject to forces satisfying the Laws, were supposed to eventually yield the true motions. But by wide assent what Newton's methods actually gave were motions referred to Galilean rather than Newtonian space-time, in which no velocity is preferred.

Relationalists like Huygens and Leibniz went further. According to Leibniz, space was "nothing at all without bodies, but the possibility of placing them," and "instants, consider'd without the things, are nothing at all; . . . they consist only in the successive order of thing" (Alexander 1986, 26–27). His most damaging criticism was that according to Newton's principles, supposedly determinate physical properties and relations were undetermined by

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any possible measurement, however indirect: not just the absolute positions but the velocities of bodies too. They could be made whatever you like—there could be no "sufficient reason" as to why they are thus and so, nor could you ever know that they are thus and so. Neither the speed nor the direction of the motion of the material universe as a whole, with respect to absolute space, could have any empirical meaning, hence, neither could the absolute speed or absolute direction of motion of any body within it.

The critique applies equally to Galilean (or for that matter to Minkowski) space-time, if taken in the substantivalist sense (I shall come back to this sense in a moment), but it is particularly telling in the context of Newton's absolute space and time. There the problem can be stated independently of Leibniz's principles and of substantivalism. In space-time terms, Newton's absolute space picks out a preferred velocity—absolute rest—but the laws of motion do not. The symmetry group of Newtonian space-time is thus smaller than the symmetry group of the laws. The theory itself says there are these absolute quantities, whose values can never be obtained, for from the laws it follows (the principle of relativity):[9]

> Corollary V.—The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forward in a right line without any circular motion. (Cajori 1934, 20)

Given that the motions of bodies among themselves—if necessary, all bodies—are all that any possible observation can ever have to go on, absolute velocities can never be determined.

Newton was clearly aware of this difficulty. For textual evidence, consider: the third edition of *Principia* contains only three hypotheses, one in Book 2 and two in Book 3, the first and third of a technical nature. The second reads:

> Hypothesis I.—That the center of the system of the world is immovable.

It was followed with the remark: "This is acknowledged by all, while some contend that the earth, others that the sun, is fixed in that center. Let us see what may from hence follow" (Cajori 1934, 419). The proposition that followed and its proof resolved Leibniz's underdetermination problem—in a manner of speaking:

> Proposition XI, Theorem XI.—That the common center of gravity of the earth, the sun, and all the planets is immovable.

9. I am taking the laws to include the requirement that mass is a scalar and that force-functions do not depend on absolute positions or velocities, for only then is Corollary V a corollary (see Barbour 1989, 31–32; Brown 2005, 37–38). From sec. 5 on I shall, for simplicity, assume force functions do not depend on relative velocities either.

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> For (by corollary iv of the Laws) that center either is at rest, or moves uniformly forward in a right line; but if that center moved, the center of the world would move also, against the Hypothesis.

From the point of view of Galilean space-time, having argued, by Corollary IV, that the center of mass of the solar system moves inertially, Newton took it as a matter of *convention* that it be considered at rest. I shall come back to this use of Corollary IV shortly.

Newton's appeal to absolute space was excoriated by Mach in his early writings on mechanics. It was considered by Hans Reichenbach and members of the Vienna Circle as purely metaphysical (one might have expected logical positivists and empiricists to look on Newton's conventionalist maneuvering with more sympathy). But philosophers of science have been more forgiving of Newton since: Howard Stein, in particular, did much to rehabilitate him.

How does Galilean space-time stand up to Leibniz's critique? A general approach to the treatment of exact symmetries is this: insist that physically real quantities, and by extension, physically real properties and relations built out of those quantities, as specified by a physical theory, be *invariant* under the symmetries of that theory.[10] In the Galilean case, that lets in relative angles and distances, (magnitudes of) relative velocities, and (magnitudes of) absolute accelerations; it disallows absolute directions and absolute speeds. The symmetry group of the space-time is bigger and now includes the symmetries of the relativity principle. The theory no longer speaks (in terms of invariants) of absolute positions or velocities but only of (magnitudes of) absolute accelerations. We then have all the quantities ordinarily thought to be measurable in NTM and get rid of the ones forever underdetermined by any possible observation.

Commitment to invariant quantities does not in itself rule on the reality of points of space-time; for example, such points can still be relationally discerned by invariant quantities (again, relative distances, angles, and intervals in time). But if space-time points are to bear spatiotemporal relations to material particles, as substantivalists assume, the difficulty recurs: such relations will be changed by symmetry transformations, actively construed—by relative drags between material bodies and points of space-time—hence, they are not physically real. The same difficulty arises for Minkowski space and for general relativistic space-times with symmetries.[11]

10. I have defended this reading of exact symmetries in physics in several places; see, e.g., Saunders (2003a, 2003b, 2013).

11. I do not myself think this difficulty is insurmountable, however, for the reasons sketched in Saunders (2003a, 2003b).

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So much is familiar ground. Only now are we are about to move on to new territory, and to the question of what Newton's laws really allow us to determine. But to that end we will need some of the machinery of affine-space theory, and it will be helpful to first set it up for the familiar space-times just encountered.[12]

An affine space has much of the structure of a vector space. It is a set in which pairs of points define vectors, in turn used to define the straight lines talked of in the Laws—the straight-line motions, real or imagined, of bodies moving inertially. Quite generally, let $\mathcal{X}$ be a set, let $\mathbb{V}^n$ be a real n-dimensional vector space, and let $+$ on $\mathcal{X}$ be an action of $\mathbb{V}^n$ on $\mathcal{X}$, which is:

- transitive; that is, for any $a, b \in \mathcal{X}$, there is a vector $\vec{c} \in \mathbb{V}^n$ such that $a + \vec{c} = b$.
- free; that is, if for some $a \in \mathcal{X}$, $a + \vec{c} = a + \vec{d}$, then $\vec{c} = \vec{d}$.

If free, the vector in $\mathbb{V}^n$ (which exists by transitivity) is unique, denoted $a - b$.

The structure $\langle \mathcal{X}, \mathbb{V}^n, + \rangle$ is an *n-dimensional affine space*, denote $\mathcal{A}^n$. A notion of parallelism for pairs of points can now be defined: a, b is *parallel* to a', b' if $a - b = a' - b'$. A transformation g: $\mathcal{A}^n \to \mathcal{A}^n$ is *affine* if it preserves parallels; that is, if $a - b = a' - b'$, then $g(a) - g(b) = g(a') - g(b')$.

If $\mathbb{V}^n$ is equipped with the Euclidean scalar product with norm $|\cdot|$, then a relative distance function h is defined on $\mathcal{A}^n \times \mathcal{A}^n$ by $h(a, b) = a - b_n$. Since $\mathcal{A}^n$ is affine, h is nondegenerate and satisfies the triangle inequality: it is a *metric* on $\mathcal{A}^n$. The resulting space $\langle \mathcal{X}, \mathbb{V}^n, +, h \rangle$ is *n-dimensional Euclidean space.*

It is now clear how to proceed. Space-time should be a four-dimensional affine space (to define straight-line inertial motions), but if it is to respect the relativity principle its Euclidean structure should be limited to simultaneous three-dimensional surfaces with no preferred velocity (so no decomposition of vectors in $\mathbb{V}^4$ into a spatial and temporal part). Define *neo-Newtonian* space-time as the structure $\langle \mathcal{X}, \mathbb{V}^4, +, \mathbb{W}, h, t \rangle$, where

- $\mathbb{V}^4$ is a four-dimensional real vector space
- $+$ is a free, transitive action of $\mathbb{V}^4$ on $\mathcal{X}$
- $\mathbb{W}$ is a three-dimensional subspace of $\mathbb{V}^4$
- h is a Euclidean inner product on $\mathbb{W}$, defining a norm $|\cdot|_3$

12. We could use the more usual perspective of a differentiable manifold, but besides keeping the mathematics as simple as possible, I am concerned to use methods as close as possible as those of *Principia* (to cast it as an "affine space plus" theory, in Stachel's [1993] terminology).

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- t is a Euclidean inner product on the quotient space $\mathbb{V}^4/W$, defining a norm $|\cdot|_t$.

Since it is a four-dimensional affine space, he notion of straight line, and straight-line parallelism, is well defined. The lines that (as vectors) lie in W connect simultaneous points in $\mathcal{X}$, that is, points a, b such that $a - b \in W$, and for such points h defines a metric $h(a, b) = |a - b|_3$ on points in $\mathcal{X}$. We could, indeed, have begun with a simultaneity relation on $\mathcal{X}$ and defined W in its terms.[13]

The affine transformations that preserve W, t, and h are the *Galilean transformations* (in which I include mirroring and time reversal). Defining a rectilinear coordinate system $\vec{x}, t$ on $\mathcal{X}$ so as to assign equal times to simultaneous points, they are

$$\begin{aligned} &\vec{x} \to R \cdot \vec{x} + \vec{u}t + \vec{d}, \\ &t \to t + s, t \to -t, \end{aligned} \tag{5}$$

where R is a matrix of determinant ± 1, $\vec{u}$, $\vec{d}$ are three-dimensional Euclidean vectors, and s is a real number. Excluding the inversions, these transformations are parameterized by 10 real numbers, the (proper, orthochronous, inhomogeneous) Galilean group. In this way we turn our "affine space plus" into a differentiable manifold (Galilean space-time). The measurable quantities—the invariants of these transformations—are time intervals, instantaneous relative distances and their derivatives, and (norms of) relative velocities and absolute accelerations and derivatives.

What then is Newtonian space-time? It is the structure $\langle \mathcal{X}, \mathbb{V}^3 \oplus \mathbb{V}^1, +, h, t \rangle$; that is, it is neo-Newtonian space-time with the additional structure that any vector in $\mathbb{V}^4$ is uniquely decomposed into the sum of a vector in $\mathbb{V}^3$ (with the Euclidean metric h) and a vector in $\mathbb{V}^1$ (with the temporal metric t). Then any two points a, b in $\mathcal{X}$ have a unique spatial separation; namely, $h(a, b) = |P(a - b)|_3$, where P is the projector from $\mathbb{V}^3 \oplus \mathbb{V}^1$ to $\mathbb{V}^3$.

Invariance of the decomposition $\mathbb{V}^4 = \mathbb{V}^3 \oplus \mathbb{V}^1$ narrows the space-time symmetries to the seven-dimensional subgroup consisting of rotations in space and translations in space and time. The relativity principle may be a symmetry of the laws, but it is not a symmetry of the space-time. Only with neo-Newtonian (or Galilean) space-time are the two correctly aligned. Or are they?

3. An Inconvenient Truth. By the late nineteenth century, analytical methods took center stage and, increasingly, group theory.[14] The concept of iner-

13. This is the method followed in Stein (1967).

14. The first part of this section draws heavily on Barbour (1989, chap. 12; 1999).

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tial coordinates was the crucial one, and that of the inertial frame. The latter term was first coined by Ludwig Lange in 1885, taking up a suggestion by Carl Neumann in his habilitation address of 1869. Neumann's work, published as *On the Principles of the Galilean-Newtonian Theory* in 1870, began the debates over the foundations of NTM by Mach and others that cleared the way for Einstein's discoveries.

Neumann's monograph was followed soon after by Ernst Mach's *History and Root of the Principle of the Conservation of Energy*. Both argued for the need for operational definitions in mechanics. The following year the mathematician Peter Tait offered the following construction (Tait 1983): let there be three force-free bodies, no two of which are at relative rest, that are noncollinear; let the sequence of equal increments in the relative distance of any two serve as a clock; then their relative distances and angles with the third, if inertial, must satisfy certain constraints over time. In fact the first three instantaneous relative configurations—"snapshots"—are unconstrained (any will do, inertial or not); only with the fourth is there a constraint, amounting to a test of whether the three bodies are really inertial.

Tait's construction was followed by several others, among them one by Lange. But its weakness as an operational definition of inertial motion was also clear, for it depended on the availability of force-free ("fundamental") bodies. Which, precisely, were these? Unlike every other force, gravitational force cannot be screened off. Tait's proposal, like Newton's appeal to absolute space, was a purely conceptual device: it had nothing to do with anything that could in practice be measured. As Bertrand Russell complained, "If motion means motion relative to fundamental bodies (and if not, their introduction is no gain from a logical point of view), then the law of gravitation becomes strictly meaningless if taken to be universal—a view which seems impossible to defend. The theory requires that there should be matter not subject to any forces, and this is denied by the law of gravitation" (Russell 1903).

The situation is different if we use Newton's methods. Rather than force-free, Newton used gravitating bodies, principally, the earth, sun, and moon. Because of the complexity of this system, he made use of an iterative procedure, but in essence the method was an operational one. Thus, to a first approximation he used the area law (Kepler's second law), itself derived from his principles for two bodies subject only to centripetal forces, to define a clock (with equal areas swept out in equal times). Time defined in this way, taking into account perturbations introduced by a third (and further bodies, notably Jupiter), is *ephemeris time*. It is time defined so as to make the observed relative angles subtended by the planets at the earth satisfy Newton's equations of motion. In the early 1950s it had acquired sufficient precision to replace sidereal time as the standard of time. It was replaced in turn by atomic clock time in 1972.

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Does that secure the status of Proposition XI, when replacing "at rest" by "inertial"? No, for the same problem returns: the solar system as a whole is no more free from gravitational forces than the "fundamental bodies" of which Russell spoke. The sun and planets are acted on by bodies beyond the solar system: in principle, arbitrarily many.[15] Newton's reasoning in deriving Proposition XI, even granting Hypothesis I, was fallacious. Here is Corollary IV to the Laws, on which the proof turns:

> The common center of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common center of gravity of all bodies acting upon each other (excluding external actions and impediments) is either at rest, or moves uniformly in a right line. (Cajori 1934, 19)

The proviso "excluding external actions" robs it of any application to local gravitating bodies, like the planets and sun, which cannot be screened from the rest of the universe.

Are the accelerations thus produced at least likely to be small, so that Proposition XI (given Hypothesis I) holds to a good approximation? The acceleration due to the stars in our local galaxy is certainly tiny—no more than 10^{-10} m sec^{-2}—but include ever-larger regions of space filled with stars, and their influence builds. It is the same as with Olber's paradox: the influence of remote stars falls off as the inverse square, but the number of stars at a given distance, if uniformly distributed, increases as the square. The result is an acceleration that scales linearly with the diameter of the universe, given an approximately uniform mass distribution.

Any elementary observation that escaped Newton should be spelled out in full.[16] Suppose, for simplicity, matter is uniformly distributed with density ρ over an enormously large sphere, and suppose the solar system is located within the sphere at distance R from its center. The gravitational forces due

15. Recognition of this point was slow. The question of the proper motions of stars among themselves was widely discussed in the mid-nineteenth century—as we learn from Mary Sommerville's *On the Connexion of the Physical Sciences* (1836)—but not much before, and not even then in terms of their action on the solar system (I am grateful to George Smith on this point). Nor was the universality of gravity taken for granted; Somerville cited the motions of binary stars as conforming to Newton's laws as evidence that the stars, and not just bodies in the solar system, truly gravitate.

16. It was also missed by Stein. Having correctly observed "Newton's theory affords a way of assigning kinematical states up to a Galilean transformation, on condition that one has succeeded in accounting completely for the relative motions by a system of action-reaction pairs, and on the further condition that there is no reason to suspect the system in question to be subject to an outside influence imparting equal accelerations to all its members," Stein (1977) continued: "Newton had the good luck to find such a system: namely, the solar system; and the skill to effect its thorough dynamical explication." It would be by good luck indeed were this to be true.

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to the matter at distance greater than R cancels, leaving only that of the matter in the interior. This mass, M, is as a function of R:

$$M(R) = \frac{4}{3}\pi R^3 \rho.$$

The gravitational action of the sphere of mass M on the solar system (by hypothesis located on the boundary of the sphere) is the same as if all the mass were concentrated at the center and hence produces an acceleration a (using the proportionality of gravitational to inertial mass):

$$a = \frac{M(R)G}{R^2} = \frac{4}{3}\pi G\rho R. \tag{6}$$

The result is not materially effected on removing the simplifying symmetries (so long as the mass density is roughly uniform). For sufficiently large R, the acceleration of the solar system will as large as you like—or do not like.

But there is always the center of mass frame for the universe as a whole, however large, so long as the total mass is finite. To this, Corollary IV applies exactly and without ambiguity. Relative to this frame, whatever it is, if stellar densities are similar to those of our local Hubble volume, the acceleration of the earth due to the rest of the universe swamps that due to the sun for a universe of size 10^{38} m and greater. As it happens, that is much larger than the visible universe (by about 12 orders of magnitude)—but there is every reason to think the actual universe is much larger, if not in fact infinite.

What if the actual universe is infinite? In that case there is a new difficulty: not so much that the "true" acceleration of the solar system is enormously large and unknown, but that it is undefined. This problem was first recognized by Hugo Seelinger in 1894 and was much discussed subsequently. By the above method, indeed, the acceleration of the solar system due to the rest of the universe can be given any number whatsoever.[17] The problem was considered so severe that some concluded the inverse square law itself should be modified. Shortly after Einstein made the same suggestion, but for a very different reason. He too assumed the universe was approximately homogeneous, but in terms of his newly discovered general theory of relativity, and in that context he saw it could not possibly be static. The solution was to modify the field equations, thus introducing his "cosmological constant." But now exactly the same reasoning applied to Newton's theory of gravity,

17. This was the inconsistency discussed by Norton (1993, 1999).

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which Einstein used as a warming-up exercise. The modification to Newton's theory of gravity eliminated the divergence of equation (6) as $R \to \infty$.[18]

Pursuit of these questions led directly to Big Bang cosmologies on the one hand (because Einstein's solution was ineffectual) and to NCT on the other (because Einstein's geometric methods were so successful). We saw in section 1 how the problem of acceleration is resolved in that setting. But let us not take our eye off the ball. There is not and never was any evidence to show the universe is less than 10^{38} m in size. Even in the favorable case, supposing we live in an island universe, by Newton's principles we never had good reason to think the solar system or any other concretely defined frame of reference was even approximately inertial. That is, the central concept of NTM, the concept of an inertial frame of reference, was never in practice operationally defined by Newton's principles, not even approximately.

4. Corollary VI. The conclusion just reached has an air of paradox. It seems self-defeating if true: how is it NTM was, in fact, successfully applied?

In only one place in *Principia* did Newton raise the question of the influence of the stars on the solar system: in Corollary II to Proposition XIV of Book 3. The proposition itself was that the aphelions and nodes of the orbits of the planets are fixed (the aphelion of an orbit is its furthest point from the sun; its nodes are the points of intersection of the plane of an orbit with the ecliptic). The result was proved in Book 1 (at Proposition XI) in the approximation in which the mutual actions of the planets and comets among themselves are neglected, but Newton stated it again in Book 3 to draw two corollaries. The first is that "the fixed stars are immovable, seeing they keep the same position to the aphelions and nodes of the planets" (Cajori 1934, 422)—meaning, the fixed stars are nonrotating with respect to the aphelions and nodes and therefore absolutely nonrotating.[19] The second reads: "And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, because of their immense distance, to produce any sensible effect in our system. Not to mention that the fixed stars, everywhere promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions, by Proposition LXX, Book 1." On the first point, we must infer Newton never made the calculation equation (6); he simply failed to see that the falloff as the inverse square of the gravitational force with distance is canceled by the quadratic increase in the number of

18. Equation (2) is replaced by: $\nabla^2\phi + \Lambda\phi = -4\pi G\rho$. This is equivalent to replacing the inverse square law by $(1/r^2)\exp(-\Lambda/r)$. Here Λ is the cosmological constant.

19. Newton had, however, in Phenomenon I, II, and II, assumed the stars to be at rest. His argument from induction for his theory of gravity was a complex one. See DiSalle (2006, chap. 2) and, for the most comprehensive study to date, Harper (2012).

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stars. But on the second the error is more subtle. Proposition LXX of Book 1 proved that the gravitational force vanishes everywhere in the interior of a uniform shell of matter; Newton's reasoning, then, would seem to be that the stars are approximately uniformly distributed ("promiscuously dispersed"), so their action may be considered as that due to a concentric system of such shells with the sun at their center proceeding to infinity. Given all this, their influence is shown to be zero.

We used Proposition LXX ourselves, in the case of an island universe, but in that case the assumption that the solar system is at the center of the universe is clearly unwarranted. In the infinite case, the assumption does not even make sense. Since the universe is homogeneous and infinite, no point is at the center. If one applies the same construction anyway, one can build a system of spheres about any point whatsoever, so about points at completely arbitrary distances from the solar system. Whereupon by the argument of equation (6) the acceleration can be anything, and we are back to Seelinger's problem.

We are also back to Einstein's problem. Such a universe cannot possibly be static. Newton recognized rather better than did Einstein that here a balancing act was impossible—that nothing short of divine providence could ensure a static universe.[20] That sounds like a reductio, but it is clear from correspondence with Bishop Bentley that Newton welcomed the conclusion. Like Bentley, Newton looked to science for arguments for the existence of God.

But our concern is with the definition of inertial frames, and on this point Newton had more to say. The remark is to be found not in the *Principia* itself but in an informal draft of Book 3, published posthumously in English translation as *The System of the World* in 1728: "It may be alleged that the

20. Newton wrote: "That there should be a central particle, so accurately placed in the middle as to be always equally attracted on all sides, and thereby continue without motion, seems to me a supposition fully as hard as to make the sharpest needle stand upright on its point upon a looking glass. For if the very mathematical center of the central particle be not accurately in the very mathematical center of the attractive power of the whole mass, the particle will not be attracted equally on all sides. And much harder is it to suppose all the particles in an infinite space should be so accurately poised one among another, as to stand still in a perfect equilibrium. For I reckon this as hard as to make, not one needle only, but an infinite number of them (so many as there are particles in infinite space) stand accurately poised upon their points. Yet I grant it possible, at least by a divine power; and if they were once to be placed, I agree with you that they would continue in that posture without motion forever, unless put into new motion by the same power" (letter dated January 17, quoted in Bentley 1809, 208). Einstein's solution (by means of a cosmological constant) required just such a balancing act; small wonder he later called it "the greatest blunder of my life" (on this point, see Bianchi and Roveli 2010).

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sun and planets are impelled by some other force equally and in the direction of parallel lines, but by such a force (by Corollary VI of the Laws of Motion) no change would happen in the situation of the planets one to another, nor any sensible effect follow: but our business is with the causes of sensible effects. "Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phenomena of the heavens" (Cajori 1934, 558).

Corollary VI to the Laws is the statement:

> If bodies moved in any manner among themselves are urged, in the direction of parallel lines by equal accelerative forces,[21] they will all continue to move among themselves, after the same manner as if thy had not been urged by those forces. (Cajori 1934, 21)

The proof was brief, and there was no comment on it in the Scholium that followed. It was used to prove Proposition III of Section II (and Propositions LVII and LXIV in later sections of Book 1); no other reference to it is to be found in *Principia*. But Proposition III was important to the argument for the inverse square law of force for gravity in Book 3. The proposition is:

> Proposition III.—Every body [A], that by a radius drawn to the center of another body [B], however moved, describes areas about that center proportional to the time, is urged by a force compounded of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled. (Cajori 1934, 44)

That is, given the premise, the force by which body A is moved is compounded of a centripetal force and of whatever force is needed to make it accelerate at the same rate as is B. But according to the Phenomena of Book 3 the moons of Jupiter do describe equal areas in equal times, so they must have those forces acting on them, so in particular (apart from the force that makes them accelerate at the same rate as Jupiter) the moons experience a centripetal force toward Jupiter. This is the first part of Newton's deduction from phenomena of the force of gravity as centripetal and falling off as the inverse square.

But how is it that all the moons of Jupiter are accelerated the same as Jupiter by the sun? The answer, of course, lies in Galileo's equivalence principle—the principle that neglecting friction, all bodies are accelerated the same by a gravitational field, whatever their composition. In Newton's terms that meant the constancy of the ratio of gravitational mass μ to inertial mass m, and it is to this that Newton immediately turned after establishing the force law of gravity, at Proposition VI of Book 3.[22] Newton argued for the

21. By this he means forces that produce equal accelerations (see sec. 5).

22. Although he had intimated it much earlier, at Definition VII of Book 1.

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universality of this ratio at length, on purely empirical grounds, with reference to experiments with pendulums that he had himself conducted.[23]

If, instead, the gravitational force is taken as given, the key to applying the Laws to the Jupiter system is not so much Corollary VI but the combination of Corollary VI with Galileo's equivalence principle, which may be stated as:

> Corollary VI*.—If bodies moved in any manner among themselves are urged in the direction of parallel lines by equal gravitational forces due to outside bodies, they will all continue to move among themselves, after the same manner as if they had not all been urged by that force.

In this by "equal gravitational forces" I mean equal force per unit gravitational mass, $\vec{g}$, so that acting on a body of inertial mass m and gravitational mass μ the force is $\mu\vec{g}$. The resulting acceleration is

$$\frac{d^2\vec{x}}{dt^2} = \frac{\mu}{m}\vec{g}.$$

From the constancy of μ/m for any material body, whatever its constitution, and given that $\vec{g}$ is uniform, the condition of Corollary VI is met. Since the gravitational force of the sun is approximately uniform over the dimensions of the Jupiter system, we can conclude that the motions of Jupiter and its moons "among themselves" will be the same as if the center of mass system were moving inertially. By Corollary V, these are the same as if the center of mass were at rest, whereupon Newton's laws apply.

It needs only a small rearrangement of this reasoning to conclude:

> Equivalence Principle.—If bodies moved in any manner among themselves are described in relation to an accelerating but nonrotating frame of reference, they will all move in relation to that frame as if acted on by uniform gravitational forces, producing the opposite acceleration.

With that we have a version of Einstein's equivalence principle—and the germ of an explanation for the universal proportionality of gravitational to inertial mass. However, we are now remote not so much from Newton's concepts, as from his purpose in Book 3 of *Principia*, which was to deduce the inverse square law from the observed motions.

Corollary VI in conjunction with Newton's dynamical determinations of a state of nonrotation and of equal time intervals[24] solves the problem of inertial frames for gravitational physics. The reason that NTM works in appli-

23. This is an argument from induction, in Newton's sense, meeting the requirements of his Rules of Reasoning in Philosophy (in particular the third).

24. At Proposition 1, Book 1, deriving Kepler's second law; see Barbour (1989, 546–56) for an illuminating discussion.

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cation to the solar system, without ever giving an operational meaning to the notion of inertial frame, is that none is needed: all that matters is that motions are referred to a freely falling nonrotating frame. That, to an excellent approximation, can be defined in terms of the dynamical behavior of the solar system—using Proposition XIV of Book 3—regardless of the distribution of matter in the rest of the universe. All that is needed is that the stars and galaxies are sufficiently distant so as to give rise to gravitational fields approximately uniform over the dimensions of the planetary orbits; whether they are changing in time—and, indeed, whether the universe is infinite—is irrelevant.

But now notice Corollary VI* together with Corollary VI also solves the problem of inertial frames for nongravitational forces—so long as they are described by Newton's laws. The difference, on going over to nongravitational physics, is that there is no analog of the constancy of gravitational to inertial mass; Corollary VI* cannot be used to justify the neglect of long-range forces other than gravity. But precisely because charge-to-mass ratios of every other force do vary, the existence of such forces would show up in the relative motions of bodies, whether referred to an inertial frame or—and this is the crucial point, by Corollary VI—to an accelerating but nonrotating frame. So long as the latter (as a material system—say, a system of gyroscopes in orbit around the earth) can be screened off from any nongravitational forces that might produce a torque, and so long as Newton's laws apply, a dynamical analysis will yield a criterion of nonrotation and thereby define a nonrotating frame. In principle, such coordinates can be extended arbitrarily and used to describe any bodies, anywhere in the universe.

What happens if the bodies of which the freely falling frame is composed cannot be screened off from nongravitational forces due to distant bodies, and if these produce a torque? If there were such forces, there would be a difficulty. It may be, in such a world, that nothing short of the center of mass frame of the entire universe would do—supposing it were a finite universe. In a world like that, it may well be that physics as we know it would not be possible. But that world, thankfully, is not ours. The only long-range force other than gravity is electromagnetism, which (except in extreme cases) can always be screened off. And anyway, because charges of opposite sign tend to cancel, and those of the same sign repel, we have good reason to believe there are no large remote sources of this kind.

But if all of this is true, is there really any need of 'real' inertial frames that are anyway operationally unavailable? Why not make do with local, freely falling, nonrotating frames, as we do in GTR? If need be, the concept could be idealized as the limiting case of such a frame. The idealization is safe, as in practice such frames can be concretely realized (gyroscopes in orbit) to enor-

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mous accuracy.[25] It is also safe even if the universe is infinite: the further the mass distribution, the more accurately its gravitational influence on the solar system is approximated by a uniform gravitational field. Newton's *Principia*, on our reading, conforms to the maxim that theories be applicable whether or not the universe is infinite.

That question carries with it another: what has any of this really to do with gravity? Gravity poses special difficulties because it is long range, always attractive, and cannot be screened off: hence the importance of the equivalence principle. But given that any rigid motion can be decomposed into a linear acceleration and a rotation, all that really matters is that the frame of reference be nonrotating. Free fall, and the appeal to gravitational physics and screening from all other forces, is simply the easiest way of materializing a nonrotating frame. Corollary VI is more fundamental than the equivalence principle.

5. Kicking Away the Ladder. Of course our reasoning up to this point has been based on the laws (Axioms, or Laws of Motion, Book 1), which suppose, by wide consent, that motions can be referred to an inertial frame. Presented analytically, as from the time of Lagrange, it is assumed there exist rectilinear inertial coordinates $\langle \vec{x},\ t \rangle$ and that the equations be written in terms of these. For N particles subject to forces F that satisfy the third law (and that for simplicity are independent of relative velocities), we then have the system of equations:[26]

$$m_j \frac{d^2 \vec{x}_j}{dt^2} = \sum_{k \neq j=1}^{N} F(\vec{x}_j - \vec{x}_k, t); j = 1, \ldots, N. \tag{7}$$

The symmetry group of this system of equations (assuming the m_j's transform as scalars) is the Galilean group (5). It would seem, then, that inertial coordinates, in the usual Galilean sense, are needed to set up the theory ab initio. It may be that in practice we can refer all the motions to local freely falling nonrotating frames, and the motions of the particles among themselves will all be the same as if referred to an inertial frame, but this inertial frame has to be defined, at least conceptually, to even write down the equations.

But the key concept to our reading of NTM as a relational theory, that of "motions among themselves," plays a role here as elsewhere. It is most naturally expressed in terms of *difference* equations for particle pairs, in terms

25. The Gravity Probe B, recently used to test for dragging of inertial frames—hence as a test of GTR—is an example.

26. Presented analytically, the theory is sensitive to the distinction between an infinite and finite universe: such a system of equations is undefined for N infinite. For this and other reasons, eventually we will have to move to a field theory.

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of the relative distance vectors

$$\vec{r}_{jk} = \vec{x}_j - \vec{x}_k. \tag{8}$$

From the N equations (7), define the $N(N-1)/2$ equations

$$\frac{d^2\vec{r}_{jk}}{dt^2} = \frac{1}{m_j}\sum_{l\neq j} F(\vec{r}_{jl}, t) - \frac{1}{m_k}\sum_{l\neq k} F(\vec{r}_{kl}, t). \tag{9}$$

Of these, only $N-1$ are linearly independent. From such a set, together with the equation

$$\sum_k m_k \frac{d^2\vec{x}_k}{dt^2} = \sum_k \sum_{j\neq k} F(\vec{r}_{jk}, t) = 0, \tag{10}$$

the N equations (7) can be derived and vice versa. Equation (10), the equation for conservation of total momentum, follows from (7) and Newton's third law (that $F(\vec{r}_{jk}, t)$ is antisymettric in j and k). It is invariant under Galilean boosts

$$\vec{x}_k \to \vec{x}_k + \vec{u}t$$

(by Corollary V), as are the $N(N-1)$ equations (9). But equations (8) and (9) and the antisymettry of F are invariant under the much wider class of symmetry transformations:

$$\vec{x}_k \to \vec{x}_k + \vec{f}(t), \tag{11}$$

where $\vec{f}$ is any twice-differentiable vector-valued function of the time.[27] The imposition of the first equality of equation (10) is what distinguishes "true" inertial frames from our nonrotating frames. If we dispense with (10) and allow that the system of equations (8) and (9) and the antisymettry of F define the entire theory—say, a new theory, different from NTM—the symmetries of the equations will be those of Corollary VI, not just Corollary V.

There is, however, something odd about this way of putting it. For was Corollary VI not already a consequence of the Laws, just as was Corollary V? Why think equation (7) is the correct expression of Newton's laws, rather than (8) and (9)? The latter are clearly better suited to define the motions of a system of "bodies among themselves." If it was a check on the correct equations for the expression of Newton's laws that they exhibit the symmetries of the laws (Corollary V), leading to (7), why is it not a check that they have the symmetries of Corollary VI, namely, equation (11), leading to equations (8) and (9) and not (7)?

27. Equation (11), together with the second of eq. (5), were called the "Newtonian group" by Ehlers (1973).

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The answer, presumably, will turn on the extent to which we can view the *Principia* as, at bottom, a theory of the relative motions of particle pairs. As a first step, one would like to see if the Definitions and Laws can be rewritten in a way that obviously translates into equations (8) and (9), rather than (7), and that preserves intact the methods of proof used in the first two books of *Principia*.

There is an obvious way to do this, namely, simply relativize them—suppose that "absolute space" refers to any rigid nonrotating frame, in the same way that in passing from Newtonian to neo-Newtonian space-time one takes "absolute space" to refer to any inertial frame. No matter which frame is chosen, the subsequent geometrical constructions (the main proof procedure used in *Principia*) go through unchanged.

But that may seem like a cheat. Is it possible, rather, to interpret the laws more directly as concerning the relative motions of particle pairs? I think that it is—and to make clear how to do this, here are the laws in modified form, keeping as close as possible to Newton's language:[28]

> *First Law**.—Every body continues in its state of relative rest, or uniform relative motion in a right line, with respect to another, unless compelled to change that state by a difference in motive forces impressed upon them.
>
> *Second Law**.—The change in the relative motion of two bodies is proportional to the difference in motive forces impressed upon them and is made in the direction of the right line in which the difference in motive forces is impressed.
>
> *Third Law*.—To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

(The third law is unchanged.)

As stated, these laws clearly support (8) and (9) and of course the antisymmetry of *F*. But now analogous modifications will have to be made in stating some of their consequences, in particular Corollary IV. From the modified laws above, it follows not that the center of mass of all bodies acting upon each is in uniform motion, "excluding external actions and impediments," but that it is at rest or in uniform motion relative to another (that is free from external actions and impediments, or for which the relative external force vanishes). However, the first part of Corollary IV stands unchanged.

28. First law: "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it." Second law: "The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed" (Cajori 1934, 12).

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But what does "uniform, straight line motion" mean, if inertial motions (in the usual sense of "inertial") are undefined? Just this: relative straight-line motions are those for which the relative distances (lengths of lines joining pairs of bodies at each instant of time) and relative angles between lines joining instantaneous bodies at various times change in a characteristic way. What way exactly? The way they change under the usual representation of force-free bodies, when moving inertially.

In any case, whatever its relation to *Principia*, the system of equations (8) and (9) together with the antisymmetry of F is of independent interest—especially if, as seems to be promised, it yields exactly the same relative motions as do (7) and (10). Can we kick away the ladder? Can we dispense with the first equality of equation (10)? *Formally*, there can be no obstacle: we are already assured that any solution to the N equations (7) will yield $N(N-1)$ difference vectors $\vec{r}_{jk}$ satisfying (9). Conversely, for any solution to the $N(N-1/2)$ equations (9) satisfying (8) (of which only $N-1$ functions $\vec{r}_{jk}$ are linearly independent), choose $\vec{f}(t)$ so that its second derivative is

$$\frac{d^2\vec{f}}{dt^2} = -\frac{1}{M}\sum_k m_k \frac{d^2\vec{x}_k}{dt^2}, \tag{12}$$

where M is the total mass, and apply the transformation (11). From the antisymmetry of F equation (10) is then satisfied, and the coordinates as functions of time then satisfy the N equations (7).

One can put it like this: the set of difference equations, equations (8) and (9), for antisymmetric F, has a gauge freedom—an overall acceleration, constant in space, as an arbitrary function of time.[29] Up to this choice of gauge, solutions to these equations are in 1:1 correspondence with solutions to (7). As Newton said, "Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phenomena of the heavens."

6. Is It Relationalism? I have urged that a key concept of *Principia* is that of "motions among themselves" and made this out in terms of difference equations. Is a theory based on equations (8) and (9) and the symmetry of F a distinctively relational theory? It *appears* so. It is certainly a theory in which no meaningful notion of the velocity or acceleration of a single body is available; there are only relative distances and relative velocities of simultaneous bodies and their time derivatives, of two or more bodies among themselves.

More than this is needed, however, if by "relationalism" we mean a genuine eliminativism of reference to points of space or space-time. But just that

29. As a result Noether's first theorem no longer applies, and the total momentum is not a conserved quantity—as has already been observed.

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is what appears to be on the table: certainly the laws, as just modified, make no mention of points of space or space-time; they speak only of quantities of motion in terms of relations among bodies.

The devil, however, lies in the details. Let us examine these quantities more carefully. As in Galilean-covariant theories, they include rates of change of the norms of the $\vec{r}_{jk}$'s, denote r_{jk}, the quantities

$$\frac{dr_{jk}}{dt}, \frac{d^2r_{jk}}{dt^2}. \tag{13}$$

These require comparisons of spatial lengths at different times. Considering three or more bodies, one also has relative angles and rates of change of relative angles, as given by (for unit r_{jk}, r_{kl}):

$$\cos^{-1}(\vec{r}_{jk} \cdot \vec{r}_{kl}), \frac{d}{dt}\cos^{-1}(\vec{r}_{jk} \cdot \vec{r}_{kl}). \tag{14}$$

Quantities of the form (13) and (14) were always acceptable to relationalists.

The traditional difficulty has lain rather with rotations—say, two bodies rotating about their common center. If the motion is purely rotational, then all quantities of the form (13) and (14) are zero. Nevertheless, the bodies have nonzero relative velocities and, referred to inertial frames, nonzero accelerations. How do we describe their motion on our relationalist reading of *Principia*? The answer is that the notion of relative velocity is independent of the principle of inertia, so it carries over unchanged, and the acceleration goes over to relative accelerations of the rotating bodies, with no need to refer it to an inertial frame. The two bodies have a relative velocity because the direction of the line connecting them changes in time, and they have a relative acceleration because the direction of this relative velocity changes in time. The space-time structure needed to make sense of rotations is that a spatial direction at one time can be compared with a spatial direction at another time, with no need of the concept of straight-line motion or departure from straight-line motion.

So much is obvious if we write the derivative as the limit of a difference equation:

$$\frac{d}{dt}\vec{r}_{jk}(t) = \lim_{\Delta t \to 0}\frac{1}{\Delta t}[\vec{r}_{jk}(t + \Delta t) - \vec{r}_{jk}(t)]. \tag{15}$$

The mistake was to think this must be expressed as

$$\begin{aligned}\frac{d}{dt}(\vec{x}_j - \vec{x}_k)(t) &= \lim_{\Delta t \to 0}\frac{1}{\Delta t}\{[\vec{x}_j(t + \Delta t) - \vec{x}_k(t + \Delta t)] - [\vec{x}_j(t) - \vec{x}_k(t)]\} \\ &= \lim_{\Delta t \to 0}\frac{1}{\Delta t}\{[\vec{x}_j(t + \Delta t) - \vec{x}_j(t)]\} - \lim_{\Delta t \to 0}\frac{1}{\Delta t}\{[\vec{x}_k(t + \Delta t) - \vec{x}_k(t)]\},\end{aligned}$$

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that is, as the difference between two velocities, each of which must be independently defined. Only the relative velocity as given by equation (15) is physically meaningful, not velocities of which it is the difference.

Of course, Machians go further and reject quantities like (15) as well. The comparison of spatial directions at different times is not needed, according to Barbour (1999), no more than comparisons of time-like vectors in spacetime. The solution to a dynamical problem in Barbour's terms is fully determined given two relative configurations of particles at different times, without information on their relative orientation. Our form of relationalism is, from a Machian point of view, a halfway house.[30]

Machians will press the epistemic question: how are these angles between particle pairs at different times determined? Clearly not in the way angles between pairs of bodies normally are—the two lines cannot be simultaneously observed. The answer is, as always: dynamically. The comparison (for a pure rotation) is in the first instance made between the relative velocities of the two bodies at neighboring times—their relative acceleration. This quantity is given by a centripetal force law as a function of their masses and relative distances. Since (for a pure rotation) their relative distance is constant in time, so is the angular velocity and the norm of their relative velocity at each time. The rate of change of relative velocity then determines the relative velocity (and the angle between the lines joining the two bodies at two times) uniquely. All these determinations (or rather their norms) are invariant under boosts to arbitrary nonrotating frames, the transformations (11), as also, for time-independent force-laws, under the transformations (5).

I suggest it was considerations like this that led Huygens to reconsider his earlier view, in reluctant agreement with Newton, that cases of pure rotation could only be understood as motions in relation to absolute space. At the end of his life, he wrote instead: "Circular motion is relative motion along parallel lines, where the direction is continually changed and the distance is kept constant through a bond. Circular motion in one body is the relative motion of the parts, while the distance remains constant owing to the bond" (Huygens 1888/1950, 507). Passages like this were quoted by Stein as evidence of the depth of Huygens's thinking (Stein 1977), but on this point there is hardly a consensus: "To analyze rotation in terms of an objective, or absolute, notion of velocity difference rather than objective, or absolute

30. It is a halfway house from the point of view of Leibniz's principles as well. Leibniz spoke of temporal orderings rather than temporal intervals, but an ordering in time is still a diachronic relation. If configurations of bodies at different times can be compared as to order, why not compare them as to spatial direction as well? There is no reason—save that, for Leibniz, all relations, temporal and spatial, are ultimately to be reduced to monadic predications (they should be "internal" relations). That is just what is provided (for temporal orderings and intervals, if not spatial relations) by Machianism in Barbour's sense.

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velocity is to possess exactly the insight Newton lacked, but it is also to reject the full-blown relational conception of motion, something that was beyond the ken of Huygens's philosophical dogmas" (Earman 1989, 71).[31] Earman seems to equate Huygens's insight with the shift from Newtonian to neo-Newtonian space-time, but the insight is deeper: only a comparison of spatial directions at different times is needed, not of velocities at the same time, still less of velocities at different times. We read Huygens as dispensing with the principle of inertia. Earman thought no theory like that was available in Newton's day.

That claim, I have shown, is debatable. And whether or not a theory based on (8) and (9) and the antisymmetry of F is a 'full-blown' relationalism, it is surely a theory in which "true and simple motion of any one whole body can in no way be conceived—what it is—and does not differ from rest of that body" (Huygens 1888/1950, fragment 8, vol. 16).

7. Newton-Huygens Space-Time. The limiting process used in section 6 shows what is needed in terms of manifold structures: a connection as a rule for comparing spatial directions at different times—a rule for the parallel transport of directions in space (compatible with the temporal and spatial metrics), not in space-time. Such a construction (with the connection an 'absolute' structure) was given by Earman; he called it "Maxwell space-time."[32] Here I conclude with a construction that goes the other way. Galilean space-time is the right manifold corresponding to neo-Newtonian space-time, an affine space; what does Maxwell space-time correspond to? It should be a weaker structure—call it 'Newton-Huygens space-time'—but yet be sufficient for the constructions used in *Principia*.

Its outline should already be clear. We are looking for a structure in which affine notions and not just metrical ones are restricted to surfaces of constant time. Define Newton-Huygens space-time as the structure $\langle \mathcal{X}, \mathbb{V}^3, \mathbb{V}^1, +_3, +_1 \rangle$, where

- $\mathbb{V}^3$ and $\mathbb{V}^1$ are real vector spaces of dimensions 3 and 1 respectively, with Euclidean inner products $\langle \cdots \rangle_3$, $\langle \cdots \rangle_1$, respectively.
- $+_3$ is a free (but not transitive) action of $\mathbb{V}^3$ on $\mathcal{X}$, that is, for any $a \in \mathcal{X}$, $\vec{b} \in \mathbb{V}^3$, $a +_3 \vec{b} \in \mathcal{X}$
- $+_1$ is a free and transitive action of $\mathbb{V}^1$ on the cosets of $\mathcal{X}$ under the action of $\mathbb{V}^3$.

31. Barbour (1989, 675) is likewise critical of Huygens on this point.

32. Earman (1989, secs. 2.3, 4.7). Earman still thought it inadequate to relationalism, however, as flouting his relational requirement R1, that space-time "cannot have structures that support absolute quantities of motion" (13). (But whether rotation is absolute or relative is part of what is in contention.)

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The action of $\mathbb{V}^3$ on $\mathcal{X}$ defines an equivalence relation on $\mathcal{X}$, "simultaneous with"; the cosets of $\mathcal{X}$ under $\mathbb{V}^3$ are time-slices. By construction, $+_3$ acts transitively on any time-slice. Hence, since $+_3$ is free, for any simultaneous $a, b \in \mathcal{X}$, there is a unique $\vec{c} \in \mathbb{V}^3$ such that $a +_3 \vec{c} = b$, denoted $b - a \in \mathbb{V}^3$. Now consider two pairs of simultaneous points in $\mathcal{X}$. If a is simultaneous with b, and a' is simultaneous with b' (possibly at a different time), the quantity:

$$\cos\theta = \frac{\langle (a-b), (a'-b') \rangle_3}{|a-b|_3 |a'-b'|_3}$$

is well-defined, where β is the angle between the directions in space $a - b$ and $a' - b'$. But there is no notion of straight line connecting nonsimultaneous points; the principle of inertia does not apply.

There remain important questions, above all, moving over to a manifold formulation: What is the relation between a theory of gravity (and other forces) formulated in Maxwell space-time and one based on Newton-Cartan space-time? In the latter there is a notion of parallelism for time-like as well as space-like vectors. That notion, we must conclude, is dispensable, to be derived, if at all, by fixing of gauge.

That takes us, again, to field theory, which is beyond our remit here. But it raises a conceptual worry. It might be thought that insofar as NCT is (approximately) empirically adequate, as go the motions of the planets and terrestrial gravitating bodies, there is de facto an empirical standard of parallelism for time-like vectors. A theory based on Maxwell space-time will simply be unable to account for this standard.

That argument, however, is too quick. We have already demonstrated the empirical equivalence of a theory based on Newton-Huygens space-time to one based on neo-Newtonian space-time, although the latter too incorporates a standard of parallelism for time-like vectors. That standard, from the perspective of Newton-Huygens space-time, is not an empirical given: it is unneeded surplus structure. We have already shown how to obtain it in the case of a finite universe by a choice of gauge, equation (12). Something similar follows from the point of view of Barbour-Bertotti space-time, in which equation (7) can be derived by a certain parameterization of the allowed curves in relative configuration space. And of course, according to that theory, the standard of parallelism of space-like vectors, with which Newton-Huygens theory is equipped, is likewise surplus structure, unnecessary to the empirical adequacy of the theory (given that the universe has zero total angular momentum and energy).

So much for the actual world. What of possible worlds, and distinctions among them drawn in NCT, invisible to ours? Take possible worlds each

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with only a single structureless particle. Depending on the connection, there will be infinitely many distinct trajectories, infinitely many distinct worlds of this kind. But in Newton-Huygens terms, as in Barbour-Bertotti theory, there is only one such world—a trivial one in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn. From the point of view of the latter theories, the fault lies with introducing a nontrivial connection—curvature—without any source, unrelated to the matter distribution. At a deeper level, it is with introducing machinery—a standard of parallelism for time-like vectors, defined even for a single particle—that from the point of view of a relationalist conception of particle motions is unintelligible.

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