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FORMAL QUALITATIVE PROBABILITY

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Abstract. Choices rarely deal with certainties; and, where assertoric logic and modal logic are insufficient, those seeking to be reasonable turn to one or more things called "probability." These things typically have a shared mathematical form, which is an arithmetic construct. The construct is often felt to be unsatisfactory for various reasons. A more general construct is that of a preordering, which may even be incomplete, allowing for cases in which there is no known probability relation between two propositions or between two events. Previous discussion of incomplete preorderings has been as if incidental, with researchers focusing upon preorderings for which quantifications are possible. This article presents formal axioms for the more general case. Challenges peculiar to some specific interpretations of the nature of probability are brought to light in the context of these propositions. A qualitative interpretation is offered for probability differences that are often taken to be quantified. A generalization of Bayesian updating is defended without dependence upon coherence. Qualitative hypothesis testing is offered as a possible alternative in cases for which quantitative hypothesis testing is shown to be unsuitable.

§1. Introduction. At least since Ellis challenged the classic theory of probability (Ellis, 1843), there has been controversy about the foundation of probability and even disagreement about the primary meaning of the word "probability." *Probability* has been equated or identified with various things and with conflations or confusions amongst them—idiosyncratic confidence, betting quotient, logical state intermediate between truth and falsehood, propensity (a generalization of causation) with its inverse, combinatoric distribution, and frequency. But I believe that there is wide, implicit agreement on what is ultimately wanted from whatever is being called "probability." Setting aside persons simply looking for mathematical recreation, what is sought are principles for determining decision-theoretic rankings amongst hypotheses and other choices, and amongst outcomes, with such rankings of choices understood to affect or effect rankings of outcomes and *vice versa*. Binary assertoric logics and three-valued modal logics are grossly insufficient for most human endeavors, in which we know relatively little to be true or to be false and need more than simply to consider everything else as possible. Whether whatever we call "probability" is itself an extension of logic or not, we want from it something that operationalizes as such. From probability, we seek a guide to life.

The extreme fruitfulness of quantification in other application, which for example helped to propel Europe from a state of relative backwardness to one of dominance of

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various sorts (Crosby, 1996), cultivates a predisposition to presume that quantification is possible in nearly every application, and informed Lord Kelvin's well-known assertion about measurement, knowledge, and science (Thomson, 1891). Regardless of whatever they might think "probability" to mean, most people—and especially most people attempting scientific study—nowadays imagine something of arithmetic properties, conforming to the Kolmogorov axioms (Kolmogorov, 1933, chap. 1, sec. 1, p. 2) or to something like them. The term "hypothesis testing" has become virtually synonymous with explicit application of quantitative methods based upon such constructs. Yet there have been discontents. *Probability* thus conceptualized has been challenged as a description of what the mind uses or even *can* use to make decisions in the face of imperfect knowledge.

History and prehistory give us at least some reason for doubt. Before Fermat and Pascal took-up the problem of Antoine Gombaud (self-appointed Chevalier de Méré) in 1654, there seems to have been comparatively little like the now familiar quantified construct (Hacking, 1975; Franklin, 2001, 2016, 2018).[1] But, though people often dealt with uncertainty in ways that now seem terrible or absurd, more typically they dealt with it well enough to survive, often to prosper, and sometimes to accomplish remarkable things. There was scientific progress in the world. Europe was not actually so dark during the Middle Ages as popular history would have it, and the Renaissance began and ended before the emergence of classical probability theory. Of course, people need not have been consciously performing arithmetic to have been behaving in a manner that approximated some arithmetic model and that were well approximated by it.

However, some notable researchers have argued that legitimate quantifications are the exception (Kries, 1886; Keynes, 1921, pt. I, chap. III, pp. 20–40; Koopman, 1940a, sec. 1, pp. 269–270; Fioretti, 2001); and some thinkers have gone so far as to contend that the relevant preorderings of propositions or of events are often incomplete (Keynes, 1921, pt. I, chap. II, sec. 4–20, pp. 22–39; Koopman, 1940a, p. 270), which is to say that some pairs have no comparative probability, in which case even spurious quantifications cannot be fitted. Thus, while formal qualitative probability has often been treated as a way-station to quantitative probability (perhaps beginning with Bernstein in 1917), it seems desirable to explore qualitative probability in the description of rational behavior and of scientific method in cases where quantifications of probability are not and *cannot* be applied. Even should the word "probability" be used for something that is by its very nature quantified, our knowledge of it may not be, nor does that knowledge necessarily provide a complete preordering.

[1] Research done in India before about AD 400, and now lost to us except for faint echoes (Godambe, 1976), might be amongst the few examples of something fitting or approximating modern quantified theory. Competent commentary on the odds of dicing may be found in *De vetula*, a poem of the mid thirteenth century (Franklin, 2018); a fragment by Galileo arrives at the same result in the same manner (Hacking, 1975, chap. 6, pp. 51–53; Franklin, 2001, chap. 11, p. 302; Franklin, 2018). An anonymous treatment in about AD 1400 (Toti Rigatelli, 1985), essentially unnoticed at the time (Franklin, 2001, chap. 11, pp. 291–296; Franklin, 2016, p. 16; Franklin, 2018), may perhaps be interpreted as implicitly employing quantitative, probabilistic reasoning (though this interpretation is disputed). Gerolamo Cardano wrote a general probabilistic treatise on games of chance (Cardano, 1663) in about 1564, but this was not published until 1663 (Hacking, 1975, chap. 6, p. 54; Franklin, 2001, chap. 11, pp. 296–300; Franklin, 2016, p. 16; Franklin, 2018).

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An inability to capture probability with precise quantities can be accommodated by associating propositions or events with *intervals*, which can even capture incompleteness of preorderings. The intervals of Boole (Boole, 1854, chap. XVII, sec. 15, pp. 267–269), of Yule (Yule, 1901, 1911b, chap. II, pp. 17–23), and of Keynes (Keynes, 1921, pt. II, chap. XV, sec. 5, pp. 160–163) are all either [0, 1] or constructed with multiplication of the unit interval by point-value quantifications; but Koopman identified the more general possibility (Koopman, 1940a, sec. 5, p. 285). However, there are pitfalls in the attempted use of intervals. Except where a set of intervals somehow fits the situation *uniquely*, it is simply a *proxy*, a matter of affixing quantitative nails so that an arithmetic hammer may be used. The use of quantifications for proxies is often mistaken for a simplification, because of familiarity, but it is an obscuring of underlying assumptions. In somewhat similar cases of quantification, peculiar artifacts of the *map* are often confused with features of the *territory* (as when some subjectivists rush to see *betting quotients* as measures *of* belief whereas they may merely be measures *conforming to* belief), and one should expect such confusion to arise in the use of intervals as proxies. Inappropriate extrapolations, interpolations, or rejections may result. Koopman did not offer a practical method for arriving at useful intervals; his concern with intervals was to find the properties of probability when it were measurable. Subsequent attempts to present principles for arriving at intervals have usually or always been generalizations from methods for arriving at measures, and have excluded apparently reasonable preorderings (Walley, 2000).

Even without such confusions, it would seem worth attempting to explore incompletely preordered qualitative probability without resort to quantification of any sort. But very little such investigation has been undertaken with formality. The definitions and axioms of Keynes (Keynes, 1921, pt. II, chap. X, sec. 4, p. 120, chap. 12, sec. 4–6, pp. 134–137) are plainly designed with a goal of reaching arithmetic probability. Koopman's definitions and axioms (Koopman, 1940a, sec. 3, pp. 275–276; Koopman, 1940b, pp. 763–767) are less so, but still turned to such purpose. He too introduced a 0 and a 1 which caused him some confusion, and his axioms omit some principles that he did not need to arrive at measures for probability but that would be applicable in the more general case. Fine (Fine, 1971; Fine, 1973b, chap. II, sec. IIB, p. 17) employed an axiom of completeness. I have encountered no more recent efforts along these lines in the published literature. The project of developing and exploring a conception of probability in terms of an incomplete preordering seems to have been sidetracked before it were fully begun.

This article attempts to resume the project, with axioms that are not selected or shaped to the purpose of arriving at measures (though these principles may nonetheless be put to that purpose by adding one or another assumption corresponding to special cases), but instead intended to identify precisely and more sufficiently what principles must be followed in probability orderings. The proposed axioms may be defended as capturing important properties in a *relatively* simple fashion and as propositions that are axioms or theoremata across systems of each and every of the major philosophical interpretations, with one proposition a possible exception which will be labored. But, that conformance notwithstanding, the axioms help to bring to light significant challenges for various interpretations of probability, as also does the aforementioned potential exception.

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§2. Formal structure.

2.1. Notation and relevant relations. Let " $(X \mid c)$" represent eventuality X in the context of condition c. (Initially, one may regard X and c as *propositions* or X and c as *events*, though in the expression of principles after the first four axioms, it becomes desirable to settle either on reference to propositions or on reference to events.) X may be said to be "*potential*" (though impossibilities are not formally disallowed), while c may be said to be *given* as evidence or by *presupposition*.[2] A natural reading of "$(X \mid c)$" would be "X given c."

The probability relations considered herein will be subsets of the Cartesian product of the set of these conditioned eventualities. Axiomatic systems often begin with a formally undefined, primitive relation of *weak supraprobability* (or perhaps one of *weak infraprobability*), which allows a mathematical elegance but has fostered some confusions of interpretation. I hope that the reader will forgive my beginning differently. Let " $\rhd$" represent *strict supraprobability*, and let " $\boxdot$" represent *equiprobability*. Thus, for example, a natural reading of " $(X \mid c_1) \rhd (Y \mid c_2)$" would be "$X$ given c_1 is more probable than Y given c_2" and " $(X \mid c_1) \boxdot (Y \mid c_2)$" could be read as "$X$ given c_1 has the probability of Y given c_2."

Formally define "weak supraprobability"

$$\trianglerighteq \stackrel{\text{def}}{=} (\rhd \cup \boxdot). \tag{D1}$$

We'll also have some use for the following[3]

$$\lhd \stackrel{\text{def}}{=} \rhd^{-1}, \tag{D2}$$

$$\trianglelefteq \stackrel{\text{def}}{=} \trianglerighteq^{-1}, \tag{D3}$$

$$\odot \stackrel{\text{def}}{=} \overline{\rhd \cup \boxdot \cup \rhd^{-1}}. \tag{D4}$$

The relation $\odot$ is empty when $\trianglerighteq$ is a complete preordering. In the absence of that assumption, measures of probability are impossible, exactly because measures are themselves completely ordered and thus impose a complete ordering on the things measured.

The first four axioms do not invite much controversy in their content, and elsewhere might be treated as if definitional.

Antisymmetry of strict supraprobability:

$$\overline{\left(\begin{array}{c}[(X \mid c_1) \rhd (Y \mid c_2)] \\ \wedge \\ [(Y \mid c_2) \rhd (X \mid c_1)]\end{array}\right)} \forall (X, Y, c_1, c_2). \tag{A1}$$

[2] The use of lowercase letters for things not considered as eventualities is simply a convenience; where " $(X \mid c)$" is meaningful, so would be " $(c \mid X)$."

[3] Throughout, an overscore will represent *complementation*—logical negation of propositions, set complementation in the case of relations.

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Reflexivity and symmetry of equiprobability:

$$\left[\begin{array}{c}[(X\,|\,c_1)\boxdot(X\,|\,c_1)]\\ \wedge\\ \left(\begin{array}{c}[(X\,|\,c_1)\boxdot(Y\,|\,c_2)]\\ \Leftrightarrow\\ [(Y\,|\,c_2)\boxdot(X\,|\,c_1)]\end{array}\right)\end{array}\right]\forall(X,Y,c_1,c_2)\,. \tag{A2}$$

Mutual exclusion of strict supraprobability and equiprobability:

$$\overline{[(X\,|\,c_1)\rhd(Y\,|\,c_2)]\wedge[(X\,|\,c_1)\boxdot(Y\,|\,c_2)]}\forall(X,Y,c_1,c_2)\,. \tag{A3}$$

Transitivity of weak supraprobability:

$$\left[\begin{array}{c}\left(\begin{array}{c}[(X\,|\,c_1)\trianglerighteq(Y\,|\,c_2)]\\ \wedge\\ [(Y\,|\,c_2)\trianglerighteq(Z\,|\,c_3)]\end{array}\right)\\ \Rightarrow\\ [(X\,|\,c_1)\trianglerighteq(Z\,|\,c_3)]\end{array}\right]\forall(X,Y,Z,c_1,c_2,c_3)\,. \tag{A4}$$

(Were $\trianglerighteq$ made primitive, instead of $\rhd$ and $\boxdot$, then (A1) and (A3) would follow from the definitions of $\rhd$ as $\trianglerighteq\backslash\trianglerighteq^{-1}$ and of $\boxdot$ as $\trianglerighteq\cap\trianglerighteq^{-1}$, as would symmetry of *equiprobability*. The remainder of (A2) would then be best replaced by an axiom that $\trianglerighteq$ were reflexive.)

2.2. Interpretation.

2.2.1. The subjects of the relations. Any event corresponds to a proposition that the event has occurred, and any proposition corresponds to the event that the proposition is true. Herein, I will treat the X and c of $(X\,|\,c\,)$ as themselves propositions, and an expression such as "$(X_1\vee X_2\,|\,c_1\wedge c_2)$" as meaningful.[4] (Any unconditional probabilities would be equivalent to those whose conditions were taken to be known with certainty, as in the case of tautologies.) Were these instead events, the equivalent expression would simply be "$(X_1\cup X_2\,|\,c_1\cap c_2\,)$." Likewise, an expression such as "$c\Rightarrow X$," representing an eventuality as logically implied by a context, would have an equivalent "$c\subseteq X$."

2.2.2. The weak relations. The formal relations above may be interpreted to fit the various interpretations of fundamental probability mentioned in the Introduction, but it is noteworthy that these relations cannot each describe a positive state of belief,[5] nor could they if the definitions were along the lines of

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[4] It might be noted that $\left(\begin{array}{c}(c\Rightarrow X_2)\\ \Rightarrow\\ [(X_1\,|\,c\,)=(X_1\wedge X_2\,|\,c\,)]\end{array}\right)\forall(X_1,X_2,c)\,.$

[5] The probability relations $\rhd$, $\boxdot$, $\trianglerighteq$, and $\odot$ all may represent *states of belief*, but *states of belief* may entail what is believed, what is rejected, and what is neither believed nor rejected.

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$$\left[\rhd \stackrel{\text{def}}{=} (\trianglerighteq \setminus \trianglelefteq)\right] \wedge \left[\lhd \stackrel{\text{def}}{=} (\trianglelefteq \setminus \trianglerighteq)\right] \wedge \left[\boxdot \stackrel{\text{def}}{=} (\trianglerighteq \cap \trianglelefteq)\right].$$

If "$(X \mid c) \trianglerighteq (Y \mid d)$" and "$(Y \mid d) \trianglelefteq (X \mid c)$" represented having *some* relative confidence and *being no less confident* in $(X \mid c)$ than in $(Y \mid d)$ *without* being either more confident in $(X \mid c)$ (which would preclude equal confidence) or being equally confident in the two (which would preclude being more confident in the one), then

$$\frac{[(X \mid c) \trianglerighteq (Y \mid d)]}{\overline{(X \mid c) \rhd (Y \mid d)} \wedge \overline{(X \mid c) \boxdot (Y \mid d)}}$$

would be somehow possible, in contradiction to the formal definition of "$\trianglerighteq$."

While a claim of form "$(X \mid c) \trianglerighteq (Y \mid d)$" does not seem peculiarly problematic as a *premise* under the aforementioned various notions of *probability*, it should cause some concern if an axiomatic system allows a claim of this form to result as a conclusion when none of the premises contain such a form (or its equivalent). In the particular case of subjectivism, it seems self-alienated to *conclude* merely that $(X \mid c) \trianglerighteq (Y \mid d)$. And if, taken as a whole, a system of axioms for subjective probability cannot proceed past a conclusion of such form, then one should ask by what justification it has excluded $(X \mid c) \odot (Y \mid d)$; a conclusion that $(X \mid c) \trianglerighteq (Y \mid d)$ requires a commitment to exactly one of the basic relations ($\rhd$ and $\boxdot$) without itself being able to make such a commitment.

2.2.3. Noncomparability. The relation $\odot$ holds between any pairs of contextualized eventualities that are not related by $\trianglerighteq$ nor by $\trianglelefteq$. $\odot$ thus implies the *absence* of a relative probability. A *pure* frequentism or a *pure* combinatoric interpretation of probability would force $\odot$ to be empty, as also (trivially) would betting quotients. But logicism and subjectivism in general do not require it to be empty. (An *impure* frequentism would be one in which $\trianglerighteq$ and $\trianglelefteq$ were always about beliefs about frequency; similarly for an impure combinatoric interpretation.)

Narrowed noncomparability. There may be some temptation to write something such as

$$[(X_1 \mid c_1) \odot (X_2 \mid c_2)] \wedge \overline{(X_1 \mid c_1)\, R\, (X_2 \mid c_2)}\,, \tag{1}$$

where $R \in \{\rhd, \boxdot, \lhd\}$, to capture the notion that the probability of $(X_1 \mid c_1)$ relative to $(X_2 \mid c_2)$ is unestablished but narrowed to two possibilities. However, the expression (1) does not capture this sort of limited exclusion; in fact, it contains a redundancy as, *ex definitione*,

$$[(X_1 \mid c_1) \odot (X_2 \mid c_2)] \Leftrightarrow \left[\begin{array}{c} \overline{(X_1 \mid c_1) \rhd (X_2 \mid c_2)} \\ \wedge \\ \overline{(X_1 \mid c_1) \boxdot (X_2 \mid c_2)} \\ \wedge \\ \overline{(X_1 \mid c_1) \lhd (X_2 \mid c_2)} \end{array}\right].$$

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The exclusion of R may be written as

$$[(X_1 \mid c_1) \odot (X_2 \mid c_2)] \vee [(X_1 \mid c_1)\, S\, (X_2 \mid c_2)] \vee [(X_1 \mid c_1)\, T\, (X_2 \mid c_2)]\,. \qquad (2)$$
$$\{S, T\} \equiv (\{\rhd, \boxdot, \lhd\} \setminus \{R\})$$

In considering a case in which it is held that one of the three relations $\rhd$, $\boxdot$, $\lhd$ may be ruled-out but either of the other two remain a possibility, it is important not to confuse the interpretation of these relations as themselves beliefs with beliefs about these relations. If we are describing, for example, an agent who is sure that two things are not equiprobable but unsure as to which is supraprobable, then we are employing a notion of *probability* as something external to the consciousness of the agent.

2.3. Remaining axioms. Though the various candidates offered as the fundamental notion or notions of probability are intertangled, the justification for any ostensible axiom would vary significantly depending upon which candidate or candidates were selected, if indeed any candidate were selected.

Even to the extent that axioms (A1) through (A4) might appear to be purely definitional, there is a question of why relations with these properties *apply* to what is of interest (knowledge, belief, frequency, or whatever). Frequentism could find $\rhd$ in $>$ and $\boxdot$ in $=$ amongst arithmetic ratios. The ostensibly intuitive logicisms of Keynes and of Koopman would claim an immediate apprehension of *supraprobability*, strict or weak (Keynes, 1921, pt. I, chap. II, pp. 10–19; Koopman, 1940a, sec. 1, p. 269; Koopman, 1940b, p. 763); a more humble logicism could claim these relations to be somehow discovered in experience. Subjectivism might claim to find them rather directly in self-examination, or by contemplating inclinations of choice (Ramsey, 1931).

Herein, I justify proposed axioms as conforming across interpretations and as capturing important properties in a relatively simple fashion; from the perspective of some philosophical positions, these propositions would be regarded as no better than that (if not worse). Some might regard the axioms herein as discovered principles of reasoning or as rationality constraints upon belief. For those readers who believe that the axioms of probability are to be known intuitively, I note that the axioms of *Subdivision*, of *Composition*, and of *Decomposition* offered below are both less complex and more general in application than those offered by Koopman (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, pp. 766–767). (These differences are especially pronounced in the case of the principle of *Subdivision*.)

Although some of the individual axioms are simpler than are corresponding principles in other systems, this system as whole is markedly more complicated than are familiar formal systems, in order that it be useable without any presumption of completeness.

Axioms (A1) through (A4) treat the subjects of the probability relations as if themselves black boxes. The remaining axioms concern how logical relations between what is given and contingencies and amongst what is given or amongst contingencies effect or affect probability relations.

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Axiom 5 (Generalized complementarity).

$$\left(\begin{array}{c}[(X \mid c_1) \trianglerighteq (Y \mid c_2)] \\ \Rightarrow \\ \left[\left(\overline{Y} \mid c_2\right) \trianglerighteq \left(\overline{X} \mid c_1\right)\right]\end{array}\right) \forall (X, Y, c_1, c_2)\,. \tag{A5}$$

Koopman calls this principle "Antisymmetry" (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, p. 766),[6] but it is not what would normally be meant in claiming that $\trianglerighteq$ were antisymmetrical. (A5) may be seen as a variation on

$$\left(\begin{array}{c}[(X \mid c) \triangleright (Y \mid c)] \\ \Rightarrow \\ \left[\left(\overline{Y} \mid c\right) \triangleright \left(\overline{X} \mid c\right)\right]\end{array}\right) \forall (X, Y, c)\,, \tag{3}$$

which is often suggested as an axiom (Fishburn, 1986).

Axiom 6 (Implication of presupposition).

$$\left(\begin{array}{c}(c_1 \Rightarrow X) \\ \Leftrightarrow \\ [(X \mid c_1) \trianglerighteq (Y \mid c_2)]\end{array}\right) \forall (X, Y, c_1, c_2)\,. \tag{A6}$$

This axiom corresponds to Koopman's Axiom of Verification (Koopman, 1940b, p. 766), which is a generalization of his earlier Axiom of Verified Contingency (Koopman, 1940a, sec. 3, p. 276), in combination with his Axiom of Implication (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, p. 766).

To some, it may seem obvious that the conditional probability of an outcome logically implied by the condition should be maximal. However, (A6) makes no allowance for *unrecognized implications*, which cannot inform judgment.

If probability relations are objective, then (A6) simply need not make allowance; probability judgments are then *about* probability, and may be mistaken in much the same senses as judgments about other things may be mistaken. On the other hand, if probability is subjective—if probability judgments simply *are* probabilities—then they cannot be wrong in some of those senses.[7] However, orthodox subjectivism acknowledges a way or ways in which probability judgments may be in error; such judgments are considered mistaken when they violate principles such as *transitivity* and *additivity*, and they are considered mistaken when they violate *coherence*. Yet, in the case of unrecognized implications, the way of being mistaken is not actually different. Were we unboundedly rational, then there would be no unrecognized implications; and, because we are not unboundedly rational, it is possible for us to be confronted

[6] Awkwardly, Koopman uses a symbol for weak *infraprobability* in one work (Koopman, 1940a, sec. 1, p. 270), but then uses the same symbol for what amounts in definition to the complement of strict *supraprobability* in another (Koopman, 1940b, p. 764), and in both cases the reversed symbol is used as that of the inverse relation, yet he often presents the same expressions using those symbols with the same names as his axioms, though these expressions must carry different meanings. Those otherwise different meanings operationalize in the same way under an assumption of completeness; I speculate that Koopman simply got ahead of himself in seeking a foundation for measures of probability.

[7] In an event-oriented framework, as favored by many subjectivists, the issue would of course be cases of sets not recognized as having a relation of form "$c_1 \subseteq X$."

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with complex problems in which we shall violate what orthodox subjectivism declares to be the rules of probability.

It might be practicable to replace (A6) with some more complicated axiom that made allowance for unrecognized implications, but that would not extract subjectivism from the more general problem.

From (A5) and (A6), one may conclude that

$$\left(\begin{array}{c} (c_1 \Rightarrow X) \\ \Rightarrow \\ \left[(Y \mid c_2) \trianglerighteq (\overline{X} \mid c_1) \right] \end{array} \right) \forall (X, Y, c_1, c_2) . \tag{4}$$

Thus, there are upper and lower bounds to probability.

To the best of my knowledge, the next two principles, (A7) and (A8), have not been axioms in any prior system. These two principles are theoremata in the familiar systems in which probability may be measured. They are not theoremata in Koopman's systems of general principles (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, pp. 766–767), yet they may be defended exactly as Koopman defended his axioms (Koopman, 1940a, sec. 3, p. 275; Koopman, 1940b, p. 767).[8]

Axiom 7 (Presuppositional interposition).

$$\left(\begin{array}{c} [(X \mid c_1) \trianglerighteq (X \mid c_1 \vee c_2)] \\ \Leftrightarrow \\ [(X \mid c_1 \vee c_2) \trianglerighteq (X \mid c_2)] \end{array} \right) \forall (X, c_1, c_2) ; \tag{A7}$$

that is to say that if one context by itself makes an outcome more probable than the joint possibility of that context and another, then that other context by itself leaves that outcome less probable than the joint possibility, and *vice versa*.

In a case in which $c_1 = (c_3 \wedge c_4)$ and $c_2 = (c_3 \wedge \overline{c_4})$,

$$\left(\begin{array}{c} [(X \mid c_3 \wedge c_4) \trianglerighteq (X \mid c_3)] \\ \Leftrightarrow \\ [(X \mid c_3) \trianglerighteq (X \mid c_3 \wedge \overline{c_4})] \end{array} \right) \forall (X, c_3, c_4) . \tag{5}$$

If one used (5) as the axiom, then (A7) could be derived from it by nearly the reverse process, but noting also that the implication in (A7) is trivially true when $c_2 = c_1$.

Many readers will recognize that

$$\left[\begin{array}{c} [(X \mid c_1) \trianglerighteq (X \mid c_2)] \\ \Rightarrow \\ \left(\begin{array}{c} [(X \mid c_1) \trianglerighteq (X \mid c_1 \vee c_2)] \\ \wedge \\ [(X \mid c_1 \vee c_2) \trianglerighteq (X \mid c_2)] \end{array} \right) \end{array} \right] \forall (X, c_1, c_2) , \tag{6}$$

which is a special case of (A9) below.

[8] I believe that the absence of (A7) and (A8) from Koopman's general system is an artifact of his preoccupation with arriving at measurable probabilities, even though he regarded such as a special case.

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Axiom 8 (Equiprobability or infraprobability of conjoined contingency).[9]

$$\left[\begin{array}{c} [(X_2 \mid c) \trianglerighteq (X_1 \wedge X_2 \mid c)] \\ \wedge \\ \left(\begin{array}{c} [(X_2 \mid c) \boxdot (X_1 \wedge X_2 \mid c)] \\ \Leftrightarrow \\ [(c \mid c) \boxdot (X_1 \mid X_2 \wedge c)] \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(X_1 \mid X_2 \wedge c) \triangleright (X_1 \wedge X_2 \mid c)] \\ \Leftrightarrow \\ \left(\begin{array}{c} [(c \mid c) \triangleright (X_2 \mid c)] \\ \wedge \\ [(X_1 \wedge X_2 \mid c) \triangleright (\overline{c} \mid c)] \end{array}\right) \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(X_1 \mid X_2 \wedge c) \boxdot (X_1 \wedge X_2 \mid c)] \\ \Leftrightarrow \\ \left(\begin{array}{c} [(c \mid c) \boxdot (X_2 \mid c)] \\ \vee \\ [(X_1 \wedge X_2 \mid c) \boxdot (\overline{c} \mid c)] \end{array}\right) \end{array}\right) \end{array}\right] \forall (X_1, X_2, c)\,. \tag{A8}$$

(The necessity claims entail some redundancy in the context of (A6), but simplify expression.)

The first two parts of (A8), that

$$\left[\begin{array}{c} [(X_1 \mid c) \trianglerighteq (X_1 \wedge X_2 \mid c)] \\ \wedge \\ \left(\begin{array}{c} [(X_1 \mid c) \boxdot (X_1 \wedge X_2 \mid c)] \\ \Leftrightarrow \\ [(c \mid c) \boxdot (X_2 \mid X_1 \wedge c)] \end{array}\right) \end{array}\right] \forall (X_1, X_2, c)\,, \tag{7}$$

are jointly a variation on Ockham's Razor. In the case where $X_1 = Y_1 \vee \overline{Y_2}$ and $X_2 = Y_1 \vee Y_2$, noting that

$$\begin{array}{rcl} \left(Y_1 \vee \overline{Y_2} \mid (Y_1 \vee Y_2) \wedge c\right) & = & \left(\left(Y_1 \vee \overline{Y_2}\right) \wedge (Y_1 \vee Y_2) \mid (Y_1 \vee Y_2) \wedge c\right) \\ & = & (Y_1 \mid (Y_1 \wedge c) \vee (Y_2 \wedge c)) \end{array}$$

and employing (A7) with (A4), with (A2), and with (A3), one arrives at

[9] $(\overline{c} \mid c) = (c \wedge \overline{c} \mid c)$; thus, the use of $(\overline{c} \mid c)$ for minimal plausibility may be viewed as an application of the Law of Noncontradiction (Aristoteles, 1933, bk. 4, chap. 3, pp. 1005b15–1005b22). Keynes and Koopman, despite their misgivings about the general applicability of complete preorderings, concerned themselves with finding conditions for just such application (and, more generally, for use of arithmetic) and moved quickly to introduce a 0 (Keynes, 1921, pt. II, chap. X, sec. 4, p. 120, def. III and chap. XII, sec. 4, p. 134, def. III; Koopman, 1940a, sec. 1, p. 270, fn. 3; Koopman, 1940b, p. 763). In Koopman's case, this introduction caused him to include needless complications in his axioms of Composition and of Decomposition, which conditions will be noted below.

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$$\left[\begin{array}{c}[(Y_1 \vee Y_2 \,|\, c) \trianglerighteq (Y_1 \,|\, c)] \\ \wedge \\ \left(\begin{array}{c}[(Y_1 \vee Y_2 \,|\, c) \boxdot (Y_1 \,|\, c)] \\ \Leftrightarrow \\ [(c \,|\, c) \boxdot (Y_1 \,|\, Y_2 \wedge c)]\end{array}\right)\end{array}\right] \forall (Y_1, Y_2, c), \tag{8}$$

which conforms to an intuition that relaxing assumptions is Ockhamistic.

The remainder of (A8) asserts that, if $c \Rightarrow X_2$, then $(X_1 \,|\, c) \boxdot (X_1 \,|\, X_2 \wedge c) \boxdot (X_1 \wedge X_2 \,|\, c)$, in keeping with $(X_1 \,|\, c) = (X_1 \,|\, X_2 \wedge c) = (X_1 \wedge X_2 \,|\, c)$; and that, otherwise, taking X_2 as given imputes greater plausibility (except in the case in which X_1 contradicts X_2). Obtaining such an imputation by a spurious presumption is of course *begging the question*. Continuing with $X_1 = Y_1 \vee \overline{Y_2}$ and $X_2 = Y_1 \vee Y_2$, one arrives at

$$\left[\begin{array}{c}\left(\begin{array}{c}[(Y_1 \,|\, (Y_1 \vee Y_2) \wedge c) \rhd (Y_1 \,|\, c)] \\ \Leftrightarrow \\ \left(\begin{array}{c}[(c \,|\, c) \rhd (Y_1 \vee Y_2 \,|\, c)] \\ \wedge \\ [(Y_1 \,|\, c) \rhd (\overline{c} \,|\, c)]\end{array}\right)\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}[(Y_1 \,|\, (Y_1 \vee Y_2) \wedge c) \boxdot (Y_1 \,|\, c)] \\ \Leftrightarrow \\ \left(\begin{array}{c}[(c \,|\, c) \boxdot (Y_1 \vee Y_2 \,|\, c)] \\ \vee \\ [(Y_1 \,|\, c) \boxdot (\overline{c} \,|\, c)]\end{array}\right)\end{array}\right)\end{array}\right] \forall (Y_1, Y_2, c), \tag{9}$$

which isn't as elegant as the transformation of (7) into (8).

Axiom 9 (Disjunctive presupposition).

$$\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}[(X \,|\, c_1) \trianglerighteq (Y \,|\, c_2)] \\ \wedge \\ [(X \,|\, c_1) \trianglerighteq (Y \,|\, c_3)]\end{array}\right) \\ \Rightarrow \\ [(X \,|\, c_1) \trianglerighteq (Y \,|\, c_2 \vee c_3)]\end{array}\right] \\ \wedge \\ \left[\begin{array}{c}\left(\begin{array}{c}[(Y \,|\, c_2) \trianglerighteq (X \,|\, c_1)] \\ \wedge \\ [(Y \,|\, c_3) \trianglerighteq (X \,|\, c_1)]\end{array}\right) \\ \Rightarrow \\ [(Y \,|\, c_2 \vee c_3) \trianglerighteq (X \,|\, c_1)]\end{array}\right]\end{array}\right) \forall (X, Y, c_1, c_2, c_3). \tag{A9}$$

This principle is not quite just a refactoring of Koopman's axiom of *Alternative Presumption* (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, p. 767),

$$\left[\begin{array}{c}\left(\begin{array}{c}[(X \,|\, c_1) \trianglerighteq (Y \,|\, c_2 \wedge d)] \\ \wedge \\ \left[(X \,|\, c_1) \trianglerighteq \left(Y \,\middle|\, c_2 \wedge \overline{d}\right)\right]\end{array}\right) \\ \Rightarrow \\ [(X \,|\, c_1) \trianglerighteq (Y \,|\, c_2)]\end{array}\right] \forall (X, Y, c_1, c_2, d), \tag{10}$$

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as nothing for the latter corresponds to what would be the case of $c_2 = c_1$ in the former. But (10) is an immediate implication of (A9).

Axiom 10 (Generalized subdivision).

$$\left[\begin{array}{c}\left(\begin{array}{c}[(X_1 \vee X_2 \mid c_1) \trianglerighteq (Y_1 \vee Y_2 \mid c_2)]\\ \wedge\\ [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \trianglerighteq (X_2 \mid c_1)]\\ \wedge\\ [(Y_1 \mid c_2) \trianglerighteq (Y_2 \mid c_2)]\end{array}\right)\\ \Rightarrow\\ [(X_1 \mid c_1) \trianglerighteq (Y_2 \mid c_2)]\end{array}\right] \forall \left(\begin{array}{c} X_1, X_2,\\ Y_1, Y_2,\\ c_1, c_2\end{array}\right). \tag{A10}$$

Koopman's presentations of his narrower principle of *Subdivision* use a combination of formal and natural expression (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, p. 767). Expressed formally and in full, it may be seen to be quite involved.

$$\left[\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}\overline{i = j}\\ \wedge\\ \left[\begin{array}{c}(i, j)\\ \in\\ \{1, 2, \ldots, n\}^2\end{array}\right]\end{array}\right)\\ \Rightarrow\\ (\overline{X_i \wedge X_j} \wedge \overline{Y_i \wedge Y_j})\end{array}\right] \forall (i, j)\\ \wedge\\ \left(\begin{array}{c}\left[\begin{array}{c}(X_{j+1} \mid \bigvee_{k=1}^{n} X_k)\\ \trianglerighteq\\ (X_j \mid \bigvee_{k=1}^{n} X_k)\end{array}\right]\\ \Leftarrow\\ [j \in \{1, 2, \ldots, n-1\}]\end{array}\right) \forall j\\ \wedge\\ \left(\begin{array}{c}\left[\begin{array}{c}(Y_{j+1} \mid \bigvee_{k=1}^{n} Y_k)\\ \trianglerighteq\\ (Y_j \mid \bigvee_{k=1}^{n} Y_k)\end{array}\right]\\ \Leftarrow\\ [j \in \{1, 2, \ldots, n-1\}]\end{array}\right) \forall j\end{array}\right]\\ \Rightarrow\\ \left[\begin{array}{c}(X_n \mid \bigvee_{k=1}^{n} X_k)\\ \trianglerighteq\\ (Y_1 \mid \bigvee_{k=1}^{n} Y_k)\end{array}\right]\end{array}\right) \forall \left(\begin{array}{c}\left\{\begin{array}{c} X_1, X_2,\\ \ldots, X_n\end{array}\right\},\\ \left\{\begin{array}{c} Y_1, Y_2,\\ \ldots, Y_n\end{array}\right\}\end{array}\right)\\ \Leftarrow\\ (n \in \mathbb{N}_1)\end{array}\right] \forall n \tag{11}$$

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But (A10) in combination with (A6) tell us that

$$\left[\begin{array}{c}\left(\begin{array}{c}\left[(X\,|\,c_1)\trianglerighteq(\overline{X}\,|\,c_1)\right]\\ \wedge\\ \left[(Y\,|\,c_2)\trianglerighteq(\overline{Y}\,|\,c_2)\right]\end{array}\right)\\ \Rightarrow\\ \left[(X\,|\,c_1)\trianglerighteq(\overline{Y}\,|\,c_2)\right]\end{array}\right]\forall(X_1,X_2,c_1,c_2)\,,$$

which provides a base for arriving at (11) inductively by further application of (A10). That the use of two jointly simpler, more general propositions was possible yet apparently unseen by Koopman should increase skepticism about intuitionism as such.

Axiom 11 (Generalized additivity).

$$\left[\begin{array}{c}\left(\begin{array}{c}[(Y_1\wedge Y_2\,|\,c_2)\trianglerighteq(X_1\wedge X_2\,|\,c_1)]\\ \wedge\\ [(X_1\,|\,c_1)\trianglerighteq(Y_1\,|\,c_2)]\\ \wedge\\ [(X_2\,|\,c_1)\trianglerighteq(Y_2\,|\,c_2)]\end{array}\right)\\ \Rightarrow\\ [(X_1\vee X_2\,|\,c_1)\trianglerighteq(Y_1\vee Y_2\,|\,c_2)]\end{array}\right]\forall\left(\begin{array}{c}X_1,X_2,\\ Y_1,Y_2,\\ c_1,c_2\end{array}\right).\qquad\text{(A11)}$$

This axiom is a straight-forward generalization of the popular principle of *Additivity* (de Finetti, 1931, sec. 7, pp. 156–157; Fishburn, 1986, sec. 1, p. 336).

(A12) and (A13) correspond to Koopman's principles of *Composition* and of *Decomposition* (Koopman, 1940a, sec. 3, p. 276; Koopman, 1940b, pp. 766–767), but are simpler and more general. So long as Koopman's principles of *Composition* and of *Decomposition* were discernible by intuition, identifying redundancies would be largely irrelevant to his project. But, as with *Subdivision*, needless complexities should provoke skepticism about their direct apprehension. Further, they have been particularly challenged as insufficiently self-evident (Shimony, 1955, p. 4), and the complexity of their expression could not have had favorable effect on that perception.

Axiom 12 (Generalized composition).

$$\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}[(X_1\,|\,c_1)\trianglerighteq(Y_1\,|\,c_2)]\\ \wedge\\ [(X_2\,|\,X_1\wedge c_1)\trianglerighteq(Y_2\,|\,Y_1\wedge c_2)]\end{array}\right)\\ \vee\\ \left(\begin{array}{c}[(X_1\,|\,c_1)\trianglerighteq(Y_2\,|\,Y_1\wedge c_2)]\\ \wedge\\ [(X_2\,|\,X_1\wedge c_1)\trianglerighteq(Y_1\,|\,c_2)]\end{array}\right)\end{array}\right]\\ \Rightarrow\\ [(X_1\wedge X_2\,|\,c_1)\trianglerighteq(Y_1\wedge Y_2\,|\,c_2)]\end{array}\right)\forall\left(\begin{array}{c}X_1,X_2,\\ Y_1,Y_2,\\ c_1,c_2\end{array}\right).\qquad\text{(A12)}$$

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Koopman's axiom of *Composition* contains two conditions not found in (A12). One is that $(Y_1 \wedge Y_2 \mid c_2)$ not have minimal probability; this condition is simply unnecessary. The other is that $(X_1 \wedge X_2 \mid c_1)$ also not have minimal probability; if (A7) is accepted, then where this condition is actually needed in (A12) it will be implied in the others. Relations were obfuscated by declaring propositions not to be 0 (rather than declaring them to be of greater than minimal probability), and resulted in the inclusion of superfluous conditions.

Axiom 13 (Generalized decomposition).

$$\left(\begin{array}{c} [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)] \\ \Rightarrow \\ \left[\begin{array}{c} \left(\begin{array}{c} [(Y_1 \mid c_2) \trianglerighteq (X_1 \mid c_1)] \\ \Rightarrow \\ [(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)] \end{array} \right) \\ \wedge \\ \left(\begin{array}{c} [(Y_1 \mid c_2) \trianglerighteq (X_2 \mid X_1 \wedge c_1)] \\ \Rightarrow \\ [(X_1 \mid c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)] \end{array} \right) \\ \wedge \\ \left(\begin{array}{c} [(Y_2 \mid Y_1 \wedge c_2) \trianglerighteq (X_1 \mid c_1)] \\ \Rightarrow \\ [(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_1 \mid c_2)] \end{array} \right) \\ \wedge \\ \left(\begin{array}{c} [(Y_2 \mid Y_1 \wedge c_2) \trianglerighteq (X_2 \mid X_1 \wedge c_1)] \\ \Rightarrow \\ [(X_1 \mid c_1) \trianglerighteq (Y_1 \mid c_2)] \end{array} \right) \end{array} \right] \end{array} \right) \forall \left(\begin{array}{c} X_1, X_2, \\ Y_1, Y_2, \\ c_1, c_2 \end{array} \right). \qquad \text{(A13)}$$

Koopman's axiom of *Decomposition* again contains a condition that $(Y_1 \wedge Y_2 \mid c_2)$ not have minimal probability, which is again an artifact of obfuscation from using " 0" to represent any proposition of minimal probability and then declaring propositions not to be 0.

2.3.1. Impossible presuppositions. Koopman makes a "tacit assumption" that no presupposition have minimal probability (Koopman, 1940b, p. 766). I do not formally exclude such presuppositions, but take the operationally equivalent position that employing them is like including contradictions amongst premises; anything can follow, and the results have no application. (Koopman's assumption was surely to disallow *arithmetic* denominators of 0.)

2.4. On consistency and on completeness of the axioms. As noted in the Introduction, all the axioms here, with the possible exception of (A6) are themselves axioms or theoremata in the systems of the major popular interpretations, and (A6) obtains under at least some of those systems. The reader can trivially verify that, while what I offer as axioms are not sufficient to imply the Kolmogorov axioms, his axioms imply (A1) through (A5) and (A7) through (A13), and that (A6) *conforms* to the Kolmogorov axioms. Thus, the axioms are at least as consistent as are those popular systems, and

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are at least as consistent as are any systems whose axioms imply those principles which are Kolmogorov's axioms.

Further, Koopman's axioms (Koopman, 1940a, sec. 3, pp. 275–276; Koopman, 1940b, p. 766) are all either axioms herein or theoremata thereof. Koopman's original axiom V (Koopman, 1940a, sec. 3, p. 275) is an implication of (A6) and, jointly, his later axiom V (Koopman, 1940b, p. 766) with his axiom I are equivalent to (A6). His original axiom R (Koopman, 1940a, sec. 3, p. 276) is an implications of (A2) with (D1); his later R (Koopman, 1940b, p. 766) follows upon that.[10] His axioms T and A are identical to (A4) and (A5) respectively. His C is generalized by (A12) with the aid of (A7); his D is simply generalized by (A13). His axiom P is an immediate implication of (A9). As explained above, his axiom S may be inductively derived from (A6) and (A10). Principles (A7), (A8), and (A11) are not implications of Koopman's axioms. Hence, the system of this article is more complete than that of Koopman.

2.5. Binary cases without conjunction or disjunction. Much or all of reasoning about qualitative probabilities may be developed from cases conceptualized in terms of just two possible outcomes or in terms of two possible presuppositions or compounds thereof. Most proposed axioms of qualitative probability are expressed in such terms. Koopman's principle of *Subdivision* (11) is an exception, but can be reduced to a theorem of (A10) in combination with (A6), each of which is expressed in binary terms.

In some important applications, one is unconcerned with conjunction and with disjunction of presuppositions and of outcomes. For example, the conventional economic theory of decision-making in the face of risk often *begins* with models in which there is some set of *lotteries*, each associating a vector of possible outcomes with a vector of probabilities, and the agent is simply taken to choose exactly one of these lotteries. Within the framework herein, choice of a lottery would correspond to some presupposition c_i.[11] Probabilities would no longer necessarily be quantified, but the probability orderings within and across lotteries would remain of interest.

Simple binary models without conjunction and without disjunction are by themselves insufficient to deal with cases in which each context may produce more than one outcome, outcomes present still further problems of uncertainty, or the aspects of

[10] Recall the point made in footnote 4 above.

[11] Drawing upon Frank Knight (Knight, 1921, I.I.26, pp. 19–20 and III.VII, pp. 197–232), prevailing convention within economics is to use "risk" to refer to those cases in which, while not all realized outcomes can be known in advance, each considered action has a knowable expected value, and to use "uncertainty" for cases in which expectations are not knowable. (This convention is violated insofar as "decision-making under uncertainty" usually refers to cases in which expected values are knowable.) However, Knight made this distinction in an attempt to explain *economic profit* on a basis that 'a *measurable* uncertainty, or "risk" proper, as we shall use the term, is so far different from an *unmeasurable* one that it is not in effect an uncertainty at all'. If one takes the *essential* distinction to be between cases in which, though realized outcomes could not be known in advance, there were nonetheless discernible best choices and cases in which there were not, then development of a theory of decision-making under *unmeasurable uncertainty* would expand the scope of *risk* and reduce that of other *uncertainty*.

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interest are shared by multiple outcomes that otherwise differ; in those cases, one must draw upon further principles, such as (A7) through (A13). Still, these simple binary cases allow expression of principles applicable more generally.

2.5.1. Cases of dichotomous outcomes. (A1) through (A6) and (A10) allow consideration of relations amongst cross-conditional cases of dichotomous outcomes. Given $(X \mid c_1)$, $\left(\overline{X} \mid c_1\right)$, $(Y \mid c_2)$, and $\left(\overline{Y} \mid c_2\right)$, and given that $\left[(X \neq Y) \wedge \left(X \neq \overline{Y}\right)\right] \vee (c_1 \neq c_2)$, there are forty six distinct possibilities (exhibited in Table A.1 of Appendix A). Cases involving $\trianglerighteq$, $\trianglelefteq$, and any other unions amongst $\rhd$, $\boxdot$, and $\odot$ are disjunctive composites of these forty six. Cases in which $c_1 \neq c_2$ of course include the dichotomous $c_2 = \overline{c_1}$. There are, trivially, just three possible cases when $\left[(X = Y) \vee \left(X = \overline{Y}\right)\right] \wedge (c_1 = c_2)$.

On the apparent intuitions of greater probability in case of dichotomous outcomes. Amongst the reasons that some may believe that probabilities are always at least implicitly quantified are that most or all of us have such intuitions as

- that the most probable realization of one context is accepted to be greater than the most probable realization of another, and can even seem to be "a lot" more probable, and, symmetrically, that the least probable realization may have less probability in one context than in another, and can seem to have "a lot" less probability;
- that the probability of a realization may seem to increase or to decrease, without switching its order with respect to its complement.

But these intuitions must be considered without losing sight of context.

When the most probable realization in one context is even more probable than the most probable in another, this is a case of

$$(X_1 \mid c_1) \rhd (X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right) \rhd \left(\overline{X_1} \mid c_1\right).$$

I would suggest that when one realization seems "a lot" more probable, this is a case of

$$\begin{array}{c} (X_1 \mid c_1) \\ \rhd \\ (X_2 \mid c_2) \rhd \cdots \rhd (X_{n-1} \mid c_{n-1}) \\ \rhd \\ (X_n \mid c_n) \rhd \left(\overline{X_n} \mid c_n\right) \\ \rhd \\ \left(\overline{X_{n-1}} \mid c_{n-1}\right) \rhd \cdots \rhd \left(\overline{X_2} \mid c_2\right) \\ \rhd \\ \left(\overline{X_1} \mid c_1\right) \end{array} .$$

And when, for example, the victory of a front-runner seems even more assured than previously, it is typically because the context has changed. One is really considering a case where X_1 and X_2 are regarded as practically identical, and

$$(X_1 \mid c_1) \rhd (X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right) \rhd \left(\overline{X_1} \mid c_1\right).$$

If an unorthodox subjectivist interpretation is employed, and it is insisted that the context has *not* changed but that the agent has simply become a more firm

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believer, the increase may nonetheless be only a change in order relative to other eventualities of other contexts, rather than a change whose explanation requires quantification.

These points apply when the set of outcomes is not initially conceptualized as dichotomous. While an agent may not consciously reconceptualize in terms of the complement of the most or least probable outcome, refactoring outcomes so that they are characterized by any necessary mutual exclusivity, still what seems to be a measured difference may more simply be an awareness of a difference in ranking.

While *measures* indeed have special significance when *unique*, note how often in life an attempt is made to understand or to convey the meaning of a quantified probability exactly by associating it with some outcome other than the one to which the quantification were assigned, or by comparing that first outcome with another outcome said to be of greater or of lesser probability; the numbers are in these cases to be understood in terms of conceptually prior relations of *equiprobability* or of *supraprobability* amongst outcomes.

The flip side is that we may apply reasoning of this sort to the relative *fairness* of coins. A coin corresponding to c_1 is necessarily more *fair* than one corresponding to c_2 if

$$(S_2 \mid c_2) \rhd (S_1 \mid c_1) \trianglerighteq (\overline{S_1} \mid c_1) \rhd (\overline{S_2} \mid c_2),$$

where "S_i" represents one side of the ith coin and "$\overline{S_i}$" the other. Under (A4), (A2), (A3), and (A5), it may be seen that

$$(S_2 \mid c_2) \rhd (S_1 \mid c_1) \rhd (\overline{S_2} \mid c_2)$$

is sufficient to show that the coin corresponding to c_1 is more *fair*. But we are not assured that $\trianglerighteq$ orders every coin, nor every pair of coins that are internally ordered by $\trianglerighteq$.

2.5.2. Cases of binary outcomes more generally. If we consider more general cases of $(X_1 \mid c_1)$, $(X_2 \mid c_1)$, $(Y_1 \mid c_2)$, and $(Y_2 \mid c_2)$, then there are many more cases; but, excluding $\odot$, they may be reduced to a canonic eleven (see Appendix B), with cases of $\odot$ filling the implied lacunæ.

Again, one might consider cases in which $c_2 = c_1$ or $c_2 = \overline{c_1}$, or cases of $X_2 = X_1$, of $Y_2 = Y_1$, or of $Y_j = X_i$.

2.6. Generalizing Bayes' Theorem. Bayes' Theorem is typically expressed and conceptualized in terms that arithmetically relate a quantitative conditional probability to its inverse in terms of two quantitative unconditional probabilities, as thus

$$prob\,(X_1 \mid X_2) = prob\,(X_2 \mid X_1) \cdot \frac{prob\,(X_1)}{prob\,(X_2)}.$$

It might seem that nothing at all like this relation may be found within the qualitative framework presented above. But, from (A12) and from (A13) one may extract

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$$\left[\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} [(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)] \\ \wedge \\ [(X_1 \mid c_1) \trianglerighteq (Y_1 \mid c_2)] \end{array}\right) \\ \Rightarrow \\ [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)] \end{array}\right] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} [(Y_1 \mid c_2) \trianglerighteq (X_1 \mid c_1)] \\ \Rightarrow \\ [(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)] \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(Y_2 \mid Y_1 \wedge c_2) \trianglerighteq (X_2 \mid X_1 \wedge c_1)] \\ \Rightarrow \\ [(X_1 \mid c_1) \trianglerighteq (Y_1 \mid c_2)] \end{array}\right) \end{array}\right] \\ \Leftarrow \\ [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)] \end{array}\right) \end{array}\right] \forall \left(\begin{array}{c} X_1, X_2, \\ Y_1, Y_2, \\ c_1, c_2 \end{array}\right). \tag{12}$$

With the application of (A2) and of (A3), this becomes

$$\left[\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} [(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)] \\ \wedge \\ [(X_1 \mid c_1) \trianglerighteq (Y_1 \mid c_2)] \end{array}\right) \\ \Rightarrow \\ [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)] \end{array}\right] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} [(Y_1 \mid c_2) \trianglerighteq (X_1 \mid c_1)] \\ \Rightarrow \\ [(X_2 \mid X_1 \wedge c_1) \triangleright (Y_2 \mid Y_1 \wedge c_2)] \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(Y_2 \mid Y_1 \wedge c_2) \trianglerighteq (X_2 \mid X_1 \wedge c_1)] \\ \Rightarrow \\ [(X_1 \mid c_1) \triangleright (Y_1 \mid c_2)] \end{array}\right) \end{array}\right] \\ \Leftarrow \\ [(X_1 \wedge X_2 \mid c_1) \triangleright (Y_1 \wedge Y_2 \mid c_2)] \end{array}\right) \end{array}\right] \forall \left(\begin{array}{c} X_1, X_2, \\ Y_1, Y_2, \\ c_1, c_2 \end{array}\right); \tag{13}$$

or, more simply,

$$\left[\begin{array}{c} \left(\begin{array}{c} \overline{(X_2 \mid X_1 \wedge c_1) \trianglerighteq (Y_2 \mid Y_1 \wedge c_2)} \\ \vee \\ \overline{(X_1 \mid c_1) \trianglerighteq (Y_1 \mid c_2)} \\ \vee \\ [(X_1 \wedge X_2 \mid c_1) \trianglerighteq (Y_1 \wedge Y_2 \mid c_2)] \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(X_2 \mid X_1 \wedge c_1) \triangleright (Y_2 \mid Y_1 \wedge c_2)] \\ \vee \\ [(X_1 \mid c_1) \triangleright (Y_1 \mid c_2)] \\ \vee \\ \overline{(X_1 \wedge X_2 \mid c_1) \triangleright (Y_1 \wedge Y_2 \mid c_2)} \end{array}\right) \end{array}\right] \forall (X_1, X_2, Y_1, Y_2, c_1, c_2), \tag{14}$$

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which is a generalization of the relation of Bayes' Theorem. In the special case of $(c_2 = c = c_1) \wedge (Y_1 = X_2) \wedge (Y_2 = X_1)$,

$$\left[\begin{array}{c} \overline{(X_1 \mid X_2 \wedge c) \trianglerighteq (X_2 \mid X_1 \wedge c)} \\ \vee \\ \overline{(X_2 \mid c) \trianglerighteq (X_1 \mid c)} \end{array} \right] \forall (X_1, X_2, c),$$

the generalization of

$$prob(X_2 \mid X_1 \wedge c) \cdot prob(X_1 \mid c) = prob(X_1 \mid X_2 \wedge c) \cdot prob(X_2 \mid c)$$

is quite obvious, and this equation is merely a rearrangement (or prearrangement) of Bayes' Theorem, with some condition c (perhaps known with certainty) taken to inform all the probabilities.

2.6.1. Updating and coherence. The point that unconditional probabilities correspond to cases in which conditions are taken to be known with perfect certainty comes to bear if we consider the usual process of Bayesian updating in which, upon the introduction of new information I_n, $prob(X \mid I_n)$ is computed and then *assumes the rôle* and names of $prob(X)$; I_n disappears into the subsequently unacknowledged background. If the presence of previously new information were not suppressed, then instead of something such as "$prob(X)$," one would write something such as "$prob\left(X \mid \bigwedge_{i=1}^{n} I_i\right)$" (or such as "$prob\left(X \mid \bigcap_{i=1}^{n} I_i\right)$").

In a framework of relations $\rhd$ and $\boxdot$, with or without completeness, updating corresponds exactly to abandonment and avoidance of those parts of the relational graph for $\trianglerighteq$ that allow the new information not to hold. One of the rules that informs the remainder of the graph is indeed (14), and to that extent updating might be regarded as Bayesian in some generalized sense. If $\trianglerighteq$ happens to be a complete preordering, then it may be proxied by many different quantifications; and, as ordinary Bayesian updating conforms to (14), it proxies that part of the relational graph that remains. One might have particular interest in the special case of (13) in which

$$\left(\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} \left[(X \mid I \wedge c) \trianglerighteq \left(X \,\middle|\, \overline{I} \wedge c\right)\right] \\ \wedge \\ \left[(I \mid c) \trianglerighteq \left(\overline{I} \mid c\right)\right] \end{array} \right) \\ \Rightarrow \\ \left[(X \wedge I \mid c) \trianglerighteq \left(X \wedge \overline{I} \mid c\right)\right] \end{array} \right] \\ \wedge \\ \left[\begin{array}{c} \left(\begin{array}{c} \left[(X \mid I \wedge c) \rhd \left(X \,\middle|\, \overline{I} \wedge c\right)\right] \\ \vee \\ \left[(I \mid c) \rhd \left(\overline{I} \mid c\right)\right] \end{array} \right) \\ \Leftarrow \\ \left[(X \wedge I \mid c) \rhd \left(X \wedge \overline{I} \mid c\right)\right] \end{array} \right] \end{array} \right) \forall (X, I, c). \qquad (15)$$

(Because of an absence of *proportionalities*, formula (15) is notably weaker than the quantified version of Bayes' Theorem in finding support for theory in evidence. We may interpret X as some body of theory and I as an outcome made more likely should the theory prove true. In this case, $(X \wedge I \mid c) \rhd \left(X \wedge \overline{I} \mid c\right)$ may indicate $(X \mid I \wedge c) \rhd \left(X \,\middle|\, \overline{I} \wedge c\right)$ or $(I \mid c) \rhd \left(\overline{I} \mid c\right)$. This weakness will surely be widely felt

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to be disappointing or unsatisfactory. But methods employing spurious quantification offer only illusory additional strength.)

I emphasize that this principle of updating is not defended by a *Dutch book* argument, but by acceptance of relevant components of the generalizations of Koopman's principles of *Composition* and of *Decomposition*, along with the reflexivity and symmetry of *equiprobability* and with mutual exclusion of *strict supraprobability* and *equiprobability*.

The menace of *Dutch books* is used to motivate *coherence* of *gambling ratios*, which are often equated with measures of belief and thence with probabilities. One might more directly *define* "probability" to mean *gambling ratio*, in which case it should immediately be acknowledged that the axioms, while conforming to *coherence*, do not imply it, and permit *gambling ratios* against which a *Dutch book* might be written. However, I have never encountered such a direct definition; and, while *gambling ratios* should conform to one's beliefs, there is in most cases no reason for them actually to *measure* those beliefs; there is no logical need to intensify or to attenuate one's beliefs when choosing gambling ratios as such. Further, one simply does not *ever* find oneself required to arrive at gambling ratios fitted to the full set of one's beliefs; and one may always or almost always refuse to participate in any game in which a *Dutch book* might be constructed (Kyburg, 1978).[12]

§3. Further remarks related to statistical inference. The use of qualitative probabilistic reasoning in inference is removed from the application that has principally drawn my own attention; nonetheless, I offer a few thoughts about inferential employment beyond the previous discussion of Bayes' Theorem.

A considerable share of contemporary attempted scientific work seeks to reason about point values that cannot be measured precisely. Such values may be conjectured physical constants, or may be statistical properties of populations that can only be sampled. (These cases would include those of supremal or infimal values of sets, such that the presence of other values within these sets compounded the difficulty of identifying those extreme values.)

Typically, an attempt is made to find an interval with a 95% probability of containing the point value; but, in a framework of purely qualitative probability, one can only make claims of forms such as "$(v \in [v_l, v_h] \mid c_1) \trianglerighteq (Y \mid c_2)$." ($Y$ might or might not be a proposition about an interval containing v; c_i might or might not be a compound proposition about a model and about evidence.) These answers may seem grossly unsatisfying, and perhaps they are. Paradoxically, they may sometimes be more sound than those generated by prevailing practices.

3.1. Confidence intervals. In a frequentist framework, an X% *confidence interval* is produced by a procedure that is believed to generate a containing interval X% of the time (Neyman, 1937) (or not less than X% of the time), across all possible values of interest. There is a wide-spread propensity to presume that an X% *confidence interval*

[12] Bayesian updating is not immediately sufficient to guarantee *coherence* unless prior gambling ratios are themselves coherent. I suspect that, even with Bayesian updating, *Dutch books* would bring us all to ruin if they had pervasive significance in the real world.

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is an interval with an $X\%$ probability of containing the associated value of interest; one may even encounter "confidence interval" defined just so. However, the proper concept does not imply that any given generated interval has an $X\%$ chance of containing the value, that the width of any given confidence interval reflects the precision of what is known, nor that the values contained in any given interval are more likely than those without (Morey, Hoekstra, Rouder, Lee, & Wagenmakers, 2016). That does not mean that any of these presumptions is necessarily false in the case of each particular confidence interval; one or more such presumption may prove to be true, but none follows logically from the concept.

In some cases, perhaps the other properties of an interval which is a confidence interval are so well understood that one can impute quantifications to the probabilities that the values of interest lie within them, though this quantification will not necessarily be that of the confidence probability. In other cases, perhaps those properties are so poorly understood that little or no reasoning is possible. In the remainder of cases, *one can reason probabilitistically from confidence intervals, but one can only reason qualitatively*, notwithstanding that the interval itself could not have been produced by a purely qualitative process.

3.2. Bayesian method. In Bayesian frameworks of various sorts, hypothesis testing is a matter of generating posterior probabilities from an interaction, by way of Bayes' Theorem, of conditional probabilities of data with prior probabilities—or with probability-like constructs treated as if prior probabilities.

A subjectivist Bayesian will use subjective probabilities in the rôles of prior probabilities. Perhaps this use is entirely reasonable, but the results cannot be rationally convincing to others except to the extent that those prior probabilities are shared. In fact, they may be qualitatively shared when not quantitatively shared, in which case a logic of qualitative probability could allow cooperative inference and development.

Objective Bayesians attempt to begin with *noninformative priors*, which seek a minimization of the effect of priors as such upon posterior measures of probability. The principles of this minimization are essentially sophisticated development of *the principle of insufficient reason*. Various criticisms have been widely expressed, perhaps especially that the appropriate *noninformative priors* may as a practical or even theoretical matter be impossible to identify in complex cases. This problem is still deeper than is normally recognized. Any dismissal of possibilities is informative, but permitting intervals as such to be used for prior probabilities would appear to be fatal to the purposes of supposedly noninformative probability measures. If no more is known of the measure of probability of a set of outcomes than that it falls in the interval $[p_1, p_2]$ or $(p_1, p_2]$, then the possible measures for any nonempty, proper subset would be $[0, p_2]$; similarly, if a set has an associated interval $[p_1, p_2)$ or $(p_1, , p_2)$, then the possible measures for any nonempty, proper subset would be $[0, p_2)$. Only an interval containing all possible prior measures of probability would be genuinely noninformative; any interval that produces different results in calculations would be *ipso facto* informative. Of course, a measure p corresponds to an interval $[p, p]$, which is not $[0, p]$ except in the case $p = 0$, and never $[0, p)$ in any case. It may be seen that the implicit purpose of a supposedly noninformative measure of probability is to inform the structure sufficiently to enable a calculation (and that even if the selection of priors is wider than a point, so long as it does not contain the full set of possibilities, the selection is informative), though

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perhaps some selected value mighty be amongst the least informative of its sort.[13] The justification for seeking to construct such trivial intervals seems to hang on a *presumption de dictu* that researchers are *licensed* to produce quantitative posterior probability measures.[14] I do not see that we are entitled to produce even qualitative judgments in every case, but I think that qualitative reasoning—note especially (15), the general principle of updating—may be justified in cases where unique quantitative results are not available, and *an ability to reason without quantification seems a very great challenge to an essential presumption of objective Bayesianism*.

§4. Future work. It may be possible to preserve the desirable features of the system of axioms herein, but to transform them into an arrangement of greater elegance. Other, perhaps more substantial shortcomings may be found and corrected; certainly it should be easier to find any such problems within a framework expressed with greater refinement.

If the relations are interpreted as those of *logical plausibility*, then the logic of maximal plausibility (certainty) is used meta-meta-propositionally by the axioms to present the logic of plausibility more generally. It may be wondered whether a usefully more integrated logic might be developed, with material implication simply a special case. (It would be possible to replace all expressions of form "$P_1 \vee P_2$" and of form "$P_1 \wedge P_2$" with expressions of form "$\left[\left(P_2 \,\middle|\, \overline{P_1}\right) \boxdot (P_3 \mid P_3)\right] \forall P_3$" and of form "$\overline{\left[\left(\overline{P_2} \mid P_1\right) \boxdot (P_3 \mid P_3)\right] \forall P_3}$" respectively, but great monsters would be encountered on that road. Something like a Łukasiewicz notation might prove very helpful towards arriving at a simpler system of formal expression.) Presumably, such a construct could be recognized from a homomorphism; though if the homomorphism were not bijective then we must consider whether it would offer new insights or illusions.

The axioms are just the most primitive of tools; from them may be assembled useful devices. The canonic binary cases and (15) may be such constructs; there should be many others with wide application.

Scientists will want practical algorithms for hypothesis testing with qualitative probabilistic reasoning. Such a devices should not be expected to have the power that has been sought and expected from quantitative methods, but could both replace some of the spuriously quantified testing that leads attempted scientific research astray, and dramatically extend the available realm of recognizably scientific study.

Interest in qualitative probability may also arise largely from problems of selecting the best actions given possible outcomes of various desirability. Thus, the "c" in "$(X \mid c)$" would often represent description of a *potential course of action* (rather than a hypothesis as such and rather than evidence) within some context, and the "X" description of an imagined outcome. An appropriate theory must consider that the possible outcomes associated with any given action are not generally all superior

[13] The arbitrary exclusion of nontrivial interval priors is especially dubious when effected within Dempster–Shafer theories.

[14] In the particular case of probabilities, symmetry conditions that are not *logically impossible* would perhaps support quantification, and an incautious thinker might thence presume that some quantification were warranted even when those conditions did not hold, and that this quantification could then be approximated by the quantification that would arise if those symmetry conditions held.

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nor all inferior to the possible outcomes associated with another,[15] and the theory must take into account both the relative desirabilities and the relative probabilities of outcomes across potential choices. In the mainstream of economic theory, agents are assumed to maximize an expectation of quantified desirability. Such models are not compatible with a notion of *weak supraprobability* as an incomplete preordering. If, additionally, *weak preferability* (the union of *strict preferability* with a relation of equal valuation) is an incomplete preordering (Mc Kiernan, 2012), then the system is completely without quantity. A treatment for such systems may be developed without reliance upon intervals as proxies.

§5. Conclusion. Exploration of a formal, qualitative framework for probabilistic thought helps to find serious challenges to orthodox subjectivism, to intuitionism, and to objective Bayesianism. It also facilitates clarification of the reasonableness of Bayesian updating, at least in some general sense. These benefits alone might vindicate the effort.

But the greater motivation is that the peculiar scope within which qualitative probabilistic reasoning applies is that in which relevant propositions are not known with certainty to be true or to be false (relevant events always to occur or never to occur), and a quantification capturing indeterminacy is not available. *This scope is most of life*—most of our choices in everyday existence, most of our questions of applied ethics, most of our problems in what are now widely recognized as sciences. A grasp of qualitative probability expands the discernible compass of reason and of science in a way that is literally immeasurable and as I think figuratively so as well. I believe that the system of axioms herein represents a set of basic tools for recognizably rational reasoning in that much larger range.

§A. Appendix: Conforming dichotomous cross-conditional claims.

[15] However, a *deontological ethics* does not separate means from consequences, and will sometimes declare that all outcomes associated with some means are *ipso facto* superior to those associated with other means.

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Table A.1. *Conforming incomplete probability orderings given that* $\left[(X \neq Y) \wedge \left(X \neq \overline{Y}\right)\right] \vee (c_1 \neq c_2)$

$(X_1 \mid c_1) \cdot \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \cdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \cdot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \cdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \cdot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \cdot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \boxdot \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \boxdot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \boxdot \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \boxdot \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \boxdot \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \boxdot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \rhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \lhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \lhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \boxdot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \boxdot \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \lhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \lhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \lhd \left(\overline{X_1} \mid c_1\right)$
$(X_1 \mid c_1) \lhd \left(\overline{X_1} \mid c_1\right)$	$(X_2 \mid c_2) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$\left(\overline{X_1} \mid c_1\right) \rhd \left(\overline{X_2} \mid c_2\right)$	$(X_1 \mid c_1) \odot \left(\overline{X_2} \mid c_2\right)$	$(X_2 \mid c_2) \odot \left(\overline{X_1} \mid c_1\right)$

Continued

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Table A.1. *Continued*

$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \boxdot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \boxdot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \lhd (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \boxdot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \rhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \rhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \boxdot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \boxdot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \boxdot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \boxdot (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \rhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \rhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \lhd (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \lhd (\overline{X_1} \mid c_1)$
$(X_1 \mid c_1) \odot (\overline{X_1} \mid c_1)$	$(X_2 \mid c_2) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (X_2 \mid c_2)$	$(\overline{X_1} \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_1 \mid c_1) \odot (\overline{X_2} \mid c_2)$	$(X_2 \mid c_2) \odot (\overline{X_1} \mid c_1)$

§B. Appendix: Conforming binary cases.

$$\left(\begin{array}{c}[(X_1 \mid c_1) \boxdot (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \boxdot (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_2 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \boxdot (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \boxdot (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_2 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \boxdot (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \boxdot (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_2 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \boxdot (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_2 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \boxdot (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_2 \mid c_1) \lhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \lhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \lhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \lhd (Y_2 \mid c_2)]\end{array}\right) \Rightarrow [(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \boxdot (Y_1 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \lhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \boxdot (Y_2 \mid c_2)]\end{array}\right) \Rightarrow [(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \lhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \boxdot (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

$$\left(\begin{array}{c}[(X_1 \mid c_1) \rhd (X_2 \mid c_1)]\\ \wedge\\ [(X_1 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_1 \mid c_2)]\\ \wedge\\ [(Y_1 \mid c_2) \rhd (Y_2 \mid c_2)]\end{array}\right) \Rightarrow \left(\begin{array}{c}[(X_1 \mid c_1) \rhd (Y_2 \mid c_2)]\\ \wedge\\ [(X_2 \mid c_1) \rhd (Y_2 \mid c_2)]\end{array}\right)$$

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