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Quantitative ergodic theorems for actions of groups of polynomial growth

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Abstract. We strengthen the maximal ergodic theorem for actions of groups of polynomial growth to a form involving jump quantity, which is the sharpest result among the family of variational or maximal ergodic theorems. As two applications, we first obtain the upcrossing inequalities with exponential decay of ergodic averages and then provide an explicit bound on the convergence rate such that the ergodic averages with strongly continuous regular group actions are metastable (or locally stable) on a large interval. Before exploiting the transference techniques, we actually obtain a stronger result—the jump estimates on a metric space with a measure not necessarily doubling. The ideas or techniques involve martingale theory, non-doubling Calderón–Zygmund theory, almost orthogonality argument, and some delicate geometric argument involving the balls and the cubes on a group equipped with a not necessarily doubling measure.

Key words: quantitative pointwise ergodic theorems, groups of polynomial growth, jump inequalities, convergence rate
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1. *Introduction*

1.1. *Background and main results.*

In the past few decades, numerous significant results related to the pointwise ergodic theorems for group actions have been established. The earliest pointwise ergodic theorems, to our knowledge, were obtained by Birkhoff [7], where he established pointwise ergodic theorems for one-parameter flow (such as a translation group on $\mathbb{R}$ or $\mathbb{Z}$). Wiener [63] extended Birkhoff's result to the case of several commuting flows. Even further, these pointwise ergodic results were generalized

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by Calderón [12] for an increasing family of compact symmetric neighbourhoods of the identity satisfying the doubling condition, which is abundant on non-commutative groups with polynomial volume growth. Calderón's works [12, 13] motivated further research on pointwise ergodic theorems, such as [6, 14, 17, 22, 26, 29, 59–61]. In particular, Breuillard [10] (see also Tessera [62]) showed that the sequence of balls with respect to any fixed word metric on groups of polynomial growth satisfy the doubling condition and are asymptotically invariant, and thus established the corresponding pointwise ergodic theorem; actually, these results apply to more general metrics such as the periodic pseudodistances as defined in [10] (and recalled in §7). This settled a long-standing problem in ergodic theory since Calderón's classical paper [12] in 1953. Lindenstrauss [46] established the pointwise ergodic theorem for tempered Følner sequences; this result resolves the problem of the existence of a Følner sequence that satisfies the pointwise ergodic theorem on an arbitrary amenable group. For more details, we refer the reader to the survey works [2, 53].

In this paper, we aim at establishing the quantitative pointwise ergodic theorems for actions of polynomial growth groups in terms of the following jump quantity. Given $\lambda > 0$, the λ-jump function of a family of measurable functions $\alpha = \{\alpha_r(x) : r \in \mathcal{I}\}$ is defined by

$$\begin{aligned}&\mathcal{N}_\lambda(\alpha)(x)\\ &\quad = \sup\{N \mid \text{there exists } r_0 < r_1 < \cdots < r_N, r_i \in \mathcal{I} : \min_{0<i\le N} |\alpha_{r_i}(x) - \alpha_{r_{i-1}}(x)| > \lambda\},\end{aligned}$$

where $\mathcal{I}$ is a subset of $(0, \infty)$ and the supremum is taken over all finite increasing sequences in $\mathcal{I}$.

Let G be a group and let d be a metric on G. For $r > 0$ and $h \in G$, we define the ball $B(h, r) = \{g \in G : d(g, h) < r\}$, and we will write it simply as B_r when $x = e$ (the identity in G). Let D be a positive integer. We recall that (G, d) satisfies the D-geometrically doubling property if every ball $B(h, r)$ can be covered by at most D balls $B(h_i, r/2)$, that is,

$$B(h, r) \subseteq \bigcup_{1\le i\le D} B(h_i, r/2). \tag{1.1}$$

Let m be a regular Borel measure such that all balls defined by d have finite and positive measure. Let $r_0 > 0$ and $\epsilon \in (0, 1]$. We say that (G, d, m) satisfies the (ϵ, r_0)-annular decay property if there exists a constant $K > 0$ such that for all $h \in G$, $r \in (r_0, \infty)$ and $s \in (0, r]$,

$$m(B(h, r + s)) - m(B(h, r)) \le K\left(\frac{s}{r}\right)^{\epsilon} m(B(h, r)). \tag{1.2}$$

Let $p \in [1, \infty)$ and $f \in L^p(G, m)$. We consider the following averages:

$$A'_r f(h) = \frac{1}{m(B(h, r))} \int_{B(h,r)} f(g)\, dm(g). \tag{1.3}$$

One main result of this paper is as follows.

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THEOREM 1.1. *Assume that (G, d, m) satisfies equations (1.1) and (1.2). Let $\mathbf{A}' = \{A'_r : r \geq r_0\}$ be the family of averaging operators given by equation (1.3). Then, the following assertions hold true.*

(i) *For any $p \in (1, \infty)$, $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}$ is of strong type (p, p) uniformly in $\lambda > 0$, that is, there exists a constant $c_p > 0$ such that*

$$\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)(g)}\|_{L^p(G,m)} \leq c_p \|f\|_{L^p(G,m)} \quad \text{for all } f \in L^p(G, m).$$

(ii) *For $p = 1$, $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}$ is of weak type $(1, 1)$ uniformly in $\lambda > 0$, that is, there exists a constant $c > 0$ such that for any $\gamma > 0$,*

$$\sup_{\lambda>0} m(\{g \in G : \lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)(g)} > \gamma\}) \leq \frac{c}{\gamma}\|f\|_{L^1(G,m)} \quad \text{for all } f \in L^1(G, m).$$

We remark that the group structure of G in effect does not play any role here and Theorem 1.1 holds in the setting of general metric measure spaces.

When d and m are assumed to be (left-) translation invariant, then conditions (1.1) and (1.2) are related to the geometric structure of group G; there exist many examples such as those introduced in [**53**, **62**], see §7 for more details. Moreover, as a consequence of Theorem 1.1, we obtain the quantitative ergodic theorems for actions of groups of polynomial growth.

Let (X, Σ, μ) be a σ-finite measure space and T an action of G on the associated L^p-spaces $L^p(X, \mu)$, under some additional assumptions recalled in later sections. In particular, if T is induced by a μ-preserving measurable transformation τ on X, then T extends to an isometric action on $L^p(X, \mu)$ for all $1 \leq p \leq \infty$, given by $T_g f(x) = f(\tau_{g^{-1}} x)$. Given an action T and a ball B_r, the associated averaging operator is given by

$$A_r f(x) = \frac{1}{m(B_r)} \int_{B_r} T_g f(x)\, dm(g). \tag{1.4}$$

THEOREM 1.2. *Assume that G is of polynomial growth with a symmetric compact generating set V and d is the associated word metric on G, that is,*

$$d(g, h) = \min\{n \in \mathbb{N},\ g^{-1}h \in V^n\}.$$

Let m be a Haar measure. Let $\mathbf{A} = \{A_r : r \in \mathbb{N}\}$ be the corresponding sequence of averaging operators with respect to an action T.

(i) *If T is an action induced by a measure-preserving measurable transformation, then $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}$ is of weak type $(1, 1)$ and of strong type (p, p) for all $1 < p < \infty$ uniformly in $\lambda > 0$.*

(ii) *If T is a strongly continuous regular action of G on $L^p(X, \mu)$ $(1 < p < \infty)$, then $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}$ is of strong type (p, p) uniformly in $\lambda > 0$.*

The notion of regular action will be recalled in §6.2.

If we take G to be the integer group $\mathbb{Z}$ and d the usual word metric, then we recover the usual ergodic average $A_n = (1/(2n + 1)) \sum_{k=-n}^{n} T^k$ for an automorphism T, as is treated

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in [9, 32]. Moreover, from the definition of jump quantity, Theorem 1.2 implies that the underlying sequence of functions $A_r f$ converges almost everywhere as $r \to \infty$, which is the pointwise ergodic theorem.

The jump quantity appeared first implicitly in [56] and then explicitly in [32], whose study was motivated by the research of variational inequality arising from probability theory [43]. For $q \in (0, \infty]$, the q-variation (semi-)norm V_q of a sequence $\{\alpha_r(x) : r \in \mathcal{I}\}$ of complex-valued functions is defined by

$$V_q(\alpha_r(x) : r \in \mathcal{I}) = \sup_{\substack{0<r_0<\cdots<r_J \\ r_j \in \mathcal{I}}} \left(\sum_{j=0}^{J} |\alpha_{r_{j+1}}(x) - \alpha_{r_j}(x)|^q \right)^{1/q}.$$

The ∞-variation norm is nothing but equivalent to the maximal norm; moreover, any q-variation norm dominates the maximal norm

$$\sup_{r \in \mathcal{I}} |\alpha_r(x)| \le |\alpha_{r_1}(x)| + V_q(\alpha_r(x) : r \in \mathcal{I}) \quad \text{for all } r_1 > 0.$$

As the jump quantity, the finite q-variation norm with $q < \infty$ also deduces immediately the pointwise convergence of the underlying sequence of operators without density argument. This idea was first exploited by Bourgain [9] to study the pointwise ergodic theorem for dynamical systems where the density argument is not available. Bourgain's work [9] inspired many studies on variational inequalities and ergodic theory, see [32, 33, 40, 41, 49, 50, 64] and the references therein.

However, by Chebychev's inequality, the q-variation norm dominates the jump quantity

$$\sup_{\lambda>0} \lambda(\mathcal{N}_\lambda(\alpha_r(x) : r \in \mathcal{I}))^{1/q} \le V_q(\alpha_r(x) : r \in \mathcal{I}).$$

However, it is shown in [35, 44] that the 2-variational inequality (and thus $q < 2$) does not hold true in general. Conversely, for any family of linear operators $\mathcal{T} = (T_t)_{t \in \mathcal{I}}$, the mapping properties of $\lambda\sqrt{\mathcal{N}_\lambda(\mathcal{T})}$ imply the corresponding q-variational inequalities for all $q > 2$ by interpolation, see e.g. [9, 34, 51]. Thus, for all $q > 2$, the q-variational version of Theorems 1.1 and 1.2, in particular, the maximal ergodic theorem, holds true.

As the first application, from Theorem 1.2, we may deduce a kind of exponential decay estimate for bounded functions by using Vitali's covering lemma and geometric argument.

THEOREM 1.3. *Assume that G is a group of polynomial growth with a symmetric finite generating set and d is the resulting word metric. Let T be an action induced by a measure-preserving measurable transformation and $\mathbf{A} = \{A_r : r \in \mathbb{N}\}$ the sequence of averaging operators with respect to the action T given by equation (1.4). Let $\lambda > 0$. Then, for every $p \in [1, \infty)$, there are two constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in (0, 1)$ depending on λ, p and group G such that for any $f \in L^p(X, \mu)$ with $\|f\|_{L^\infty(X,\mu)} \le 1$,*

$$\mu(\{x \in X : \mathcal{N}_\lambda(\mathbf{A}f)(x) > n\}) \le \tilde{c}_1 \tilde{c}_2^n \|f\|^p_{L^p(X,\mu)}.$$

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This result immediately yields the upcrossing inequalities with exponential decay. Recall that for two real numbers a and b with $b > a$, the upcrossings of a family of real-valued functions $\alpha = \{\alpha_r(x) : r \in \mathcal{I}\}$, $\mathcal{N}_{a,b}(\alpha)(x)$ is defined by

$$\sup\{N \in \mathbb{N} | \text{ there exists } s_1 < r_1 < \cdots < s_N < r_N, r_i, s_i \in \mathcal{I} \text{ such that } \alpha_{s_i}(x) < a \text{ and } \alpha_{r_i}(x) > b\}.$$

Taking $\lambda = b - a$, it is easy to see that $\mathcal{N}_{a,b}(\alpha)(x) \leq 2\mathcal{N}_{\lambda/2}(\alpha)(x)$. Thus, we obtain the following corollary.

COROLLARY 1.4. *Let a and b be two real numbers with $b > a$. Then, for every $p \in [1, \infty)$, there are two constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in (0, 1)$ depending on λ, p and group G such that for any real-valued function $f \in L^p(X, \mu)$ with $\|f\|_{L^\infty(X,\mu)} \leq 1$,*

$$\mu(\{x \in X : \mathcal{N}_{a,b}(\mathbf{A}f)(x) > n\}) \leq \tilde{c}_1 \tilde{c}_2^n \|f\|^p_{L^p(X,\mu)}. \tag{1.5}$$

A variant of equation (1.5) for non-negative functions has been recently proved by Moriakov [52], as a generalization of Kalikow and Weiss's exponential estimate for $G = \mathbb{Z}$ [36]. Note that our method here is completely different from the one used by Moriakov that is based on a generalization of the Vitali covering theorem. However, Kalikow and Weiss's estimate was motivated by Bishop's fundamental result [8] and the similar estimates in the martingale setting [20, 21]. Recently, upcrossing inequalities with exponential decay have aroused a surge of interest and found many applications to stationary process, ergodic theory and information theory, see e.g. [5, 27].

To present the second application, let us recall briefly the related background. Krengel [42] has shown that the convergence speed can be arbitrarily slow in the sense that given any sequence of real positive numbers $(a_r)_{r\in\mathbb{N}}$ converging to 0 and any ergodic automorphism of the unit interval equipped with Lebesgue measure, there exists a subset A of $[0, 1]$ such that

$$\limsup_{r\to\infty} \frac{1}{a_r} |(A_r(\mathbb{1}_A))(x) - \mu(A)| = \infty$$

almost everywhere $x \in [0, 1]$, where $\mathbb{1}_A$ stands for the characteristic function of A and A_r terms are the standard one-sided ergodic averages. Instead, people have been interested in the research program on the explicit bound of the rate of convergence such that the ergodic averages are locally stable on a large interval. This program starts with a 'metastable' description of the mean or pointwise ergodic theorem (see e.g. [5, 57, 58]); for our purpose in the present paper, we are mainly concerned with the pointwise ergodic theorem. More precisely, let (X, μ) be a probability space and T be an operator on $L^1(X)$ induced by a measure-preserving transformation τ, that is, $Tf = f \circ \tau$. Birkhoff's pointwise ergodic theorem is equivalent to the following metastable description (see e.g. [5, Theorem 3.3]):

$$\begin{aligned} &\text{for all } \lambda_1, \lambda_2 > 0, \Theta : \mathbb{N} \to \mathbb{N} \text{ with } \Theta(r) \geq r \text{ for all } r \in \mathbb{N}, \ f \in L^1(X, \mu), \\ &\quad \text{there exists } r_1 \in \mathbb{N} \text{ such that } \mu(\{x \in X : \max_{r_1 \leq r \leq \Theta(r_1)} |A_r f(x) - A_{r_1} f(x)| > \lambda_1\}) \leq \lambda_2. \end{aligned} \tag{1.6}$$

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If the maximal ergodic theorem is available, Birkhoff's pointwise ergodic theorem is reduced to the validity of equation (1.6) for f belonging to some dense subset of $L^1(X)$, for instance, $f \in L^2(X)$ when (X, μ) is a probability space.

If the dynamical system is explicitly presented and equation (1.6) holds, then for any $\lambda_1, \lambda_2 > 0$, Θ and f, one can compute a witness to the existential quantifier in equation (1.6) by simply trying values of r until one satisfies

$$\mu(\{x \in X : \max_{r \le s \le \Theta(r)} |A_s f(x) - A_r f(x)| > \lambda_1\}) \le \lambda_2.$$

Thus, the metastable description in equation (1.6) has an inherent computational interpretation. Indeed, by first dealing with the mean ergodic theorem, Avigad *et al* [**5**, Theorem 3.14] were able to extract an explicit or a computable bound on the convergence rate r_1, expressed in terms of $\lambda_1, \lambda_2, \Theta, \|f\|_{L^2}$. In the past twenty years, many authors have been taking part in the research program in describing the explicit convergence rate of ergodic averages. For more details on this topic, the reader is referred to a recent survey work [**39**].

Along this research line, we provide an explicit bound on the convergence rate of ergodic averages with strongly continuous regular group actions.

THEOREM 1.5. *Suppose that G is a group of polynomial growth with a symmetric finite generating set and d is the resulting word metric. Let (X, μ) be a measure space and T a strongly continuous regular action of G on $L^2(X, \mu)$. Let f be a function in $L^2(X, \mu) \cap L^\infty(X, \mu)$, $\lambda_1 > 0$, $\lambda_2 > 0$, and Θ be a function satisfying $\Theta(r) \ge r$ for every $r \in \mathbb{N}$. Let C be a given constant such that Theorem 1.2(ii) holds in the case $p = 2$. Set $N = [128C/\lambda_1^3\lambda_2\|f\|_{L^\infty}\|f\|_{L^2}^2]$. Then, there exists one r_1 satisfying $1 \le r_1 \le \Theta^N(1)$ and*

$$\mu(\{x \in X : \max_{r_1 \le r \le \Theta(r_1)} |A_r f(x) - A_{\Theta(r_1)} f(x)| > \lambda_1\}) \le \lambda_2.$$

Remark 1.6. Unlike the proof of [**5**, Theorem 3.14], where the authors needed first to obtain an explicit bound of the convergence rate of the mean ergodic theorem, our Theorem 1.5 is deduced directly from the jump inequalities—Theorem 1.2. However, the iteration number N in Theorem 1.5 is much bigger than that in [**5**, Theorem 3.14], but our measure space (X, μ) is more general without any restriction.

1.2. *Methods and more results.* The theorems rely on several key results obtained in this paper.

The first step to show Theorem 1.1 is now standard, that is, to control the jump quantity by the 'dyadic' jump and the short variation operator, see e.g. [**34**]. More precisely, let $f \in L^p(G, m)$. We dominate the jump quantity $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)}$ in the following way:

$$\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)} \le 2\lambda\sqrt{\mathcal{N}_{\lambda/6}(A'_{\delta^n} f : n > n_{r_0})} + 16\Big(\sum_{n \ge n_{r_0}} V_2(A'_r f : r \in [\delta^n, \delta^{n+1}))^2\Big)^{1/2},$$

where $\delta > 1$ is a constant depending on G that will be determined in Proposition 2.2 and n_{r_0} is the unique integer such that $\delta^{n_{r_0}} < r_0 \le \delta^{n_{r_0}+1}$. Moreover, compared with

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the resulting martingale $\mathbb{E}f = (\mathbb{E}_n f : n \in \mathbb{N})$ (see Definition 2.3), the dyadic jump is controlled by

$$96\sqrt{2}\Big(\sum_{n>n_{r_0}} |A'_{\delta^n} f - \mathbb{E}_n f|^2\Big)^{1/2} + 2\sqrt{2}\lambda\sqrt{\mathcal{N}_{\lambda/24}(\mathbb{E}_n f : n > n_{r_0})}.$$

Thus, we obtain the following pointwise estimate:

$$\begin{aligned}\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)} \le\ & 96\sqrt{2}\Big(\sum_{n>n_{r_0}} |A'_{\delta^n} f - \mathbb{E}_n f|^2\Big)^{1/2} \\ & + 16\Big(\sum_{n\ge n_{r_0}} V_2(A'_r f : r \in [\delta^n, \delta^{n+1}))^2\Big)^{1/2} + 2\sqrt{2}\lambda\sqrt{\mathcal{N}_{\lambda/24}(\mathbb{E}_n f : n > n_{r_0})}.\end{aligned} \tag{1.7}$$

This inequality implies that to bound $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}' f)}$, it suffices to estimate the three parts on the right-hand side, respectively. Note that the boundedness of the jump operator $\lambda\sqrt{\mathcal{N}_\lambda(\mathbb{E})}$ was proved in [32, 56], see Lemma 2.4; and so for our purposes, we only need to estimate the first two terms on the right-hand side of inequality (1.7). For abbreviation, we denote

$$S(f) := \Big(\sum_{n>n_{r_0}} |A'_{\delta^n} f - \mathbb{E}_n f|^2\Big)^{1/2}$$

and

$$SV(f) := \Big(\sum_{n\ge n_{r_0}} V_2(A'_r f : r \in [\delta^n, \delta^{n+1}))^2\Big)^{1/2}.$$

For these two operators, we will show more results including the (L^∞, BMO) estimate which is also necessary to obtain the result for $2 < p < \infty$ (see §5 for the definition of BMO). In what follows, L^∞_c denotes the compactly supported L^∞ functions.

THEOREM 1.7. *Assume that (G, d, m) satisfies conditions (1.1) and (1.2), then:*

(i) *for every $p \in (1, \infty)$, there exists a constant $c_p > 0$ such that for every $f \in L^p(G, m)$,*

$$\|S(f)\|_{L^p(G,m)} \le c_p \|f\|_{L^p(G,m)}; \tag{1.8}$$

(ii) *there exists a constant $c > 0$ such that for every $f \in L^1(G, m)$,*

$$m(\{g \in G : S(f)(g) > \gamma\}) \le \frac{c}{\gamma}\|f\|_{L^1(G,m)} \quad \text{for all } \gamma > 0; \tag{1.9}$$

(iii) *and for every $f \in L^\infty_c(G, m)$,*

$$\|S(f)\|_{\mathrm{BMO}} \le c\|f\|_{L^\infty_c(G,m)}. \tag{1.10}$$

The same results hold still true for the short variation operator SV.

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THEOREM 1.8. *Assume that (G, d, m) satisfies conditions (1.1) and (1.2), then:*

(i) *for every $p \in (1, \infty)$, there exists a constant $c_p > 0$ such that for every $f \in L^p(G, m)$,*

$$\|SV(f)\|_{L^p(G,m)} \leq c_p \|f\|_{L^p(G,m)}; \tag{1.11}$$

(ii) *there exists a constant $c > 0$ such that for every $f \in L^1(G, m)$,*

$$m(\{g \in G : SV(f)(g) > \gamma\}) \leq \frac{c}{\gamma}\|f\|_{L^1(G,m)} \quad \text{for all } \gamma > 0; \tag{1.12}$$

(iii) *and for every $f \in L^\infty_c(G, m)$,*

$$\|SV(f)\|_{\mathrm{BMO}} \leq c\|f\|_{L^\infty_c(G,m)}. \tag{1.13}$$

The main idea in showing parts (ii), (i) ($p = 2$ and thus $1 < p < 2$) of both Theorems 1.7 and 1.8 originates from the fundamental work [**32**, **33**], which was further explored in [**30**, **34**]. While the idea of obtaining part (iii) and thus part (i) ($2 < p < \infty$) mainly comes from [**28**, **30**], which in turn is motivated by [**32**]. However, note that under conditions (1.2) and (1.1), (G, d, m) is not necessarily a doubling metric measure space; this induces several new difficulties in showing Theorems 1.7 and 1.8, such as, that the 'dyadic cubes' constructed in Proposition 2.2 may not admit the small boundary property (cf. [**15**, Theorem 11]) and that the standard Calderón–Zygmund decomposition for homogeneous space is not enough, etc. For these reasons, we have to pay great attention to the geometric argument involving the cubes and the balls, and explore the non-doubling Calderón–Zygmund theory, etc.

Remark 1.9. Let us mention that in [**50**], the authors presented a novel approach to the L^p-boundedness of the short variation operator SV. More precisely, by using [**50**, (2.4)], the operator SV can be dominated by some square function, and then the L^2-estimate becomes easier; moreover, the L^p-estimate could be also obtained via some sharper L^2-estimate and a bootstrapping argument. We leave the details to the interested reader.

To deduce Theorem 1.2 from Theorem 1.1, there need to be two transference principles: the first one is for actions induced by measure-preserving measurable transforms, while another one is for regular actions. We refer the reader to equation (6.7) for the definition of regular actions and the related constant $\|\cdot\|_r$. A group G is called amenable if it admits a Følner sequence $(F_n)_{n\in\mathbb{N}}$, that is, for every $g \in G$,

$$\lim_{n\to\infty} \frac{m((F_n g) \triangle F_n)}{m(F_n)} = 0, \tag{1.14}$$

where $\triangle$ denotes the usual symmetric difference of two sets.

THEOREM 1.10. *Let G be an amenable group equipped with invariant metric d and a right-Haar measure m, and T an action on $L^p(X, \mu)$. Let $\mathbf{A}' = \{A'_r : r \in \mathcal{I}\}$ and $\mathbf{A} = \{A_r : r \in \mathcal{I}\}$ be two sequences of averaging operators given by equation (1.3) and (1.4), respectively.*

(i) *Let $p \in [1, \infty)$. If T is an action induced by a measure-preserving measurable transformation and $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}$ is of weak (respectively strong) type (p, p) uniformly in $\lambda > 0$, then $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}$ is of weak (respectively strong) type (p, p) uniformly in $\lambda > 0$*

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and, moreover, $\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}\|_{L^p\to L^{p,\infty}} \le \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}\|_{L^p\to L^{p,\infty}}$ *(respectively* $\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}\|_{L^p\to L^p} \le \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}\|_{L^p\to L^p}$*).*

(ii) *Let* $p \in (1,\infty)$*. If* T *is a strongly continuous regular action of* G *on* $L^p(X)$ *and* $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}$ *is of strong type* (p,p) *uniformly in* $\lambda>0$*, then* $\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}$ *is of strong type* (p,p) *uniformly in* $\lambda>0$ *and, moreover, there exists a constant* $c_p>0$ *such that* $\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A})}\|_{L^p\to L^p} \le c_p \sup_{h\in G} \|T_h\|_r^2 \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}\|_{L^p\to L^p}$.

It is a little surprising that Theorem 1.10(i) holds true due to the fact that $\lambda\sqrt{\mathcal{N}_\lambda(\cdot)}$ is *a priori* not a norm; while to prove Theorem 1.10(ii), in addition to the use of the fact that $\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\cdot)}\|_p$ is equivalent to a norm (cf. [**51**]) for $p>1$, the argument is subtle due to the appearance of the supremum over λ outside L^p norm.

An outline of this paper is as follows. In §2, we first recall some necessary preliminaries concerning the definition of 'dyadic cubes', which was constructed by Hytönen and Kairema [**31**] in metric space; and then present some technical lemmas, which are essential in showing Theorems 1.7 and 1.8. The last two theorems will be proved in §§3–5. In §6, we prove the transfer principles, namely Theorem 1.10. In §7, we discuss the (ϵ, r_0)-annular decay property, providing examples and formulating problems; in particular, the balls with respect to a word metric over groups of polynomial growth satisfy conditions (1.1), (1.2) and (1.14), and thus we obtain Theorem 1.2. In §§8 and 9, we prove Theorems 1.3 and 1.5, respectively.

Throughout this paper, we always denote C by a positive constant that may vary from line to line, while c_p denotes a positive constant possibly depending on the subscripts.

2. *Preliminaries and some technical lemmas*

In this section, we first recall the 'dyadic cubes' constructed on the measure space (G,d,m) satisfying conditions (1.1) and (1.2) which might not be a measure doubling metric space, and the resulting martingale inequalities will play a key role in the probabilistic approach to jump or variational inequalities. We then collect several technical lemmas that involve or rely on the estimates of the boundaries of 'dyadic cubes' or balls or a certain configuration among them; these estimates are subtle, and thus we will set and fix in the whole paper several constants such as $k_1, n_0, n_1, c_0, C_0, C_1, L_0, L_1, \delta$ which depend on conditions (1.1) and (1.2) and the construction. These preliminary results or technical lemmas, collected in such a way, will facilitate much of the presentation of the proof of Theorems 1.7 and 1.8.

We will exploit the system of 'dyadic cubes' constructed in the setting of geometrically doubling metric measure spaces, which means that the spaces satisfy condition (1.1) (cf. [**31**, Theorem 2.2]). It is well known that the geometrically doubling property is a generalization of the doubling measure property (cf. [**16**]). Hence, the measure space (G,d,m) satisfying conditions (1.1) and (1.2) might not be measure doubling. However, by a simple computation, the (ϵ, r_0)-annular decay property—condition (1.2)—implies the measure doubling condition for large balls, that is, for every $x\in G$ and $r_0<r\le R<\infty$,

$$\frac{m(B(x,R))}{m(B(x,r))} \le (K+1)\Big(\frac{R}{r}\Big)^{\epsilon}. \tag{2.1}$$

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This estimation will be used later. Moreover, from the geometrically doubling condition, one can easily deduce the following property.

PROPOSITION 2.1. *Let (G, d, m) satisfy the conditions of Theorem 1.1. Let $0 < r \leq R$. Any ball $B(x, R)$ can be covered by no more than $D^{\log_2[R/r]+1}$ balls of radius r.*

For more information about the geometrically doubling property, we refer the reader to [16].

PROPOSITION 2.2. [31, Theorem 2.2] *Let (G, d, m) satisfy the conditions of Theorem 1.1. Fix constants $0 < c_0 < C_0 < \infty$ and $\delta > 1$ such that*

$$18C_0\delta^{-1} \leq c_0.$$

Let I_k ($k \in \mathbb{Z}$) be an index set and $\{z_\alpha^k \in G : \alpha \in I_k, k \in \mathbb{Z}\}$ be a collection of points with the properties that

$$d(z_\alpha^k, z_\beta^k) \geq c_0\delta^k \ (\alpha \neq \beta), \quad \min_\alpha d(x, z_\alpha^k) < C_0\delta^k \quad \text{for all } x \in G,\ k \in \mathbb{Z}. \tag{2.2}$$

Then, there exist a family of sets $\{Q_\alpha^k\}_{\alpha\in I_k}$ associating with $\{z_\alpha^k\}_{\alpha\in I_k}$, and constants $a_0 := c_0/3$ and $C_1 := 2C_0$ such that:

(i) *for all $k \in \mathbb{Z}$, $\bigcup_{\alpha\in I_k} Q_\alpha^k = G$;*
(ii) *if $k \leq l$, then either $Q_\alpha^k \subset Q_\beta^l$ or $Q_\alpha^k \cap Q_\beta^l = \emptyset$;*
(iii) *for each (k, α) and each $k < n$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^n$, and for $n = k + 1$, we call such Q_β^{k+1} the parent of Q_α^k;*
(iv) $B(z_\alpha^k, a_0\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k)$.

We remark that the geometrically doubling property ensures that the minimum of the second inequality in equation (2.2) is attained.

For $k \in \mathbb{Z}$, let $\mathcal{F}_k$ be the σ-algebra generated by the 'dyadic cubes' $\{Q_\alpha^k : \alpha \in I_k\}$. We recall the following notions associated to the (reverse) martingale theory.

Definition 2.3. Let $f : G \to \mathbb{C}$ be a locally integrable function and $k \in \mathbb{Z}$. The conditional expectation of f with respect to $\mathcal{F}_k$ is defined by

$$\mathbb{E}_k f(x) = \sum_{\alpha\in I_k} \frac{1}{m(Q_\alpha^k)} \int_{Q_\alpha^k} f(y)\, dm(y) \mathbb{1}_{Q_\alpha^k}(x); \tag{2.3}$$

the resulting martingale difference operator $\mathbb{D}_k$ is defined as

$$\mathbb{D}_k f = \mathbb{E}_{k-1} f - \mathbb{E}_k f.$$

We check at once $\mathbb{E}_k \circ \mathbb{E}_j = \mathbb{E}_{\max(j,k)}$ and that for $f \in L^2(G, d, m)$, $f = \sum_{k\in\mathbb{Z}} \mathbb{D}_k f$ and

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\mathbb{D}_k f|^2 \right)^{1/2} \right\|_{L^2} = \|f\|_{L^2}.$$

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Denote $\mathbb{E} = \{\mathbb{E}_k : k \in \mathbb{Z}\}$. We remark that the strong type (p, p) with $1 < p < \infty$ and the weak type (1,1) estimates for operator $\lambda\sqrt{\mathcal{N}_\lambda(\mathbb{E})}$ were given implicitly in [56] and explicitly in [32]. We state the results as follows.

LEMMA 2.4. *Let $\mathbb{E} = \{\mathbb{E}_k : k \in \mathbb{Z}\}$ be defined as above.*

(i) *When $p \in (1, \infty)$, there is a constant $c_p > 0$ such that for all $f \in L^p(G, m)$,*

$$\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbb{E}f)}\|_{L^p(G,m)} \le c_p \|f\|_{L^p(G,m)}.$$

(ii) *For $p = 1$, there is a constant $c > 0$ such that for every $\gamma > 0$ and $f \in L^1(G, m)$,*

$$\sup_{\lambda>0} m(\{x \in G : \lambda\sqrt{\mathcal{N}_\lambda(\mathbb{E}f)(x)} > \gamma\}) \le \frac{c}{\gamma}\|f\|_{L^1(G,m)}.$$

Since (G, d, m) might not be a measure doubling metric space, the small boundary property of the 'dyadic cubes' constructed from equation (2.2) (see e.g. [15]) does not hold in general. However, from the (ϵ, r_0) annular decay property, we do have some boundary property—Lemma 2.6, which will be enough for our purpose in the present paper.

Set

$$L_0 = [\log_\delta(12/c_0)] + 1,\ L_1 = [\log_\delta(36r_0/c_0)] + 1.$$

LEMMA 2.5. *Let $k, L \in \mathbb{Z}$ satisfy $L_0 < L < k + L_0 - L_1$ and $\alpha \in I_k$. Then, we have*

$$m(\{x \in Q_\alpha^k : d(x, G \setminus Q_\alpha^k) \le \delta^{k-L}\}) \le \frac{(K+1)^2}{(L - L_0 + 1)}\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(Q_\alpha^k).$$

Proof. For a fixed point $x \in Q_\alpha^k$ with $d(x, G \setminus Q_\alpha^k) \le \delta^{-L}\delta^k$, we claim that there exists a chain $Q_{\sigma_{k-L+L_0}}^{k-L+L_0} \subset \cdots \subset Q_{\sigma_{k-1}}^{k-1} \subset Q_{\sigma_k}^k = Q_\alpha^k$ such that $x \in Q_{\sigma_{k-L+L_0}}^{k-L+L_0}$ and

$$d(z_{\sigma_j}^j, z_{\sigma_i}^i) \ge c_0\delta^i/12 \quad \text{for all } k - L + L_0 \le j < i \le k. \tag{2.4}$$

Indeed, let $x \in Q_\alpha^k$, by Proposition 2.2(i)–(iii), for any $n \le k$, there exists a 'dyadic cube' Q_σ^n such that $x \in Q_\sigma^n \subseteq Q_\alpha^k$. Hence, there exists a chain $Q_{\sigma_{k-L+L_0}}^{k-L+L_0} \subset \cdots \subset Q_{\sigma_{k-1}}^{k-1} \subset Q_{\sigma_k}^k = Q_\alpha^k$ with $x \in Q_{\sigma_{k-L+L_0}}^{k-L+L_0}$. Now, we verify equation (2.4). First, by Proposition 2.2(iv), for every $k - L + L_0 \le j \le k$, we have $x \in B(z_{\sigma_j}^j, C_1\delta^j)$. We also have $B(z_{\sigma_i}^i, a_0\delta^i) \subset Q_{\sigma_i}^i \subset Q_\alpha^k$. If equation (2.4) were not true, then

$$\begin{aligned} a_0\delta^i &\le d(z_{\sigma_i}^i, G \setminus Q_\alpha^k) \le d(z_{\sigma_i}^i, z_{\sigma_j}^j) + d(z_{\sigma_j}^j, x) \\ &\quad + d(x, G \setminus Q_\alpha^k) < \frac{c_0}{12}\delta^i + C_1\delta^j + \delta^{-L}\delta^k \\ &\le \frac{a_0}{4}\delta^i + \frac{a_0}{3}\delta^{j+1} + \delta^{-L_0}\delta^i \le \frac{a_0}{4}\delta^i + \frac{a_0}{3}\delta^i + \frac{a_0}{4}\delta^i < a_0\delta^i, \end{aligned}$$

since $a_0 = c_0/3$, $C_1 = 2C_0$, $3C_1 \le a_0\delta$ and $\delta^{-L_0} \le \delta^{-\log_\delta(12/c_0)} = c_0/12$. This leads to a contradiction and the claim is proved.

For every 'dyadic cube' Q_β^m, we denote it briefly by (m, β). Let $E = \{x \in Q_\alpha^k : d(x, G \setminus Q_\alpha^k) \le \delta^{-L}\delta^k\}$. For each $x \in E$, there exists a chain of pair $(i, \beta(i, x))$ with the

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properties proved in the first paragraph. Set $S_i = \bigcup_{x\in E}\{z^i_{\beta(i,x)}\}$ for $k - L + L_0 \le i \le k$. In the following, we abbreviate $z^i_{\beta(i,x)}$ to $z_{i(x)}$. We have the following observation: for $z_{i(x)} \ne z_{j(y)}$,

$$\begin{aligned} &B(z_{i(x)}, c_0\delta^i/36) \cap B(z_{j(y)}, c_0\delta^j/36) = \emptyset \\ &\qquad \text{for all } \ z_{i(x)} \in S_i,\ z_{j(y)} \in S_j,\ k - L + L_0 \le i, j \le k. \end{aligned} \tag{2.5}$$

For $i = j$, by equation (2.2), the above assertion is trivially right. For $i \ne j$, without loss of generality, we assume $i > j$. Note that by the definition of S_j, for each $z_{j(y)} \in S_j$, there exists a point $\zeta \in E$ such that $d(z_{j(y)}, G \setminus Q^k_\alpha) \le d(z_{j(y)}, \zeta) + d(\zeta, G \setminus Q^k_\alpha) \le C_1\delta^j + \delta^{-L}\delta^k$. It follows that if $B(z_{i(x)}, c_0\delta^i/36) \cap B(z_{j(y)}, c_0\delta^j/36) \ne \emptyset$, then there exists a point $z \in B(z_{i(x)}, c_0\delta^i/36) \cap B(z_{j(y)}, c_0\delta^j/36)$ such that

$$\begin{aligned} a_0\delta^i &\le d(z_{i(x)}, G \setminus Q^k_\alpha) \le d(z_{i(x)}, z_{j(y)}) + d(z_{j(y)}, G \setminus Q^k_\alpha) \\ &\le d(z_{i(x)}, z) + d(z_{j(y)}, z) + d(z_{j(y)}, G \setminus Q^k_\alpha) \\ &\le \frac{c_0}{36}\delta^i + \frac{c_0}{36}\delta^j + C_1\delta^j + \delta^{-L}\delta^k \le \frac{c_0}{18}\delta^i + C_1\delta^j + \delta^{-L}\delta^k \le \frac{a_0}{6}\delta^i + \frac{a_0}{3}\delta^{j+1} + \delta^{-L_0}\delta^i \\ &\le \frac{a_0}{6}\delta^i + \frac{a_0}{3}\delta^i + \frac{a_0}{4}\delta^i < a_0\delta^i, \end{aligned}$$

where we used $a_0 = c_0/3$, $C_1 = 2C_0$, $3C_1 \le a_0\delta$ and $\delta^{-L_0} \le \delta^{-\log_\delta(12/c_0)} = c_0/12$ again. This leads to a contradiction and so equation (2.5) holds.

We now prove the desired result. In the following, we write z_i for $z_{i(x)}$. Setting $G_i = \bigcup_{z_i\in S_i} B(z_i, c_0\delta^i/36)$, for any $k - L + L_0 \le i \le k$, we have

$$\begin{aligned} m(E) &\le \sum_{z_{k-L+L_0}\in S_{k-L+L_0}} m(B(z_{k-L+L_0}, C_1\delta^{k-L+L_0})) \\ &\le (K+1)\left(\frac{36C_1}{c_0}\right)^\epsilon \sum_{z_{k-L+L_0}\in S_{k-L+L_0}} m(B(z_{k-L+L_0}, c_0\delta^{k-L+L_0}/36)) \\ &= (K+1)\left(\frac{36C_1}{c_0}\right)^\epsilon \sum_{w_i\in S_i} \sum_{\substack{z_{k-L+L_0}\preceq w_i,\\ z_{k-L+L_0}\in S_{k-L+L_0}}} m(B(z_{k-L+L_0}, c_0\delta^{k-L+L_0}/36)) \\ &\le (K+1)\left(\frac{36C_1}{c_0}\right)^\epsilon \sum_{w_i\in S_i} m(B(w_i, C_1\delta^i)) \\ &\le (K+1)^2\left(\frac{36C_1}{c_0}\right)^{2\epsilon} \sum_{w_i\in S_i} m(B(w_i, c_0\delta^i/36)) = (K+1)^2\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(G_i), \end{aligned}$$

where $z_{k-L+L_0} \preceq w_i$ in the third line means that the inclusion of the corresponding 'dyadic cubes' $Q^{k-L+L_0}_{\beta(k-L+L_0)} \subset Q^i_{\beta(i)}$, and we used equation (2.1) in the second and last inequalities since $\delta^i \ge \delta^{k-L+L_0} \ge \delta^{L_1} > 36r_0/c_0$. The equality in the third line follows from equation (2.5), as does the equality in the last line. From the above inequalities and the disjointness of the sets G_i, we obtain

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$$m(E) \le \frac{(K+1)^2}{(L-L_0+1)}\left(\frac{72C_0}{c_0}\right)^{2\epsilon} \sum_{i=k-L+L_0}^{k} m(G_i) \le \frac{(K+1)^2}{(L-L_0+1)}\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(Q_\alpha^k),$$

and the lemma follows. □

Set

$$L_2 = [\log_\delta(4C_0+1)]+1, \quad L_3 = \left[2(K+1)^2\left(\frac{72C_0}{c_0}\right)^{2\epsilon}\right] + L_0 + L_2,$$
$$\eta = (\log_\delta 2)/L_3, \quad C_2 = 4(K+1)^2(72C_0/c_0)^{2\epsilon}, \quad C_2' = (K+1)(3C_1/a_0)^{\epsilon}.$$

LEMMA 2.6. *Under the assumptions of Lemma 2.5, we have*

$$m(\{x \in Q_\alpha^k : d(x, G\setminus Q_\alpha^k) \le \delta^{k-L}\}) \le C_2\delta^{-L\eta}m(Q_\alpha^k),$$
$$m(\{x \in G\setminus Q_\alpha^k : d(x, Q_\alpha^k) \le \delta^{k-L}\}) \le C_2C_2'\delta^{-L\eta}m(Q_\alpha^k).$$

Proof. Let us focus on the first inequality. Let $\ell \in \mathbb{N}$ and set

$$E_{-\ell}(Q_\alpha^k) = \{Q_\beta^{k-\ell} \subset Q_\alpha^k : d(Q_\beta^{k-\ell}, G\setminus Q_\alpha^k) \le (C_1+1)\delta^{k-\ell}\},$$

where $d(Q_\beta^{k-\ell}, G\setminus Q_\alpha^k) = \inf_{x\in Q_\beta^{k-\ell}} d(x, G\setminus Q_\alpha^k)$. We denote by

$$e_{-\ell}(Q_\alpha^k) = \{x : x \in Q_\beta^{k-\ell} \text{ with } Q_\beta^{k-\ell} \in E_{-\ell}(Q_\alpha^k)\}$$

the underlying point set. We proceed to show that

$$\{x \in Q_\alpha^k : d(x, G\setminus Q_\alpha^k) \le \delta^{k-\ell}\} \subseteq e_{-\ell}(Q_\alpha^k) \subseteq \{x \in Q_\alpha^k : d(x, G\setminus Q_\alpha^k) \le \delta^{k-\ell+L_2}\}. \tag{2.6}$$

Fix $x \in Q_\alpha^k$ such that $d(x, G\setminus Q_\alpha^k) \le \delta^{k-\ell}$. There exists 'dyadic cube' $Q_\beta^{k-\ell}$ such that $x \in Q_\beta^{k-\ell} \subset Q_\alpha^k$, and then

$$d(Q_\beta^{k-\ell}, G\setminus Q_\alpha^k) \le C_1\delta^{k-\ell} + d(x, G\setminus Q_\alpha^k) \le (C_1+1)\delta^{k-\ell}.$$

However, fix a point $x \in e_{-\ell}(Q_\alpha^k)$. Then there exists $Q_\beta^{k-\ell} \in E_{-\ell}(Q_\alpha^k)$ such that

$$d(x, G\setminus Q_\alpha^k) \le C_1\delta^{k-\ell} + d(Q_\beta^{k-\ell}, G\setminus Q_\alpha^k) \le (2C_1+1)\delta^{k-\ell} \le \delta^{k-\ell+L_2},$$

since $C_1 = 2C_0$, and equation (2.6) is proved.

To achieve our goal, we split L into two cases: $L_0 < L \le 2L_3$ and $L > 2L_3$. We first prove the case $L > 2L_3$. Set $M_0 = [L/L_3]$; here and below, $[t]$ denotes the integer part of a real number t. Let $F_1(Q_\alpha^k) = E_{-L_3}(Q_\alpha^k)$ and

$$F_n(Q_\alpha^k) = \bigcup_{Q_\beta^{k-(n-1)L_3} \in F_{n-1}(Q_\alpha^k)} E_{-L_3}(Q_\beta^{k-(n-1)L_3})$$

for $2 \le n \le M_0$. Let $f_n(Q_\alpha^k)$ be the underlying point set. It is easy to check that

$$e_{-nL_3}(Q_\alpha^k) \subset f_n(Q_\alpha^k). \tag{2.7}$$

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Moreover, for each $Q_\beta^{k-(n-1)L_3} \in F_{n-1}(Q_\alpha^k)$ and $1 \le n \le M_0$, plugging equation (2.6) into Lemma 2.5, we obtain

$$\begin{aligned} m(e_{-L_3}(Q_\beta^{k-(n-1)L_3})) &\le m\{x \in Q_\beta^{k-(n-1)L_3} : d(x, G \setminus Q_\beta^{k-(n-1)L_3}) \le \delta^{k-nL_3+L_2}\} \\ &\le \frac{(K+1)^2}{(L_3-L_2-L_0+1)}\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(Q_\beta^{k-(n-1)L_3}) \le \frac{1}{2}m(Q_\beta^{k-(n-1)L_3}), \end{aligned}$$

since $nL_3 \le L < k + L_0 - L_1$. Then,

$$m(f_n(Q_\alpha^k)) \le \tfrac{1}{2}m(f_{n-1}(Q_\alpha^k)),$$

and iterating the above inequality, we have $m(f_{M_0}(Q_\alpha^k)) \le 2^{-M_0}m(Q_\alpha^k)$. From this inequality and equation (2.7), one has

$$\begin{aligned} m(\{x \in Q_\alpha^k : &d(x, G \setminus Q_\alpha^k) \le \delta^{k-L}\}) \le m(e_{-L}(Q_\alpha^k)) \le m(e_{-M_0L_3}(Q_\alpha^k)) \\ &\le m(f_{M_0}(Q_\alpha^k)) \le 2^{-M_0}m(Q_\alpha^k) \le 2^{-L/L_3+1}m(Q_\alpha^k) = 2\delta^{-\eta L}m(Q_\alpha^k), \end{aligned}$$

where the second inequality in the first line follows from the definition of $e_{-\ell}(Q_\alpha^k)$.

For the case $L_0 < L \le 2L_3$, using Lemma 2.5 again, we have

$$\begin{aligned} m(\{x \in Q_\alpha^k : d(x, G \setminus Q_\alpha^k) \le \delta^{k-L}\}) &\le (K+1)^2\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(Q_\alpha^k) \\ &\le \delta^{-\eta L}\delta^{2\eta L_3}(K+1)^2\left(\frac{72C_0}{c_0}\right)^{2\epsilon} m(Q_\alpha^k) \\ &= 4(K+1)^2\left(\frac{72C_0}{c_0}\right)^{2\epsilon} \delta^{-\eta L}m(Q_\alpha^k). \end{aligned}$$

We now prove the second inequality. Let $\widetilde{E} = \{x \in G \setminus Q_\alpha^k : d(x, Q_\alpha^k) \le \delta^{k-L}\}$. Define the set $\widetilde{I} = \{\beta : Q_\beta^k \cap \widetilde{E} \ne \emptyset, \beta \in I_k\}$ and $e(Q_\beta^k) = \{x \in Q_\beta^k : d(x, G \setminus Q_\beta^k) \le \delta^{k-L}\}$. Then, by Proposition 2.2(i), we obtain $\widetilde{E} = \bigcup_{\beta\in\widetilde{I}} e(Q_\beta^k)$.

Fix $\beta \in \widetilde{I}$. There exists a point $y_0 \in Q_\beta^k \cap \widetilde{E}$. By Proposition 2.2(iv), for every $y \in Q_\beta^k$, we have

$$d(y, z_\alpha^k) \le d(y, y_0) + d(y_0, z_\alpha^k) \le C_1\delta^k + \delta^{k-L} + C_1\delta^k \le 3C_1\delta^k.$$

Hence, $Q_\beta^k \subseteq B(z_\alpha^k, 3C_1\delta^k)$. By Proposition 2.2(ii), the 'dyadic cubes' Q_β^k are disjoint; hence, $\bigcup_{\beta\in\widetilde{I}} Q_\beta^k \subseteq B(z_\alpha^k, 3C_1\delta^k)$. Then, by the first inequality and equation (2.1), we obtain

$$\begin{aligned} m(\widetilde{E}) &\le m\left(\bigcup_{\beta\in\widetilde{I}} e(Q_\beta^k)\right) \le \sum_{\beta\in\widetilde{I}} m(e(Q_\beta^k)) \\ &\le \sum_{\beta\in\widetilde{I}} C_2\delta^{-L\eta}m(Q_\beta^k) \le C_2\delta^{-L\eta}m\left(\bigcup_{\beta\in\widetilde{I}} Q_\beta^k\right) \le C_2\delta^{-L\eta}m(B(z_\alpha^k, 3C_1\delta^k)) \\ &\le C_2(K+1)(3C_1/a_0)^\epsilon m(B(z_\alpha^k, a_0\delta^k)) \\ &\le C_2(K+1)(3C_1/a_0)^\epsilon m(Q_\alpha^k), \end{aligned}$$

which completes the proof. □

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Given a ball B_{δ^n} and a 'dyadic cube' Q_α^k, define

$$\mathcal{H}(B_{\delta^n}, Q_\alpha^k) = \{x \in Q_\alpha^k : B(x, \delta^n) \cap (Q_\alpha^k)^c \neq \emptyset\}.$$

Set

$$n_0 = \max\{L_1 - L_0, 0\}.$$

LEMMA 2.7. *Let $n > n_0$ and $k > L_0$. Then, for every 'dyadic cube' Q_α^{n+k}, we have*

$$m(\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k})) \leq C_2 \delta^{-k\eta} m(Q_\alpha^{n+k}).$$

Proof. We check at once that for every $x \in \mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k})$, the distance $d(x, (Q_\alpha^{n+k})^c)$ is not bigger than δ^n, and so

$$\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k}) \subseteq \{x \in Q_\alpha^{n+k} : d(x, G \setminus Q_\alpha^{n+k}) \leq \delta^n\}.$$

Note that $k > L_0$ and $n > L_1 - L_0$, then by Lemma 2.6, we obtain

$$m(\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k})) \leq m(\{x \in Q_\alpha^{n+k} : d(x, G \setminus Q_\alpha^{n+k}) \leq \delta^{n+k-k}\}) \leq C_2 \delta^{-k\eta} m(Q_\alpha^{n+k}),$$

which is the desired conclusion. □

Remark 2.8. Set $\widetilde{\mathcal{H}}(B_{\delta^n}, Q_\alpha^k) = \{x \in G \setminus Q_\alpha^k : B(x, \delta^n) \cap (Q_\alpha^k)^c \neq \emptyset\}$. Similar to Lemma 2.7, by Lemma 2.6, we have

$$m(\widetilde{\mathcal{H}}(B_{\delta^n}, Q_\alpha^{n+k})) \leq C_2 \delta^{-k\eta} m(Q_\alpha^{n+k}).$$

Set

$$K_\epsilon = (2^\epsilon + 1)K + 2^\epsilon.$$

LEMMA 2.9. *Let $r \geq 2r_0$, then for every $s \in (0, r]$, we have*

$$m(B(x, r+s)) - m(B(x, r-s)) \leq K_\epsilon \left(\frac{s}{r}\right)^\epsilon m(B(x, r)).$$

Proof. We only need to estimate the part $m(B(x, r)) - m(B(x, r-s))$, since by the (ϵ, r_0)-annular decay property—equation (1.2)—we have $m(B(x, r+s)) - m(B(x, r)) \leq K(s/r)^\epsilon m(B(x, r))$. In the following, we split the s into two cases $s \in (0, r/2)$ and $s \in [r/2, r)$. For $s \in (0, r/2)$, since $r \geq 2r_0$, then $r - s > r/2 \geq r_0$, and applying the (ϵ, r_0)-annular decay property again, we have

$$\begin{aligned} m(B(x, r)) - m(B(x, r-s)) &\leq K\left(\frac{s}{r-s}\right)^\epsilon m(B(x, r-s)) \\ &\leq K\left(\frac{r}{r-s}\right)^\epsilon \left(\frac{s}{r}\right)^\epsilon m(B(x, r)) \leq K 2^\epsilon \left(\frac{s}{r}\right)^\epsilon m(B(x, r)). \end{aligned}$$

For $s \in [r/2, r)$, since $1/2 \leq s/r < 1$, so $2^\epsilon (s/r)^\epsilon \geq 1$ and $m(B(x, r)) - m(B(x, r-s)) \leq 2^\epsilon (s/r)^\epsilon m(B(x, r))$. Combining the two cases, we obtain

$$m(B(x, r+s)) - m(B(x, r-s)) \leq ((2^\epsilon + 1)K + 2^\epsilon)\left(\frac{s}{r}\right)^\epsilon m(B(x, r)),$$

and the lemma follows. □

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Given a ball $B(x, r)$ and an integer n, we define

$$\mathcal{I}(B(x,r),n) = \bigcup_{\alpha}\{Q_\alpha^n \cap B(x,r) : Q_\alpha^n \cap \partial B(x,r) \neq \emptyset\}.$$

Set

$$n_1 = \min\{n \in \mathbb{N} : \delta^n \geq 2r_0\}, \quad k_1 = \max\{k \in \mathbb{Z} : C_1\delta^k \leq 1\}.$$

Unless otherwise stated, we assume that $n > n_1$, $k < k_1$ in the following three lemmas.

LEMMA 2.10. *For any $x \in G$, we have*

$$\sup_{r\in[\delta^n,\delta^{n+1}]} \frac{m(\mathcal{I}(B(x,r),n+k))}{m(B(x,r))} \leq K_\epsilon C_1^\epsilon \delta^{\epsilon k}.$$

Proof. Note that for $k < k_1$ and $r \in [\delta^n, \delta^{n+1}]$, $\mathcal{I}(B(x,r),n+k)$ is contained in the annulus $B(x, r + C_1\delta^{n+k}) \setminus B(x, r - C_1\delta^{n+k})$. For every $n > n_1$ and $r \in [\delta^n, \delta^{n+1}]$, Lemma 2.9 yields

$$\begin{aligned} m(\mathcal{I}(B(x,r),n+k)) &\leq m(B(x,r+C_1\delta^{n+k}) \setminus B(x,r-C_1\delta^{n+k})) \\ &\leq K_\epsilon \left(\frac{C_1\delta^{n+k}}{r}\right)^\epsilon m(B(x,r)) \\ &\leq K_\epsilon C_1^\epsilon \delta^{\epsilon k} m(B(x,r)), \end{aligned}$$

which is the desired conclusion. □

To state the next technical lemmas, we need the following estimate (cf. [1, Theorem 3.5]). Recall that A'_r is the averaging operator given by equation (1.3).

PROPOSITION 2.11. *Let $r>0$, then for every $p\in[1,\infty]$, we have $\|A'_r\|_{L^p(G,m)\to L^p(G,m)} \leq D^{1/p}$.*

With $\mathcal{I}(B(x,r),n+k)$ being defined as above, we define

$$M_{n+k}f(x) = \sup_{r\in[\delta^n,\delta^{n+1}]} \left|\frac{1}{m(B(x,r))}\int_{\mathcal{I}(B(x,r),n+k)} f(y)\,dm(y)\right|.$$

LEMMA 2.12. *Let $p \in [1,\infty]$ and p' be its conjugate index. There exists a constant $D_p = ((K+1)(\delta+1)^\epsilon)^{1/p} D^{1/p}(K_\epsilon C_1^\epsilon)^{1/p'}$ such that for all $f \in L^p(G,m)$,*

$$\|M_{n+k}f\|_{L^p(G,m)} \leq D_p \delta^{\epsilon k/p'} \|f\|_{L^p(G,m)}.$$

Proof. For $p = \infty$, using Lemma 2.10, the conclusion is obvious. For $p \in [1,\infty)$, fix $x \in G$ and note that $\mathcal{I}(B(x,r),n+k) \subseteq B(x, r + C_1\delta^{n+k})$. Then, using the Hölder inequality and Lemma 2.10, we obtain

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$$M_{n+k}f(x) \le \sup_{r\in[\delta^n,\delta^{n+1}]} \frac{m(\mathcal{I}(B(x,r),n+k))^{1/p'}}{m(B(x,r))}\left(\int_{\mathcal{I}(B(x,r),n+k)} |f(y)|^p dm(y)\right)^{1/p}$$
$$\le (K_\epsilon C_1^\epsilon)^{1/p'}\delta^{\epsilon k/p'}\left(\frac{1}{m(B(x,\delta^n))}\int_{B(x,\delta^{n+1}+C_1\delta^{n+k})} |f(y)|^p dm(y)\right)^{1/p},$$

and by inequality (2.1), the above inequality yields

$$M_{n+k}f(x)$$
$$\le (K_\epsilon C_1^\epsilon)^{1/p'}\delta^{\epsilon k/p'}\left(\frac{(K+1)(\delta+1)^\epsilon}{m(B(x,\delta^{n+1}+C_1\delta^{n+k}))}\int_{B(x,\delta^{n+1}+C_1\delta^{n+k})} |f(y)|^p\, dm(y)\right)^{1/p}.$$

Then, applying Proposition 2.11, we have

$$\|M_{n+k}f\|_{L^p} \le ((K+1)(\delta+1)^\epsilon)^{1/p} D^{1/p}(K_\epsilon C_1^\epsilon)^{1/p'}\delta^{\epsilon k/p'}\|f\|_{L^p}$$

and the lemma follows. □

Given an annulus $B(x,r)\setminus B(x,s)$ and integers k, n, define

$$\mathcal{I}(B(x,r)\setminus B(x,s),n)$$
$$= \bigcup_\alpha \{Q_\alpha^n \cap (B(x,r)\setminus B(x,s)) : Q_\alpha^n \cap \partial(B(x,r)\setminus B(x,s)) \ne \emptyset\}$$

and

$$M_{n+k}^S f(x) = \sup_{\delta^n\le r_0<\cdots r_J\le\delta^{n+1}}\left(\sum_{i=1}^J \left|\frac{1}{m(B(x,r_i))}\int_{\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)} f(y)\, dm(y)\right|^2\right)^{1/2}.$$

LEMMA 2.13. *There exists a constant $C_3 = (2(K+1)D\delta^\epsilon C_1^\epsilon K_\epsilon)^{1/2}$ such that for all $f\in L^2(G,m)$,*

$$\|M_{n+k}^S f\|_{L^2(G,m)} \le C_3\delta^{\epsilon k/2}\|f\|_{L^2(G,m)}.$$

Proof. Let $x\in G$. Fixing $\delta^n \le r_{i-1} < r_i \le \delta^{n+1}$, by the Cauchy–Schwarz inequality and Lemma 2.10, we have

$$\left|\frac{1}{m(B(x,r_i))}\int_{\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)} f(y)\, dm(y)\right|^2$$
$$\le \frac{m(\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k))}{m(B(x,r_i))^2}\int_{\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)} |f(y)|^2\, dm(y)$$
$$\le \frac{2C_1^\epsilon K_\epsilon\delta^{\epsilon k}}{m(B(x,\delta^n))}\int_{\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)} |f(y)|^2\, dm(y),$$

where we used the fact that $\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)\subseteq \mathcal{I}(B(x,r_i),n+k)\cup \mathcal{I}(B(x,r_{i-1}),n+k)$ in the last inequality. From this, inequality (2.1) and the observation that $\bigcup_i \mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}),n+k)\subseteq B(x,\delta^{n+1})$, it follows that

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$$M^S_{n+k} f(x) \le \left(\frac{2C_1^\epsilon K_\epsilon \delta^{\epsilon k}}{m(B(x,\delta^n))} \int_{B(x,\delta^{n+1})} |f(y)|^2 \, dm(y) \right)^{1/2}$$
$$\le \left(\frac{2(K+1)\delta^\epsilon C_1^\epsilon K_\epsilon \delta^{\epsilon k}}{m(B(x,\delta^{n+1}))} \int_{B(x,\delta^{n+1})} |f(y)|^2 \, dm(y) \right)^{1/2}.$$

Then, using Proposition 2.11, we conclude

$$\|M^S_{n+k} f\|_{L^2} \le (2(K+1)D\delta^\epsilon C_1^\epsilon K_\epsilon)^{1/2} \delta^{\epsilon k/2} \|f\|_{L^p}. \qquad \square$$

Let f be a locally integrable function on G. We define

$$S_n(f) = |A'_{\delta^n} f - \mathbb{E}_n f|, \ SV_n(f) = V_2(A'_r f : r \in [\delta^n, \delta^{n+1})).$$

Fixing $\delta^n \le r_{i-1} < r_i < \delta^{n+1}$, we have

$$\left| \frac{1}{m(B(x,r_i))} \int_{B(x,r_i)} f(y) \, dm(y) - \frac{1}{m(B(x,r_{i-1}))} \int_{B(x,r_{i-1})} f(y) \, dm(y) \right|$$
$$\le \left| \frac{1}{m(B(x,r_i))} \int_{B(x,r_i)\setminus B(x,r_{i-1})} f(y) \, dm(y) \right|$$
$$+ \left| \left(\frac{1}{m(B(x,r_i))} - \frac{1}{m(B(x,r_{i-1}))} \right) \int_{B(x,r_{i-1})} f(y) \, dm(y) \right|.$$

From the above observation and the triangle inequality of ℓ^2-norm, we obtain

$$SV_n(f)(x) \le SV_I(f)(x) + SV_{II}(f)(x), \tag{2.8}$$

where

$$SV_I(f)(x) = \left(\sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \frac{1}{m(B(x,r_i))^2} \left| \int_{B(x,r_i)\setminus B(x,r_{i-1})} f(y) \, dm(y) \right|^2 \right)^{1/2}$$

and

$$SV_{II}(f)(x)$$
$$= \left(\sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \left| \left(\frac{1}{m(B(x,r_{i-1}))} - \frac{1}{m(B(x,r_i))} \right) \int_{B(x,r_{i-1})} f(y) \, dm(y) \right|^2 \right)^{1/2}.$$

Moreover, since the ℓ^1 norm is not less than the ℓ^2 norm, then

$$SV_n(f)(x) \le \sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \frac{1}{m(B(x,r_i))} \left| \int_{B(x,r_i)\setminus B(x,r_{i-1})} f(y) \, dm(y) \right|$$
$$+ \sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \left| \left(\frac{1}{m(B(x,r_{i-1}))} - \frac{1}{m(B(x,r_i))} \right) \int_{B(x,r_{i-1})} f(y) \, dm(y) \right|$$
$$\le \frac{1}{m(B(x,\delta^n))} \int_{B(x,\delta^{n+1})\setminus B(x,\delta^n)} |f(y)| \, dm(y)$$

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$$+\left(\frac{1}{m(B(x,\delta^n))}-\frac{1}{m(B(x,\delta^{n+1}))}\right)\int_{B(x,\delta^{n+1})}|f(y)|\,dm(y)$$
$$\le\frac{2}{m(B(x,\delta^n))}\int_{B(x,\delta^{n+1})}|f(y)|\,dm(y). \tag{2.9}$$

By equation (2.1) and Proposition 2.11, the above discussions imply the following lemma.

LEMMA 2.14. *Let $p\in[1,\infty]$. There exists a constant $C_4=2(K+1)D^{1/p}\delta^{\epsilon}$ such that*

$$\sup_{n\ge n_{r_0}}\|S_n\|_{L^p(G,m)\to L^p(G,m)}\le D^{1/p}+1,\qquad \sup_{n\ge n_{r_0}}\|SV_n\|_{L^p(G,m)\to L^p(G,m)}\le C_4.$$

Finally, we state the following version of the almost orthogonality principle, see e.g. [33] for a proof.

LEMMA 2.15. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of sub-linear operators from L^2 to L^2 on some σ-finite measure space. Let $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$ be two sequences of L^2 functions. Let N be a positive integer number. Assume that for $n\ge N$, there exists a sequence of positive constant $\{a(j)\}_{j\in\mathbb{Z}}$ with $w=\sum_{k\in\mathbb{Z}}a(k)<\infty$ such that*

$$\|T_n(u_{k+n})\|_{L^2}\le a(k)\|v_{k+n}\|_{L^2}, \tag{2.10}$$

then

$$\sum_{n\ge N}\left\|\sup_{j,m}\left|T_n\Big(\sum_{j\le k\le m}u_{k+n}\Big)\right|\right\|_{L^2}^2\le w^2\sum_{n\in\mathbb{Z}}\|v_n\|_{L^2}^2.$$

Furthermore, if $T_*:=\max_{0<n\le N}\|T_n\|_{L^2\to L^2}<\infty$, T_n is strongly continuous for every $n\in\mathbb{N}$, $f=\sum_{n\in\mathbb{Z}}u_n$ and $\sum_{n\in\mathbb{Z}}\|v_n\|_{L^2}^2\le C\|f\|_{L^2}^2$, then we have

$$\sum_{n\in\mathbb{N}}\|T_nf\|_{L^2}^2\le(Cw^2+NT_*^2)\|f\|_{L^2}^2.$$

3. *Strong type* (2, 2) *estimates*

In this section, we prove that the square function $S(f)$ and the short variation operator $SV(f)$ are of strong type (2, 2). We begin with the square function $S(f)$.

Proof of equation (1.8) in the case where $p=2$. Fix $f\in L^2(G,m)$. Recall that

$$S(f)=\left(\sum_{n>n_{r_0}}|A'_{\delta^n}f-\mathbb{E}_nf|^2\right)^{1/2}.$$

Set $n_2=\max\{n_0,n_1,n_{r_0}\}$. By Lemma 2.14, we first have

$$\left\|\left(\sum_{n_{r_0}<n\le n_2}|A'_{\delta^n}f-\mathbb{E}_nf|^2\right)^{1/2}\right\|_{L^2}\le 2(n_2-n_{r_0})\|f\|_{L^2}.$$

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Note that $f = \sum_{n\in\mathbb{Z}} \mathbb{D}_n f$. Set $T_n = A'_{\delta^n} - \mathbb{E}_n$, $u_n = v_n = \mathbb{D}_n f$, $N = n_2$ in Lemma 2.15; it suffices for our purposes to prove that for every $n > n_2$ and $k \in \mathbb{Z}$, there exists a sequence $a(k)$ of positive number with $w = \sum_{k\in\mathbb{Z}} a(k) < \infty$ such that

$$\|A'_{\delta^n}\mathbb{D}_{n+k}f - \mathbb{E}_n\mathbb{D}_{n+k}f\|_{L^2} \le a(k)\|\mathbb{D}_{n+k}f\|_{L^2}. \tag{3.1}$$

To achieve our goal, we first set

$$k_2 = \max\{L_0 + 1, |k_1|, \min\{k \in \mathbb{N} : a_0\delta^{k-1} > 1\}\}, \tag{3.2}$$

and then divide k into three cases $-k_2 \le k \le k_2$, $k > k_2$ and $k < -k_2$.

Case $-k_2 \le k \le k_2$. Applying Lemma 2.14 for function $\mathbb{D}_{n+k}f$, we obtain

$$\|A'_{\delta^n}\mathbb{D}_{n+k}f - \mathbb{E}_n\mathbb{D}_{n+k}f\|_{L^2} \le (D^{1/2} + 1)\|\mathbb{D}_{n+k}f\|_{L^2}.$$

Case $k > k_2$. Note that $\mathbb{E}_n\mathbb{D}_{n+k}f = \mathbb{D}_{n+k}f$, so we write

$$\|A'_{\delta^n}\mathbb{D}_{n+k}f - \mathbb{E}_n\mathbb{D}_{n+k}f\|^2_{L^2(G,m)} = \sum_\alpha \int_{Q_\alpha^{n+k-1}} |A'_{\delta^n}\mathbb{D}_{n+k}f(x) - \mathbb{D}_{n+k}f(x)|^2\, dm(x).$$

Fix a 'dyadic cube' Q_α^{n+k-1}. Note that $\mathbb{D}_{n+k}f(x)$ is a constant valued function on Q_α^{n+k-1}, and so it follows that on Q_α^{n+k-1}, $A'_{\delta^n}\mathbb{D}_{n+k}f(x) - \mathbb{D}_{n+k}f(x) \ne 0$ only if x belongs to the set $\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})$. Applying Lemma 2.7, we have

$$m(\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})) \le C_2\delta^{(1-k)\eta}m(Q_\alpha^{n+k-1}). \tag{3.3}$$

Let $x \in \mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})$ and define the set $I_x = \{\beta : B(x, \delta^n) \cap Q_\beta^{n+k-1} \ne \emptyset\}$. Set $I_\alpha = \bigcup_{x\in\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})} I_x$. Fix $\beta \in I_\alpha$, so there exist a point $x \in \mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})$ and $y_0 \in Q_\beta^{n+k-1}$ such that $y_0 \in B(x, \delta^n) \cap Q_\beta^{n+k-1}$. Then, for every $y \in Q_\beta^{n+k-1}$, by Proposition 2.2(iv), we have

$$d(y, z_\alpha^{n+k-1}) \le d(y, y_0) + d(y_0, x) + d(x, z_\alpha^{n+k-1}) \le 3C_1\delta^{n+k-1} + \delta^n.$$

It follows that for every $\beta \in I_\alpha$, $Q_\beta^{n+k-1} \subseteq B(z_\alpha^{n+k-1}, 3C_1\delta^{n+k-1} + \delta^n))$ and $Q_\alpha^{n+k-1} \subseteq B(z_\beta^{n+k-1}, 3C_1\delta^{n+k-1} + \delta^n))$. By Proposition 2.2(ii), the 'dyadic cubes' Q_β^k are disjoint, and hence $\bigcup_{\beta\in I_\alpha} Q_\beta^{n+k-1} \subseteq B(z_\alpha^k, 3C_1\delta^{n+k-1} + \delta^n)$. Moreover, it is clear that if $\beta \notin I_\alpha$, then $\alpha \notin I_\beta$. However, by Proposition 2.1, one can see that $B(z_\alpha^{n+k-1}, 3C_1\delta^{n+k-1} + \delta^n))$ is covered by at most $D^{\log_2[(3C_1+1)/a_0]+1}$ balls of radius $a_0\delta^{n+k-1}$. Since by Proposition 2.2(iv), we know that each Q_β^{n+k-1} contains a ball $B(z_\beta^{n+k-1}, a_0\delta^{n+k-1})$, then

$$\#\{I_\alpha\} \le D^{\log_2[(3C_1+1)/a_0]+1}, \tag{3.4}$$

here and in what follows, $\#\{A\}$ stands for the number of the set A. Set

$$m_\alpha = \max_{\beta\in I_\alpha} |\mathbb{D}_{n+k}f(z_\beta^{n+k-1})|,$$

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where z_β^{n+k-1} is the point associated with the corresponding 'dyadic cube' Q_β^{n+k-1}. Since $\mathbb{D}_{n+k}f(x)$ is a constant-valued function on Q_β^{n+k-1}, it follows that

$$m_\alpha^2 \le \sum_{\beta\in I_\alpha} \frac{1}{m(Q_\beta^{n+k-1})} \int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k}f(x)|^2\, dm(x). \tag{3.5}$$

By the above inequalities, Proposition 2.2(iv), and equations (3.3) and (2.1), we conclude

$$\begin{aligned}
&\sum_\alpha \int_{Q_\alpha^{n+k-1}} |A'_{\delta^n}\mathbb{D}_{n+k}f(x) - \mathbb{D}_{n+k}f(x)|^2\, dm(x)\\
&\quad= \sum_\alpha \int_{\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1})} |A'_{\delta^n}\mathbb{D}_{n+k}f(x) - \mathbb{D}_{n+k}f(x)|^2\, dm(x)\\
&\quad\le 4\sum_\alpha m_\alpha^2 m(\mathcal{H}(B_{\delta^n}, Q_\alpha^{n+k-1}))\\
&\quad\le 4C_2\delta^{(1-k)\eta}\sum_\alpha m_\alpha^2 m(Q_\alpha^{n+k-1})\\
&\quad\le 4C_2\delta^{(1-k)\eta}\sum_\alpha\sum_{\beta\in I_\alpha} \frac{m(Q_\alpha^{n+k-1})}{m(Q_\beta^{n+k-1})}\int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k}f(x)|^2\, dm(x)\\
&\quad\le 4C_2\delta^{(1-k)\eta}\sum_\alpha\sum_{\beta\in I_\alpha} \frac{m(B(z_\beta^{n+k-1}, 3C_1\delta^{n+k-1}+\delta^n))}{m(B(z_\beta^{n+k-1}, a_0\delta^{n+k-1}))}\int_{Q_\beta^{n+k-1}} |\mathbb{D}_{n+k}f(x)|^2\, dm(x)\\
&\quad\le 4C_2(K+1)((3C_1+1)/a_0)^\epsilon D^{\log_2[(3C_1+1)/a_0]+1}\delta^{(1-k)\eta}\int_G |\mathbb{D}_{n+k}f(x)|^2\, dm(x).
\end{aligned}$$

Case $k < -k_2$. Since $\mathbb{E}_n\mathbb{D}_{n+k}f = 0$ and $\int_{Q_\alpha^{k+n}} \mathbb{D}_{n+k}f = 0$ for every $Q_\alpha^{k+n} \in \mathcal{F}_{k+n}$, then, for any $x \in G$,

$$\begin{aligned}
|A'_{\delta^n}\mathbb{D}_{n+k}f(x)| &= \left|\frac{1}{m(B(x,\delta^n))}\sum_\alpha \int_{Q_\alpha^{k+n}\cap B(x,\delta^n)} \mathbb{D}_{n+k}f(y)\, dm(y)\right|\\
&= \left|\frac{1}{m(B(x,\delta^n))}\int_{\mathcal{I}(B(x,\delta^n),n+k)} \mathbb{D}_{n+k}f(y)\, dm(y)\right|\\
&\le M_{n+k}\mathbb{D}_{n+k}f(x).
\end{aligned}$$

From this and Lemma 2.12, we obtain

$$\|A'_{\delta^n}\mathbb{D}_{n+k}f\|_{L^2} \le \|M_{n+k}\mathbb{D}_{n+k}f\|_{L^2} \le D_2\delta^{k\epsilon/2}\|\mathbb{D}_{n+k}f\|_{L^2}.$$

Therefore, for every $n > n_2$, we determine

$$a(k) = \begin{cases} (4C_2(K+1)((3C_1+1)/a_0)^\epsilon D^{\log_2[(3C_1+1)/a_0]+1}\delta^\eta)^{1/2}\delta^{-\eta k/2}, & k > k_2,\\ D^{1/2}+1, & -k_2 \le k \le k_2,\\ D_2\delta^{\epsilon k/2}, & k < -k_2, \end{cases}$$

and $\sum_{k\in\mathbb{Z}} a(k) < \infty$, which completes the proof. □

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The proof of strong type (2, 2) estimate for $SV(f) = (\sum_{n\geq n_{r_0}} |SV_n(f)|^2)^{1/2}$ is similar in spirit to that of the square function $S(f)$.

Proof of equation (1.11) in the case where $p = 2$. First, by Lemma 2.14, we have

$$\left\| \left(\sum_{n_{r_0}\leq n\leq n_2} |SV_n(f)|^2 \right)^{1/2} \right\|_{L^2} \leq 2(n_2 - n_{r_0} + 1)C_4 \|f\|_{L^2}.$$

Set $T_n = SV_n$, $u_n = v_n = \mathbb{D}_n$, $N = n_2$ in Lemma 2.15. For every $n > n_2$, we divide k into three cases $-k_2 \leq k \leq k_2$, $k > k_2$ and $k < -k_2$, where k_2 is defined in equation (3.2).

Case $-k_2 \leq k \leq k_2$. For function $\mathbb{D}_{n+k}f$, using Lemma 2.14 again, we have

$$\|V_2(A'_r\mathbb{D}_{n+k}f : r \in [\delta^n, \delta^{n+1}))\|_{L^2} \leq C_4\|\mathbb{D}_{n+k}f\|_{L^2}.$$

Case $k > k_2$. We write

$$\begin{aligned}&\|V_2(A'_r\mathbb{D}_{n+k}f : r \in [\delta^n, \delta^{n+1}))\|^2_{L^2(G,m)}\\ &= \sum_\alpha \int_{Q_\alpha^{n+k-1}} |V_2(A'_r\mathbb{D}_{n+k}f(x) : r \in [\delta^n, \delta^{n+1}))|^2\, dm(x).\end{aligned}$$

Fix a 'dyadic cube' Q_α^{n+k-1}. Recall that $\mathbb{D}_{n+k}f(x)$ is a constant valued function on Q_α^{n+k-1}. It follows that for every $\delta^n \leq r_i < r_{i+1} < \delta^{n+1}$ and $x \in Q_\alpha^{n+k-1}$, $|A'_{r_{i+1}}\mathbb{D}_{n+k}f(x) - A'_{r_i}\mathbb{D}_{n+k}f(x)| \neq 0$ only if there exists at least one ball $B(x, r_i)$ or $B(x, r_{i+1})$ intersecting with $(Q_\alpha^{n+k-1})^c$. So $V_2(A'_r\mathbb{D}_{n+k}f : r \in [\delta^n, \delta^{n+1}))$ is supported on $\mathcal{H}(B_{\delta^{n+1}}, Q_\alpha^{n+k-1})$. By Lemma 2.7, we have

$$m(\mathcal{H}(B_{\delta^{n+1}}, Q_\alpha^{n+k-1})) \leq C_2\delta^{(2-k)\eta} m(Q_\alpha^{n+k-1}). \tag{3.6}$$

However, by equation (2.8), we know that $V_2(A'_r\mathbb{D}_{n+k}f(x) : r \in [\delta^n, \delta^{n+1}))$ is controlled by the sum of $SV_I(\mathbb{D}_{n+k}f)(x)$ and $SV_{II}(\mathbb{D}_{n+k}f)(x)$, where

$$\begin{aligned}&SV_I(\mathbb{D}_{n+k}f)(x)\\ &= \left(\sup_{\delta^n\leq r_0<\cdots<r_J<\delta^{n+1}} \sum_{i=1}^J \frac{1}{m(B(x, r_i))^2} \left| \int_{B(x,r_i)\setminus B(x,r_{i-1})} \mathbb{D}_{n+k}f(y)\, dm(y) \right|^2 \right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&SV_{II}(\mathbb{D}_{n+k}f)(x) =\\ &\left(\sup_{\delta^n\leq r_0<\cdots<r_J<\delta^{n+1}} \sum_{i=1}^J \left| \left(\frac{1}{m(B(x, r_{i-1}))} - \frac{1}{m(B(x, r_i))} \right) \int_{B(x,r_{i-1})} \mathbb{D}_{n+k}f(y)\, dm(y) \right|^2 \right)^{1/2}.\end{aligned}$$

Let $x \in \mathcal{H}(B_{\delta^{n+1}}, Q_\alpha^{n+k-1})$ and set $I_x = \{\beta : B(x, \delta^{n+1}) \cap Q_\beta^{n+k-1} \neq \emptyset\}$. We define the set $I_\alpha = \bigcup_{x\in\mathcal{H}(B_{\delta^{n+1}}, Q_\alpha^{n+k-1})} I_x$. Similar to equation (3.4), we have $\#\{I_\alpha\} \leq D^{\log_2[(3C_1+1)/a_0]+1}$. Write

$$m_\alpha = \max_{\beta\in I_\alpha} |\mathbb{D}_{n+k}f(z_\beta^{n+k-1})|.$$

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It follows from equation (2.1) that

$$\begin{aligned}
&SV_I(\mathbb{D}_{n+k}f)^2(x)\\
&\quad\le \frac{m(B(x,\delta^{n+1})\setminus B(x,\delta^n))}{m(B(x,\delta^n))^2}\sup_{\delta^n\le r_0<\cdots<r_J<\delta^{n+1}}\sum_{i=1}^{J}\int_{B(x,r_i)\setminus B(x,r_{i-1})}|\mathbb{D}_{n+k}f(y)|^2\,dm(y)\\
&\quad\le \left(\frac{m(B(x,\delta^{n+1})\setminus B(x,\delta^n))}{m(B(x,\delta^n))}\right)^2 m_\alpha^2 \le (K+1)^2\delta^{2\epsilon}m_\alpha^2
\end{aligned}$$

and

$$\begin{aligned}
&SV_{II}(\mathbb{D}_{n+k}f)^2(x)\\
&\quad\le m_\alpha^2 m(B(x,\delta^{n+1}))^2\left(\sup_{\delta^n\le r_0<\cdots<r_J<\delta^{n+1}}\sum_{i=1}^{J}\left|\frac{1}{m(B(x,r_{i-1}))}-\frac{1}{m(B(x,r_i))}\right|\right)^2\\
&\quad\le m_\alpha^2 m(B(x,\delta^{n+1}))^2\left(\frac{1}{m(B(x,\delta^n))}-\frac{1}{m(B(x,\delta^{n+1}))}\right)^2 \le (K+1)^2\delta^{2\epsilon}m_\alpha^2.
\end{aligned}$$

Combining the above two inequalities with equations (3.5) and (3.6), we have

$$\begin{aligned}
&\int_{Q_\alpha^{n+k-1}}|V_2(A_r'\mathbb{D}_{n+k}f(x):r\in[\delta^n,\delta^{n+1}))|^2\,dm(x)\\
&\quad=\int_{\mathcal{H}(B_{\delta^{n+1}},Q_\alpha^{n+k-1})}|V_2(A_r'\mathbb{D}_{n+k}f(x):r\in[\delta^n,\delta^{n+1}))|^2\,dm(x)\\
&\quad\le 2\int_{\mathcal{H}(B_{\delta^{n+1}},Q_\alpha^{n+k-1})}SV_I(\mathbb{D}_{n+k}f)^2(x)+SV_{II}(\mathbb{D}_{n+k}f)^2(x)\,dm(x)\\
&\quad\le 4(K+1)^2\delta^{2\epsilon}m(\mathcal{H}(B_{\delta^{n+1}},Q_\alpha^{n+k-1}))m_\alpha^2\\
&\quad\le 4C_2(K+1)^2\delta^{2\epsilon}\delta^{(2-k)\eta}\sum_{\beta\in I_\alpha}\frac{m(Q_\alpha^{n+k-1})}{m(Q_\beta^{n+k-1})}\int_{Q_\beta^{n+k-1}}|\mathbb{D}_{n+k}f(x)|^2\,dm(x)\\
&\quad\le 4C_2(K+1)^2((3C_1+1)/a_0)^\epsilon\delta^{2\epsilon}\delta^{(2-k)\eta}\sum_{\beta\in I_\alpha}\int_{Q_\beta^{n+k-1}}|\mathbb{D}_{n+k}f(x)|^2\,dm(x),
\end{aligned}$$

and summing over all α shows

$$\begin{aligned}
&\|V_2(A_r'\mathbb{D}_{n+k}f:r\in[\delta^n,\delta^{n+1}))\|_{L^2}\\
&\quad\le(4C_2(K+1)^2((3C_1+1)/a_0)^\epsilon D^{\log_2[(3C_1+1)/a_0]+1}\delta^{2\epsilon}\delta^{(2-k)\eta})^{1/2}\|\mathbb{D}_{n+k}f\|_{L^2}.
\end{aligned}$$

Case $k<-k_2$. Let $x\in G$. Note that for every $\delta^n\le r_{i-1}<r_i\le\delta^{n+1}$,

$$\begin{aligned}
&\int_{B(x,r_i))\setminus B(x,r_{i-1})}\mathbb{D}_{n+k}f(y)\,dm(y)=\int_{\mathcal{I}(B(x,r_i))\setminus B(x,r_{i-1}),n+k)}\mathbb{D}_{n+k}f(y)\,dm(y),\\
&\int_{B(x,r_{i-1})}\mathbb{D}_{n+k}f(y)\,dm(y)=\int_{\mathcal{I}(B(x,r_{i-1}),n+k)}\mathbb{D}_{n+k}f(y)\,dm(y),
\end{aligned}$$

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so $SV_I(\mathbb{D}_{n+k}f)(x) = M^S_{n+k}(\mathbb{D}_{n+k}f)(x)$. Moreover, using the estimate (2.1) and the fact that the ℓ^1-norm is greater than the ℓ^2-norm, we obtain

$$\begin{aligned}
&SV_{II}(\mathbb{D}_{n+k}f)(x)\\
&\leq \sup_{\delta^n \leq r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \left| \left(\frac{1}{m(B(x, r_{i-1}))} - \frac{1}{m(B(x, r_i))} \right) \right.\\
&\qquad\qquad \left. \int_{\mathcal{I}(B(x,r_{i-1}),n+k)} \mathbb{D}_{n+k}f(y)\, dm(y) \right|\\
&\leq (K+1)\delta^{\epsilon} \sup_{\delta^n \leq r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \left| \frac{m(B(x, \delta^{n+1}))}{m(B(x, r_{i-1}))} - \frac{m(B(x, \delta^{n+1}))}{m(B(x, r_i))} \right| M_{n+k}(\mathbb{D}_{n+k}f)(x)\\
&\leq (K+1)^2 \delta^{2\epsilon} M_{n+k}(\mathbb{D}_{n+k}f)(x).
\end{aligned}$$

Using Lemmas 2.13 and 2.12, we have

$$\begin{aligned}
\|V_2(A'_r \mathbb{D}_{n+k}f : r \in [\delta^n, \delta^{n+1}))\|_{L^2} &\leq \|SV_I(\mathbb{D}_{n+k}f)\|_{L^2} + \|SV_{II}(\mathbb{D}_{n+k}f)\|_{L^2}\\
&\leq \|M^S_{n+k}(\mathbb{D}_{n+k}f)\|_{L^2} + (K+1)^2\delta^{2\epsilon}\|M_{n+k}(\mathbb{D}_{n+k}f)\|_{L^2}\\
&\leq (C_3 + (K+1)^2\delta^{2\epsilon} D_2)\delta^{\epsilon k/2}\|\mathbb{D}_{n+k}f\|_{L^2}.
\end{aligned}$$

Hence, for every $n > n_2$, we determine

$$a(k) = \begin{cases} (4C_2(K+1)^2((3C_1+1)/a_0)^{\epsilon} D^{\log_2[(3C_1+1)/a_0]+1}\delta^{2(\epsilon+\eta)})^{1/2}\delta^{-k\eta/2}, & k > k_2,\\ C_4, & -k_2 \leq k \leq k_2,\\ (C_3 + (K+1)^2\delta^{2\epsilon} D_2)\delta^{\epsilon k/2}, & k < -k_2, \end{cases}$$

and $\sum_k a(k) < \infty$, which completes the proof. □

4. *Weak type* (1, 1) *estimates*

In this section, we prove that the square function $f \to S(f)$ and the short variation operator $f \to SV(f)$ are of weak type (1, 1).

Under the conditions (1.2) and (1.1), the space (G, d, m) might not be a doubling measure space and the usual Calderón–Zygmund decomposition does not work any more. We need the following version of Calderón–Zygmund decomposition, which is motivated by Gundy's decomposition from martingale theory. This decomposition was constructed in [47] for non-doubling measure on $\mathbb{R}^d$, and construction works without alterations for the space (G, d, m).

Let $f : G \to \mathbb{C}$ be a locally integrable function. The 'dyadic' maximal function is defined by

$$M_d f(x) = \sup_{k \in \mathbb{Z}} \mathbb{E}_k(|f|)(x).$$

Given a 'dyadic cube' Q^k_α, let $\widetilde{Q}^k_\alpha = \{y \in G : d(y, z^k_\alpha) \leq 3C_1\delta^{k+1}\}$ and $\widehat{Q}^k_\alpha$ be its parent. We denote by $\widehat{z}^k_\alpha$ the corresponding point of $\widehat{Q}^k_\alpha$. Set $\langle f \rangle_{Q^k_\alpha} = (1/m(Q^k_\alpha)) \int_{Q^k_\alpha} f$.

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LEMMA 4.1. *Let $f \in L^1(G, m)$ and $\gamma > 0$. Let*

$$\{x \in G : M_d f(x) > \gamma\} = \bigcup_{k\in\mathbb{Z}} \Omega_k,$$

where

$$\Omega_k = \{x \in G : \mathbb{E}_k(|f|)(x) > \gamma, \mathbb{E}_j(|f|)(x) \le \gamma,\ j > k\}.$$

Then, we decompose

$$\Omega = \bigcup_{k\in\mathbb{Z}} \Omega_k = \bigcup_{k\in\mathbb{Z}} \Big(\bigcup_{\alpha\in\Lambda_k} Q_\alpha^k \Big)$$

into a disjoint union of maximal 'dyadic cubes' Q_α^k, where $\{\Lambda_k\}_k$ stands for the sequence of the corresponding index set. Let

$$g(x) = f\mathbb{1}_{\Omega^c} + \sum_k \sum_{\alpha\in\Lambda_k} \langle f\rangle_{\widehat{Q}_\alpha^k} \mathbb{1}_{Q_\alpha^k}(x) + \sum_k \sum_{\alpha\in\Lambda_k} (\langle f\rangle_{Q_\alpha^k} - \langle f\rangle_{\widehat{Q}_\alpha^k}) \frac{m(Q_\alpha^k)}{m(\widehat{Q}_\alpha^k)} \mathbb{1}_{\widehat{Q}_\alpha^k}(x),$$

$$b(x) = \sum_k b_k = \sum_k \sum_{\alpha\in\Lambda_k} b_\alpha^k(x) = \sum_k \sum_{\alpha\in\Lambda_k} (f(x) - \langle f\rangle_{Q_\alpha^k}) \mathbb{1}_{Q_\alpha^k}(x),$$

$$\xi(x) = \sum_k \xi_k = \sum_k \sum_{\alpha\in\Lambda_k} \xi_\alpha^k(x) = \sum_k \sum_{\alpha\in\Lambda_k} (\langle f\rangle_{Q_\alpha^k} - \langle f\rangle_{\widehat{Q}_\alpha^k}) \Big(\mathbb{1}_{Q_\alpha^k}(x) - \frac{m(Q_\alpha^k)}{m(\widehat{Q}_\alpha^k)} \mathbb{1}_{\widehat{Q}_\alpha^k}(x) \Big).$$

Then:

(i) $f = g + b + \xi$;

(ii) *let $p \in [1, \infty)$ and $m = [p] + 1$, the function g satisfies*

$$\|g\|_{L^p}^p \le 3 \cdot 2^p (m!)^{(p-1)/(m-1)} \gamma^{p-1} \|f\|_{L^1};$$

(iii) *the function b satisfies*

$$\int_G b_\alpha^k = 0,\ \|b\|_{L^1} = \sum_k \sum_{\alpha\in\Lambda_k} \|b_\alpha^k\|_{L^1} \le 2\|f\|_{L^1};$$

(iv) *the function ξ satisfies*

$$\int_G \xi_\alpha^k = 0,\ \|\xi\|_{L^1} = \sum_k \sum_{\alpha\in\Lambda_k} \|\xi_\alpha^k\|_{L^1} \le 4\|f\|_{L^1}.$$

With the above decomposition, the almost orthogonality principle, which was exploited well in [32] or [30], does not seem to work. Thereby, we have to provide another method to achieve our goal. A new input is the observation that the operators S and SV are essentially dyadic operator, in particular, they are perfect dyadic in the small scale—small cubes *versus* large balls.

We first deal with the operator S.

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Proof of equation (1.9). Fix $f \in L^1(G, m)$ and $\gamma > 0$. Keeping the notation in Lemma 4.1, we get $f = g + b + \xi$. We first have

$$\begin{aligned} m(\{x \in G : S(f)(x) > \gamma\}) \leq\ & m(\{x \in G : S(g)(x) > \gamma/3\}) \\ & + m(\{x \in G : S(b)(x) > \gamma/3\}) \\ & + m(\{x \in G : S(\xi)(x) > \gamma/3\}). \end{aligned}$$

By the L^2-boundedness of the operator S and Lemma 4.1(ii), the first term on the right-hand side is controlled by

$$\begin{aligned} m(\{x \in G : S(g)(x) > \gamma/3\}) &\leq \frac{9}{\gamma^2} \int_G |S(g)(x)|^2\, dm(x) \leq \frac{9c_2}{\gamma^2} \int_G |g(x)|^2\, dm(x) \\ &\leq \frac{108\sqrt{6}c_2}{\gamma} \int_G |f(x)|\, dm(x). \end{aligned}$$

It remains to handle the other two terms, we set

$$k_3 = \min\{k : a_0\delta^k > r_0\},\ \widetilde{\Omega} = \Big(\bigcup_{k<k_3} \Omega_k \Big) \cup \Big(\bigcup_{k\geq k_3} \widetilde{\Omega}_k \Big),$$

where $\widetilde{\Omega}_k = \bigcup_{\alpha\in\Lambda_k} \widetilde{Q}^k_\alpha$. By equation (2.1) and the fact that M_d is of weak type $(1, 1)$, $m(\widetilde{\Omega})$ is controlled by

$$\begin{aligned} m(\widetilde{\Omega}) &= \sum_{k<k_3} m(\Omega_k) + \sum_{k\geq k_3} m(\widetilde{\Omega}_k) \leq \sum_{k<k_3} \sum_{\alpha\in\Lambda_k} m(Q^k_\alpha) + \sum_{k\geq k_3} \sum_{\alpha\in\Lambda_k} m(\widetilde{Q}^k_\alpha) \\ &\leq \sum_{k<k_3} \sum_{\alpha\in\Lambda_k} m(Q^k_\alpha) + \sum_{k\geq k_3} \sum_{\alpha\in\Lambda_k} \frac{m(\widetilde{Q}^k_\alpha)}{m(Q^k_\alpha)} m(Q^k_\alpha) \\ &\leq \sum_{k<k_3} \sum_{\alpha\in\Lambda_k} m(Q^k_\alpha) + (K+1)(3C_1\delta/a_0)^\epsilon \Big(\sum_{k\geq k_3} \sum_{\alpha\in\Lambda_k} m(Q^k_\alpha) \Big) \\ &\leq \frac{(K+1)(3C_1\delta/a_0)^\epsilon}{\gamma} \|f\|_{L^1}. \end{aligned}$$

We now focus on the term $m(\{x \in G \setminus \widetilde{\Omega} : S(\xi)(x) > \gamma/3\})$. Recall that $n_2 = \max\{n_0, n_1\}$, we decompose

$$\begin{aligned} & m(\{x \in G \setminus \widetilde{\Omega} : S(\xi)(x) > \gamma/3\}) \\ & \leq m\Bigg(\Bigg\{ x \in G \setminus \widetilde{\Omega} : \Bigg(\sum_{n_{r_0}<n\leq n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2 \Bigg)^{1/2} > \gamma/6 \Bigg\} \Bigg) \\ & \quad + m\Bigg(\Bigg\{ x \in G \setminus \widetilde{\Omega} : \Bigg(\sum_{n>n_2} |A_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2 \Bigg)^{1/2} > \gamma/6 \Bigg\} \Bigg). \end{aligned}$$

For the first part of the right-hand side of the above inequality, using Lemmas 2.14 and 4.1(iv), we have

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$$
\begin{aligned}
&m\left(\left\{x \in G \setminus \widetilde{\Omega} : \left(\sum_{n_{r_0}<n\le n_2} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2\right)^{1/2} > \gamma/6\right\}\right)\\
&\le \sum_{n_{r_0}<n\le n_2} \frac{6}{\gamma}\int_{G\setminus\widetilde{\Omega}} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|\, dm(x)\\
&\le \frac{12(n_2 - n_{r_0})}{\gamma}\|\xi\|_{L^1(G,m)} \le \frac{48(n_2 - n_{r_0})}{\gamma}\|f\|_{L^1(G,m)}.
\end{aligned}
$$

For the second part, we first have

$$
\begin{aligned}
&m\left(\left\{x \in G \setminus \widetilde{\Omega} : (\sum_{n>n_2} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2)^{1/2} > \gamma/6\right\}\right)\\
&\le \frac{6}{\gamma}\int_{G\setminus\widetilde{\Omega}} \left(\sum_{n>n_2} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2\right)^{1/2} dm(x)\\
&\le \frac{6}{\gamma}\int_{G\setminus\widetilde{\Omega}} \sum_{n>n_2} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|\, dm(x)\\
&\le \frac{6}{\gamma}\sum_{n>n_2}\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\Lambda_{n+k}}\int_{G\setminus\widetilde{\Omega}} |A'_{\delta^n}\xi^{n+k}_\alpha(x) - \mathbb{E}_n\xi^{n+k}_\alpha(x)|\, dm(x). \qquad (4.1)
\end{aligned}
$$

We now deal with the integral term $\int_{G\setminus\widetilde{\Omega}} |A_{\delta^n}\xi^{n+k}_\alpha(x) - \xi^{n+k}_\alpha(x)|\, dm(x)$. Let

$$
k_4 = \max\{k : k < 0 \ \&\ 2C_1\delta^{k+1} \le 1\}, \quad k_5 = \max\{|k_4|, k_3\}.
$$

We split the k into three cases: $-k_5 \le k \le k_5$, $k > k_5$ and $k < -k_5$. We will prove that

$$
\int_{G\setminus\widetilde{\Omega}} |A'_{\delta^n}\xi^{n+k}_\alpha(x) - \mathbb{E}_n\xi^{n+k}_\alpha(x)|\, dm(x) = a(k)\|\xi^{n+k}_\alpha\|_{L^1(G,m)},
$$

where

$$
a(k) = \begin{cases} (D+1), & -k_5 \le k \le k_5;\\ 0, & k > k_5;\\ 4^\epsilon(K+1)^2C_1^\epsilon DK_\epsilon\delta^{\epsilon k}, & k < -k_5. \end{cases} \qquad (4.2)
$$

Assume this result momentarily. Combining equations (4.1), (4.2) with Lemma 4.1(iv), one can see immediately that

$$
\begin{aligned}
&m\left(\left\{x \in G \setminus \widetilde{\Omega} : \left(\sum_{n>n_2} |A'_{\delta^n}\xi(x) - \mathbb{E}_n\xi(x)|^2\right)^{1/2} > \gamma/6\right\}\right)\\
&\le \frac{6}{\gamma}\sum_{n>n_2}\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\Lambda_{n+k}} a(k)\|\xi^{n+k}_\alpha\|_{L^1(G,m)}\\
&\le \frac{6}{\gamma}\left(\sum_{k\in\mathbb{Z}} a(k)\right)\left(\sum_{n\in\mathbb{Z}}\sum_{\alpha\in\Lambda_n}\|\xi^n_\alpha\|_{L^1(G,m)}\right) \le \frac{C_{\epsilon,\delta}}{\gamma}\|f\|_{L^1(G,m)}.
\end{aligned}
$$

We now prove equation (4.2).

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Case $-k_5 \le k \le k_5$. Using Lemma 2.14 for function ξ_α^{n+k}, we have

$$\int_{G\setminus\widetilde{\Omega}} |A'_{\delta^n}\xi_\alpha^{n+k}(x) - \mathbb{E}_n\xi_\alpha^{n+k}(x)|\, dm(x) \le \int_G |A'_{\delta^n}\xi_\alpha^{n+k}(x) - \mathbb{E}_n\xi_\alpha^{n+k}(x)|\, dm(x) \le (D+1)\|\xi_\alpha^{n+k}\|_{L^1(G,m)}.$$

Case $k > k_5$. Note that ξ_α^{n+k} is supported on $\widehat{Q}_\alpha^{n+k}$. Recall that $\widetilde{Q}_\alpha^{n+k} = \{y \in G : d(y, z_\alpha^{n+k}) \le 3C_1\delta^{n+k+1}\}$. Let $y \in \widehat{Q}_\alpha^{n+k}$. We have $d(y, z_\alpha^{n+k}) \le d(y, \widehat{z}_\alpha^{n+k+1}) + d(z_\alpha^{n+k}, \widehat{z}_\alpha^{n+k+1}) \le 2C_1\delta^{n+k+1}$. This gives $\widehat{Q}_\alpha^{n+k} \subseteq \widetilde{Q}_\alpha^{n+k}$.

Fix $x \in G \setminus \widetilde{\Omega}$. There exists an unique 'dyadic cube' Q_β^n containing x. We claim that

$$B(x, \delta^n) \cap \widehat{Q}_\alpha^{n+k} = \emptyset, \;\; Q_\beta^n \cap \widehat{Q}_\alpha^{n+k} = \emptyset. \tag{4.3}$$

Since

$$A'_{\delta^n}\xi_\alpha^{n+k}(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)\cap\widehat{Q}_\alpha^{n+k}} \xi_\alpha^{n+k}(y)\, dm(y)$$

and

$$\mathbb{E}_n\xi_\alpha^{n+k}(x) = \frac{1}{m(Q_\beta^n)}\int_{Q_\beta^n\cap\widehat{Q}_\alpha^{n+k}} \xi_\alpha^{n+k}(y)\, dm(y),$$

it follows from the claim that $A'_{\delta^n}\xi_\alpha^{n+k} = \mathbb{E}_n\xi_\alpha^{n+k} = 0$.

We now prove the claim. If $Q_\beta^n \cap \widehat{Q}_\alpha^{n+k} \neq \emptyset$, by Proposition 2.2(ii), it follows that $x \in Q_\beta^n \subseteq \widehat{Q}_\alpha^{n+k} \subseteq \widetilde{Q}_\alpha^{n+k}$, which is a contradiction with $x \in G \setminus \widetilde{\Omega}$. However, if there exists a point $z \in B(x, \delta^n) \cap \widehat{Q}_\alpha^{n+k}$, then $d(x, \widehat{z}_\alpha^{n+k+1}) \le d(x, z) + d(z, \widehat{z}_\alpha^{n+k+1}) \le \delta^n + C_1\delta^{n+k+1} < 2C_1\delta^{n+k+1}$, and so it follows that

$$d(x, z_\alpha^{n+k}) \le d(x, \widehat{z}_\alpha^{n+k+1}) + d(z_\alpha^{n+k}, \widehat{z}_\alpha^{n+k+1}) < 3C_1\delta^{n+k+1},$$

which is in contrast to $x \in G \setminus \widetilde{\Omega}$, and equation (4.3) is proved. This gives the conclusion.

Case $k < -k_5$. We also have

$$\mathbb{E}_n\xi_\alpha^{n+k}(x) = 0 \quad \text{for all } x \in G \setminus \widetilde{\Omega}.$$

Indeed, let $x \in Q_\beta^n$. If $Q_\beta^n \cap \widehat{Q}_\alpha^{n+k} \neq \emptyset$, then by Proposition 2.2(ii), it follows that $\widehat{Q}_\alpha^{n+k} \subseteq Q_\beta^n$ and

$$\mathbb{E}_n\xi_\alpha^{n+k}(x) = \int_{\widehat{Q}_\alpha^{n+k}} \xi_\alpha^{n+k}(y)\, dm(y) = 0.$$

Given a ball $B(x, r)$ and 'dyadic cube' Q_α^k, we define the set $\mathcal{I}(B(x,r), Q_\alpha^k) = \{Q_\alpha^k \cap B(x,r) : Q_\alpha^k \cap \partial B(x,r) \neq \emptyset\}$. Since ξ_α^{n+k} is supported on $\widehat{Q}_\alpha^{n+k}$ and $\int_{\widehat{Q}_\alpha^{n+k}} \xi_\alpha^{n+k} = 0$, it follows that for all $x \in G$,

$$|A_{\delta^n}\xi_\alpha^{n+k}(x)| = \left|\frac{1}{m(B(x,\delta^n))}\int_{\mathcal{I}(B(x,\delta^n),\widehat{Q}_\alpha^{n+k})} \xi_\alpha^{n+k}(y)\, dm(y)\right| \le \frac{1}{m(B(x,\delta^n))}\int_{\delta^n - 2C_1\delta^{n+k+1} \le d(x,y) \le \delta^n + 2C_1\delta^{n+k+1}} |\xi_\alpha^{n+k}(y)|\, dm(y).$$

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Then, by the above estimate, we obtain

$$\begin{aligned}
&\int_G |A_{\delta^n}\xi_\alpha^{n+k}(x)|\,dm(x)\\
&\quad\le \int_G \frac{1}{m(B(x,\delta^n))}\int_{\delta^n-2C_1\delta^{n+k+1}\le d(x,y)\le\delta^n+2C_1\delta^{n+k+1}} |\xi_\alpha^{n+k}(y)|\,dm(y)\,dm(x)\\
&\quad\le \int_G |\xi_\alpha^{n+k}(y)|\cdot\int_G \frac{\mathbb{1}_{B(y,\delta^n+2C_1\delta^{n+k+1})\setminus B(y,\delta^n-2C_1\delta^{n+k+1})}(x)}{m(B(x,\delta^n))}\,dm(x)\,dm(y). \qquad (4.4)
\end{aligned}$$

Set $A(y)=B(y,\delta^n+2C_1\delta^{n+k+1})\setminus B(y,\delta^n-2C_1\delta^{n+k+1})$. Fix $y\in G$. By equation (2.1), we have

$$\frac{\mathbb{1}_{A(y)}(x)}{m(B(x,\delta^n))}\le 2^\epsilon(K+1)\frac{\mathbb{1}_{A(y)}(x)}{m(B(x,\delta^n+2C_1\delta^{n+k+1}))}\mathbb{1}_{B(y,\delta^n+2C_1\delta^{n+k+1})}(x).$$

By the same argument as in the proof of [1, Theorem 3.5], we have the following properties. For any $0<\varepsilon<1$, there exist M-points $\{u_i:1\le i\le M\}$ with $M\le D$ in $B(y,\delta^n+2C_1\delta^{n+k+1})$ such that $B(y,\delta^n+2C_1\delta^{n+k+1})\setminus\bigcup_{i=1}^M B(u_i,\delta^n+2C_1\delta^{n+k+1})=\emptyset$. Fix $x\in B(y,\delta^n+2C_1\delta^{n+k+1})$, and let j be the first index such that

$$x\in B(u_j,\delta^n+2C_1\delta^{n+k+1}),\ m(B(u_j,\delta^n+2C_1\delta^{n+k+1}))\le(1+\varepsilon)m(B(x,\delta^n+2C_1\delta^{n+k+1})).$$

By the above discussions, we first have

$$\begin{aligned}
&\int_G \frac{\mathbb{1}_{A(y)}(x)}{m(B(x,\delta^n))}\,dm(x)\\
&\quad\le 2^\epsilon(K+1)\int_G \frac{\mathbb{1}_{A(y)}(x)}{m(B(x,\delta^n+2C_1\delta^{n+k+1}))}\mathbb{1}_{B(y,\delta^n+2C_1\delta^{n+k+1})}(x)\,dm(x)\\
&\quad\le 2^\epsilon(K+1)\int_G \sum_{i=1}^M \frac{(1+\varepsilon)\mathbb{1}_{A(y)\cap B(u_i,\delta^n+2C_1\delta^{n+k+1})}(x)}{m(B(u_i,\delta^n+2C_1\delta^{n+k+1}))}\,dm(x). \qquad (4.5)
\end{aligned}$$

Note that each $u_i\in B(y,\delta^n+2C_1\delta^{n+k+1})$, then $B(y,\delta^n)\subseteq B(u_i,2\delta^n+2C_1\delta^{n+k+1})$. It follows from equation (2.1) that

$$\frac{m(B(y,\delta^n))}{m(B(u_i,\delta^n+2C_1\delta^{n+k+1}))}\le 2^\epsilon(K+1).$$

Moreover, by Lemma 2.9, we have $m(A(y))/m(B(y,\delta^n))\le K_\epsilon C_1^\epsilon\delta^{\epsilon k}$. Combining these two estimates with equation (4.5), we conclude

$$\int_G \frac{\mathbb{1}_{A(y)}(x)}{m(B(x,\delta^n))}\,dm(x)\le(1+\varepsilon)4^\epsilon(K+1)^2C_1^\epsilon DK_\epsilon\delta^{\epsilon k}. \qquad (4.6)$$

Combining equation (4.6) with equation (4.4), and then letting $\varepsilon\to 0$, we obtain $\|A_{\delta^n}\xi_\alpha^{n+k}\|_{L^1}\le 6^\epsilon(K+1)^2C_1^\epsilon DK_\epsilon\delta^{\epsilon k}\|\xi_\alpha^{n+k}\|_{L^1}$. Together with the above three cases for k, equation (4.2) is proved.

The estimate of $m(\{x\in G\setminus\widetilde{\Omega}:S(b)(x)>\gamma/3\})$ is similar and so will only be indicated briefly. Fix $n>n_2$. We first prove that

$$\mathbb{E}_n b(x)=0\quad\text{for all } x\in G\setminus\widetilde{\Omega}. \qquad (4.7)$$

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Note that $b = \sum_k \sum_{\alpha\in\Lambda_k} b_\alpha^k$. By the linearity of operator $\mathbb{E}_n$, we only need to prove that for each b_α^k, $\mathbb{E}_n b_\alpha^k = 0$.

Fix $x \in G \setminus \widetilde{\Omega}$. Let $Q_\beta^n \ni x$. Since b_α^k is supported on Q_α^k, it follows that $\mathbb{E}_n b_\alpha^k(x) = \int_{Q_\alpha^k \cap Q_\beta^n} b_\alpha^k(y)\, dm(y)$. If $Q_\alpha^k \cap Q_\beta^n \neq \emptyset$, we split k into two cases: $k \geq n$ and $k < n$. For $k \geq n$, by Proposition 2.2(ii), it follows that $x \in Q_\beta^n \subseteq Q_\alpha^k$, this leads to a contradiction since $Q_\alpha^k \subseteq \widetilde{\Omega}$. For $k < n$, using Proposition 2.2(ii) again, we have $Q_\alpha^k \subseteq Q_\beta^n$. It follows immediately that

$$\mathbb{E}_n b_\alpha^k(x) = \frac{1}{m(Q_\beta^n)} \int_{Q_\alpha^k} b_\alpha^k(y)\, dm(y) = 0.$$

This gives equation (4.7). Similar to equation (4.2), we can also establish the following inequality:

$$\int_{G\setminus\widetilde{\Omega}} |A_{\delta^n} b_\alpha^{n+k}(x)|\, dm(x) \leq \begin{cases} D\|b_\alpha^{n+k}\|_{L^1}, & -k_5 \leq k \leq k_5; \\ 0, & k > k_5; \\ 4^\epsilon (K+1)^2 C_1^\epsilon D K_\epsilon \delta^{\epsilon k} \|b_\alpha^{n+k}\|_{L^1}, & k < -k_5. \end{cases} \tag{4.8}$$

Note that for $-k_5 \leq k \leq k_5$, the estimate holds by Proposition 2.11. For $k > k_5$, the estimate holds true by equation (4.3) and the fact that b_α^{n+k} is supported on Q_α^{n+k}. For $k < -k_5$, we first have

$$\begin{aligned} |A_{\delta^n} b_\alpha^{n+k}(x)| &= \left| \frac{1}{m(B(x,\delta^n))} \int_{\mathcal{I}(B(x,\delta^n), Q_\alpha^{n+k})} b_\alpha^{n+k}(y)\, dm(y) \right| \\ &\leq \frac{1}{m(B(x,\delta^n))} \int_{\delta^n - 2C_1\delta^{n+k} \leq d(x,y) \leq \delta^n + 2C_1\delta^{n+k}} |b_\alpha^{n+k}(y)|\, dm(y). \end{aligned}$$

By the same estimates as equations (4.4) and (4.6), the above inequality yields

$$\begin{aligned} &\int_{G\setminus\Omega} |A_{\delta^n} b_\alpha^{n+k}(x)|\, dm(x) \\ &\leq \frac{1}{m(B(x,\delta^n))} \int_G \int_{\delta^n - 2C_1\delta^{n+k} \leq d(x,y) \leq \delta^n + 2C_1\delta^{n+k}} |b_\alpha^{n+k}(y)|\, dm(y)\, dm(x) \\ &\leq 4^\epsilon (K+1)^2 C_1^\epsilon D K_\epsilon \delta^{\epsilon k} \|b_\alpha^{n+k}\|_{L^1(G,m)}; \end{aligned}$$

together with the above two cases, this proves equation (4.8) and the proof is complete. □

The proof of weak type (1, 1) estimate for operator SV is similar to S. Let us explain it briefly.

Proof of equation (1.12). Fix $f \in L^1(G,m)$. Decompose $f = g + b + \xi$. The desired estimate for g is true by the fact that SV is of strong type (2, 2). In what follows, we only state the proof for ξ since the proof for b is similar.

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By similar arguments as in the previous proof, we mainly need to prove the following inequality for $n > n_2$:

$$\int_{G\setminus\widetilde{\Omega}} V_2(A'_r\xi_\alpha^{n+k}(x) : r \in [\delta^n, \delta^{n+1}))\, dm(x) \le a(k)\|\xi_\alpha^{n+k}\|_{L^1(G,m)},$$

where

$$a(k) = \begin{cases} C_4, & -k_5 \le k \le k_5; \\ 0, & k > k_5; \\ 2\cdot 4^\epsilon(\delta+1)^{2\epsilon}(K+1)^3 C_1^\epsilon D K_\epsilon \delta^{\epsilon k}, & k < -k_5. \end{cases}$$

We now focus on the above inequality. Fix $n > n_2$. Note that for $-k_5 \le k \le k_5$, by Lemma 2.14, the estimate is true. For $k > k_5$, similar to equation (4.3), we can prove that for every $x \in G \setminus \widetilde{\Omega}$ and $r \in [\delta^n, \delta^{n+1}]$, $B(x,r) \cap \widehat{Q}_\alpha^{n+k} = \emptyset$; hence, $V_2(A'_r\xi_\alpha^{n+k} : r \in [\delta^n, \delta^{n+1})) = 0$. It remains to prove the case $k < -k_5$.

Given an annular $B(x,R)\setminus B(x,r)$ and 'dyadic cube' Q_α^k, we define the set $\mathcal{I}(B(x,R)\setminus B(x,r), Q_\alpha^k) = \{Q_\alpha^k \cap (B(x,R)\setminus B(x,r)) : Q_\alpha^k \cap \partial(B(x,R)\setminus B(x,r)) \ne \emptyset\}$. Since $\int_{\widehat{Q}_\alpha^{n+k}} \xi_\alpha^{n+k} = 0$, it follows that

$$\begin{aligned} &V_2(A'_r\xi_\alpha^{n+k}(x) : r \in [\delta^n, \delta^{n+1})) \\ &\le \sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \frac{1}{m(B(x,r_i))} \left| \int_{\mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}), \widehat{Q}_\alpha^{n+k})} \xi_\alpha^{n+k}(y)\, dm(y) \right| \\ &+ \sup_{\delta^n \le r_0 < \cdots < r_J < \delta^{n+1}} \sum_{i=1}^{J} \left| \left(\frac{1}{m(B(x,r_{i-1}))} - \frac{1}{m(B(x,r_i))} \right) \int_{\mathcal{I}(B(x,r_i), \widehat{Q}_\alpha^{n+k})} \xi_\alpha^{n+k}(y)\, dm(y) \right|. \end{aligned}$$

Let $l_1 \in [\delta^n, \delta^{n+1})$ be the radius depending on x such that $\widehat{Q}_\alpha^{n+k} \cap \partial B(x,l_1) \ne \emptyset$ (note that if such a radius l_1 does not exist, then the desired estimate is valid). It follows that $\bigcup_i \mathcal{I}(B(x,r_i)\setminus B(x,r_{i-1}), \widehat{Q}_\alpha^{n+k}) \subseteq B(x, l_1 + 2C_1\delta^{n+k+1})\setminus B(x, l_1 - 2C_1\delta^{n+k+1})$ and for every $r_i \in [\delta^n, \delta^{n+1})$, one has $\mathcal{I}(B(x,r_i)), \widehat{Q}_\alpha^{n+k}) \subseteq B(x, l_1 + 2C_1\delta^{n+k+1})\setminus B(x, l_1 - 2C_1\delta^{n+k+1})$. It follows from equation (2.1) that

$$\begin{aligned} &V_2(A'_r\xi_\alpha^{n+k}(x) : r \in [\delta^n, \delta^{n+1})) \\ &\quad \le \frac{2(K+1)\delta^\epsilon}{m(B(x,l_1))} \int_{B(x,l_1+2C_1\delta^{n+k+1})\setminus B(x,l_1-2C_1\delta^{n+k+1})} |\xi_\alpha^{n+k}(y)|\, dm(y). \end{aligned}$$

Therefore, using the same argument as equations (4.4) and (4.6), we get

$$\begin{aligned} &\int_{G\setminus\widetilde{\Omega}} V_2(A'_r\xi_\alpha^{n+k}(x) : r \in [\delta^n, \delta^{n+1}))\, dm(x) \\ &\quad \le 2\cdot 4^\epsilon(\delta+1)^\epsilon(K+1)^2 C_1^\epsilon D K_\epsilon \delta^{\epsilon k}\|\xi_\alpha^{n+k}\|_{L^1(G,m)}, \end{aligned}$$

which completes the proof. □

5. *(L^∞, BMO) estimates*

In this section, we prove that both operator S and operator SV map L_c^∞ to dyadic BMO space.

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Given a locally integrable function $f \in L^1_{\mathrm{loc}}(G, m)$, the dyadic sharp maximal function is defined by

$$M^\sharp f(x) = \sup_{(\alpha,k),x\in Q^k_\alpha} \inf_c \frac{1}{m(Q^k_\alpha)} \int_{Q^k_\alpha} |f(y) - c|\, dm(y),$$

and then the dyadic BMO space is defined as $\mathrm{BMO} = \{f \in L^1_{\mathrm{loc}}(G, m) : \|M^\sharp f\|_{L^\infty} < \infty\}$ with the norm $\|f\|_{\mathrm{BMO}} = \|M^\sharp f\|_{L^\infty}$.

We first handle operator S, which was defined as

$$S(f) = \Big(\sum_{n>n_{r_0}} |A'_{\delta^n} f - \mathbb{E}_n f|^2 \Big)^{1/2}.$$

As in dealing with the weak type (1, 1) estimate, we also need the non-doubling analysis.

Proof of equation (1.10). Fix a 'dyadic cube' Q^k_β. Recall that $\widetilde{Q}^k_\beta = \{y \in G : d(y, z^k_\alpha) \le 3C_1\delta^{k+1}\}$ and $k_3 = \min\{k : a_0\delta^k > r_0\}$. Set

$$Q^{k*}_\beta = \begin{cases} \widetilde{Q}^k_\beta, & k > k_3; \\ Q^k_\beta, & k \le k_3, \end{cases} \tag{5.1}$$

and

$$f(x) = f(x)\mathbb{1}_{Q^{k*}_\beta}(x) + f(x)\mathbb{1}_{G\setminus Q^{k*}_\beta}(x) := f_1(x) + f_2(x).$$

By the triangle inequality, it follows that

$$\begin{aligned} \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f)(y) - c|\, dm(y) &\le \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f)(y) - S(f_2)(y)|\, dm(y) \\ &\quad + \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f_2)(y) - c|\, dm(y) \\ &\le \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f_1)(y)|\, dm(y) + \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f_2)(y) - c|\, dm(y). \end{aligned}$$

We now focus on the first term of the right-hand side. By the Cauchy–Schwarz inequality, equations (5.1), (2.1) and the fact that the operator S is of strong type (2, 2), we have

$$\begin{aligned} \frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f_1)(y)|\, dm(y) &\le \left(\frac{1}{m(Q^k_\beta)} \int_{Q^k_\beta} |S(f_1)(y)|^2\, dm(y) \right)^{1/2} \\ &\le \left(\frac{c_2^2}{m(Q^k_\beta)} \int_{Q^{k*}_\beta} |f(y)|^2\, dm(y) \right)^{1/2} \\ &\le c_2 (3C_1\delta/a_0)^{\epsilon/2} \|f\|_{L^\infty}. \end{aligned}$$

It remains to handle the term $(1/m(Q^k_\beta)) \int_{Q^k_\beta} |S(f_2)(y) - c|\, dm(y)$. Taking $c = S(f_2)(z^k_\beta)$, we first claim that

$$\mathbb{E}_n f_2(x) - \mathbb{E}_n f_2(z^k_\beta) = 0 \quad \text{for all } x \in Q^k_\beta.$$

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Indeed, let $Q_\alpha^n \ni x$. If $n \le k$, by Proposition 2.2(ii), then $Q_\alpha^n \subseteq Q_\beta^k$. Since f_2 is supported on $G \setminus Q_\alpha^{k\,*}$, it follows that $\mathbb{E}_n f_2(x) = \mathbb{E}_n f_2(z_\beta^k) = 0$. If $n > k$, using Proposition 2.2(ii) again, we have $Q_\beta^k \subseteq Q_\alpha^n$ and $x, z_\beta^k \in Q_\alpha^n$. It follows that $\mathbb{E}_n f_2(x) = \mathbb{E}_n f_2(z_\beta^k)$, then the claim is proved.

However, let $n_3 = k_3 + [\log_\delta(2C_1/a_0)] + 2$. By the triangle inequality, we first have

$$\begin{aligned}\Big(\sum_{n>n_{r_0}} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2\Big)^{1/2} &\le \Big(\sum_{n_{r_0}<n\le n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2\Big)^{1/2} \\ &\quad + \Big(\sum_{n>n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2\Big)^{1/2} \\ &\le 2(n_3 - n_{r_0})\|f\|_{L^\infty} + \Big(\sum_{n>n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2\Big)^{1/2}. \end{aligned} \tag{5.2}$$

Let $d(Q_\beta^k)$ be the diameter of Q_β^k. Set $n_4 = \min\{n : \delta^n > d(Q_\beta^k)\}$. We now consider two cases for n_3: $n_3 \ge n_4$ and $n_3 < n_4$.

Case $n_3 \ge n_4$. Note that $\delta^{n_3} > d(Q_\beta^k)$. By equation (5.2), the proof will be completed if we proved the following inequality:

$$\Big(\sum_{n>n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2\Big)^{1/2} \le \frac{2(K + 2^\epsilon K(K+1))}{1-\delta^{-\epsilon}}\|f\|_{L^\infty} \quad \text{for all } x \in Q_\beta^k.$$

We now focus on the above inequality. By the triangle inequality, we have

$$\begin{aligned}&|A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)| \\ &\quad = \left|\frac{1}{m(B(x,\delta^n))}\int_{B(x,\delta^n)} f_2(y)\,dm(y) - \frac{1}{m(B(z_\beta^k,\delta^n))}\int_{B(z_\beta^k,\delta^n)} f_2(y)\,dm(y)\right| \\ &\quad \le \left|\frac{1}{m(B(x,\delta^n))}\int_{B(x,\delta^n)} f_2(y)\,dm(y) - \frac{1}{m(B(x,\delta^n))}\int_{B(z_\beta^k,\delta^n)} f_2(y)\,dm(y)\right| \\ &\qquad + \left|\frac{1}{m(B(x,\delta^n))}\int_{B(z_\beta^k,\delta^n)} f_2(y)\,dm(y) - \frac{1}{m(B(z_\beta^k,\delta^n))}\int_{B(z_\beta^k,\delta^n)} f_2(y)\,dm(y)\right| \\ &\quad \le \frac{m(B(x,\delta^n)\triangle B(z_\beta^k,\delta^n))}{m(B(x,\delta^n))}\|f\|_{L^\infty} + \left|\frac{m(B(z_\beta^k,\delta^n))}{m(B(x,\delta^n))} - 1\right|\|f\|_{L^\infty} \\ &\quad \le 2\frac{m(B(x,\delta^n)\triangle B(z_\beta^k,\delta^n))}{m(B(x,\delta^n))}\|f\|_{L^\infty}.\end{aligned}$$

Note that

$$\begin{aligned}B(x,\delta^n)\triangle B(z_\beta^k,\delta^n) &\subseteq (B(x, d(x,z_\beta^k)+\delta^n) \setminus B(x,\delta^n)) \\ &\quad \cup (B(z_\beta^k, d(x,z_\beta^k)+\delta^n) \setminus B(z_\beta^k,\delta^n)).\end{aligned}$$

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For $n > n_3$, we have $\delta^n > r_0$. Note that $x \in Q_\beta^k$, then by Proposition 2.2(iv), we have $B(z_\beta^k, \delta^n) \subseteq B(x, \delta^n + d(Q_\beta^k))$. It follows from equation (2.1) that

$$\frac{m(B(z_\beta^k, \delta^n))}{m(B(x, \delta^n))} \leq \frac{m(B(x, \delta^n + d(Q_\beta^k)))}{m(B(x, \delta^n))} \leq 2^\epsilon (K+1).$$

Combining the above inequality with condition (1.2), we have

$$\begin{aligned}
&\frac{m(B(x, \delta^n) \triangle B(z_\beta^k, \delta^n))}{m(B(x, \delta^n))} \\
&\leq \frac{m(B(x, d(x, z_\beta^k) + \delta^n) \setminus B(x, \delta^n))}{m(B(x, \delta^n))} \\
&\quad + \frac{m(B(z_\beta^k, d(x, z_\beta^k) + \delta^n) \setminus B(z_\beta^k, \delta^n))}{m(B(z_\beta^k, \delta^n))} \frac{m(B(z_\beta^k, \delta^n))}{m(B(x, \delta^n))} \\
&\leq (K + 2^\epsilon K(K+1)) \left(\frac{d(x, z_\beta^k)}{\delta^n} \right)^\epsilon.
\end{aligned}$$

Therefore,

$$|A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)| \leq 2(K + 2^\epsilon K(K+1)) \left(\frac{d(Q_\beta^k)}{\delta^n} \right)^\epsilon \|f\|_{L^\infty}. \tag{5.3}$$

By the above inequality, we have

$$\begin{aligned}
\left(\sum_{n > n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2 \right)^{1/2} &\leq 2(K + 2^\epsilon (K+1)) \|f\|_{L^\infty} \sum_{n : \delta^n > d(Q_\beta^k)} \left(\frac{d(Q_\beta^k)}{\delta^n} \right)^\epsilon \\
&\leq \frac{2(K + 2^\epsilon K(K+1))}{1 - \delta^{-\epsilon}} \|f\|_{L^\infty}.
\end{aligned}$$

Case $n_3 < n_4$. Note that $d(Q_\beta^k) \geq \delta^{n_3 - 1}$. It follows from Proposition 2.2(iv) that

$$a_0 \delta^k \geq (a_0 / 2C_1) \delta^{(n_3 - 1)} > \delta^{k_3} \geq r_0$$

and $f_2 = f \mathbb{1}_{G \setminus \widetilde{Q_\beta^k}}$. We claim that for $n_3 < n \leq n_4$,

$$A'_{\delta^n} f_2(x) = 0 \quad \text{for all } x \in Q_\beta^k. \tag{5.4}$$

Indeed, fix $x \in Q_\beta^k$ and let $y \in B(x, \delta^n)$, then $d(y, z_\beta^k) \leq d(y, x) + d(z_\beta^k, x) \leq \delta^n + C_1 \delta^k \leq d(Q_\beta^k) + C_1 \delta^k \leq 3C_1 \delta^k$. From this, we have $B(x, \delta^n) \subset \widetilde{Q}_\beta^k$. Then, the claim follows from the fact that f_2 is supported on $G \setminus \widetilde{Q_\beta^k}$.

Combining equation (5.4) with a similar argument as in the previous proof, we also have

$$\left(\sum_{n > n_3} |A'_{\delta^n} f_2(x) - A'_{\delta^n} f_2(z_\beta^k)|^2 \right)^{1/2} \leq \frac{2(K + 2^\epsilon K(K+1))}{1 - \delta^{-\epsilon}} \|f\|_{L^\infty} \quad \text{for all } x \in Q_\beta^k,$$

and the proof is complete. □

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The proof of $(L_c^\infty, \mathrm{BMO})$ boundedness of SV is similar to S. Let us give a sketch of the proof.

Proof of equation (1.13). Let f be a L_c^∞ function. Recall that the short variation operator is defined by $SV(f) = (\sum_{n\geq n_{r_0}} V_2(A'_r f : r \in [\delta^n, \delta^{n+1}))^2)^{1/2}$. Fix a 'dyadic cube' Q_β^k and let $= f\mathbb{1}_{Q_\beta^{k*}} + f\mathbb{1}_{G\setminus Q_\beta^{k*}} := f_1 + f_2$. We only state the proof for the case where $n_3 \geq n_4$, namely $\delta^{n_3} > d(Q_\beta^k)$, since the proof for the case where $n_3 < n_4$ is similar. By an argument similar to that in the proof for operator S, it is sufficient to prove that there exists a constant $C_{\epsilon,\delta,K} > 0$ such that for all $x \in Q_\beta^k$,

$$\left(\sum_{n>n_3} V_2(A'_r f_2(x) - A'_r f_2(z_\beta^k) : r \in [\delta^n, \delta^{n+1}))^2 \right)^{1/2} \leq C_{\epsilon,\delta,K} \|f\|_{L^\infty}.$$

Fix $n > n_3$ and $x \in Q_\beta^k$. By the definition of short variation operator, there exists a sequence $\{r_i\} \subseteq [\delta^n, \delta^{n+1})$ such that

$$\begin{aligned} &V_2(A'_r f_2(x) - A'_r f_2(z_\beta^k) : r \in [\delta^n, \delta^{n+1})) \\ &\quad \leq 2\left(\sum_i |A'_{r_i} f_2(x) - A'_{r_i} f_2(z_\beta^k) - (A'_{r_{i-1}} f_2(x) - A'_{r_{i-1}} f_2(z_\beta^k))|^2 \right)^{1/2}. \end{aligned} \tag{5.5}$$

We split such sequence $\{r_i\}$ into two cases: $J_1 = \{i : r_i - r_{i-1} \leq d(Q_\beta^k)^\epsilon / \delta^{(\epsilon-1)n}\}$ and $J_2 = \{i : r_i - r_{i-1} > d(Q_\beta^k)^\epsilon / \delta^{(\epsilon-1)n}\}$. Therefore, the right-hand side of the above inequality is controlled by the sum of

$$SV_{I_n}(f_2)(x) = \left(\sum_{i:i\in J_1} |A'_{r_i} f_2(x) - A'_{r_i} f_2(z_\beta^k) - (A'_{r_{i-1}} f_2(x) - A'_{r_{i-1}} f_2(z_\beta^k))|^2 \right)^{1/2}$$

and

$$SV_{II_n}(f_2)(x) = \left(\sum_{i:i\in J_2} |A'_{r_i} f_2(x) - A'_{r_i} f_2(z_\beta^k) - (A'_{r_{i-1}} f_2(x) - A'_{r_{i-1}} f_2(z_\beta^k))|^2 \right)^{1/2}.$$

We now focus on $SV_{I_n}(f_2)(x)$. By the triangle inequality, we have

$$\begin{aligned} SV_{I_n}(f_2)(x) \leq &\left(\sum_{i:i\in J_1} |A'_{r_i} f_2(x) - A'_{r_{i-1}} f_2(x)|^2 \right)^{1/2} \\ &+ \left(\sum_{i:i\in J_1} |A'_{r_i} f_2(z_\beta^k) - A'_{r_{i-1}} f_2(z_\beta^k)|^2 \right)^{1/2}. \end{aligned}$$

Note that for any $z \in Q_\beta^k$, we have

$$\begin{aligned} |A'_{r_i} f_2(z) - A'_{r_{i-1}} f_2(z)| \leq &\frac{1}{m(B(z, r_i))} \int_{B(z,r_i)\setminus B(z,r_{i-1})} |f_2(y)|\, dm(y) \\ &+ \left(\frac{1}{m(B(z, r_{i-1}))} - \frac{1}{m(B(z, r_i))} \right) \int_{B(z,r_{i-1})} |f_2(y)|\, dm(y) \end{aligned}$$

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$$\leq \frac{m(B(z,r_i)) - m(B(z,r_{i-1}))}{m(B(z,r_i))}\|f_2\|_{L^\infty} + \left(\frac{m(B(z,r_{i-1}))}{m(B(z,r_{i-1}))} - \frac{m(B(z,r_{i-1}))}{m(B(z,r_i))}\right)\|f_2\|_{L^\infty}$$
$$\leq 2\left(\frac{m(B(z,r_i)) - m(B(z,r_{i-1}))}{m(B(z,r_i))}\right)\|f_2\|_{L^\infty}$$
$$\leq 2\|f\|_{L^\infty}\int_{m(B(z,r_{i-1}))}^{m(B(z,r_i))} \frac{1}{u}\,du$$

by condition (1.2), and the integral term of the above inequality is controlled by

$$\int_{m(B(z,r_{i-1}))}^{m(B(z,r_i))} \frac{1}{u}\,du \leq (m(B(z,r_i)) - m(B(z,r_{i-1})))^{1/2}\left(\int_{m(B(z,r_{i-1}))}^{m(B(z,r_i))} \frac{1}{u^2}\,du\right)^{1/2}$$
$$\leq \left(\frac{m(B(z,r_i)) - m(B(z,r_{i-1}))}{m(B(z,r_{i-1}))}\right)^{1/2}\left(m(B(z,r_{i-1}))\int_{m(B(z,r_{i-1}))}^{m(B(z,r_i))} \frac{1}{u^2}\,du\right)^{1/2}$$
$$\leq K\left(\frac{r_i - r_{i-1}}{r_{i-1}}\right)^{\epsilon/2}\left(\int_{m(B(z,r_{i-1}))}^{m(B(z,r_i))} \frac{m(B(z,\delta^{n+1}))}{u^2}\,du\right)^{1/2}.$$

Then, using equation (2.1), the above inequality yields

$$\left(\sum_{i:i\in J_1} |A'_{r_i}f_2(x) - A'_{r_{i-1}}f_2(x)|^2\right)^{1/2}$$
$$\leq 2K\|f\|_{L^\infty}\left(\sum_{i:i\in J_1}\left(\frac{r_i - r_{i-1}}{r_{i-1}}\right)^{\epsilon}\int_{m(B(x,r_{i-1}))}^{m(B(x,r_i))} \frac{m(B(x,\delta^{n+1}))}{u^2}\,du\right)^{1/2}$$
$$\leq 2K\|f\|_{L^\infty}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon^2/2}\left(\int_{m(B(x,\delta^n))}^{m(B(x,\delta^{n+1}))} \frac{m(B(x,\delta^{n+1}))}{u^2}\,du\right)^{1/2}$$
$$\leq 2K((K+1)\delta^\epsilon)^{1/2}\|f\|_{L^\infty}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon^2/2}.$$

By a similar argument, we have

$$\left(\sum_{i:i\in J_1} |A'_{r_i}f_2(z_\beta^k) - A'_{r_{i-1}}f_2(z_\beta^k)|^2\right)^{1/2} \leq 2K((K+1)\delta^\epsilon)^{1/2}\|f\|_{L^\infty}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon^2/2}.$$

Therefore,

$$SV_{I_n}(f_2)(x) \leq 4K((K+1)\delta^\epsilon)^{1/2}\|f\|_{L^\infty}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon^2/2}.$$

For the part $SV_{II_n}(f_2)(x)$, using the triangle inequality, we first have

$$SV_{II_n}(f_2)(x) \leq \left(\sum_{i:i\in J_2} |A'_{r_i}f_2(x) - A'_{r_i}f_2(z_\beta^k)|^2\right)^{1/2}$$
$$+ \left(\sum_{i:i\in J_2} |A'_{r_{i-1}}f_2(x) - A'_{r_{i-1}}f_2(z_\beta^k)|^2\right)^{1/2}.$$

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Note that $\#\{J_2\} \le (\delta-1)\delta^{\epsilon n}/d(Q_\beta^k)^\epsilon$. Similar to equation (5.3), for any $z \in Q_\beta^k$ and $r \in [\delta^n, \delta^{n+1})$, we have

$$|A_r' f_2(z) - A_r' f_2(z_\beta^k)| \le 2(K + 2^\epsilon K(K+1))\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^\epsilon \|f\|_{L^\infty}.$$

Combining the above inequalities, we have

$$\begin{aligned} SV_{II_n}(f_2)(x) &\le 4(K + 2^\epsilon K(K+1))\|f\|_{L^\infty}\left((\delta-1)\left(\frac{\delta^n}{d(Q_\beta^k)}\right)^\epsilon\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{2\epsilon}\right)^{1/2} \\ &= 4(K + 2^\epsilon K(K+1))(\delta-1)^{1/2}\|f\|_{L^\infty}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon/2}. \end{aligned}$$

Finally, together, the estimates of $SV_{I_n}(f_2)(x)$ and $SV_{II_n}(f_2)(x)$ with equation (5.5) give

$$\begin{aligned} &\left(\sum_{n>n_3} V_2(A_r' f_2(x) - A_r' f_2(z_\beta^k) : r \in [\delta^n, \delta^{n+1}))^2\right)^{1/2} \\ &\quad\le 8(K + 2^\epsilon K(K+1))(\delta-1)^{1/2}\|f\|_{L^\infty}\left(\sum_{n:\delta^n > d(Q_\beta^k)}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^\epsilon\right)^{1/2} \\ &\qquad + 8K((K+1)\delta^\epsilon)^{1/2}\|f\|_{L^\infty}\left(\sum_{n:\delta^n > d(Q_\beta^k)}\left(\frac{d(Q_\beta^k)}{\delta^n}\right)^{\epsilon^2}\right)^{1/2} \\ &\quad\le 8(K + 2^\epsilon K(K+1))\left(\frac{\delta-1}{1-\delta^{-\epsilon}}\right)^{1/2}\|f\|_{L^\infty} + 8K\left(\frac{(K+1)\delta^\epsilon}{1-\delta^{-\epsilon^2}}\right)^{1/2}\|f\|_{L^\infty}, \end{aligned}$$

and the proof is complete. □

6. *Transference principles*

In this section, we establish the transference principles for the jump operator. Recall that a sequence of compact sets $\{F_n\}_{n\in\mathbb{N}}$ with positive measures in a locally compact group G is called a Følner sequence if for every $g \in G$,

$$\lim_i \frac{m((F_n g)\triangle F_n)}{m(F_n)} = 0,$$

or equivalently, for all compact set K in G,

$$\lim_n \frac{m(F_n K)}{m(F_n)} = 1. \tag{6.1}$$

A group G is called amenable if it admits such a Følner sequence. It is well known that if G is a group with polynomial volume growth, then it is amenable (cf. [25]) and, in particular, the family of balls $\{B_r\}_{r>0}$ generated by any word metric on G is a Følner sequence (cf. [10, 53, 62]). For more information about Følner sequences and amenable groups, we refer the reader to [54].

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6.1. *Weak type inequalities.* We now prove Theorem 1.10*(i)*.

Proof of Theorem 1.10(i). We only give the proof of transference principle for weak type inequalities, since the strong type one can be done verbatim. Let $p \in [1, \infty)$. Given an $f \in L^p(X, \mu)$, let T be an action induced by a μ-preserving measurable transformation τ, that is, for all $g \in G$, $T_g f(x) = f(\tau_{g^{-1}} x)$. Fix $x \in X$ and a compact set A. Define

$$F_A(g) = \begin{cases} T_g f(x), & g \in A; \\ 0, & g \notin A. \end{cases}$$

Let N be an integer large enough and K a compact set such that for $r \leq N$, we have $B_r \subseteq K$. Clearly, $F_{AK}(g) = T_g f(x) \mathbb{1}_{AK}(g)$. Moreover, if $h \in A$ and $k \in K$, we have

$$T_h T_k f(x) = T_{hk} f(x) = F_{AK}(hk).$$

It follows that for all $h \in A$ and $r \in (0, N] \cap \mathcal{I}$, we have

$$T_h A_r f(x) = \frac{1}{m(B_r)} \int_{B_r} T_{hg} f(x) \mathbb{1}_{AK}(hg)\, dm(g) = \frac{1}{m(B_r)} \int_{B_r} F_{AK}(hg)\, dm(g). \tag{6.2}$$

Let $\mathbf{A}_N f = \{A_r f : r \in (0, N] \cap \mathcal{I}\}$ and $\mathbf{A}'_N F = \{A'_r F : r \in (0, N] \cap \mathcal{I}\}$. For all $h \in G$, set $T_h \mathbf{A}_N f = \{T_h A_r f : r \in (0, N] \cap \mathcal{I}\}$; here and subsequently, for a sequence of measurable functions $\alpha = \{\alpha_r : r \in \mathcal{I}\}$, $T\alpha$ stands for $\{T\alpha_r : r \in \mathcal{I}\}$. From equation (6.2), we have

$$\lambda \sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)} = \lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N F_{AK})(h)}. \tag{6.3}$$

Fix $\gamma > 0$ and define the set

$$\mathcal{D}(\gamma) = \left\{(h, x) \in A \times X : \lambda \sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)} > \gamma\right\}.$$

Fix $h \in A$ and define the set

$$\mathcal{D}^h(\gamma) = \left\{x \in X : \lambda \sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)} > \gamma\right\}.$$

Since $T_h(\mathbb{1}_{\mathcal{D}^e(\gamma)})(x) = \mathbb{1}_{\mathcal{D}^h(\gamma)}(x)$, it follows that

$$\int_X \mathbb{1}_{\mathcal{D}^h(\gamma)}(x)\, d\mu(x) = \int_X T_h(\mathbb{1}_{\mathcal{D}^e(\gamma)})(x)\, d\mu(x) = \int_X \mathbb{1}_{\mathcal{D}^e(\gamma)}(x)\, d\mu(x). \tag{6.4}$$

However, fix $x \in X$ and define the set

$$\mathcal{D}_x(\gamma) = \left\{h \in A : \lambda \sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)} > \gamma\right\}.$$

By equation (6.3), one can see that

$$\mathcal{D}_x(\gamma) = \left\{h \in A : \lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N F_{AK})(h)} > \gamma\right\}.$$

Moreover, using the assumption that the jump operator is of weak type (p, p), we have

$$m(\mathcal{D}_x(\gamma)) \leq \frac{\|\lambda \sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p \to L^{p,\infty}}}{\gamma^p} \|F_{AK}\|^p_{L^p(G,m)}.$$

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It follows from the above inequality that

$$\int_X\int_A \mathbb{1}_{\mathcal{D}_x(\gamma)}(h)\,dm(h)\,d\mu(x) \le \frac{\|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p\to L^{p,\infty}}}{\gamma^p}\int_X\int_G |F_{AK}(h)|^p\,dm(h)\,d\mu(x)$$
$$= \frac{\|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p\to L^{p,\infty}}}{\gamma^p} m(AK)\int_X |f(x)|^p\,d\mu(x). \tag{6.5}$$

By the Fubini theorem, we can see that

$$\int_X\int_A \mathbb{1}_{\mathcal{D}_x(\gamma)}(h)\,dm(h)\,d\mu(x) = \int_{G\times X}\mathbb{1}_{\mathcal{D}(\gamma)}(h,x)\,dm(h)\,d\mu(x)$$
$$= \int_A\int_X \mathbb{1}_{\mathcal{D}^h(\gamma)}(x)\,d\mu(x)\,dm(h). \tag{6.6}$$

Using equation (6.4), we have

$$\int_A\int_X \mathbb{1}_{\mathcal{D}^h(\gamma)}(x)\,d\mu(x)\,dm(h) = m(A)\int_X \mathbb{1}_{\mathcal{D}^e(\gamma)}(x)\,d\mu(x)$$
$$= m(A)\mu(\{x\in X : \lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)(x)} > \gamma\}).$$

Combining equations (6.5), (6.6) with the above inequality, we conclude

$$\mu(\{x\in X : \lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)(x)} > \gamma\}) \le \frac{\|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p\to L^{p,\infty}}}{\gamma^p}\frac{m(AK)}{m(A)}\int_X |f(x)|^p\,d\mu(x).$$

Since G is an amenable group, by equation (6.1), for any $\varepsilon > 0$, we can choose the above subset A such that $m(AK)/m(A) \le (1+\varepsilon)$. By the arbitrariness of ε and the monotone convergence theorem, letting $N\to\infty$, we have

$$\mu(\{x\in X : \lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}f)(x)} > \gamma\}) \le \frac{\|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}\|^p_{L^p\to L^{p,\infty}}}{\gamma^p}\int_X |f(x)|^p\,d\mu(x),$$

which is the desired conclusion. □

6.2. *Strong type inequalities.* In this subsection, we assume that the action T is a strongly continuous regular action of G on $L^p(X,\mu)$. Before stating the result to be proved, we give some notation and lemmas. The following lemma was proved in [51].

LEMMA 6.1. *Let $\alpha = \{\alpha_r(x), r\in\mathcal{I}\}$ be a sequence of measurable functions on measurable space (X,μ). For every $p\in(1,\infty)$ and $\theta\in(0,1)$, there exists a positive constant $c_{p,\theta}$ such that*

$$c^{-1}_{p,\theta}\sup_{\lambda>0}\|\lambda\sqrt{\mathcal{N}_\lambda(\alpha)}\|_{L^p} \le [L^\infty(X;V_\infty), L^{\theta p}(X;V_{2\theta})]_{\theta,\infty}(\alpha) \le c_{p,\theta}\sup_{\lambda>0}\|\lambda\sqrt{\mathcal{N}_\lambda(\alpha)}\|_{L^p}.$$

Moreover, if $\max\{1/p, 1/2\} < \theta < 1$, then the vector-valued interpolation space $[L^\infty(X;V_\infty), L^{\theta p}(X;V_{2\theta})]_{\theta,\infty}$ admits an equivalent norm; in particular, if $p>1$, $\sup_{\lambda>0}\|\lambda\sqrt{\mathcal{N}_\lambda(\cdot)}\|_{L^p}$ admits an equivalent norm.

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See [51] for the definition of vector-valued interpolation spaces and more details about jump quasi-norms.

An operator $T : L^p(X, \mu) \to L^p(X, \mu)$ is called regular if there exists a constant $C > 0$ such that

$$\| \sup_{k\geq 1} |T(f_k)| \|_{L^p} \leq C \| \sup_{k\geq 1} |f_k| \|_{L^p} \tag{6.7}$$

for any finite sequence $\{f_k : k \geq 1\}$ in $L^p(X, \mu)$. Let us denote by $\|T\|_r$ the smallest C for which this holds. Let $\mathfrak{B}$ be a Banach space. If T is a regular operator on $L^p(X, \mu)$, then the tensor product operator $T \otimes \mathrm{id}_{\mathfrak{B}} : L^p(X, \mu) \otimes \mathfrak{B} \to L^p(X, \mu) \otimes \mathfrak{B}$ extends to a bounded operator $\widetilde{T \otimes \mathrm{id}_{\mathfrak{B}}}$ from the Bochner space $L^p(X; \mathfrak{B})$ to $L^p(X; \mathfrak{B})$, and

$$\|\widetilde{T \otimes \mathrm{id}_{\mathfrak{B}}}\|_{L^p(X;\mathfrak{B})\to L^p(X;\mathfrak{B})} \leq \|T\|_r. \tag{6.8}$$

For more information on regular operators, we refer the reader to [55]. A group action T of G on $L^p(X, \mu)$ is called regular if for any $g \in G$, T_g is regular and $\sup_{g\in G} \|T_g\|_r < \infty$.

Moreover, using Lemma 6.1 together with equation (6.8), one can obtain the following lemma.

LEMMA 6.2. *Fix $p \in (1, \infty)$. Let $T : L^p(X, \mu) \to L^p(X, \mu)$ be a regular operator. Given a sequence of measurable functions $\alpha = \{\alpha_r(x) : r \in \mathcal{I}\}$ in $L^p(X, \mu)$, there exists a constant $c_p > 0$ such that*

$$\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(T\alpha)}\|_{L^p} \leq c_p \|T\|_r \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\alpha)}\|_{L^p}.$$

In what follows, we state the strong type (p, p) transference principle for strongly continuous regular group actions.

Proof of Theorem 1.10(ii). Let $p \in (1, \infty)$ and $f \in L^p(X, \mu)$. Fix $x \in X$ and a compact set A, and define $F_A(g) = T_g f(x) \mathbb{1}_A(g)$. Let N be an integer large enough and K a compact set such that for every $r \leq N$, we have $B_r \subseteq K$. Clearly, $F_{AK}(h) = T_h f(x) \mathbb{1}_{AK}(h)$.

Keeping the notation introduced in the proof of weak type inequalities and using equation (6.3), we have

$$\begin{aligned}\int_A |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, dm(h) &= \int_A |\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N F_{AK})(h)}|^p \, dm(h) \\ &\leq \int_G |\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N F_{Ak})(h)}|^p \, dm(h).\end{aligned}$$

Using the strong type (p, p) jump inequality which is assumed for the translation action, we obtain

$$\begin{aligned}\int_A |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, dm(h) &\leq \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p\to L^p} \int_G |F_{Ak}(h)|^p \, dm(h) \\ &= \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p\to L^p} \int_{AK} |T_h f(x)|^p \, dm(h).\end{aligned}$$

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Moreover, integrating both sides of the above inequality over X and using the Fubini theorem, we have

$$\begin{aligned}
&\int_X \int_A |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, dm(h) \, d\mu(x) \\
&\quad \le \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p \to L^p} \int_{AK} \int_X |T_h f(x)|^p \, d\mu(x) \, dm(h) \\
&\quad = \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p \to L^p} \sup_{h\in G} \|T_h\|_r^p m(AK) \int_X |f(x)|^p \, d\mu(x). \qquad (6.9)
\end{aligned}$$

However, by the assumption that T is a strongly continuous regular action of G on $L^p(X, \mu)$ and Lemma 6.2, we have

$$\begin{aligned}
\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)}\|_{L^p} &= \inf_{h\in A} \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(T_{h^{-1}} T_h \mathbf{A}_N f)}\|_{L^p} \\
&\le c_p \sup_{h\in G} \|T_h\|_r \inf_{h\in A} \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)}\|_{L^p}.
\end{aligned}$$

By the above inequality, we have

$$\begin{aligned}
&\sup_{\lambda>0} \int_X |\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)(x)}|^p \, d\mu(x) = \frac{1}{m(A)} \int_A \sup_{\lambda>0} \int_X |\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)(x)}|^p \, d\mu(x) \, dm(h) \\
&\le c_p \sup_{h\in G} \|T_h\|_r^p \frac{1}{m(A)} \int_A \sup_{\lambda>0} \inf_{h\in A} \int_X |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, d\mu(x) \, dm(h) \\
&= c_p \sup_{h\in G} \|T_h\|_r^p \sup_{\lambda>0} \frac{1}{m(A)} \int_A \inf_{h\in A} \int_X |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, d\mu(x) \, dm(h) \\
&\le c_p \sup_{h\in G} \|T_h\|_r^p \sup_{\lambda>0} \frac{1}{m(A)} \int_A \int_X |\lambda\sqrt{\mathcal{N}_\lambda(T_h \mathbf{A}_N f)(x)}|^p \, d\mu(x) \, dm(h).
\end{aligned}$$

Using the Fubini theorem and equation (6.9), the above inequality yields

$$\begin{aligned}
&\sup_{\lambda>0} \int_X |\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}_N f)(x)}|^p \, d\mu(x) \\
&\quad \le c_p \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}'_N)}\|^p_{L^p \to L^p} \sup_{h\in G} \|T_h\|_r^{2p} \frac{m(AK)}{m(A)} \int_X |f(x)|^p \, d\mu(x).
\end{aligned}$$

By equation (6.1) and a similar argument as in the proof of weak type inequalities, letting $N \to \infty$, we have

$$\sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A} f)}\|_{L^p} \le c_p \sup_{\lambda>0} \|\lambda\sqrt{\mathcal{N}_\lambda(\mathbf{A}')}\|_{L^p \to L^p} \sup_{h\in G} \|T_h\|_r^2 \|f\|_{L^p},$$

which is the desired conclusion. □

7. *Annular decay property*

In this section, we discuss the annular decay property. We first recall the $(\epsilon, 1)$-annular decay property of word metrics and verify that this property is stable under $(1, C)$-quasi isometry (recalled below), and thus obtain the quantitative ergodic theorems, including Theorem 1.2, on the polynomial growth group equipped with a metric that is $(1, C)$-quasi isometric to a word metric. We then check that all the known examples of a periodic metric,

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which was introduced by Breuillard [10], satisfy some (ϵ, r_0)-annular decay property. At the moment of writing, we do not know how to verify this property for all periodic metrics.

Let G be a polynomial growth group with a symmetric compact generating set V. Recall that the word metric d is defined by

$$\text{for all } x, y \in G, \quad d(x, y) = \inf\{n \in \mathbb{N}, x^{-1}y \in V^n\}.$$

It is clear that d is a (left-) invariant metric on G. Let $r > 0$ and B_r be the ball generated by word metric d of radius r in G. It is well known that there exist two constants $C_V > 0$ and $D_G > 0$ such that for every $r \in (0, \infty)$,

$$C_V^{-1} r^{D_G} \leq m(B_r) \leq C_V r^{D_G}. \tag{7.1}$$

See [23, 48]. From the above inequality, it is easy to check that (G, d, m) satisfies the measure doubling condition, so it follows that such (G, d, m) satisfies condition (1.1). By an argument the same as in the proof of [62, Theorem 4], we have the following proposition.

PROPOSITION 7.1. *Let G be a polynomial growth group with a symmetric compact generating set V and $\{B_r\}_{r>0}$ be the balls given by the corresponding word metric. Then, there exist two constants $\theta = \log_2(1 + 1/C_V^2 10^{D_G})$ and $c_V = (1 + 1/C_V^2 10^{D_G})^3$ such that for all $r \in [1, \infty)$ and $s \in (0, r]$,*

$$m(B_{r+s} \setminus B_r) \leq c_V \left(\frac{s}{r}\right)^{\theta} m(B_r). \tag{7.2}$$

Remark 7.2. In terms of our terminology, the above ball annular decay property is just the $(\epsilon, 1)$-annular decay property in equation (1.2) of (G, d, m). Note that all the groups of polynomial growth are amenable (cf. [25]), and thus Theorem 1.2 follows from Theorem 1.1 by the transference principles.

In fact, in [62, Theorem 4], the $(\epsilon, 1)$-annular decay property is established for every metric measure space satisfying the measure doubling condition and Property (M). Recall that a metric space $(\mathcal{X}, d, \mu)$ is said to satisfy Property (M) if there exists a constant $C > 0$ such that the Hausdorff distance between any pair of balls with the same centre and any radii between r and $r + 1$ is less than C. In other words, for all $x \in \mathcal{X}$, $r > 0$ and $y \in B(x, r + 1)$, we have $d(y, B(x, r)) \leq C$. Property (M) is equivalent to the property: there exists a constant $C < \infty$ such that for all $r > 0$, $s \geq 1$, $y \in B(x, r + s)$,

$$d(y, B(x, r)) \leq Cs; \tag{7.3}$$

and is also equivalent to that the metric space $(\mathcal{X}, d, \mu)$ admits monotone geodesics, see e.g. [62, Proposition 2] for the relevant definitions and proofs. Moreover, Property (M) is invariant under Hausdorff equivalence but unstable under quasi-isometry in the sense of [62, p. 50], where one can find the relevant counterexamples.

We prove that Property (M) is invariant under the $(1, C)$-quasi-isometry. Two metrics d_1 and d_2 on $\mathcal{X}$ are called $(1, C)$-quasi-isometric if there exists a constant $C > 0$ such that for any $x, y \in \mathcal{X}$, $|d_1(x, y) - d_2(x, y)| \leq C$.

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PROPOSITION 7.3. *Let d_1 and d_2 be two metrics on $\mathcal{X}$. Assume that d_1 is $(1, C_{\mathcal{X}})$-quasi-isometric to d_2 and $(\mathcal{X}, d_1)$ satisfies Property (M), then $(\mathcal{X}, d_2)$ satisfies Property (M).*

Proof. Fix a point $x \in \mathcal{X}$, $s \geq 1$ and $r > 0$. We denote by $B_1(x, r)$ and $B_2(x, r)$ the balls generated by d_1 and d_2, respectively. Since d_1 is $(1, C_{\mathcal{X}})$-quasi-isometric to d_2, it follows that $B_2(x, r) \subseteq B_1(x, r + C_{\mathcal{X}})$. Let $z \in B_2(x, r + s)$. We split r into two cases: $r > C_{\mathcal{X}}$ and $r \leq C_{\mathcal{X}}$. For $r > C_{\mathcal{X}}$, we have $B_1(x, r - C_{\mathcal{X}}) \subseteq B_2(x, r)$ and it follows that

$$d_2(z, B_2(x, r)) \leq d_1(z, B_1(x, r - C_{\mathcal{X}})) + C_{\mathcal{X}}.$$

Combining equation (7.3) for d_1 with the fact that $s \geq 1$ and $z \in B_1(x, r + s + C_{\mathcal{X}})$, the above inequality yields

$$d_2(z, B_2(x, r)) \leq C(s + 2C_{\mathcal{X}}) + C_{\mathcal{X}} \leq (C + 2CC_{\mathcal{X}} + C_{\mathcal{X}})s.$$

For $r \leq C_{\mathcal{X}}$, it is easy to check that $d_2(z, B_2(x, r)) \leq r + s \leq (C_{\mathcal{X}} + 1)s$. Combining this case with the case $r > C_{\mathcal{X}}$, we prove that $(\mathcal{X}, d_2)$ satisfies equation (7.3), which completes the proof. □

Remark 7.4. By the above observations, we can see that any left-invariant metric, defined on a polynomial volume growth group G which is $(1, C)$-quasi-isometric to a word metric, satisfies the $(\epsilon, 1)$-annular decay property. Thus, Theorem 1.2 holds for any metric that is $(1, C)$-quasi-isometric to a word metric.

Motivated by the notion—Property (M)—we can introduce Property (M_{r_0}), namely there exists a positive constant $C < \infty$ such that the Hausdorff distance between any pair of balls with the same centre and any radii belonging to $[r, r + 1]$ with $r > r_0$ is less than C. Similar to [62, Theorem 4], one can show that for a doubling metric measure space (X, d, μ) with Property (M_{r_0}), then there exists $\theta > 0$ and a constant $C > 0$ such that for all $x \in X$, $r \in [r_0, \infty)$ and $s \in (0, r]$, we have $\mu(B_{r+s} \setminus B_r) \leq C(s/r)^{\theta}\mu(B_r)$. As shown in Proposition 7.3, Property (M_{r_0}) is also stable under $(1, C)$-quasi-isometry.

If a metric measure space satisfies the (ϵ, r_0)-annular decay property for all $r_0 > 0$, then we call it as satisfying the ϵ-annular decay property. This terminology, to our best knowledge, was introduced by Buckley [11, (1.1)] on a metric space. A slight variant on a manifold was introduced by Colding and Minicozzi [18], which they called the ϵ-volume regularity property. In recent years, the ϵ-annular decay property has been widely exploited in harmonic analysis, see [3, 4, 19, 37, 38, 45, 65] for more details.

The following example tells us that there are numerous metric measure spaces only satisfying the (ϵ, r_0)-annular decay property for some r_0, but not all $r_0 > 0$.

Example 7.5. Fix a positive integer r_0. Let $\mathcal{X} = \mathbb{Z}$ endowed with the counting measure μ. The metric d is given by

$$d(x, y) = \begin{cases} 0, & x = y, \\ r_0/2, & 0 < |x - y| \leq r_0/2, \\ \max\{|x - y|, r_0\}, & |x - y| > r_0/2. \end{cases}$$

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One can check that for any $k \in (-\infty, -1) \cap \mathbb{Z}$, $\mu(B(0, r_0) \setminus B(0, r_0 - 2^k)) = r_0$, it follows that this metric space does not satisfy the ϵ-annular decay property, but satisfies the (ϵ, r_0)-annular decay property.

In [11], Buckley proved that the metric space $(\mathcal{X}, d, \mu)$ has the ϵ-annular decay property when it satisfies the measure doubling condition and the (α, β)-chain ball property. Lin, Nakai and Yang [45] established the ϵ-annular decay property for the metric space $(\mathcal{X}, d, \mu)$ if $(\mathcal{X}, d, \mu)$ satisfies the measure doubling condition and the weak geodesic property (that is, equation (7.3) holds for all $s > 0$). Actually, Lin *et al* proved that the weak geodesic property is equivalent to the (α, β)-chain ball property and is also equivalent to the monotone geodesic property. A typical class of metric spaces having the (α, β)-chain ball property (or the weak geodesic property) is the length spaces. The following proposition was established in [11, Corollary 2.2].

PROPOSITION 7.6. *If $(\mathcal{X}, d, \mu)$ is a length space (a metric space in which the distance between any two points is the infimum of the lengths of all curves joining the two points) and satisfies the measure doubling condition, then $(\mathcal{X}, d, \mu)$ satisfies the ϵ-annular decay property.*

Basic examples of length spaces include the graph which is defined in [62, p. 51], the homogeneous groups endowed with homogeneous metrics (such as $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$), the Riemannian metric spaces, the Finsler metric spaces, the sub-Riemannian metric and the sub-Finsler metric spaces (also called the Carnot–Carathéodory metric spaces). For more examples, we refer the reader to the book [24].

In addition to length spaces, the RD-spaces with condition (H_α), $\alpha \in (0, 1]$ satisfy also the ϵ-annular decay property, see e.g. [45, Example 4.1(iii)] for the assertion and the related notions.

Finally, we focus on the periodic metric spaces. Recall that a pseudodistance d on locally compact group G is called a periodic metric if it satisfies the following properties:

(i) d is invariant under left translations by a closed co-compact subgroup H, meaning that for all $x, y \in G$ and all $h \in H$, $d(hx, hy) = d(x, y)$;
(ii) d is locally bounded and proper;
(iii) d is asymptotically geodesic.

For more information about the periodic metrics, we refer the reader to [10, §4]. Moreover, in [10], the following result is proved. There exist two constants $d_G > 0$ and $c_d > 0$ such that

$$\lim_{r \to \infty} \frac{m(B_r)}{r^{d_G}} = c_d,$$

where B_r is a ball given by the periodic metric of radius r in a polynomial growth group G. From the above identity, we can see that equipped with a left-invariant periodic metric d, (G, d, m) satisfies the doubling measure condition and equation (1.14). However, condition (1.2) seems quite inaccessible from the above estimate. Nevertheless, all the periodic metrics provided in [10, §4] satisfy condition (1.2).

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Example 7.7.

(i) Let G be a polynomial volume growth group with a compact symmetric generating set. The corresponding word metric is a periodic metric.

(ii) Let G be a simply connected nilpotent Lie group. Let Γ be a finitely generated torsion free nilpotent group which is embedded as a co-compact discrete subgroup of G. We denote by V the generating set of Γ and d_V the word metric on Γ. The metric d on G which is defined by $d(x, y) = d_V(h_x, h_y)$ is a periodic metric, where $x \in h_x F$ and $y \in h_y F$ and F is some fixed fundamental domain for Γ in G.

(iii) Let G/Γ be a nilmanifold with universal cover G and fundamental group Γ. Let d be a Riemannian metric on G/Γ. The Riemannian metric $\tilde{d}$ on G, which is extended by Riemannian metric d, is a periodic metric.

(iv) Let G be a connected Lie group. The left-invariant Carnot–Carathéodory metric or left-invariant Riemannian metric on G is a periodic metric.

Note that examples (i) and (ii) are the spaces endowed with word metrics, then by Proposition 7.1, such spaces satisfy the $(\epsilon, 1)$-annular decay property; examples (iii) and (iv) are actually length spaces, and so by Proposition 7.6, such spaces satisfy the ϵ-annular decay property.

Based on the above examples, one cannot expect the ϵ-annular decay property for all periodic metrics, but the following conjecture is still expected; if this were the case, we would be able to obtain the quantitative ergodic theorems for all periodic metrics on polynomial growth groups.

Question 7.8. Let G be a polynomial growth group endowed with a left-invariant periodic metric d and a Haar measure m. Then, there exists one $r_0 > 0$ such that (G, d, m) satisfies the (ϵ, r_0)-annular decay property.

Combining Proposition 7.3 with [**10**, proof of Corollary 1.6], we can see that to prove the (ϵ, r_0)-annular decay property for polynomial growth group G endowed with left-invariant periodic metric, the question is left to prove the case when G is a simply connected solvable Lie group of polynomial growth.

8. *Application one: exponential decay estimates*

In this section, we establish the jump estimates with exponential decay, namely Theorem 1.3. Let G be a group of polynomial growth with a symmetric finite generating set V in this section, and $\mathbf{A}' = \{A'_r : r \in \mathbb{N}\}$ be the sequence of averaging operators given by equation (1.3). It is certainly interesting to establish Theorem 1.3 in the more general setting. However, we do not know how to prove it.

We start with several lemmas. The first one is a transference principle, which can be established verbatim using the arguments as in the proof of Theorem 1.10(i). We omit the details.

LEMMA 8.1. *Let $p \in [1, \infty)$. Let T be an action induced by a μ-preserving measurable transformation τ on X. Let $\mathbf{A} = \{A_r : r \in \mathbb{N}\}$ be the sequence of averaging operators given*

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by equation (1.4). If there exist two constants $C_\lambda > 0$ and $c_\lambda \in (0,1)$ such that for every $n \in \mathbb{N}$ and $F \in L^p(G,m)$,

$$m(\{g \in G : \mathcal{N}_\lambda(\mathbf{A}'F)(g) > n\}) \le C_\lambda c_\lambda^n \|F\|^p_{L^p(G,m)},$$

then for every $n \in \mathbb{N}$ and $f \in L^p(X,\mu)$,

$$\mu(\{x \in X : \mathcal{N}_\lambda(\mathbf{A}f)(x) > n\}) \le C_\lambda c_\lambda^n \|f\|^p_{L^p(X,\mu)}.$$

With the above transference principle, it suffices to show Theorem 1.3 when $X = G$. The second lemma that we need is a trivial jump estimate following from Theorem 1.2.

LEMMA 8.2. *For every $p \in [1,\infty)$, there exists a constant $c_p > 0$ such that for all $F \in L^p(G,m)$,*

$$m(\{g \in G : \mathcal{N}_\lambda(\mathbf{A}'F)(g) > n\}) \le \frac{c_p}{(\lambda\sqrt{n})^p}\|F\|^p_{L^p(G,m)}.$$

To present the next two lemmas, we fix $\lambda > 0$ and $F \in L^p(G,m)$ with $\|F\|_{L^\infty} \le 1$. For $q \in \mathbb{N}$, define

$$\mathcal{F}_q = \mathcal{F}_q(\lambda, F) = \{x \in G : \mathcal{N}_\lambda(\mathbf{A}'F)(x) > q\}$$

and

$$\mathcal{F}'_q(F,\lambda) = \{(x, r_0) : x \in G, \text{ there exists } 1 \le r_0 < r_1 < \cdots < r_q \text{ such that } \min_{1\le i\le q} |A'_{r_i}F(x) - A'_{r_{i-1}}F(x)| > \lambda\}.$$

Let $\mathcal{G}'_q = \mathcal{G}'_q(F,\lambda) = \{B(x,r_0) : (x,r_0) \in \mathcal{F}'_q(F,\lambda)\}$ and $\mathcal{G}_q = \mathcal{G}_q(F,\lambda) = \bigcup_{B\in\mathcal{G}'_q} B$. It is clear that $\mathcal{F}_q \subseteq \mathcal{G}_q$.

Set

$$C_{V,\lambda} = \min\{1/(8c_V), 1/\lambda\},$$
$$\Phi(q) = 2^p \cdot 3^{D_G} c_p C_V^4 C_{V,\lambda}^{-D_G/\theta} \lambda^{-D_G/\theta - p} q^{-p/2},$$

where the constants D_G, C_V, c_V and θ were given in §7.

LEMMA 8.3. *For any $q \in \mathbb{N}$, one has*

$$m(\mathcal{G}_q(F,\lambda)) \le \Phi(q)\|F\|^p_{L^p(G,m)}.$$

Proof. We first prove the following inequality:

$$m(B(x,r)) \le C_V^2 (C_{V,\lambda}\lambda)^{D_G/\theta} m(B(x, (C_{V,\lambda}\lambda)^{1/\theta} r)) \quad \text{for all } r \in \mathbb{N}. \tag{8.1}$$

Indeed, if $(C_{V,\lambda}\lambda)^{1/\theta} r < 1$, since m is a counting measure, then $m(B(x,(C_{V,\lambda}\lambda)^{1/\theta}r)) = 1$. By equation (7.1), we have $m(B(x,r)) \le C_V (C_{V,\lambda}\lambda)^{-D_G/\theta} m(B(x,(C_{V,\lambda}\lambda)^{1/\theta}r))$. If $(C_{V,\lambda}\lambda)^{1/\theta} r \ge 1$, using equation (7.1) again, we have $m(B(x,r)) \le C_V^2 (C_{V,\lambda}\lambda)^{-D_G/\theta} m(B(x,(C_{V,\lambda}\lambda)^{1/\theta}r))$, and so equation (8.1) is proved.

Applying the Vitali covering lemma, we can select a subset

$$\{B_{j_1}, B_{j_2}, \ldots, B_{j_n}, \ldots\} \subseteq \mathcal{G}'_q$$

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of pairwise disjoint balls satisfying

$$\mathcal{G}_q \subseteq \bigcup_i 3B_{j_i}. \tag{8.2}$$

For each ball $B_{j_i} = B(x_{j_i}, r_{j_i})$ selected, by the definition of $\mathcal{G}'_q$, there exists a sequence $1 \le r_{j_i} < r_{j_i+1} < \cdots < r_{j_i+q}$ such that

$$|A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k-1}} F(x_{j_i})| > \lambda \quad \text{for all } 1 \le k \le q.$$

Now, we fix one ball $B(x_{j_i}, r_{j_i})$ and a sequence $1 \le r_{j_i} < r_{j_i+1} < \cdots < r_{j_i+q}$. We claim that for all $y \in B(x_{j_i}, (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i})$ and $1 \le k \le q$,

$$|A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| > \lambda/2,$$

namely, $B(x_{j_i}, (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i}) \in \mathcal{F}_q(\lambda/2, F)$.

Assume this claim momentarily. We have

$$\begin{aligned} m\Big(\bigcup_i B_{j_i}\Big) &= \sum_i m(B_{j_i}) \le C_V^2 (C_{V,\lambda}\lambda)^{-D_G/\theta} \sum_i m(B(x_{j_i}, (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i})) \\ &\le C_V^2 (C_{V,\lambda}\lambda)^{-D_G/\theta} m\Big(\bigcup_i B(x_{j_i}, (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i})\Big) \\ &\le C_V^2 (C_{V,\lambda}\lambda)^{-D_G/\theta} m\Big(\mathcal{F}_q\Big(\frac{\lambda}{2}, F\Big)\Big) \le C_V^2 (C_{V,\lambda}\lambda)^{-D_G/\theta} \frac{2^p c_p}{(\lambda\sqrt{q})^p} \|F\|_{L^p}^p, \end{aligned}$$

where the equality follows from the disjointness of the balls B_{j_i}, the first inequality follows from equation (8.1), the second inequality follows from the fact $C_{V,\lambda}\lambda \le 1$ and the disjointness of the balls B_{j_i}, the third inequality follows from the claim, and the last inequality follows from Lemma 8.2.

Moreover, combining the above inequality with equation (8.2), we have

$$\begin{aligned} m(\mathcal{G}_q) &\le m\Big(\bigcup_i 3B_{j_i}\Big) \le 3^{D_G} C_V^2 \sum_i m(B_{j_i}) \\ &\le 2^p \cdot 3^{D_G} c_p C_V^4 C_{V,\lambda}^{-D_G/\theta} \lambda^{-D_G/\theta - p} q^{-p/2} \|F\|_{L^p}^p, \end{aligned}$$

and the conclusion is proved.

We now prove the claim. Fix $y \in B(x_{j_i}, (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i})$. By a simple geometric argument, we can check at once that for every $0 \le k \le q$, $B(x_{j_i}, r_{j_i+k}) \triangle B(y, r_{j_i+k})$ is contained in

$$\begin{aligned} &(B(x_{j_i}, r_{j_i+k} + (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i}) \setminus B(x_{j_i}, r_{j_i+k})) \\ &\quad \cup (B(y, r_{j_i+k} + (C_{V,\lambda}\lambda)^{1/\theta} r_0) \setminus B(y, r_{j_i+k})). \end{aligned}$$

Using the inequality (7.2) and the fact that the measure m is invariant under the translation, the above inequality implies

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$$\begin{aligned} m(B(x_{j_i}, r_{j_i+k}) \triangle B(y, r_{j_i+k})) &\le m(B(x_{j_i}, r_{j_i+k} + (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i}) \setminus B(x_{j_i}, r_{j_i+k})) \\ &\quad + m(B(y, r_{j_i+k} + (C_{V,\lambda}\lambda)^{1/\theta} r_{j_i}) \setminus B(y, r_{j_i+k})) \\ &\le 2c_V \left(\frac{(C_{V,\lambda}\lambda)^{1/\theta} r_{j_i}}{r_{j_i+k}} \right)^{\theta} m(B_{r_{j_i+k}}) \\ &\le \frac{\lambda}{4} m(B_{r_{j_i+k}}). \end{aligned}$$

Combining the above inequality with $\|F\|_{L^\infty} \le 1$, one has

$$\begin{aligned} \left| \int_{B(y, r_{j_i+k})} F(z)\, dm(z) - \int_{B(x_{j_i}, r_{j_i+k})} F(z)\, dm(z) \right| &= \left| \int_{B(x_{j_i}, r_{j_i+k}) \triangle B(y, r_{j_i+k})} F(z)\, dm(z) \right| \\ &\le \frac{\lambda}{4} m(B_{r_{j_i+k}}). \end{aligned} \tag{8.3}$$

It follows that

$$|A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k}} F(y)| \le \frac{\lambda}{4} \quad \text{for all } 0 \le k \le q.$$

Using the triangle inequality, for every $1 \le k \le q$, we have

$$\begin{aligned} |A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k-1}} F(x_{j_i})| &\le |A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| + |A'_{r_{j_i+k}} F(x_{j_i}) - A'_{r_{j_i+k}} F(y)| \\ &\quad + |A'_{r_{j_i+k-1}} F(x_{j_i}) - A'_{r_{j_i+k-1}} F(y)| \\ &\le |A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| + \lambda/2. \end{aligned}$$

By the above inequality, we have $|A'_{r_{j_i+k}} F(y) - A'_{r_{j_i+k-1}} F(y)| > \lambda/2$ and the claim is proved. □

LEMMA 8.4. *For positive integers p and q, one has*

$$m(\mathcal{G}_{(p+1)q}(F, \lambda)) \le 3^{D_G} C_V^2 \Phi(q) m(\mathcal{G}_{pq}(F, \lambda)).$$

The proof of this lemma is inspired by the proof of [32, (5.7)].

Proof. For every ball $B = B(x, r) \in \mathcal{G}'_{(p+1)q}$, by the definition of $\mathcal{G}'_{(p+1)q}$, there exists a sequence $r = r_0 < r_1 < \cdots < r_{(p+1)q}$ such that

$$|A'_{r_k} F(x) - A'_{r_{k-1}} F(x)| > \lambda \quad \text{for all } 1 \le k \le (p+1)q.$$

So we have $B(x, r_q) \in \mathcal{G}'_{pq}$. Write $\widetilde{B} = B(x, r_q)$. Set $\mathcal{B} = \mathcal{G}'_{(p+1)q}$ and

$$\mathcal{B}' = \{B' : B' \in \mathcal{G}'_{pq} \text{ satisfies } B' = \widetilde{B} \text{ for some } B \in \mathcal{B}\}.$$

Note that $\mathcal{B}' \subseteq \mathcal{G}'_{pq}$, then by the definition of $\mathcal{G}_{(p+1)q}$, the proof is finished if we show

$$m\left(\bigcup_{B \in \mathcal{B}} B \right) \le 3^{D_G} C_V^2 \Phi(q) m\left(\bigcup_{B' \in \mathcal{B}'} B' \right). \tag{8.4}$$

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We now focus on the above inequality. Before proving this estimate, we introduce some new notation. We assume that B_1 is the maximal size ball of $\mathcal{B}'$. We set $\mathcal{B}_1 = \mathcal{B}$, $\mathcal{B}'_1 = \mathcal{B}'$ and define the sets

$$\mathcal{I}_1 = \{B | B \in \mathcal{B}_1, \widetilde{B} \cap B_1 \neq \emptyset\},$$
$$\mathcal{I}'_1 = \{B' | B' \in \mathcal{B}'_1, B' \cap B_1 \neq \emptyset\}.$$

We first prove the following estimate:

$$m\Big(\bigcup_{B\in\mathcal{I}_1} B\Big) \leq \Phi(q) m(3B_1), \tag{8.5}$$

where $3B_1$ denotes the ball with the same centre as B_1 and its radius is 3 times that of B_1.

Fix $B = B(x, r) \in \mathcal{I}_1$. Since $B(x, r) \in \mathcal{G}'_{(p+1)q}$, then there exists $r = r_0 < r_1 < \cdots < r_{(p+1)q}$ such that for all $1 \leq k \leq (p+1)q$,

$$|A'_{r_k} F(x) - A'_{r_{k-1}} F(x)| > \lambda.$$

Since B_1 is a maximal size ball of $\mathcal{B}'$ and $B_1 \cap B(x, r_q) \neq \emptyset$, so for all $0 \leq k \leq q$, $B(x, r_k) \subseteq 3B_1$, it follows that

$$|A'_{r_k}(F\mathbb{1}_{3B_1})(x) - A'_{r_{k-1}}(F\mathbb{1}_{3B_1})(x)| > \lambda \quad \text{for all } 1 \leq k \leq q.$$

Hence, $B \in \mathcal{G}'_q(F\mathbb{1}_{3B_1}, \lambda)$. Combining this observation with Lemma 8.3 and equation (7.1), we have

$$\begin{aligned} m\Big(\bigcup_{B\in\mathcal{I}_1} B\Big) &\leq m\Big(\bigcup_{B\in\mathcal{G}'_q(F\mathbb{1}_{3B_1},\lambda)} B\Big) = m(\mathcal{G}_q(F\mathbb{1}_{3B_1}, \lambda)) \leq \Phi(q)\|F\mathbb{1}_{3B_1}\|^p_{L^p(G,m)} \\ &\leq \Phi(q) m(3B_1) \leq 3^{D_G} C_V^2 \Phi(q) m(B_1). \end{aligned}$$

So, equation (8.5) is proved.

Let $\mathcal{B}_2 = \mathcal{B}_1 \setminus \mathcal{I}_1$ and $\mathcal{B}'_2 = \mathcal{B}'_1 \setminus \mathcal{I}'_1$. Select $B_2 \in \mathcal{B}'_2$ is a maximal size ball of $\mathcal{B}'_2$. Note that $B_1 \in \mathcal{I}'_1$, so B_2 is disjoint from B_1. We define the sets

$$\mathcal{I}_2 = \{B | B \in \mathcal{B}_2, \widetilde{B} \cap B_2 \neq \emptyset\},$$
$$\mathcal{I}'_2 = \{B' | B' \in \mathcal{B}'_2, B' \cap B_2 \neq \emptyset\}.$$

By similar discussions of equation (8.5), we have

$$m\Big(\bigcup_{B\in\mathcal{I}_2} B\Big) \leq \Phi(q) m(3B_2) \leq 3^{D_G} C_V^2 \Phi(q) m(B_2).$$

Repeating the above process, then we can select the pairwise disjoint balls $B_1, B_2, \ldots$ which belongs to $\mathcal{B}'$ and the sets $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}'_1, \mathcal{I}'_2, \ldots$ with the properties

$$\bigcup_i \mathcal{I}_i = \mathcal{B}, \ \bigcup_i \mathcal{I}'_i = \mathcal{B}', \ m\Big(\bigcup_{B\in\mathcal{I}_i} B\Big) \leq 3^{D_G} C_V^2 \Phi(q) m(B_i).$$

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Summing i for the latter inequality and using the property that the balls $\{B_i\}_i$ are pairwise disjoint, we have

$$m\Big(\bigcup_{B\in\mathcal{B}} B\Big) = m\Big(\bigcup_i \bigcup_{B\in\mathcal{I}_i} B\Big) \le \sum_i m\Big(\bigcup_{B\in\mathcal{I}_i} B\Big) \le 3^{D_G} C_V^2 \Phi(q) \sum_i m(B_i)$$
$$= 3^{D_G} C_V^2 \Phi(q) m\Big(\bigcup_i B_i\Big) \le 3^{D_G} C_V^2 m\Big(\bigcup_{B'\in\mathcal{B}'} B'\Big).$$

We obtained equation (8.4) and the lemma follows. □

Proof of Theorem 1.3. By Lemma 8.1, it suffices for our purpose to show that for any $\lambda > 0$, there exist two constants $\tilde{c}_1$ and $\tilde{c}_2 \in (0,1)$ such that for any $n \in \mathbb{N}$ and $F \in L^p(G,m)$ with $\|F\|_{L^\infty} \le 1$,

$$m(\{g \in G : \mathcal{N}_\lambda(\mathbf{A}'F)(g) > n\}) \le \tilde{c}_1 \tilde{c}_2^n \|F\|^p_{L^p(G,m)}.$$

From now on, fix one $\lambda > 0$ and one $F \in L^p(G,m)$ with $\|F\|_{L^\infty} \le 1$. First, by Lemma 8.3, we set $q_0 = \min\{q \in \mathbb{N} : \Phi(q) \le \frac{1}{2}\}$. For each $n > 0$, compared with q_0, we divide the n into two cases: $n \ge q_0$ and $1 \le n < q_0$.

We first consider the case $1 \le n < q_0$. By Lemma 8.2, we can set $\tilde{c}_1 = 2c_p/\lambda^p$, $\tilde{c}_2 = (1/2)^{1/q_0}$. It remains to consider the case $n \ge q_0$. Write $n = sq_0 + r$ with $0 \le r < q_0$. Using Lemma 8.4, we have

$$m(\mathcal{F}_n) \le m(\mathcal{F}_{sq_0}) \le m(\mathcal{G}_{sq_0}) \le (\tfrac{1}{2})^{s-1} m(\mathcal{G}_{q_0}).$$

However, using Lemma 8.3 again, we have

$$m(\mathcal{G}_{q_0}) \le \Phi(q_0)\|F\|^p_{L^p} \le \tfrac{1}{2}\|F\|^p_{L^p}.$$

Note that $s = (n-r)/q_0$, so by the above discussions, we have

$$m(\mathcal{F}_n) \le (\tfrac{1}{2})^{-r/q_0} (\tfrac{1}{2})^{n/q_0} \|F\|^p_{L^p},$$

and so we can set $\tilde{c}_1 = 2$ and $\tilde{c}_2 = (1/2)^{1/q_0}$ in this case.

Finally, we can take

$$\tilde{c}_1 = \max\{2, 2c_p/\lambda^p\},\ \tilde{c}_2 = (1/2)^{1/q_0},$$

and the proof is complete. □

9. *Application two: an explicit bound of the convergence rate*

In this section, we apply the jump inequalities to acquire an explicit bound on the convergence rate of ergodic averages for a strongly continuous regular action of a group of polynomial growth.

Proof of Theorem 1.5. Fix a function Θ with the properties that $\Theta(r) \ge r$ for every $r \in \mathbb{N}$. Let $M \in \mathbb{N}$. Consider the sequence of $M+1$ intervals

$$\mathrm{I}_1 = [1, \Theta(1)], \quad \mathrm{I}_2 = [\Theta(1), \Theta(\Theta(1))], \quad \ldots, \quad \mathrm{I}_{M+1} = [\Theta^M(1), \Theta^{M+1}(1)].$$

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Fix $\lambda_1 > 0, \lambda_2 > 0$ and $f \in L^2(X) \cap L^\infty(X)$. Without loss of generality, we may assume $\lambda_1 \le 2\|f\|_{L^\infty}$. Let $1 \le i \le M$ and $1 \le j \le [4\|f\|_{L^\infty}/\lambda_1]$. Set

$$\mathcal{K}_{i,j} = \Big\{ x \in X : \text{ there exists } r, s \in I_i \text{ such that } A_s f(x) < -\|f\|_{L^\infty} + \frac{j}{2}\lambda_1, \ A_r f(x) > -\|f\|_{L^\infty} + \frac{(j+1)}{2}\lambda_1 \Big\}$$

and

$$\mathcal{L}_i = \big\{ x \in X : \max_{r_i \in I_i} |A_{r_i} f(x) - A_{\Theta^i(1)} f(x)| > \lambda_1 \big\}.$$

We claim that if for each $1 \le i \le M$,

$$\mu(\mathcal{L}_i) > \lambda_2,$$

then $M \le (128C/\lambda_1^3\lambda_2)\|f\|_{L^\infty}\|f\|_{L^2}^2 - 1$.

Assume this claim momentarily. Given $N > 128C/\lambda_1^3\lambda_2\|f\|_{L^\infty}\|f\|_{L^2}^2 - 1$, then there exists one $1 \le i \le N$ such that

$$\mu\big(\big\{ x \in X : \max_{r_i \in I_i} |A_{r_i} f(x) - A_{\Theta^i(1)} f(x)| > \lambda_1 \big\}\big) \le \lambda_2.$$

Choose such an i and let $r_1 = \Theta^i(1)$, then the theorem is concluded.

We now focus on the claim. Fix i. It is clear that $\mathcal{L}_i \subseteq \bigcup_{1 \le j \le [4\|f\|_{L^\infty}/\lambda_1]} \mathcal{K}_{i,j}$. It follows that

$$\sum_{i=1}^{M+1} \sum_{j=1}^{[4\|f\|_{L^\infty}/\lambda_1]} \mu(\mathcal{K}_{i,j}) \ge \sum_{i=1}^{M+1} \mu\Big(\bigcup_{1 \le j \le [4\|f\|_{L^\infty}/\lambda_1]} \mathcal{K}_{i,j} \Big) \ge \sum_{i=1}^{M+1} \mu(\mathcal{L}_i) > (M+1)\lambda_2.$$

This inequality yields that there exists some $1 \le j \le [4\|f\|_{L^\infty}/\lambda_1]$ such that

$$\sum_{i=1}^{M+1} \mu(\mathcal{K}_{i,j}) > \frac{(M+1)\lambda_1\lambda_2}{4\|f\|_{L^\infty}}. \tag{9.1}$$

Let $\mathbf{A} = \{A_r : r \in \mathbb{N}\}$. Fixing such a j, and letting $a = -\|f\|_{L^\infty} + (j/2)\lambda_1$ and $b = -\|f\|_{L^\infty} + ((j+1)/2)\lambda_1$, we have

$$\begin{aligned} \sum_{i=1}^{M+1} \mu(\mathcal{K}_{i,j}) &\le \int_X \sum_{i=1}^{M+1} \mathbb{1}_{\mathcal{K}_{i,j}}(x)\, d\mu(x) \le \int_X \mathcal{N}_{a,b}(\mathbf{A}f)(x)\, d\mu(x) \\ &= \frac{1}{(b-a)^2} \int_X |(b-a)\sqrt{\mathcal{N}_{a,b}(\mathbf{A}f)}(x)|^2\, d\mu(x) \\ &\le \frac{32C}{\lambda_1^2}\|f\|_{L^2(X)}^2, \end{aligned} \tag{9.2}$$

where we used Theorem 1.2(ii) in the last inequality. Then, the claim is obtained directly from equations (9.1) and (9.2), which completes the proof. □

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