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A note concerning the global 2-forms on a $\varkappa = 0$ threefold

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Let V be a smooth complex algebraic threefold with $\kappa(V) = 0$. Consider the space of holomorphic 2-forms on V, and let s denote the rank of the subsheaf of Ω_V^2 generated by these global forms. It was conjectured by Ueno in [8] that $h^0(V, \Omega_V^2) \leqslant 3$ – here as always, h^i denotes the dimension of the corresponding cohomology group H^i.

A minimal model for V is a birationally equivalent projective threefold X whose singularities are $\mathbb{Q}$-factorial and terminal and such that the $\mathbb{Q}$-Cartier divisor K_X is *nef*. Recall that a $\mathbb{Q}$-Cartier divisor D is said to be *nef* if $D.C \geqslant 0$ for all curves C on X. For the definitions and more details on terminal singularities, see [7]. The existence of such minimal models is still conjectural, although significant progress has been made here by Kawamata [1, 2] based on the fundamental work of Mori [5].

The purpose of this paper is to note that the results of Mabuchi [3] and a conjecture of Miyaoka concerning the semipositivity of the cotangent sheaf on a non-uniruled threefold (cf. [4]) together imply the following stronger form of Ueno's conjecture for $\kappa = 0$ threefolds with a minimal model.

CONJECTURE. *Let V be a smooth complex algebraic threefold with $\kappa = 0$ and with a minimal model in the sense above. Then $h^0(V, \Omega_V^2)$ equals the rank s of the subsheaf of Ω_V^2 generated by its global sections; in particular $h^0(V, \Omega_V^2) \leqslant 3$.*

We now fix some notations; let X denote a minimal model of V and $f: \tilde{X} \to X$ be a desingularization of X which restricts to an isomorphism from $\tilde{X} - f^{-1}$ (singular points) to the smooth locus of X. Set $t = h^0(V, \Omega_V^2) = h^2(V, \mathcal{O}_V)$ – the numbers t and s are birational invariants and so are the same for every smooth model.

The result of Mabuchi

We define a rational map $\phi: \tilde{X} \dashrightarrow Gr(t, t-s)$, given generically by $\phi(x) = \{\omega \in H^0(\tilde{X}, \Omega_{\tilde{X}}^2);\ \omega(x) = 0\}$. According to theorem 8·4·1 of [3] we have two possibilities; either (*a*) $s = t$, or (*b*) $s = 1$ or 2 and the rational image $\phi(\tilde{X})$ has dimension at least 2. We aim to show therefore that the case (*b*) cannot occur. If it did, note that either $t = 3$ and $s = 1$ or that $t \geqslant 4$.

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If (b) holds and $s = 1$, then we have a divisor $\tilde{N}$ on $\tilde{X}$ with $\mathcal{O}_{\tilde{X}}(\tilde{N})$ a full subsheaf of $\Omega^2_{\tilde{X}}$ and such that ϕ corresponds to the linear system $|\tilde{N}|$. We therefore have a torsion-free rank one quotient sheaf $\tilde{Q}$ of $\Omega^1_{\tilde{X}}$ with $c_1(\tilde{Q}) = K_{\tilde{X}} = \tilde{N}$. This in turn gives rise to a torsion-free quotient sheaf Q of $f^* \Omega^1_{\tilde{X}}$ (in fact a subsheaf of $f_* \tilde{Q}$) with $c_1(Q) = K_X - N$, where N is the $\mathbb{Q}$-Cartier divisor on X corresponding to $\tilde{N}$ on $\tilde{X}$.

For the $s = 2$ case in (b), we note that $\Lambda^2 \Omega^2_{\tilde{X}}$ is isomorphic to $\Omega^1_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})$. Therefore the global sections of $\Omega^2_{\tilde{X}}$ determine a divisor $\tilde{L}$ on $\tilde{X}$ such that $\mathcal{O}_{\tilde{X}}(\tilde{L})$ is a full subsheaf of $\Omega^1_{\tilde{X}}$ and the map ϕ corresponds to a linear subsystem of $|\tilde{L} + K_{\tilde{X}}|$. Letting L denote the $\mathbb{Q}$-Cartier divisor on X corresponding to $\tilde{L}$ on $\tilde{X}$, we note that $f_* \Omega^1_{\tilde{X}}$ has a torsion free rank two quotient sheaf Q with $c_1(Q) = K_X - L$.

The irregular case

This is straightforward. As we noted in [**10**], by the theory of the Albanese map (see for instance [**9**]) V has a smooth minimal model X on which K_X is numerically trivial, and hence $\chi(\mathcal{O}_V) = \chi(\mathcal{O}_X) = -K_X . c_2(X)/24 = 0$. In particular we see that $t \leqslant q$, where q is the irregularity of V. From the classification of irregular $\kappa = 0$ threefolds [**9**], we know that $q \leqslant 3$ with equality if and only if X is an abelian threefold. Thus $t \leqslant 3$ with equality if and only if X is an abelian threefold, in which case $s = 3$ as required. From the result of Mabuchi therefore, the Theorem is true for irregular threefolds.

The numerical Kodaira dimension

We can now restrict ourselves to the case when V is regular with a minimal model X. It was shown in [**10**] that if X is Gorenstein, then $\chi(\mathcal{O}_V) = 0$ and hence $h^2(\mathcal{O}_V) = 0$. There are, however, many examples of $\kappa = 0$ threefolds with $\chi(\mathcal{O}_V) > 0$ (see [**8**]); for instance the Kummer threefold has $\chi = 4$ and $h^2 = 3$. Let r denote the index of X (see [**6**]); i.e. the smallest positive integer m for which mK_X is a Cartier divisor. Thus X is Gorenstein if and only if $r = 1$.

For X a minimal model, we define the numerical Kodaira dimension $\kappa_{\text{num}}(X)$ to be the minimum i such that K^i_X is not numerically trivial on X (or equivalently that $K^i_X . H^{3-i} \neq 0$ for H ample on X). Since $\kappa(X) = 0$, it is easily seen that $\kappa_{\text{num}}(X) < 3$ (see [**4**] or [**11**], lemma 1·3). It is conjectured that if $\kappa = 0$, then K_X is numerically trivial on a minimal model, but the proof of this seems some way off. If $\kappa_{\text{num}}(X) = 2$, we deduce that

$$h^0(nrK_X) - h^1(nrK_X) = nrK_X . c_2/12 + \chi(\mathcal{O}_V)$$

where $K_X . c_2$ denotes $(f^* rK_X) . c_2(\tilde{X})/r$, a rational number independent of the resolution (see [**4**] or [**11**], proposition 3·1). The results of [**10**] or [**4**] show that $K_X . c_2 \geqslant 0$, and hence we deduce that $K_X . c_2 = 0$ and $\chi(\mathcal{O}_V) \leqslant 1$ (since $h^0(nrK_X) \leqslant 1$ for all $n > 0$). In particular, since V is now assumed to be regular, $h^2(\mathcal{O}_V) \leqslant 1$ and the result follows trivially. We may assume therefore that $\kappa_{\text{num}}(X) = 0$ or 1.

The conjecture of Miyaoka and the $\kappa_{\text{num}} = 0$ case

The following conjecture is partially proved in [**4**].

CONJECTURE. *Let X be a normal complex projective variety of dimension 3, and H_1, H_2 ample divisors on X. Let $f: \tilde{X} \to X$ be a desingularization of X. If $\tilde{X}$ is not uniruled, then for every torsion-free quotient sheaf Q of $f_* \Omega^1_{\tilde{X}}$, the inequality $c_1(Q) . H_1 . H_2 \geqslant 0$ holds.*

Let us now consider the $\kappa_{\text{num}}(X) = 0$ case. We have seen that there is a torsion-free

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quotient sheaf Q of $f_* \Omega^1_{\tilde{X}}$ with $c_1(Q) = K_X - N$ in the $s = 1$ case, and $c_1(Q) = K_X - L$ in the $s = 2$ case.

Let H denote any very ample divisor on X; clearly $H^2.N > 0$ in the $s = 1$ case, and $H^2.(L+K_X) > 0$ in the $s = 2$ case. Since however K_X is numerically trivial, we deduce in both cases that $c_1(Q).H^2 < 0$, contradicting the conjecture of Miyaoka.

The $\kappa_{\text{num}} = 1$ case

We consider first the subcase when $s = 1$. Let $|\tilde{N}_0|$ be the mobile part of $|\tilde{N}|$ on $\tilde{X}$, and let N_0 be the corresponding $\mathbb{Q}$-Cartier divisor on X. As above, let H denote a very ample divisor on X. Since $\phi(\tilde{X})$ has dimension at least 2, it follows that $N_0^2.H > 0$. Thus for any $n \geqslant 0$, $N_0^2.(H+nK_X) > 0$.

We now apply the Hodge index theorem on a suitable multiple of the ample divisor $H+nK_X$, obtaining an inequality

$$\{H^2.(H+nK_X)\}\{N_0^2.(H+nK_X)\} \leqslant \{H.N_0.(H+nK_X)\}^2.$$

Since the left-hand side is unbounded, the same is true of the right-hand side. Since K_X is nef, it follows that $K_X.N_0.H > 0$, and hence also that $K_X.N.H > 0$.

Recall now that we have a torsion free quotient sheaf Q of $f_* \Omega^1_{\tilde{X}}$ with $c_1(Q) = K_X - N$. Setting $H_1 = H$ and $H_2 = H+nK_X$ (both clearly ample), we see that

$$c_1(Q).H_1.H_2 = (K_X - N).H.(H+nK_X) < 0$$

for n sufficiently large (using $\kappa_{\text{num}}(X) = 1$). This then contradicts the conjecture of Miyaoka.

The subcase when $s = 2$ is similar. We have a divisor $\tilde{L}_0$ on $\tilde{X}$ such that $|\tilde{L}_0 + K_{\tilde{X}}|$ is the mobile part of $|\tilde{L}+K_{\tilde{X}}|$; let L_0 be the corresponding $\mathbb{Q}$-Cartier divisor on $\tilde{X}$. As above, we have $(L_0+K_X)^2.H > 0$, and hence applying the index theorem that $K_X.(L_0+K_X).H > 0$. Therefore $K_X.L_0.H > 0$ since $\kappa_{\text{num}}(X) = 1$, and $K_X.L.H > 0$ since K_X is nef. We have a torsion-free quotient sheaf Q of $f_* \Omega^1_{\tilde{X}}$ with $c_1(Q) = K_X - L$, and the contradiction follows in the same way as before.

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