Ergod. Th. & Dynam. Sys. (2002), **22**, 1215–1232
DOI: 10.1017/S0143385702000639
© *2002 Cambridge University Press*
Printed in the United Kingdom

A generalized Lefschetz fixed point theorem and symbolic dynamics in delay equations

BERNHARD LANI-WAYDA† and ROMAN SRZEDNICKI‡

† *Mathematisches Institut der Universität Giessen, Arndtstrasse 2, 35392 Giessen, Germany*
(e-mail: Bernhard.Lani-Wayda@math.uni-giessen.de)
‡ *Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland*
(e-mail: Roman.Srzednicki@im.uj.edu.pl)

(*Received* 15 *April* 2000 *and accepted in revised form* 19 *October* 2001)

Abstract. We prove a generalized version of the Lefschetz fixed point theorem, and use it to obtain a variety of periodic and aperiodic solutions for differential delay equations; in particular, of the type $\dot{x}(t) = f(x(t-1))$. Here $f : \mathbb{R} \to \mathbb{R}$ is odd and two-periodic, and we obtain both strictly periodic solutions and solutions periodic modulo a multiple of two. The qualitative behavior of solutions can be coded by symbol sequences containing the sequence of levels about which these solutions oscillate.

1. *Introduction and main result*

The existence of infinitely many periodic orbits in a dynamical system is an expression of complicated dynamical behavior. Classical results of this type for ordinary differential equations are due to Poincaré, Birkhoff and Šil'nikov [**1, 11, 13**].

In the present paper, we study scalar delay equations of the type

$$\dot{x}(t) = f(x(t-1)), \tag{f}$$

and we use an appropriate version of the Lefschetz fixed point theorem to obtain a rich variety of qualitatively different periodic and aperiodic solutions. The nonlinearities $f : \mathbb{R} \to \mathbb{R}$ considered here are sine-like in the sense that they are periodic (the period is normalized to two), and between one zero z of f and the zero $z + 2$ there is exactly one more zero.

Such equations are used as models for the phase shift between oscillators which are supposed to synchronize; see [**19**]. In this interpretation, a constant solution corresponds to perfect synchronization, and oscillations around some constant level correspond to a still roughly synchronized operation, but with variations in the phase shift. Finally, solutions

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

wandering between different levels correspond to so-called 'cycle slippings', i.e. an at least temporarily total loss of synchronization.

In connection with periodic nonlinearities f, say, with period $T > 0$, a special kind of solution is of interest: so-called periodic solutions of the second kind. A solution $y : \mathbb{R} \to \mathbb{R}$ of equation (f) is called a *periodic solution of the second kind* if there exist $k \in \mathbb{Z}$ and $\omega > 0$ (called a *generalized period*) such that $y(t+\omega) = y(t) + k \cdot T$ for $t \in \mathbb{R}$. Thus, periodic solutions of the second kind are periodic modulo a multiple of the period of the nonlinearity f.

An early mathematical study of sine-like equations of type (f) as a model for frequency generators was given in [**7**], where periodic solutions of the second kind were obtained from the Schauder fixed point theorem.

Bifurcation of periodic solutions of $\dot{x}(t) = -\alpha \sin(x(t-1))$ for $\alpha > \pi/2$ was studied numerically in [**5**], and solutions wandering up and down between different levels in an irregular manner were observed to be typical for larger values of α.

Later, in [**10**], existence of such 'wandering' solutions following arbitrarily prescribed symbol sequences was established for a piecewise linear function $-\alpha \cdot \check{s}$ with period two, and for small perturbations of the corresponding equation $(-\alpha\check{s})$, with $\alpha = (9/e)(\log(9) - 1)$. The function $\check{s}$ is defined by

$$\check{s}(x) := x \text{ for } |x| \le 0.5, \quad \check{s}(x) := -x + 1 \text{ for } x \in (0.5, 1.5),$$

and by two-periodic continuation to all of $\mathbb{R}$. Thus, $\check{s}$ is a piecewise linear version of the function $x \mapsto (1/\pi)\sin(\pi x)$. We briefly describe the class of admissible, that is sufficiently 'small', perturbations of this particular nonlinearity. (Note that $\check{s}$ is not differentiable at the points of $1/2 + \mathbb{Z}$, so we cannot directly talk about C^1-small perturbations.) In the sequel, for a Banach space X we denote by $BC^1(X, \mathbb{R})$ the space of bounded C^1-functions $X \to \mathbb{R}$ with bounded derivative and by $\mathbf{C}$ we denote the space of continuous maps $[-1, 0] \to \mathbb{R}$ (both spaces are endowed with the usual norm). First, we consider perturbed equations of the form $\dot{x}(t) = g(x(t-1))$ with $g \in BC^1(\mathbb{R}, \mathbb{R})$. Roughly speaking, a two-periodic function g is in the admissible class if:

(a) g is close enough to $-\alpha\check{s}$ in the C^0-norm; and

(b) g is also close enough to $-\alpha\check{s}$ in the C^1-norm, except for a small neighborhood of the set $1/2 + \mathbb{Z}$, where $-\alpha\check{s}$ is not differentiable.

More precisely, there exist numbers $\theta_1, \theta_2 > 0$ such that every $g \in BC^1(\mathbb{R}, \mathbb{R})$ is admissible if it satisfies the following conditions:

(i) $g(x+2) = g(x)$ $(x \in \mathbb{R})$;

(ii) $\|g'\|_{C^0} < \alpha + 1$;

(iii) $\|g - (-\alpha\check{s})\|_{C^0} < \theta_1$;

(iv) $\|[g - (-\alpha\check{s})]|_{\mathbb{R}\setminus[1/2+(-\theta_2,\theta_2)+\mathbb{Z}]}\|_{C^1} < \theta_1$.

(Compare [**10**], Theorem 4.15 and Remark 4.9; we have obviously $|(-\alpha\check{s})'(x)| \le \alpha$ for $x \in \mathbb{R} \setminus [1/2 + \mathbb{Z}]$.)

Second, we mention more general perturbed equations. Any $g \in BC^1(\mathbb{R}, \mathbb{R})$ defines a functional $\hat{g} : \mathbf{C} \to \mathbb{R}$, which is in $BC^1(\mathbf{C}, \mathbb{R})$, by $\hat{g}(\psi) := g(\psi(-1))$. As usual, we denote by x_t the segment of x at time t, that is, $x_t(\theta) = x(t+\theta)$, $\theta \in [-1, 0]$. With this notation, the equations $\dot{x}(t) = g(x(t-1))$ and $\dot{x}(t) = \hat{g}(x_t)$ are equivalent. We say that a

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

functional $F : \mathbf{C} \to \mathbb{R}$ is two-periodic if $F(\psi + 2) = F(\psi)$ for all $\psi \in \mathbf{C}$, where $\psi + 2$ is the function $[-1, 0] \ni t \mapsto \psi(t) + 2$. If now $g \in BC^1(\mathbb{R}, \mathbb{R})$ satisfies conditions (i)–(iv) above, and if $F \in BC^1(\mathbf{C}, \mathbb{R})$ is two-periodic and sufficiently close to the functional $\hat{g}$ in the C^1-norm, then the equation

$$\dot{x}(t) = F(x_t) \tag{F}$$

is also an admissible perturbation of the piecewise linear example. Such equations generate a semiflow $\Phi_F : \mathbb{R}_0^+ \times \mathbf{C} \to \mathbf{C}$, $(t, \psi) \mapsto x_t^\psi$, where $x^\psi : [-1, \infty) \to \mathbb{R}$ denotes the solution of equation (F) with initial segment $x_0^\psi = \psi$.

From now on we shall refer to the above class of equations which are sufficiently small perturbations of $(-\alpha\check{s})$ by the symbol $\mathcal{F}$. We will be interested in solutions of such equations which follow a prescribed level sequence; this notion is defined precisely below.

Definition. Assume that $I \subset \mathbb{R}$ is an interval, that $y \in C^0(I, \mathbb{R})$, and that $k \in \mathbb{Z}$. Let $J \subset I$ be a subinterval.

(a) We say that y *oscillates about* k *on* J if $y(J) \cap \mathbb{Z} = \{k\}$.

(b) Now let $k, l \in \mathbb{Z}$ with $k \neq l$. We say that y *wanders from* k *to* l *on* J if there exist subintervals J_k, J_{kl}, J_l of J with $J = J_k \cup J_{kl} \cup J_l$, with $J_k \leq J_{kl} \leq J_l$, and such that y oscillates about k on J_k, y oscillates about l on J_l, and

$$y(J_{kl}) \cap \mathbb{Z} = \{m \in \mathbb{Z} : \min\{k, l\} < m < \max\{k, l\}\}.$$

(c) Given a sequence $\mathbf{l} = (\ldots, l_{-1}, l_0, l_1, \ldots) \in \mathbb{Z}^{\mathbb{Z}}$, we say that $y \in C^0(\mathbb{R}, \mathbb{R})$ *follows the level sequence* $\mathbf{l}$ if there exist intervals J_j $(j \in \mathbb{Z})$ such that $\sup_{j\in\mathbb{Z}}(\sup J_j - \inf J_j) < \infty$, $\bigcup_{j\in\mathbb{Z}} J_j = \mathbb{R}$, $\inf J_j < \inf J_{j+1}$, $\sup J_j < \sup J_{j+1}$ $(j \in \mathbb{Z})$, and y wanders from l_j to l_{j+1} on J_j $(j \in \mathbb{Z})$.

One central result of [**10**] was, roughly speaking, that for every equation from the class $\mathcal{F}$ and for every level sequence $\mathbf{l} = (\ldots, l_{-1}, l_0, l_1, l_2, \ldots)$ with $l_0 = -1$, $|l_{k+1} - l_k| = 2$ $(k \in \mathbb{Z})$, there exists a solution defined on all of $\mathbb{R}$ which follows this sequence in the sense of the above definition. The method of proof, a version of the Smale horseshoe method in combination with compactness arguments, did *not yield uniqueness* of the solution following a prescribed level sequence.

Consider now the case of a periodic level sequence $\mathbf{l}$, say, with period $K \in \mathbb{N}$. In this case, the construction from [**10**] does not necessarily give periodic time intervals J_j (in the above notation), and, as already remarked, the solution following the periodic level sequence is not necessarily unique. For these reasons, using only the results from [**10**] one cannot argue that the solution following the periodic sequence $\mathbf{l}$ must be periodic.

In the present work we shall use statements from [**10**], which describe the dynamics in terms of return maps, and a generalized Lefschetz fixed point theorem, in order to obtain both strictly periodic solutions and periodic solutions of the second kind. We also obtain aperiodic solutions following prescribed level sequences.

Our method does not provide the geometric information that the initial values of the solutions obtained are located on graphs of Lipschitz maps, as was the case in [**10**]. However, the technique of our work is a new alternative, and we obtain additional information on periodic solutions.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

Let us remark that the periodic solutions obtained by Furumochi for equations of the type (f) are of the second kind, and such that within one period they always move *up* an integer multiple of the period of f. Also, the periodic solutions of the second kind obtained in [**17**], as a consequence of transverse heteroclinic connections in sine-like delay equations, always move up within each period.

In contrast to these results, in the present paper we obtain periodic solutions of the second kind moving up and down in accordance with any prescribed level sequence. Also, this seems to be the first application of Lefschetz-type theorems in the context of delay equations. Our main result is the following.

THEOREM 1.1. *For every functional differential equation from the class $\mathcal{F}$ and for every level sequence $\mathbf{l} = (\dots, l_{-1}, l_0, l_1, l_2, \dots)$ with $l_0 = -1$, $|l_{k+1} - l_k| = 2$ $(k \in \mathbb{Z})$, there exists a solution y which follows this sequence. If $\mathbf{l}$ is such that the sequence $\{s_n\}$ defined by $s_n := l_{n+1} - l_n$ $(n \in \mathbb{Z})$ is periodic with minimal period $K \in \mathbb{N}$, then there exists a periodic solution y of the second kind following $\mathbf{l}$. If, in this case, P is the minimal generalized period of y, then y satisfies $y(t + P) = y(t) + l_K - l_0$ $(t \in \mathbb{R})$.*

In §2, we prove a fixed point theorem for compositions of maps (Theorem 2.1). We also formulate a particular case of this theorem (Corollary 2.1) which directly generalizes the Lefschetz theorem, and which may be of interest away from the context of the present paper. In §3, we apply Theorem 2.1 to obtain a result on the existence of chaotic dynamics in the sense of semiconjugacy with a subshift of finite type (Theorem 3.1). In §4, we describe the relevant properties of return maps associated with the delay equations under consideration. Combining these tools, we prove the main theorem in §5.

2. *A fixed point theorem*

In this section we establish a fixed point theorem which later yields a result on the existence of chaotic dynamics. Actually, our result is a minor modification of Theorem 2 in [**16**] where compact pairs of spaces were considered. Taking into account other possible applications, we formulate the result in a slightly more general setting than necessary for the rest of the paper.

Let (X, A) be a topological pair. In the case $A \neq \emptyset$, we denote by X/A the quotient space obtained by the collapsing of A to a point; that point in X/A is denoted by $[A]$. If $A = \emptyset$, then X/A denotes the disjoint union of X and a point $*$ outside of X and we put and $[A] := *$. In both cases, $[x]$ denotes the equivalence class of $x \in X$ in X/A.

We state the basic definitions related to the Lefschetz number (cf. [**3, 8**] for a detailed exposition). Let V be a vector space over $\mathbb{Q}$. For an endomorphism $\phi : V \to V$ define the generalized kernel of ϕ by

$$\operatorname{gker}\phi := \bigcup_{n=1}^{\infty} \phi^{-n}(0).$$

ϕ induces an endomorphism $\tilde{\phi} : V/\operatorname{gker}\phi \to V/\operatorname{gker}\phi$. The map ϕ is called *admissible* if $V/\operatorname{gker}\phi$ has finite dimension. If ϕ is admissible then the *generalized trace* of ϕ is defined by

$$\operatorname{Tr}\phi := \operatorname{tr}\tilde{\phi},$$

where tr denotes the ordinary trace.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

Now let $V = \{V_n\}_{n\in\mathbb{Z}}$ be a graded vector space over $\mathbb{Q}$, and let $\phi = \{\phi_n\}_{n\in\mathbb{Z}}$ be an endomorphism, where $\phi_n : V_n \to V_n$. The map ϕ is called *admissible* provided ϕ_n is admissible for every n and $\operatorname{gker}\phi_n = V_n$ for all except finitely many n. For an admissible endomorphism ϕ, define its *generalized Lefschetz number* by

$$\Lambda(\phi) := \sum_{n=0}^{\infty}(-1)^n \operatorname{Tr}\phi_n.$$

By H^* (respectively $\widetilde{H}^*$) we denote the Alexander–Spanier (respectively the reduced Alexander–Spanier) cohomology functor with coefficients in $\mathbb{Q}$. A continuous map $f : (X, A) \to (X, A)$ is called a *Lefschetz map* provided $H^*(f) : H^*(X, A) \to H^*(X, A)$ is admissible. Slightly abusing notation, for such a map we define its *generalized Lefschetz number* by

$$\Lambda(f) := \Lambda(H^*(f)).$$

If $f : X \to X$ is a Lefschetz map then $\widetilde{H}^*(f) : \widetilde{H}^*(X) \to \widetilde{H}^*(X)$ is admissible, and we define

$$\widetilde{\Lambda}(f) := \Lambda(\widetilde{H}^*(f)).$$

Now let $f : X \to X'$ be a continuous map between topological spaces X and X'. Assume that (C, E) is a pair of subsets of X, and that (C', E') is a pair of subsets of X'. We associate to f the map $f^{\#}_{(C,E),(C',E')} : C/E \to C'/E'$ (denoted for brevity by $f^{\#}$) defined as

$$f^{\#}([x]) := \begin{cases} [f(x)] & \text{if } x \in C \setminus E,\ f(x) \in C' \setminus E', \\ [E'] & \text{if } x \in E \text{ or } f(x) \in (X' \setminus C') \cup E'. \end{cases}$$

For the case $(C, E) = (C', E')$, the above map $f^{\#}$ was introduced in [**12**] in order to define index pairs in the discrete time Conley index theory. Below we provide a sufficient condition for the continuity of $f^{\#}$. It was used in [**15**] in a slightly more restrictive context.

PROPOSITION 2.1. *Let (C, E) and (C', E') be pairs of closed sets in X and X', respectively. If*

$$f(E) \cap C' \subset E', \quad C' \cap \overline{f(C) \setminus C'} \subset E',$$

then the induced map $f^{\#} : C/E \to C'/E'$ is continuous.

Proof. Let $x \in C$. If $f(x) \in C' \setminus E'$, then necessarily $x \in C \setminus E$ and $f(x) \notin \overline{f(C) \setminus C'}$. Let W be an open neighborhood of $f(x)$ in $X' \setminus E'$ such that

$$W \cap \overline{f(C) \setminus C'} = \emptyset,$$

and let U be an open neighborhood of x in $C \setminus E$ such that $f(U) \subset W$. Define $V := W \cap C'$. Then V is an open neighborhood of $f(x)$ in $C' \setminus E'$. It follows that $f(U) \subset V$. Indeed, if $z \in U$ and $f(z) \in f(C) \setminus C'$ then $f(z) \notin W$, which is impossible. Since $C/E \setminus [E]$ is homeomorphic to $C \setminus E$ and the same holds for the second pair, $f^{\#}$ is continuous at $[x]$.

Assume now that $f(x) \in E' \cup (X' \setminus C')$, hence $f^{\#}([x]) = [E']$. Let N be an open neighborhood of $[E']$ in C'/E'. It follows that

$$N = \{[y] \in C'/E' : y \in C' \setminus G\}$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

for some closed subset $G \subset C'$ such that $G \cap E' = \emptyset$. Since C' is closed, the set $X' \setminus G$ is an open neighborhood of E' in X'. Further, $f(E) \subset X' \setminus G$, because $f(E) \cap C' \subset E'$. Thus, the set

$$M = \{[x] \in C/E : f(x) \in X' \setminus G\}$$

is an open set in C/E (actually: a neighborhood of $[E]$), and $(f^{\#})^{-1}(N) = M$. □

Now we establish the main result of this section. Let k be a positive integer. We write $\mathbb{Z}_k$ for $\mathbb{Z}$ modulo k. For $j \in \mathbb{Z}_k$, let X_j be a topological space, let (C_j, E_j) be a pair of subsets of X_j, and let $f_j : C_j \to X_{j+1}$ be a continuous map.

THEOREM 2.1. *Assume that for every $j \in \mathbb{Z}_k$, X_j is a normed space, C_j and E_j are closed in X_j, and $f_j^{\#} := (f_j)^{\#}_{(C_j,E_j),(C_{j+1},E_{j+1})}$ is continuous. Moreover, assume that C_0 and E_0 are absolute neighborhood retracts (ANRs), f_{j_0} is compact and X_{j_0+1} is complete (i.e. a Banach space) for some $j_0 \in \mathbb{Z}_k$. Then $f_{k-1}^{\#} \circ \cdots \circ f_0^{\#}$ is a Lefschetz map, and if $\widetilde{\Lambda}(f_{k-1}^{\#} \circ \cdots \circ f_0^{\#}) \neq 0$ then there exist $x_0 \in \overline{C_0 \setminus E_0}$ and $x_j \in C_j$, $j = 1, \ldots, k-1$, such that*

$$f_j(x_j) = x_{j+1}.$$

In particular, x_0 is a fixed point of $f_{k-1} \circ \cdots \circ f_0$.

Proof. (The proof follows the proof of Theorem 2 in [**16**], with minor changes.) Put

$$e_j : E_j \ni x \to (x, 1) \in X_j \times S^1,$$

where S^1 is the unit circle in the complex plane. Define the adjunction

$$M_j := C_j \cup_{e_j} (X_j \times S^1) := (C_j \times 1 \cup (X_j \times S^1) \times 0)/\sim,$$

where $\sim$ is the minimal equivalence relation which identifies $(x, 1) \in E_j \times 1$ with $((x, 1), 0) \in (X_j \times S^1) \times 0$. For $z \in C_j$ or $z \in X_j \times S^1$ and $l \in \{0, 1\}$, let $[z, l]$ denote the equivalence class of (z, l) in the relation $\sim$. We define the following subsets of the adjunction M_j:

$$\begin{aligned} C_j^* &:= \{[x, 1] \in M_j : x \in C_j\}, \\ D_j^* &:= \{[z, 0] \in M_j : z \in X_j \times S^1\}, \\ E_j^* &:= \{[x, 1] \in M_j : x \in E_j\} = C_j^* \cap D_j^*. \end{aligned}$$

Obviously, all the above sets are closed in M_j. By the Tietze–Dugundji Theorem (see [**6**]), there exists a continuous map $\Psi_j : X_j \to X_{j+1}$ such that $\Psi_j|_{C_j} = f_j$ and $\Psi_j(X_j) \subset \operatorname{Conv} f_j(C_j)$, where Conv denotes the closed convex hull. In particular, Ψ_{j_0} is compact by the Mazur theorem (since X_{j_0+1} is a Banach space). Let $\phi_j : X_j \to [0, 1]$ be a Urysohn Lemma function such that $\phi^{-1}(0) = E_j$. Define $g_j : M_j \to M_{j+1}$ by

$$g_j([x, 1]) := \begin{cases} [f_j(x), 1] & \text{if } x \in C_j \setminus E_j,\ f_j(x) \in C_{j+1} \setminus E_{j+1}, \\ [(f_j(x), 1), 0] & \text{if } x \in E_j \text{ or } f_j(x) \in (X_{j+1} \setminus C_{j+1}) \cup E_{j+1}, \end{cases}$$

$$g_j([(x, \alpha), 0]) := [(\Psi_j(x), \alpha \exp(\pi i \phi_j(x)/k)), 0].$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

One can check that

$$(g_{k-1} \circ \cdots \circ g_0)([x, l]) = [x, l]$$

if and only if

$$x \in C_0, \quad l = 1, \quad (f_{k-1} \circ \cdots \circ f_0)(x) = x, \quad (f_j \circ \cdots \circ f_0)(x) \in C_{j+1}.$$

In order to prove the theorem we have to show the existence of a fixed point of $g_{k-1} \circ \cdots \circ g_0$ in $\overline{C_0^* \setminus E_0^*}$. To this purpose it suffices to prove that:

(I) g_j is continuous for every $j \in \mathbb{Z}_k$;

(II) M_0 and D_0^* are ANRs;

(III) the map $g_{k-1} \circ \cdots \circ g_0 : (M_0, D_0^*) \to (M_0, D_0^*)$ is compact; and

(IV) $\Lambda(g_{k-1} \circ \cdots \circ g_0) = \widetilde{\Lambda}(f_{k-1}^{\#} \circ \cdots \circ f_0^{\#})$.

Indeed, if (I)–(IV) hold, the required assertion follows by a fixed point theorem of Bowszyc (see [**3**, Theorem (4.5)]).

Ad (I). The restriction $g_j|_{D_j^*}$ is continuous, hence it suffices to prove that $g_j|_{C_j^*}$ is continuous. Let $[x, 1] \in C_j^*$. If $g_j([x, 1]) \in C_{j+1}^* \setminus E_{j+1}^*$, then $x \in C_j \setminus E_j$ by the definition of g_j. Since for every $i \in \mathbb{Z}_k$ the map $[y] \mapsto [y, 1]$ is a homeomorphism $C_i/E_i \setminus [E_i] \to C_i^* \setminus E_i^*$ and $C_i^* \setminus E_i^*$ is open in M_i, the continuity of g_j at $[x, 1]$ follows from the continuity of $f_j^{\#}$ at $[x]$.

Assume now that $g_j([x, 1]) \in E_{j+1}^*$. Let Z be an open neighborhood of $g_j([x, 1])$ in M_{j+1}. Then, for some open neighborhood U of $f_j(x)$ in C_{j+1}, and some open neighborhood V of $(f_j(x), 1)$ in $X_{j+1} \times S^1$,

$$Z = \{[y, 1] \in M_{j+1} : y \in U\} \cup \{[z, 0] \in M_{j+1} : z \in V\}.$$

Let U' and V' be open subsets of X_{j+1} such that $U = U' \cap C_{j+1}$, $f_j(x) \in V'$ and $V' \times \{1\} \subset V$. There exists a neighborhood W of $x \in C_j$ such that $f_j(W) \in U' \cap V'$. It follows that $g_j([y, 1]) \subset Z$ for every $y \in W$, hence g_j is continuous at $[x, 1]$.

Finally, let $g_j([x, 1]) \in M_{j+1} \setminus C_{j+1}^*$. It follows that $f_j(x) \in X_{j+1} \setminus C_{j+1}$. Let Z be a neighborhood of $g_j([x, 1])$ in M_{j+1}. One can assume that $Z := \{[z, 0] \in M_{j+1} : z \in V\}$ for some open set $V \subset X_{j+1} \times S^1$ such that

$$V \cap C_{j+1} \times \{1\} = \emptyset.$$

Let V' be an open neighborhood of $f_{j+1}(x)$ such that $V' \times \{1\} \subset V$, and let N be a neighborhood of x in C_j such that $f_j(N) \subset V'$. Then $g_j([y, 1]) \in Z$ for every $y \in N$, hence the continuity of g_j follows.

Ad (II). This follows from [**2**, Theorem (IV.6.1)].

Ad (III). It suffices to prove that g_{j_0} is compact. We have already proved that Ψ_{j_0} is compact, hence $g_{j_0}|_{D_{j_0}^*}$ is compact. Since C_{j_0+1} is closed in X_{j_0+1} and f_{j_0} is compact, $g_{j_0}|_{C_{j_0}^*}$ is also compact and the result follows.

Ad (IV). Since the map

$$C_j/E_j \setminus [E_j] \ni [x] \mapsto [x, 1] \in M_j \setminus D_j^*$$

is a homeomorphism for every $j \in \mathbb{Z}_k$, by the strong excision property (see [**14**, Theorem 6.6.5]) the maps

$$H^*(g_{k-1} \circ \cdots \circ g_0) : H^*(M_0, D_0^*) \to H^*(M_0, D_0^*),$$
$$H^*(f_{k-1}^{\#} \circ \cdots \circ f_0^{\#}) : H^*(C_0/E_0, [E_0]) \to H^*(C_0/E_0, [E_0])$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

are conjugate. The latter map is conjugate to $\widetilde{H}^*(f^\#_{k-1} \circ \cdots \circ f^\#_0)$. The result follows by properties of the generalized Lefschetz number (see [**3**, Lemma (2.2)]). □

In the case $k = 1$ the above result is a direct generalization of the Lefschetz fixed point theorem.

COROLLARY 2.1. *Let X be a Banach space, let (C, E) be a pair of closed ANRs in X, and let $f : C \to X$ be a compact continuous map. Assume that the induced map $f^\# : C/E \to C/E$ is continuous. Then $f^\#$ is a Lefschetz map, and if*

$$\widetilde{\Lambda}(f^\#) \neq 0,$$

then there exists an $x_0 \in \overline{C \setminus E}$ such that $f(x_0) = x_0$.

3. *Chaotic dynamics*

Let $\Sigma_r := \{1, \dots, r\}^{\mathbb{Z}}$ denote the set of bi-infinite sequences with symbols $1, \dots, r$, and let $\sigma : \Sigma_r \to \Sigma_r$ be the shift map, i.e. $\sigma(\{s_n\}_{n\in\mathbb{Z}})_k := s_{k+1}$. The space Σ_r is compact in the cartesian product topology, and σ is continuous in that topology. The map σ (or, more formally, the pair (Σ_r, σ)) is called the *full shift* on r symbols. Let $A = (A_{ij})_{i,j=1,\dots,r}$ be a square matrix, all entries of which are equal to 0 or 1. Define

$$\Sigma_A := \{\{s_n\}_{n\in\mathbb{Z}} \in \Sigma_r : A_{s_n s_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\}.$$

Denote $\sigma_A : \Sigma_A \to \Sigma_A$ the restriction of σ. σ_A is called the *subshift of finite type* generated by the matrix A. The matrix A is called *irreducible* if A^k has all non-zero entries for some positive integer k (cf. [**18**]). In that case the corresponding subshift σ_A generates a dynamical system on Σ_A which is chaotic in Devaney's sense (see [**4**]). In particular, the set of periodic points of σ_A is dense in Σ_A.

For a map $f : D \to X$, where $D \subset X$, and a subset T of D invariant under f, let $\mathrm{traj}(f, T)$ be the space of trajectories in T, i.e. sequences $\{x_n\}_{n\in\mathbb{Z}}$ in $T^{\mathbb{Z}}$ such that $f(x_n) = x_{n+1}$ for every $n \in \mathbb{Z}$. f induces the map $f^{\mathbb{Z}} : \mathrm{traj}(f, T) \to \mathrm{traj}(f, T)$, which is equal to the shift

$$f^{\mathbb{Z}}(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_0, x_1, x_2, \dots).$$

As a corollary of Theorem 2.1 we obtain the following statement on the existence of chaotic dynamics.

THEOREM 3.1. *Let X be a Banach space, for $p = 1, \dots, r$ let (C_p, E_p) be a pair of closed ANRs in X, and let $C_p \cap C_q = \emptyset$ provided $p \neq q$. Assume that for some n and every p*

$$\widetilde{H}^n(C_p/E_p) = \mathbb{Q}, \quad \widetilde{H}^i(C_p/E_p) = 0 \quad \text{for all } i \neq n.$$

Let $f : \bigcup_{p=1}^r C_p \to X$ be a compact continuous map and let $A = (A_{ij})_{i,j=1,\dots,r}$ be an irreducible matrix. If, for every p and q such that $A_{pq} = 1$, the induced map

$$f^\#_{pq} := f^\#_{(C_p,E_p),(C_q,E_q)} : C_p/E_p \to C_q/E_q$$

is continuous, and the homomorphism $\widetilde{H}^*(f^\#_{pq}) : \widetilde{H}^*(C_q/E_q) \to \widetilde{H}^*(C_p/E_p)$ is an isomorphism, then there exists a compact set $T \subset C_1 \cup \cdots \cup C_r$ invariant with respect

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

to f, and a continuous surjective map $\phi : \mathrm{traj}(f, T) \to \Sigma_A$ such that $\sigma_A \circ \phi = \phi \circ f^{\mathbb{Z}}$. Moreover, the counterimage of each K-periodic sequence in Σ_A contains at least one K-periodic trajectory of f.

Proof. Let $s \in \Sigma_A$ be a K-periodic sequence. In particular, $s_K = s_0$. It follows that $\widetilde{H}(f^{\#}_{s_{K-1}s_0} \circ \cdots \circ f^{\#}_{s_0 s_1})$ is an isomorphism, hence its Lefschetz number is non-zero since the reduced cohomologies of C_{s_0}/E_{s_0} are one-dimensional. By Theorem 2.1, there exists a sequence

$$(x^s_0, \ldots, x^s_{K-1}) \in C_{s_0} \times \cdots \times C_{s_{K-1}}$$

such that $f(x^s_{K-1}) = x^s_0$ and $f(x^s_i) = x^s_{i+1}$ for $i = 0, \ldots, K-2$. Put

$$T := \overline{\{x^s_i \in X : i = 0, \ldots, K-1,\ s \text{ is a } K\text{-periodic sequence in } \Sigma_A,\ K \in \mathbb{N}\}}.$$

As the closure of an invariant set, T is also an invariant subset of X. Further, T is compact since it is contained in $\overline{f(\bigcup_{i=1}^{r} C_p)}$. Note that $\mathrm{traj}(f, T)$ is a closed (hence compact) subset of $T^{\mathbb{Z}}$. Let the map $\phi : \mathrm{traj}(f, T) \to \Sigma_A$ be defined by $\phi(\{x_n\}) = \{s_n\}$ if and only if $x_n \in C_{s_n}$ for every $n \in \mathbb{Z}$. The map ϕ is continuous and contains the set of periodic sequences of σ_A in its image. Since the latter set is dense in Σ_A, we see that ϕ is surjective and the result is proved. □

4. *Sine-like delay equations and return maps*

In [**10**, Theorem 5.4] it is shown that equation $(-\alpha\check{s})$ with $\alpha = (9/e)(\log(9) - 1)$ possesses a heteroclinic solution $x : \mathbb{R} \to \mathbb{R}$ joining the equilibrium -1 to the equilibrium $+1$. The linearized equation at both equilibria is

$$\dot{v}(t) = \alpha v(t-1), \tag{v}$$

and has precisely one eigenvalue λ in the right half plane, which is real. The eigenvalues with negative real parts occur in complex conjugate pairs, and there is a decomposition of the state space $\mathbf{C}$ invariant under equation (v),

$$\mathbf{C} = \tilde{S} \oplus S_2 \oplus \mathbb{R} \cdot \exp(\lambda \cdot),$$

where $\tilde{S}$ is infinite dimensional, S_2 is the two-dimensional space corresponding to the leading pair $\mu, \bar{\mu}$ of stable eigenvalues, and $\exp(\lambda\cdot)$ means the corresponding function on $[-1, 0]$. Thus, in particular, the local unstable manifold at -1 is one dimensional, and its global continuation finally hits the stable manifold of $+1$. The global unstable manifold of -1 coincides with the orbit of the heteroclinic solution x.

Let us now describe the construction of a return map χ on a codimension-one surface of section. We cannot repeat all the details from [**10**], but we give the most relevant ideas. The construction is sketched here only for the unperturbed equation, to avoid technical complications. We denote the projection onto the ith factor in a product by pr_i.

Let μ be the leading stable eigenvalue of the linearized equation (v), with $\Im(\mu) > 0$. There is an isomorphism

$$\tilde{S} \times \mathbb{C} \times \mathbb{R} \to \mathbf{C} = \tilde{S} \oplus S_2 \oplus \mathbb{R} \cdot \exp(\lambda \cdot),$$
$$(\tilde{x}, w, z) \mapsto \tilde{x} + w \exp(\mu \cdot) + \bar{w} \exp(\bar{\mu} \cdot) + z \cdot \exp(\lambda \cdot).$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

We denote the inverse isomorphism by $\iota : \mathbf{C} \to \tilde{S} \times \mathbb{C} \times \mathbb{R}$. The map χ consists of a local and a global part.

The local part. In the coordinates given by ι, the local dynamics at the equilibrium -1 is given by a semigroup of linear operators $T(t) : \tilde{S} \times \mathbb{C} \times \mathbb{R} \to \tilde{S} \times \mathbb{C} \times \mathbb{R}$ $(t \geq 0)$, which has the form

$$T(t)(\tilde{x}, w, z) = (\tilde{T}(t)\tilde{x}, \exp(\mu t)w, \exp(\lambda t)z),$$

where $\{\tilde{T}(t)\}_{t\geq 0}$ denotes a semigroup of linear operators on $\tilde{S}$ satisfying $|\tilde{T}(t)| \leq K \exp(\gamma t)$, with constants $K > 0$ and $\gamma < \Re(\mu) < 0$. The relation between the semigroup and the local dynamics at -1 is as follows: consider $\psi \in \mathbf{C}$ such that the solution x^ψ with initial segment $x_0^\psi = \psi$ stays in the region around -1 in which the unperturbed equation $(-\alpha\check{s})$ is linear for all times t in an interval $[-1, \bar{t}]$, where $\bar{t} > 0$. Set $(\tilde{x}, w, z) := \iota(\psi - (-1)) \in \tilde{S} \times \mathbb{C} \times \mathbb{R}$. Then for all $t \in [0, \bar{t}]$ one has

$$x_t^\psi = -1 + \iota^{-1}T(t)(\tilde{x}, w, z) = -1 + \iota^{-1}(\tilde{T}(t)\tilde{x}, \exp(\mu t)w, \exp(\lambda t)z).$$

For a suitably chosen number $z_0 > 0$, and for $\psi = -1 + \iota^{-1}(\tilde{x}, w, z)$ such that $0 < |z| \leq z_0$, the time until the third component of the solution reaches z_0 in absolute value is given by

$$\tau(z) := \frac{1}{\lambda} \log\left(\frac{z_0}{|z|}\right).$$

Using the time map τ, one defines a 'local' map

$$\mathcal{T}_0 : \tilde{S} \times \mathbb{C} \times [-z_0, z_0] \setminus \{0\} \to \tilde{S} \times \mathbb{C} \times \{-z_0, z_0\}$$

by

$$\mathcal{T}_0(\tilde{x}, w, z) := T(\tau(z))(\tilde{x}, w, z) = (\tilde{T}(\tau(z))\tilde{x}, \exp(\mu\tau(z)w), \operatorname{sign}(z) \cdot z_0)$$

(cf. the corresponding construction in $\mathbb{R}^3$ in [**9**, p. 320]).

The global part. With an appropriately chosen number $R > 0$, one defines surfaces of section at the equilibria -1 and $+1$ by the equations $|\mathrm{pr}_2\iota(\psi - (-1))| = R$ and $|\mathrm{pr}_2\iota(\psi - 1)| = R$, respectively. The heteroclinic solution x hits the surface at $+1$ at some time $t_+ > 0$, i.e. $|\mathrm{pr}_2\iota(x_{t_+} - 1)| = R$ (see also Figure 1, which is explained in more detail below). Moreover, the parametrization of x is chosen such that there exists $t_- < -1$ with $\mathrm{pr}_3\,\iota(x_{t_-} - (-1)) = z_0$. From the implicit function theorem, one obtains solutions starting from initial values ψ close to x_{t_-} which also hit the surface of section at $+1$ after a unique time $\tilde{\tau}(\psi)$ close to $t_+ - t_-$. With the semiflow $\Phi : \mathbb{R}_0^+ \times \mathbf{C} \to \mathbf{C}$, we thus have $|\mathrm{pr}_2\iota[\Phi(\tilde{\tau}(\psi), \psi) - 1]| = R$ for such ψ. The fact that $x(t)$ is in the region of linearity of $\check{s}$ at -1 for $t \leq t_-$ implies that $x(t) = -1 + c \cdot \exp(\lambda t)$ for these t, with a constant $c > 0$. Hence we have $\mathrm{pr}_1\,\iota[x_{t_-} - (-1)] = 0$, $\mathrm{pr}_2\,\iota[x_{t_-} - (-1)] = 0$, and $\mathrm{pr}_3\,\iota[x_{t_-} - (-1)] = z_0$. Now for $(\tilde{y}, w) \in \tilde{S} \times \mathbb{C}$ close to 0, and with $\psi := -1 + \iota^{-1}(\tilde{y}, w, z_0)$, we can define a 'global' map $\mathcal{T}_1$ by setting

$$\mathcal{T}_1(\tilde{y}, w, z_0) := \iota[\Phi(\tilde{\tau}(\psi), \psi) - 1] \in \tilde{S} \times \mathbb{C} \times \mathbb{R}.$$

(Then, of course, $|\mathrm{pr}_2\iota\mathcal{T}_1(\tilde{y}, w, z_0)| = R$.) Further, $\mathcal{T}_1$ is extended to points with third component equal to $-z_0$ by setting

$$\mathcal{T}_1(\tilde{y}, w, -z_0) := -\mathcal{T}_1(-\tilde{y}, -w, z_0).$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

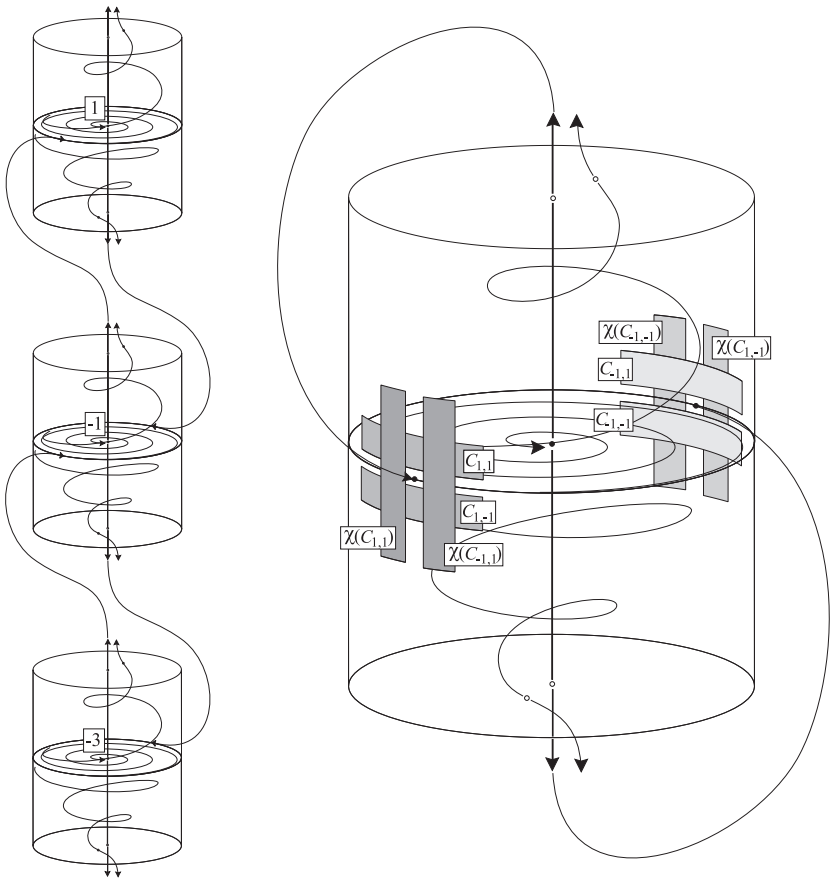


FIGURE 1. The left-hand side shows a simplified image of the heteroclinic trajectories connecting the constant solutions -3, -1, and $+1$, and the surfaces of sections around those solutions. The right-hand side shows the reduction (mod 2) of neighborhoods of the constant solutions with the sets $C_{i,j}$ and $\chi(C_{i,j})$ (case $I_1 \cap I_{-1} = \emptyset$).

(For the case of perturbed equations which may not be odd, the corresponding definition is more complicated.) The map χ is essentially obtained as a composition of $\mathcal{T}_1$ with $\mathcal{T}_0$, in appropriate coordinates. Consider $(\tilde{x}, w, z) \in \tilde{S} \times \mathbb{C} \times \mathbb{R}$ such that $0 < |z| < z_0$, and that $|w| = R$. If $|z|$ is small enough then $\mathrm{pr}_1 \mathcal{T}_0(\tilde{x}, w, z)$ and $\mathrm{pr}_2 \mathcal{T}_0(\tilde{x}, w, z)$ are close to zero, and

$$(x', w', z') := (\mathcal{T}_1 \circ \mathcal{T}_0)(\tilde{x}, w, z)$$

is well defined. Moreover, there exist unique $\theta, \theta' \in [0, 2\pi)$ with $w = R\exp(i\theta)$ and $w' = \mathrm{pr}_2(\mathcal{T}_1 \circ \mathcal{T}_0)(\tilde{x}, w, z) = R\exp(i\theta')$. The map χ is now defined by sending $(\tilde{x}, \theta, z)$ to $(x', \theta', z') \in \tilde{S} \times \mathbb{R} \times \mathbb{R}$.

As already remarked, the construction of such maps χ is possible not only for equation $(-\alpha\check{s})$, but for all nonlinearities from the class $\mathcal{F}$. We do not denote the dependence of χ on the nonlinearity; all such maps χ have in common the properties (1)–(5) which are listed below, and which will allow us to apply Theorem 3.1.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

From now on we write S for the space $\tilde{S} \times \mathbb{R}$, with the max-norm. Next, we give a more detailed description of the domain of χ. In [**10**], closed balls $I_{-1}, I_1 \subset S$ and intervals $J_{i,j} = [a_{i,j}, b_{i,j}]$ are defined for $i, j \in \{-1, 1\}$ which satisfy the following conditions:

(a) either $I_1 = I_{-1}$ or $\operatorname{dist}(I_1, I_{-1}) > 0$;

(b) $a_{i,-1} < b_{i,-1} < 0 < a_{i,1} < b_{i,1}$ for $i = \pm 1$; and

(c) $J_{1,j} = J_{-1,j}$ $(j = \pm 1)$ in case $I_{-1} = I_1$.

For $i, j \in \{-1, 1\}$, we set

$$C_{i,j} := I_i \times J_{i,j}, \quad E_{i,j} := I_i \times \{a_{i,j}, b_{i,j}\}, \quad \Delta := \bigcup_{i,j \in \{-1,1\}} C_{i,j}.$$

The map χ is defined on an open neighborhood of Δ in $S \times \mathbb{R}$ and is of class C^1.

We do not reproduce the detailed definitions of the sets $C_{i,j}$ from [**10**], but we give an indication of how they are constructed. Ignoring the first component in the decomposition $\iota(\mathbf{C}) = \tilde{S} \times \mathbb{C} \times \mathbb{R}$, one can visualize the essential part of the dynamics of the unperturbed equation $(-\alpha\check{s})$ in a three-dimensional picture, see Figure 1. The figure is simplified by identification modulo 2, which turns the heteroclinic connections into a pair of homoclinic loops, as can be seen in the right-hand side of Figure 1. In this three-dimensional projection, the surface of the section defined by $|\mathrm{pr}_2 \iota(\psi - (-1))| = 1$ is a cylinder surface which has the unstable direction as its axis.

The upper and lower homoclinic solutions hit this surface at some points ψ_+ and ψ_-, respectively. The coordinates given by the map ι of these points have the form $\iota(\psi_\pm - (-1)) = (\tilde{x}_\pm, R\exp(i\Phi_\pm), 0)$. We may regard $(\tilde{x}_\pm, \Phi_\pm, 0)$ as (generalized) cylinder coordinates of the points $\psi_\pm$. Now the sets $I_{\pm 1}$ are defined as the products $\tilde{I}_\pm \times [\Phi_\pm - \delta, \Phi_\pm + \delta]$, where $\tilde{I}_\pm$ is a small closed ball around $\tilde{x}_\pm$ in $\tilde{S}$, and $\delta > 0$ is a suitably chosen small number. In this sense, $I_{\pm 1}$ is a neighborhood of the first two cylinder coordinates of $\psi_\pm$ in $\tilde{S} \times \mathbb{R}$.

Next, for $i, j = -1, 1$, the intervals $J_{i,j} = [a_{i,j}, b_{i,j}]$ must be chosen such that:

(i) $b_{i,-1} < 0 < a_{i,1}$, $i = -1, 1$;

(ii) the time spent by points with cylinder coordinates $\xi = (\tilde{x}, \varphi) \in I_i$ and $z \in J_{i,j}$ in the region around -1, where the equation is linear and long enough (to make the part in $\tilde{S}$ decay strongly enough);

(iii) if $\xi \in I_i$, and if z varies between $a_{i,j}$ and $b_{i,j}$, then the image point $\chi(\xi, z)$ (which is obtained following the upper or lower homoclinic orbit) has a second component that varies over all intervals $J_{i,j}$, in a strictly monotonic fashion.

The precise details are given in [**10**], §3. Property (iii) can be achieved using the behavior of the local map $\mathcal{T}_0$, and a transversality type condition requiring that, in the decomposition $\tilde{S} \times \mathbb{C} \times \mathbb{R}$, the derivative $D_2\mathcal{T}_1(0, 0, z_0)$ sends the space $\mathbb{C}$ to a space with non-zero projection to the third coordinate. This condition follows from a corresponding condition on the variational equation along the heteroclinic solution x, which is shown to hold in [**10**], Lemma 5.3.

Remark. Whether $I_1 = I_{-1}$ or not depends on whether $\Phi_+ = \Phi_-$ or not. In the example from [**10**], §5, the symmetries implied $\Phi_+ = \Phi_- + \pi$, so that $\operatorname{dist}(I_1, I_{-1}) > 0$, which has to be regarded as the typical case. Since the case $I_{-1} = I_1$ may occur in other examples, we decided to include it.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

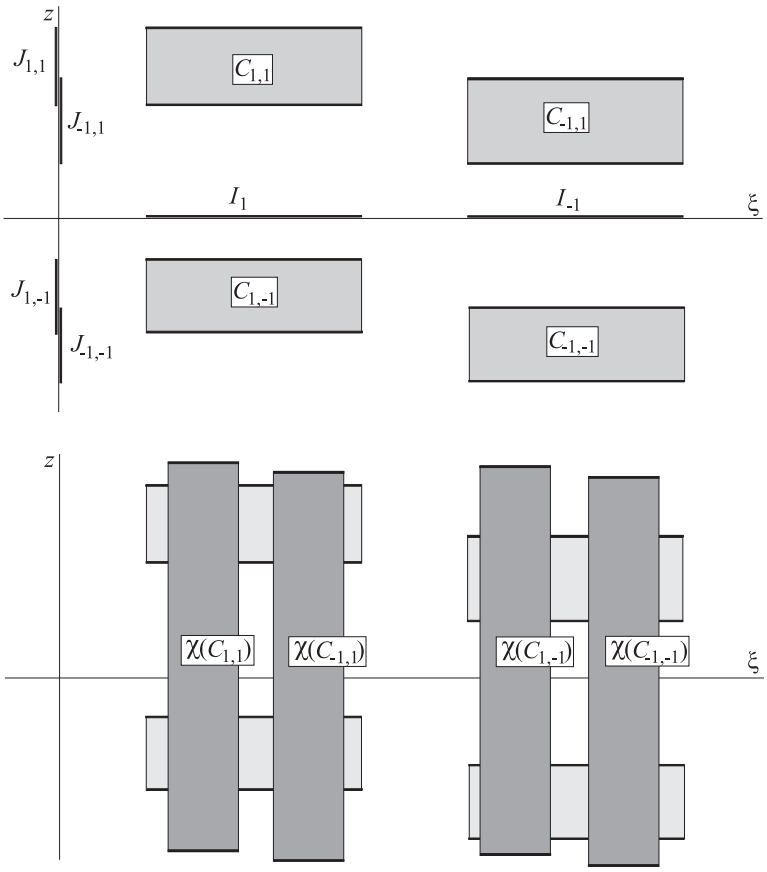


FIGURE 2. A schematic diagram of the sets $C_{i,j}$ and their images under χ. Horizontal sides of the rectangles represent the sets $E_{i,j}$ and $\chi(E_{i,j})$.

The reader should note that the sets labeled $C_{i,j}$ in Figure 1 really consist of points of the form $(R\exp(i\Phi), z)$ coming from points with generalized cylinder coordinates $(\tilde{x}, \Phi, z) \in C_{i,j}$. The actual sets $C_{i,j}$ are shown in Figure 2. Moreover, the flow lines in Figure 1 indicate the motion of the corresponding segments $-1 + \iota^{-1}(\tilde{x}, R\exp(i\Phi), z)$ under the semiflow in $\mathbf{C}$.

For the remaining part of the paper, the following properties of the map χ are relevant.

(1) The set $\overline{\chi(\Delta)}$ is compact.

(2) To every trajectory $\{(\xi_n, z_n)\}_{n\in\mathbb{Z}}$ of χ in Δ, there corresponds a unique solution $y : \mathbb{R} \to \mathbb{R}$ of the underlying equation, which follows the level sequence $\mathbf{l} = \{l_n\}_{n\in\mathbb{Z}}$ defined by $l_0 = -1, l_{n+1} = l_n + 2\,\mathrm{sign}(z_n)$.

(3) For all $(\xi, z) \in \Delta$ one has the implications

$$z > 0 \Longrightarrow \mathrm{pr}_1\,\chi(\xi, z) \in I_1, \quad z < 0 \Longrightarrow \mathrm{pr}_1\,\chi(\xi, z) \in I_{-1}.$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

(4) With $B := \max_{i,j=-1,1}\{|a_{i,j}|, |b_{i,j}|\}$ one has for $i, j = -1, 1$ and $\xi \in I_i$

$$|\mathrm{pr}_2\chi(\xi, a_{i,j})| > B, \quad |\mathrm{pr}_2\chi(\xi, b_{i,j})| > B.$$

(5) $\operatorname{sign}\mathrm{pr}_2\,\chi(\xi, a_{i,j}) = -\operatorname{sign}\mathrm{pr}_2\,\chi(\xi, b_{i,j})$ for $x \in I_i$, $i, j = -1, 1$.

These properties are in fact weaker than the properties shown in [**10**], Lemma 3.2 and Theorem 2.4.

In the following result we list some properties of the pairs $(C_{i,j}, E_{i,j})$, and of the corresponding maps induced by χ. This prepares the application of Theorem 3.1 in the proof of Theorem 1.1 in the last section.

PROPOSITION 4.1. *For $i, j \in \{-1, 1\}$, $(C_{i,j}, E_{i,j})$ is a pair of closed ANRs in the Banach space $S \times \mathbb{R}$, and*

$$\widetilde{H}^1(C_{i,j}/E_{i,j}) = \mathbb{Q}, \quad \widetilde{H}^k(C_{i,j}/E_{i,j}) = 0 \quad \textit{for all } k \neq 1.$$

Moreover, for $i, j, i', j' \in \{-1, 1\}$ the induced map $\chi^{\#}_{(i,j),(i',j')} : C_{i,j}/E_{i,j} \to C_{i',j'}/E_{i',j'}$ is equal to the constant map $[E_{i',j'}]$ provided $i' \neq j$ and $I_1 \cap I_{-1} = \emptyset$, and it is continuous and $\widetilde{H}^*(\chi^{\#}_{(i,j),(i',j')})$ is an isomorphism in the opposite case.

Proof. Since $C_{i,j}/E_{i,j}$ has the homotopy type of the one-dimensional sphere, its cohomologies have the required value. It follows directly from the definition that $\chi^{\#}_{(i,j),(i',j')}$ is constant in the case $i' \neq j$ and I_1 disjoint from I_{-1} since property (3) is satisfied.

Assume now that $i' = j$ or $I_1 = I_{-1}$. In order to simplify notation let C, E, a, b, I, and J denote, respectively, $C_{i,j}$, $E_{i,j}$, $a_{i,j}$, $b_{i,j}$, I_i, and $J_{i,j}$. We apply a similar convention to the symbols with primes, i.e. $C_{i',j'}$ is denoted by C', $a_{i',j'}$ by a', etc.

Using Proposition 2.1 we show that $\chi^{\#} : C/E \to C'/E'$ is continuous. Indeed, $\chi(E) \cap C' = \emptyset$ by property (4) and $\chi(C)\setminus C'$ is contained in $I' \times ((-\infty, a) \cup (b, \infty))$ by (3), hence the intersection of its closure with C' must be contained in E'. Let $r : S \times \mathbb{R} \to S \times \mathbb{R}$ be defined by

$$r(\xi, z) := \begin{cases} (\xi, z) & \text{if } z \in J', \\ (\xi, a') & \text{if } z \leq a', \\ (\xi, b') & \text{if } z \geq b'. \end{cases}$$

By property (3) the image of the pair (C, E) under the map $r \circ \chi$ is contained in (C', E'). In the following diagram, the vertical arrows are induced by the quotient maps:

$$\begin{array}{ccc} (C, E) & \xrightarrow{r\circ\chi} & (C', E') \\ \downarrow & & \downarrow \\ (C/E, [E]) & \xrightarrow{\chi^{\#}} & (C'/E', [E']) \end{array}$$

Since $\chi^{\#} = (r \circ \chi)^{\#}$, the diagram commutes. By the strong excision property, the vertical arrows induce isomorphisms in the Alexander–Spanier cohomologies; hence, in order to prove that $\widetilde{H}^*(\chi^{\#})$ is an isomorphism, it suffices to prove that

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

$H^*(r \circ \chi) : H^*(C', E') \to H^*(C, E)$ is an isomorphism. Let ξ_0 be a point in I and let $e : J \to C$ be given by $z \mapsto (\xi_0, z)$. Define a map ψ as the composition

$$\psi : (J, \{a, b\}) \xrightarrow{e} (C, E) \xrightarrow{r \circ \chi} (C', E') \xrightarrow{\mathrm{pr}_2} (J', \{a', b'\}).$$

The first and third arrows induce isomorphisms. Indeed, $\mathrm{pr}_2 \circ e$ and $e \circ \mathrm{pr}_2$ are equal and, respectively, homotopic to the identity. It suffices to prove that ψ induces an isomorphism. By property (5), ψ is homotopic to the linear homeomorphism $(J, \{a, b\}) \to (J', \{a', b'\})$, hence the result follows. □

5. *Proof of Theorem 1.1*

We will apply Theorem 3.1 to the sets and maps introduced in Section 4. To this aim, we should slightly change the notation: the symbol -1 replaces 2 in the shift on two symbols, i.e. Σ_2 is now defined as $\{1, -1\}^{\mathbb{Z}}$ and $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ now replace 1, 2, 3, and 4, respectively, in the shift on four symbols Σ_4.

Define a matrix

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

A is irreducible since all entries in A^2 are equal to 1. The sequence $\{(i_n, j_n)\}_{n \in \mathbb{Z}} \in \Sigma_4$ belongs to Σ_A if and only if $i_{n+1} = j_n$.

LEMMA 5.1. *The map*

$$\omega : \Sigma_A \ni \{(i_n, j_n)\}_{n \in \mathbb{Z}} \to \{j_n\}_{n \in \mathbb{Z}} \in \Sigma_2$$

is continuous with respect to the product topologies and surjective, and the counterimage of each K-periodic sequence in Σ_2 contains a K-periodic sequence of Σ_A.

Proof. (Continuity is clear) Let $s := \{s_n\} \in \Sigma_2$. Define $t := \{t_n\} \in \Sigma_A$ by $t_n := (s_{n-1}, s_n)$. Then $s = \omega(t)$, and if s is K-periodic, then t is also K-periodic. □

By combining property (1), Theorem 3.1, Proposition 4.1, and, in addition, Lemma 5.1 if $I_1 \cap I_{-1} = \emptyset$, we obtain the following assertion.

PROPOSITION 5.1. *For every sequence $\{s_n\} \in \Sigma_2$ there exists a trajectory $\{(\xi_n, z_n)\}_{n \in \mathbb{Z}}$ of χ in Δ such that $\mathrm{sign}(z_n) = s_n$ for each $n \in \mathbb{Z}$. Moreover, if the sequence $\{s_n\}$ is K-periodic then the corresponding trajectory $\{(\xi_n, z_n)\}$ can also be chosen to be K-periodic.*

Now we are ready to prove the main result of this paper.

Proof of Theorem 1.1. Let a level sequence $\mathbf{l} = (\dots, l_{-1}, l_0, l_1, \dots)$ with $l_0 = -1$ and with $|l_{n+1} - l_n| = 2$ $(n \in \mathbb{Z})$ be given. Define $s_n := \mathrm{sign}(l_{n+1} - l_n)$ for $n \in \mathbb{Z}$. Applying Proposition 5.1, we obtain a trajectory $\{(\xi_n, z_n)\}_{n \in \mathbb{Z}}$ such that $\mathrm{sign}(z_n) = s_n$ $(n \in \mathbb{Z})$. From property (2) of χ we obtain a corresponding solution $y : \mathbb{R} \to \mathbb{R}$ following the level sequence as described in property (2), and it is obvious that this sequence coincides with $\mathbf{l}$.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

Now if $\mathbf{l}$ is such that the associated sequence $\{s_n\}$ has a minimal period $K \in \mathbb{N}$, then Proposition 5.1 shows that the trajectory $\{(\xi_n, z_n)\}_{n\in\mathbb{Z}}$ can be chosen to be K-periodic. Let $y : \mathbb{R} \to \mathbb{R}$ be the corresponding solution. If $\xi_n = (\tilde{x}_n, \theta_n) \in \tilde{S} \times \mathbb{R}$ $(n \in \mathbb{Z})$, there exists a sequence of times $t_n \in \mathbb{R}$ $(n \in \mathbb{Z})$ with $t_0 = 0$ such that $y_{t_n} = l_n + \iota^{-1}(\tilde{x}_n, R\exp(i\theta_n), z_n)$ $(n \in \mathbb{Z})$, and the sequence $\{\tau_n\}_{n\in\mathbb{Z}}$ defined by $\tau_n := t_{n+1} - t_n$ is also K-periodic, and $\tau_n > 1$ for all $n \in \mathbb{Z}$ (see [**10**, Proposition 4.12 and Lemma 3.2]).

Now $(\xi_{n+K}, z_{n+K}) = (\xi_n, z_n)$ $(n \in \mathbb{Z})$ implies that $y_{t_{n+K}} = y_{t_n} + (l_{n+K} - l_n)$ $(n \in \mathbb{Z})$; with K-periodicity of the sequence $\{s_n\}$ we conclude $y_{t_{n+K}} = y_{t_n} + l_K - l_0$ $(n \in \mathbb{Z})$. Since the differential equation is autonomous with the two-periodic right-hand side, and since $l_K - l_0 \in 2\mathbb{Z}$, for $n \in \mathbb{Z}$ the function $y^n := y(t_n + \cdot) + l_K - l_0$ is also a solution of the equation, and $y^n_0 = y_{t_{n+K}}$. Forward uniqueness of solutions implies that, for $n \in \mathbb{Z}$ and $t \geq 0$, one has $y(t_n + t) + l_K - l_0 = y^n(t) = y(t_{n+K} + t)$.

Now let $s \in \mathbb{R}$. Since the differences τ_n satisfy $\tau_n > 1$ for $n \in \mathbb{Z}$, there exist $j \in \mathbb{Z}$ and $t \geq 0$ such that $t_j + t = s$. With $P := t_K - t_0$, periodicity of the sequence $\{\tau_n\}$ implies $P = t_{j+K} - t_j$. We obtain

$$\begin{aligned} y(s + P) &= y(t_j + t + P) = y(t_j + t + t_{j+K} - t_j) = y(t + t_{j+K}) \\ &= y(t_j + t) + l_K - l_0 = y(s) + l_K - l_0. \end{aligned}$$

Hence, y is a periodic solution of the second kind, with generalized period P. It remains to be shown that P is minimal with this property. In the proof of that fact we will use the following property of the solution y and the sequence of times $\{t_n\}_{n\in\mathbb{Z}}$:

$$y_{t_n}([-1, 0]) \cap \mathbb{Z} = \{l_n\} \quad (n \in \mathbb{Z}) \tag{$*$}$$

(see the beginning of the proof of Claim (4.12.3) in [**10**]).

Assume that y has a generalized period $\tilde{P}$ with $\tilde{P} \in (0, P)$. Since $P = t_K - t_0 = t_K$, there exists a unique $n \in \{0, \ldots, K - 1\}$ such that $\tilde{P} \in [t_n, t_{n+1})$. We have that y wanders from l_n to l_{n+1} on $[t_n - 1, t_{n+1}]$; cf. [**10**, Claim (4.12.3)] and $(*)$. Further, by the definition of generalized period and by $(*)$, there exists $m \in \mathbb{Z}$ with $y_{\tilde{P}} = y_0 + 2m$, which implies $y_{\tilde{P}}([-1, 0]) \cap \mathbb{Z} = \{-1 + 2m\}$. Together, we obtain that $2m - 1 \in \{l_n, l_{n+1}\}$.

Case 1. $2m - 1 = l_n$. Then y wanders

- from l_{n-1} to l_n on $[t_{n-1} - 1, \tilde{P}]$;
- from l_n to l_{n+1} on $[\tilde{P} - 1, t_{n+1}]$;
- from l_{n+1} to l_{n+2} on $[t_{n+1} - 1, t_{n+2}]$;

and so on. Hence $y(\tilde{P} + \cdot)$ follows the level sequence $(l_{n+j})_{j\in\mathbb{Z}}$. On the other hand, the facts that $y(\tilde{P} + \cdot) = y + 2m$ and that y follows $\mathbf{l}$ together imply that $y(\tilde{P} + \cdot)$ follows the level sequence $(l_j + 2m)_{j\in\mathbb{Z}}$. Thus we have $l_{n+j} = l_j + 2m$ $(j \in \mathbb{Z})$, which implies that $s_{n+j} = l_{n+1+j} - l_{n+j} = l_{j+1} - l_j = s_j$ $(j \in \mathbb{Z})$, so that n is a period of the sequence (s_j). This contradicts the minimality of K as a period of (s_j).

Case 2. $2m - 1 = l_{n+1}$. Then an argument analogous to the first case yields that (s_j) has period $n + 1$, which is again a contradiction in the case $n + 1 < K$. In the case $n + 1 = K$ we have

$$y_{\tilde{P}}([-1, 0]) \cap \mathbb{Z} = 2m - 1 = l_{n+1} = l_K = y_{t_K}([-1, 0]) \cap \mathbb{Z} = y_P([-1, 0]) \cap \mathbb{Z}.$$

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

Recalling that y wanders from l_n to $l_{n+1} = l_K$ on $[t_n - 1, t_{n+1}] = [t_n - 1, t_K]$, one obtains that $y([\tilde{P} - 1, P]) \cap \mathbb{Z} = \{l_K\}$. Since both P and $\tilde{P}$ are generalized periods of y, there exists $\tilde{m} \in \mathbb{Z}$ with $y_P = y_{\tilde{P}} + 2\tilde{m}$, and together with the last equality one sees that $\tilde{m} = 0$. It follows from forward uniqueness of solutions that $y(t + (P - \tilde{P})) = y(t)$ for $t \geq \tilde{P}$, and so $y([\tilde{P}, \infty)) \cap \mathbb{Z} = \{l_K\}$. This contradicts the facts that y follows the sequence $\mathbf{l}$ and that $|l_{k+1} - l_k| = 2$ $(k \in \mathbb{Z})$.

Thus we have obtained a contradiction in all cases, and P must be the minimal generalized period of y. □

6. *Concluding remarks*

(1) In the dynamics of delay equations as considered here, one can find not only one but many different sets Δ as above; in the case of a heteroclinic connection (which exists, for example, for the piecewise linear example $(-\alpha\check{s})$), even a countably infinite set of such regions. We did not include these technical details in the main theorem.

(2) The properties (1)–(5) of the map χ, which made the application of Theorem 3.1 possible, are considerably weaker than the corresponding conditions from [**10**]. Hence there is hope that the method of our paper may find further applications, although at this point only the presented application is available.

Acknowledgement. RS was supported by KBN, grant P03A 028 17.

References

[1] G. D. Birkhoff. On the periodic motions of dynamical systems. *Acta Math.* **50** (1927), 359–379.

[2] K. Borsuk. *Theory of Retracts (Monografie Matematyczne, 44)*. PWN, Warsaw, 1967.

[3] C. Bowszyc. Fixed point theorems for the pairs of spaces. *Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys.* **16** (1968), 845–850.

[4] R. Devaney. *Chaotic Dynamical Systems*. Addison-Wesley, New York, 1989.

[5] P. Dormayer and B. Lani-Wayda. Floquet multipliers and secondary bifurcations in functional differential equations. Numerical and analytical results. *Z. Angew. Math. Phys.* **46** (1995), 823–858.

[6] J. Dugundji. An extension of Tietze's Theorem. *Pacific J. Math.* **1** (1951), 353–367.

[7] T. Furumochi. Existence of periodic solutions of one-dimensional differential-delay equations. *Tôhoku Math. J.* **30** (1978), 13–35.

[8] A. Granas. The Leray–Schauder index and the fixed point theory for arbitrary ANRs. *Bull. Soc. Math. France* **100** (1972), 209–228.

[9] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, New York, 1986.

[10] B. Lani-Wayda. Wandering solutions of equations with sine-like feedback. *Mem. Amer. Math. Soc.* **718** (2001).

[11] H. Poincaré. *Méthodes nouvelles de la Mécanique Céleste*, vol. III. Gauthier-Villars, Paris, 1899.

[12] J. Robbin and D. Salamon. Dynamical systems, shape theory and the Conley index. *Ergod. Th. & Dynam. Sys.* **8*** (1988), 375–393.

[13] L. P. Šil'nikov. A case of the existence of a denumerable set of periodic motions. *Sov. Math. Dokl.* **6** (1965), 163–66.

[14] E. H. Spanier. *Algebraic Topology*. McGraw-Hill, New York, 1966.

[15] R. Srzednicki. Generalized Lefschetz Theorem and fixed point index formula. *Topology Appl.* **81** (1997), 207–224.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press

[16] R. Srzednicki. A generalization of the Lefschetz fixed point theorem and detection of chaos. *Proc. Amer. Math. Soc.* **128**(4) (2000), 1231–1239.

[17] H.-O. Walther. Hyperbolic periodic solutions, heteroclinic connections and transversal homoclinic points in autonomous differential delay equations. *Mem. Amer. Math. Soc.* **402** (1989).

[18] S. Wiggins. *Global Bifurcations and Chaos*. Springer, New York, 1988.

[19] W. Wischert, A. Wunderlin, A. Pelster, M. Olivier and J. Groslambert. Delay-induced instabilities in nonlinear feedback systems. *Phys. Rev.* E **49**(1) (1994), 203–219.

https://doi.org/10.1017/S0143385702000639 Published online by Cambridge University Press