B.A.J. **4**, III, 615-636 (1998)

A NOTE ON THE JUMP-EQUILIBRIUM MODEL

BY P. P. HUBER

ABSTRACT

This paper considers some of the properties of the jump-equilibrium model suggested by Smith (1996). A more detailed description of the model's derivation is provided. The model is represented in a discrete time format and a number of statistics are derived and discussed.

KEYWORDS

Stochastic Asset Models; Jump-Equilibrium Model; Financial Economics

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1. INTRODUCTION

1.1 *The Model's Objectives*

1.1.1 This paper considers some of the properties of the jump-equilibrium stochastic asset model proposed by Smith (1996).

1.1.2 The primary objective of this model appears to be to describe parsimoniously the behaviour of the nominal and real returns on the major asset classes. In addition to this general objective, Smith (1996, p1102) required a model that fully describes the fixed-interest and index-linked yield curves, allows for occasional price 'jumps', incorporates a risk-neutral law, and uses the same mathematical structure to describe all the asset classes or is symmetrical. The 'jumps' were required so that the model would replicate market crashes, such as the October 1987 crash. The risk-neutral law was required so that the model could be used to solve certain optimisation problems and to price various derivative securities. Furthermore, Smith (1996, p1089) argued that, for most actuarial applications, it is prudent to assume that markets are efficient. This is because historical inefficiencies are likely to be exploited by rational investors in the future (see Dyson & Exley, 1995). These features of the model represent an advance on other actuarial models, such as Wilkie's (1995b) model, and potentially enable it to be used in a wider range of applications.

1.1.3 However, Smith (1996, p1103) also acknowledged some of the drawbacks of the jump-equilibrium model:

> "it is often observed that markets tend to have bursts of high volatility alternating with more stable periods, sometimes referred to as an ARCH effect, which I have ignored. Neither have I allowed for mean reverting or error-correction effects. I have little doubt that the gamma

https://doi.org/10.1017/S1357321700000143 Published online by Cambridge University Press

distribution can be shown to be a poor fit to the various series where I have used it. Over a short time period, I only allow the yield curves to make parallel shifts ... The model also ... allow[s] negative yields as a possibility for all asset classes ... Finally, many of the fitting techniques I have used are of questionable validity."

This paper aims to further explore the properties of the jump-equilibrium model.

1.2 *Outline*

Section 2 describes the model and its derivation. This section provides a more detailed description of the model's derivation than that given in Smith (1996). Furthermore, the model is expressed in a discrete time format. Section 3 considers some of the statistical properties of the model. Section 4 concludes. Proofs of the equations used are provided in the Appendix.

2. DESCRIPTION

2.1 *Model Derivation*

2.1.1 The fundamental model for asset class j is given by the following equation (for $j=\mathrm{x}\equiv$sterling, $\mathrm{q}\equiv$consumer goods, $\mathrm{s}\equiv$equity securities with dividends reinvested, and $\mathrm{p}\equiv$property with income reinvested, $t>0$):

$$X_j(t) = f_j(t)\cdot\prod_{k=1}^{n}\exp\left[\beta_{jk}\cdot G_k(t) - \lambda_{jk}\cdot\int_0^t G_k(w)dw\right] \tag{2.1.1}$$

where $X_j(t)$ represents the capital value of a unit of asset class j denominated in a notional non-depreciating risk-neutral currency at time t, $f_j(t)$ are deterministic functions of t, $n=4$ represents the number of basic asset classes, $\exp[\cdot]$ represents the exponential function, and $G_j(t)$ represent independent compound Poisson processes so that $G_j(t+\tau) - G_j(t) \sim \Gamma(\tau\cdot\theta_j,1)$ and $G_j(0)=0$. The parameters β_{jk} and λ_{jk} are constrained so that $\beta_{jk}<1$ and $\lambda_{jk}\geq 0$. These constraints ensure that bond prices are finite and positive (see equation (2.1.3)).

2.1.2 The $X_j(t)$ series are not observable because they are denominated in a notional currency. However, as each asset class is denominated in the same notional currency, observable quantities can be derived by taking ratios. For example (for $t>0$):

$$R_{\mathrm{q}}(t) = \frac{X_{\mathrm{q}}(t)}{X_{\mathrm{x}}(t)} \tag{2.1.2}$$

where $R_{\mathrm{q}}(t)$ represents the actual numerical value of the retail, or consumer, price index at time t.

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2.1.3 Bond prices can be derived from equation (2.1.1) by taking expectations, because the notional currency is assumed to be risk neutral. Therefore, at time t, the price in units of asset class j of a τ-duration zero-coupon bond paying one unit of asset class j is given by (see Section A.1, for $t>0$):

$$\frac{\mathrm{E}_t[X_j(t+\tau)]}{X_j(t)} = \left(\frac{f_j(t+\tau)}{f_j(t)}\right) \cdot \prod_{k=1}^{n} \left\{ \exp[\tau \cdot (\theta_k - \lambda_{jk} \cdot G_k(t))] \cdot \left(\frac{(1-\beta_{jk})^{1-\beta_{jk}}}{(1-\beta_{jk}+\lambda_{jk}\cdot\tau)^{1-\beta_{jk}+\lambda_{jk}\cdot\tau}} \right)^{\frac{\theta_k}{\lambda_{jk}}} \right\} \tag{2.1.3}$$

where $\mathrm{E}_t[\cdot]$ represents the expectations operator conditional on all information available at time t. If $\lambda_{jk}=0$, then the limiting form of the above equation is used, which involves replacing the term inside the product with $(1-\beta_{jk})^{-\theta_k \cdot \tau}$.

2.1.4 Equation (2.1.3) can then be used to determine the functions $f_j(t)$ from the observed initial zero-coupon yield curve for each asset class. Define $\delta_j(t,\tau)$ to be the continuously compounded τ-year spot interest rate, or the τ-year zero-coupon yield, for asset class j at time t (see Hull, 1993, p81). Hence, $\delta_x(t,\tau)$ represents the fixed-interest zero-coupon yield curve and $\delta_q(t,\tau)$ represents the index-linked zero-coupon yield curve. For equity securities and property $\delta_j(t,\tau)$ is not meaningful, as these asset classes include re-invested cash flows. The functions $f_j(t)$, for $t>0$, can then be derived from the following identity (for $\tau>0$):

$$\exp[-\tau \cdot \delta_j(0,\tau)] = \frac{\mathrm{E}_0[X_j(\tau)]}{X_j(0)}. \tag{2.1.4}$$

2.1.5 Furthermore, from equation (2.1.1), $f_j(0)=X_j(0)$. Numerical values of $X_j(0)$ can be obtained by arbitrarily setting $X_x(0)=1$, then, from equation (2.1.2), $X_q(0)=R_q(0)$. For equity securities and property $X_j(0)$ is equal to the value of an appropriate total return index at time 0. Therefore, the functions $f_j(t)$ are given by (for $t\geq 0$):

$$f_j(t) = X_j(0) \cdot \exp[-t \cdot \delta_j(0,t)] \cdot \prod_{k=1}^{n} \left\{ \exp[-t \cdot \theta_k] \cdot \left(\frac{(1-\beta_{jk}+\lambda_{jk}\cdot t)^{1-\beta_{jk}+\lambda_{jk}\cdot t}}{(1-\beta_{jk})^{1-\beta_{jk}}} \right)^{\frac{\theta_k}{\lambda_{jk}}} \right\}. \tag{2.1.5}$$

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The limiting form of the above equation is used if $\lambda_{jk}=0$ (see ¶2.1.3).

2.1.6 Equation (2.1.3) can also be used to derive total return indices, which are given by (see Section A.2, for $t>0$):

$$RX_j(t,\tau) = RX_j(0,\tau)\cdot\prod_{k=1}^{n}\exp[(\beta_{jk}-\lambda_{jk}\cdot\tau)\cdot G_k(t)]\cdot(1-\beta_{jk}+\lambda_{jk}\cdot\tau)^{\theta_k\cdot t} \qquad (2.1.6)$$

where $RX_j(t,\tau)$ represents the total return index of a constantly rebalanced portfolio of τ-duration zero-coupon asset class j bonds denominated in the notional currency.

2.1.7 Note that the equity and property asset classes include re-invested cash flows, so that $X_s(t)$ and $X_p(t)$ already represent total return indices denominated in the notional currency. Hence $RX_s(t,\,0)=X_s(t)$ and $RX_p(t,\,0)=X_p(t)$, which implies that $\lambda_{sk}=\lambda_{pk}=0$ for all k, and that $\delta_s(0,\,t)=\delta_p(0,\,t)=0$ for all $t>0$.

2.1.8 The observed sterling total return indices are then given by (for $j=$ m ≡ short-term fixed-interest securities or cash, b ≡ long-term fixed-interest securities, r ≡ index-linked securities, s ≡ equity securities, and p ≡ property, $t>0$):

$$R_j(t) = \frac{RX_j(t,\tau_j)}{X_x(t)} \qquad (2.1.7)$$

where $R_j(t)$ represents the observed sterling total return index for asset class j at time t, $RX_m(\cdot,\,\cdot)\equiv RX_b(\cdot,\,\cdot)\equiv RX_x(\cdot,\,\cdot)$, $RX_r(\cdot,\,\cdot)\equiv RX_q(\cdot,\,\cdot)$, and $\tau_m=\tau_s=\tau_p=0$. Hence, the total return indices for short-term and long-term fixed-interest securities are obtained from the sterling total return index, with durations of zero and τ_b respectively. The index-linked total return index is obtained from the consumer goods total return index with a duration of τ_r.

2.2 *Discrete Time Representation*

2.2.1 The jump-equilibrium model was calibrated to annual rate of return data using a discrete time representation given by (for $j=$ m, b, r, s and p, $t>0$):

$$r_j(t,u) = \mu_j(t,u)+\zeta_j(t,u) \qquad (2.2.1)$$

$$\mu_j(t,u) = t\cdot\delta_x(0,t)-(t-u)\cdot\delta_x(0,t-u)$$
$$+\sum_{k=1}^{n}\theta_k\cdot u\cdot\left(\beta_{jk}-\beta_{xk}-\lambda_{jk}\cdot\tau_j+\lambda_{xk}\cdot\left(t-\frac{u}{2}\right)+1+\log_e\left(\frac{1-\beta_{jk}+\lambda_{jk}\cdot\tau_j}{\psi_{xk}(t,u)}\right)\right) \qquad (2.2.2)$$

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$$\begin{aligned}\zeta_j(t,u) &= \sum_{k=1}^{n}(\beta_{jk}-\beta_{xk}-\lambda_{jk}\cdot\tau_j)\cdot(G_k(t)-G_k(t-u)-\theta_k\cdot u)\\ &\quad+\lambda_{xk}\cdot\left(\int_{t-u}^{t}G_k(w)dw-\theta_k\cdot u\cdot\left(t-\frac{u}{2}\right)\right)\end{aligned} \tag{2.2.3}$$

where:

$$\psi_{jk}(t,\tau)=\left(\frac{(1-\beta_{jk}+\lambda_{jk}\cdot t)^{1-\beta_{jk}+\lambda_{jk}\cdot t}}{(1-\beta_{jk}+\lambda_{jk}\cdot(t-\tau))^{1-\beta_{jk}+\lambda_{jk}\cdot(t-\tau)}}\right)^{\frac{1}{\lambda_{jk}\cdot\tau}} \tag{2.2.4}$$

where $r_j(t,u)=\log_e R_j(t)-\log_e R_j(t-u)$ represents the continuously compounded rate of return on asset class j over the interval $t-u$ to t. The limiting form of $\psi_{jk}(t,\tau)$ at $\lambda_{jk}=0$ is $\exp[1]\cdot(1-\beta_{jk})$. As in equation (2.1.7), $\beta_{mk}=\beta_{bk}=\beta_{xk}$, $\lambda_{mk}=\lambda_{bk}=\lambda_{xk}$, $\beta_{rk}=\beta_{qk}$, and $\lambda_{rk}=\lambda_{qk}$.

2.2.2 Note that, as $G_k(t)$ is an increasing function of t, the model's error terms $\zeta_j(t,u)$ become increasingly dominated over time by the second term in equation (2.2.3) involving the integral. This suggests that the returns implied by the model become increasingly autocorrelated over time. Furthermore, as this term is identical for all securities, it suggests that the model's expected returns become increasingly cross-correlated over time.

2.2.3 The price inflation jump-equilibrium model can be described in discrete time by the following equations (for $t>0$):

$$r_q(t,u)=\mu_q(t,u)+\zeta_q(t,u) \tag{2.2.5}$$

$$\begin{aligned}\mu_q(t,u) &= t\cdot(\delta_x(0,t)-\delta_q(0,t))-(t-u)\cdot(\delta_x(0,t-u)-\delta_q(0,t-u))\\ &\quad+\sum_{k=1}^{n}\theta_k\cdot u\cdot\left(\beta_{qk}-\beta_{xk}-(\lambda_{qk}-\lambda_{xk})\cdot\left(t-\frac{u}{2}\right)+\log_e\left(\frac{\psi_{qk}(t,u)}{\psi_{xk}(t,u)}\right)\right)\end{aligned} \tag{2.2.6}$$

$$\begin{aligned}\zeta_q(t,u) &= \sum_{k=1}^{n}(\beta_{qk}-\beta_{xk})\cdot(G_k(t)-G_k(t-u)-\theta_k\cdot u)\\ &\quad-(\lambda_{qk}-\lambda_{xk})\cdot\left(\int_{t-u}^{t}G_k(w)dw-\theta_k\cdot u\cdot\left(t-\frac{u}{2}\right)\right).\end{aligned} \tag{2.2.7}$$

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2.3 *Parameter Values*

2.3.1 The model's parameter values were determined by equating the means of the model's annual risk premiums, calculated using the equations in Section 2.2, to values implied by theoretical considerations and by equating the risk premiums' other moments to values implied by historical data (see Smith, 1996, pp1130-1138). These recommended parameter values are given in Table 2.3.1. Recommended durations for general applications were $\tau_b = 15$ and $\tau_r = 10$. Note that $\beta_{xk} = \beta_{qk}$ for all k, which implies that the price inflation model (equations (2.2.6) and (2.2.7)) can be simplified.

2.3.2 The risk premiums used to determine the model's parameter values were calculated relative to the rate of return on short-term fixed-interest securities. The means and covariances, or variances if $j=l$, of these risk premiums are independent of time, and are given by (for $j, l =$ b, r, s, and p):

$$\mu_j(t,u) - \mu_m(t,u) = \sum_{k=1}^{n} \theta_k \cdot u \cdot \left(\beta_{jk} - \beta_{xk} - \lambda_{jk} \cdot \tau_j + \log_e \left(\frac{1 - \beta_{jk} + \lambda_{jk} \cdot \tau_j}{1 - \beta_{xk}} \right) \right) \tag{2.3.1}$$

$$\operatorname{cov}[r_j(t) - r_m(t), r_l(t) - r_m(t)] = \sum_{k=1}^{n} \theta_k \cdot u \cdot (\beta_{jk} - \beta_{xk} - \lambda_{jk} \cdot \tau_j) \cdot (\beta_{lk} - \beta_{xk} - \lambda_{lk} \cdot \tau_l) \tag{2.3.2}$$

where cov[·, ·] represents the covariance function.

2.3.3 As these risk premiums are independent of time, the jump-equilibrium model cannot generally be used to determine dynamic investment strategies that result in above average returns, on a risk-adjusted basis (see Smith, 1996, p1091).

2.3.4 The assumptions used to determine the theoretical risk premiums included that a representative investor has a power utility function and optimally

Table 2.3.1. Parameter values for the jump-equilibrium model

Parameter	$k=1$	$k=2$	$k=3$	$k=4$
β_{xk}	0.0623	0.1185	0.0326	0.0029
β_{qk}	0.0623	0.1185	0.0326	0.0029
β_{sk}	0.0410	0.0596	0.0326	0.0029
β_{pk}	0.0630	0.1048	−0.0212	0.0029
λ_{xk}	0.0033	0.0005	0	0
λ_{qk}	0.0016	0.0004	0.0002	0.0019
λ_{sk}	0	0	0	0
λ_{pk}	0	0	0	0
θ_k	4.9284	10	6.7772	10

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holds the market portfolio of 15% long-term fixed-interest securities, 5% index-linked securities, 60% equity securities, and 20% property. The returns for this portfolio were measured relative to the returns on short-term fixed-interest securities. This method for determining risk premiums has also been suggested by Black & Litterman (1992, p.32). It has been termed the 'equilibrium' method, because it ensures that a 'representative investor' would optimally hold the market portfolio. The 'equilibrium' method is not unique to the jump-equilibrium model, and it could potentially be used to parameterise other models, including Wilkie's (1995b) model.

2.4 *Neutral Initial Yield Curves*

2.4.1 The equations in Section 2.2 show that the jump-equilibrium models can be represented in discrete time by the sum of a mean term $\mu_j(t,u)$ and an error term $\zeta_j(t,u)$. The mean terms partly reflect the expected returns that are implied by the initial yield curves, which are given by, for nominal returns on asset class j in year t (see equation (2.2.2)):

$$t \cdot \delta_x(0,t) - (t-u) \cdot \delta_x(0,t-u). \tag{2.4.1}$$

This equation reflects the expectations hypothesis of the term structure, which was used by Dyson & Exley (1995). The balance of the mean terms can be interpreted as an allowance for a risk premium on different types of assets, including different duration fixed-interest securities.

2.4.2 Furthermore, the mean terms can be used to determine 'neutral' initial zero-coupon yield curves, which imply constant expected continuously compounded rates of return. These 'neutral' yield curves are merely useful for illustrating the properties to the jump-equilibrium model. In practical applications of the model the actual observed yield curves should always be used.

2.4.3 Figure 2.4.1 plots the 'neutral' initial zero-coupon fixed-interest yield curve that would result in a constant expected, continuously compounded, nominal rate of return on short-term fixed-interest securities of $\log_e(1.0375^2)$, using the parameters in Table 2.3.1. This chart shows that the 'neutral' fixed-interest yield curve increases from 7.36% for $\tau=0$ to 9.29% for $\tau=46$ and decreases thereafter, becoming negative after duration 157.

2.4.4 Similarly, 'neutral' initial index-linked yield curves can be calculated from the expected real rates of return implied by the model, which are given by, for asset class j in year t (for $j=$ m, b, r, s, and p, $t>0$):

$$\mu_j(t,u) - \mu_q(t,u) = t \cdot \delta_q(0,t) - (t-u) \cdot \delta_q(0,t-u) \tag{2.4.2}$$

$$+\sum_{k=1}^{n} \theta_k \cdot u \cdot \left(\beta_{jk} - \beta_{qk} - \lambda_{jk} \cdot \tau_j + \lambda_{qk} \cdot \left(t - \frac{u}{2} \right) + 1 + \log_e \left(\frac{1 - \beta_{jk} + \lambda_{jk} \cdot \tau_j}{\psi_{qk}(t,u)} \right) \right).$$

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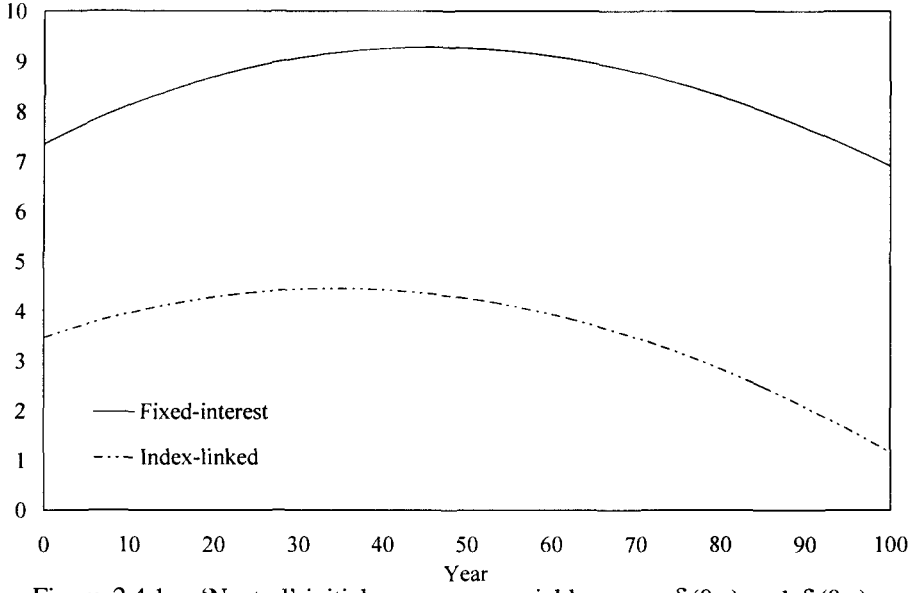


Figure 2.4.1. 'Neutral' initial zero-coupon yield curves, $\delta_x(0,\tau)$ and $\delta_q(0,\tau)$

2.4.5 Using the parameters in Table 2.3.1, Figure 2.4.1 also plots the 'neutral' initial zero-coupon index-linked yield curve that would result in a constant expected, continuously compounded, real rate of return on short-term fixed-interest securities of $\log_e(1.0175^2)$. The 'neutral' index-linked yield curve increases from 3.47% for $\tau=0$ to 4.45% for $\tau=34$ and decreases thereafter, becoming negative after duration 110.

2.4.6 Although these 'neutral' yield curves appear to be reasonable for durations of less than 35 years, their long-term shape is difficult to justify. In particular, they are not consistent with the liquidity preference hypothesis (see Hull, 1993, p87). If the initial yield curves increased continually with increasing duration, then the model would imply that expected nominal and real rates of return increase to positive infinity over time. However, this feature of the model is only significant for applications with very long time horizons.

2.4.7 If the model is used in these applications, then it is important that the initial yield curves are chosen carefully. If it is assumed that $\tau_b=15$, $\tau_r=10$, $\delta_x(0,\tau)=\log_e(1.0375^2)$ for all τ, as Smith (1996, p1152) did, and using the parameters in Table 2.3.1, the expected nominal rate of return on all assets decreases for the first 31 years by a total of 2.5% and increases thereafter to infinity. Similar results are obtained for real returns, which decrease for the first 23 years by a total of 1% and increase thereafter to infinity. Furthermore, these assumptions imply that the expected rate of price inflation decreases over time

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from 3.86% in the first year, becoming negative after year 121. These results concur with Lee's remark in the discussion of Smith (1996).

2.4.8 The price inflation model's mean terms $\mu_q(t,u)$, imply that investors generally require a risk premium on fixed-interest securities over index-linked securities with the same duration, if rational expectations are assumed (see Huang & Litzenberger, 1988). If no risk premium was required, as assumed by Wilkie (1995a, p 273) and Dyson & Exley (1995, p498), then investors' expectations of the rate of price inflation over the interval $t-u$ to t would be equal to:

$$t \cdot (\delta_x(0,t) - \delta_q(0,t)) - (t-u) \cdot (\delta_x(0,t-u) - \delta_q(0,t-u)). \tag{2.4.3}$$

Therefore, the summation term in equation (2.2.6) can be interpreted as an allowance for a risk premium on fixed-interest securities over index-linked securities. Using the parameters in Table 2.3.1, this summation term decreases continuously over time and tends to negative infinity. This is reflected in the 'neutral' initial yield curves, which diverge with increasing duration (see Figure 2.4.1).

2.5 *Spot Rate Yield Curves*

2.5.1 The spot rate yield curve for asset class j, implied by the jump-equilibrium model in year t, is given by (see equation (2.1.3), for $j=$ x, q, $\tau>0$, and $t\geq 0$):

$$\delta_j(t,\tau) = \frac{(t+\tau)\cdot\delta_j(0,t+\tau) - t\cdot\delta_j(0,t)}{\tau} + \sum_{k=1}^{n}\theta_k \cdot \left(\log_e\left(\frac{\psi_{jk}(\tau,\tau)}{\psi_{jk}(t+\tau,\tau)}\right) + \lambda_{jk}\cdot t \right) + \sum_{k=1}^{n}\lambda_{jk}\cdot(G_k(t) - \theta_k \cdot t) \tag{2.5.1}$$

where $\psi_{jk}(t,\tau)$ is defined in equation (2.2.4).

2.5.2 This equation implies that spot interest rates are calculated from the initial forward interest rates allowing for a term premium and a stochastic term. These term premiums are the same as those implied by the 'neutral' initial yield curves (see Figure 2.4.1). After allowing for a term premium, as noted by Smith (1996, p1103), the jump-equilibrium model only permits 'parallel' shifts in the yield curves. This is because the stochastic term, represented by the second summation term in equation (2.5.1), is independent of duration. Note that the variance of the expected spot interest rate is a linear function of time.

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3. STATISTICAL PROPERTIES

3.1 *Asset Class Returns*

3.1.1 Using the parameters in Table 2.3.1 and assuming 'neutral' initial yield curves, Table 3.1.1 reports the mean, standard deviation, skewness and kurtosis of the annual nominal rate of return on all the asset classes in various years (see Section A.3). Broadly similar statistics are obtained if real rates of return are considered. The means of the rates of return increase marginally over time, even though 'neutral' initial yield curves were used. This is purely a function of the definition used to obtain the 'neutral' curves. Note that, if the 'neutral' yield curves had been constrained to ensure that the annual rates of return were a constant function of time, then these curves would have had a similar shape to those illustrated in Figure 2.4.1. The variance of the rates of return also increases over time and tends to infinity. This reflects the increasing dominance of the term involving the integral in equation (2.2.3) (see ¶2.2.2). Thus, the funnel of doubt concerning the rate of return on all securities continues to increase. This assumption is not included in Wilkie's (1995b) model. Wilkie (1995b, p.779) stressed that, for long-term applications, "interest rates, or at least real interest rates, must be modelled as statistically stationary series."

Table 3.1.1. Distribution of the annual nominal rate of return in various years

		Year						
Asset class	Statistic	1	2	5	10	20	50	100
Short-term	Mean	7.64	7.64	7.65	7.67	7.70	7.79	7.94
	Std dev	0.47	0.93	1.68	2.47	3.56	5.70	8.10
	Skewness	1.12	0.72	0.45	0.35	0.29	0.28	0.31
	Kurtosis	5.07	3.83	3.33	3.20	3.15	3.15	3.18
Long-term	Mean	10.37	10.37	10.38	10.40	10.43	10.52	10.68
	Std dev	11.51	11.54	11.63	11.78	12.07	12.92	14.23
	Skewness	−0.50	−0.49	−0.47	−0.43	−0.37	−0.21	−0.03
	Kurtosis	3.17	3.17	3.16	3.14	3.11	3.07	3.06
Index-linked	Mean	8.87	8.88	8.88	8.90	8.93	9.02	9.18
	Std dev	7.42	7.47	7.60	7.82	8.24	9.40	11.07
	Skewness	−0.29	−0.28	−0.25	−0.20	−0.12	0.04	0.19
	Kurtosis	3.05	3.05	3.04	3.04	3.03	3.05	3.11
Equity	Mean	16.72	16.73	16.74	16.75	16.79	16.89	17.05
	Std dev	21.24	21.26	21.32	21.41	21.60	22.17	23.09
	Skewness	−0.04	−0.03	−0.03	−0.02	0.00	0.06	0.14
	Kurtosis	2.80	2.80	2.80	2.81	2.82	2.86	2.94
Property	Mean	10.87	10.87	10.88	10.90	10.93	11.02	11.18
	Std dev	15.55	15.57	15.64	15.76	15.99	16.65	17.72
	Skewness	−0.24	−0.23	−0.22	−0.21	−0.17	−0.09	0.03
	Kurtosis	2.90	2.90	2.90	2.90	2.91	2.93	2.98

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Table 3.1.2. Distribution of the nominal rate of return on equities over the first interval

Statistic	Daily	Weekly	Monthly	Half-yearly	Yearly	5-yearly
Mean	0.04	0.30	1.30	8.05	16.72	115.58
Std dev	0.95	2.53	5.32	13.90	21.24	87.78
Skewness	−10.34	−3.84	−1.73	−0.41	−0.04	0.94
Kurtosis	158.90	24.31	7.14	3.01	2.80	4.38

3.1.2 The distributions of the returns on all the asset classes, except for short-term fixed-interest securities, are initially negatively skewed. This implies that large negative returns are more likely to occur than large positive returns. This also suggests that interest rates are likely to be positively skewed, which concurs with the assumption made by Pentikäinen *et al.* (1995). The skewness in equity returns is not substantial.

3.1.3 The distribution of returns on short-term and long-term fixed-interest securities and index-linked securities are slightly leptokurtic. This implies that the model is more likely to produce large 'shocks' for these asset classes than if the returns were assumed to be normally distributed. The distributions of the returns on equity and property securities are initially marginally platykurtic, which suggests that the model does not generally produce large annual equity or property price 'shocks'.

3.1.4 However, over both shorter and longer averaging intervals, equity returns are leptokurtic (see Table 3.1.2). Broadly similar statistics are obtained for the other asset classes, except for the skewness and kurtosis of the returns on short-term fixed-interest securities, which are far more extreme over the shorter intervals. The statistics in Table 3.1.2 suggest that the model generates large negative equity returns over relatively short time intervals, but these large infrequent negative returns cancel out with more typical positive returns over annual intervals. The kurtosis of the distribution of daily returns appears to be too high compared to historical evidence. Anderson & Breedon (1996) estimated that the kurtosis of daily equity returns over the interval 1946-95 was 13.273. The kurtosis of the distribution of monthly returns is of a similar order to the value of 8.1889 for monthly returns over the interval 1946-95 calculated by Anderson & Breedon (1996). The skewness of the return distributions at shorter intervals is far lower than the values of −0.1155 and −0.0419 for daily and monthly equity returns over the interval 1946-95 determined by Anderson & Breedon (1996). Over longer averaging intervals, the equity return distribution is positively skewed and leptokurtic. Hence, the large negative equity returns only have a slight impact on long-term rates of return.

3.1.5 Using the parameters in Table 2.3.1, rates of return on short-term fixed-interest securities are highly positively autocorrelated (see Section A.3). The correlation between the annual nominal rate of return on short-term fixed-interest securities in year 1 and year 2 is 0.75. Rates of return on the other asset classes are initially negatively autocorrelated, but these autocorrelations increase over

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Table 3.2.1. Distribution of the annual rate of price inflation in various years

Statistic	Year 1	2	5	10	20	50	100
Mean	3.97	3.97	3.98	3.99	4.02	4.10	4.23
Std dev	0.43	0.85	1.54	2.26	3.26	5.21	7.40
Skewness	−0.30	−0.17	−0.07	−0.01	0.04	0.12	0.19
Kurtosis	3.70	3.25	3.07	3.03	3.02	3.03	3.06

time and tend to one. This result confirms the observation made in ¶2.2.2. The correlation between the annual nominal rate of return on long-term fixed-interest, index-linked, equity, and property securities in year 1 and year 2 are −0.072, −0.053, −0.018, and −0.001, respectively.

3.2 *Price Inflation*

3.2.1 Using the parameters in Table 2.3.1 and assuming 'neutral' initial yield curves, Table 3.2.1 reports the mean, standard deviation, skewness, and kurtosis of the annual rate of expected price inflation in various years (see Section A.3). This table shows that the jump-equilibrium model assumes that next year's rate of price inflation is known with relative certainty and that the level of uncertainty about future price inflation increases rapidly over time. The model assumes that virtually no information exists on the rate of price inflation at infinity. The assumption, that the variance of price inflation tends to infinity, is also incorporated in some integrated econometric price inflation models (see, for example, Clare *et al.*, 1994). This implies that the degree of confidence in price inflation forecasts decreases the further into the future these forecasts are made; the funnel of doubt continues to increase. However, the issue as to whether price inflation is stationary has not been resolved, and others, such as Wilkie (1995b), have found that stationary models adequately describe the historical data.

3.2.2 Over the first month, the jump-equilibrium model implies that the mean rate of inflation is 0.32% with a standard deviation of only 0.01%. The standard deviation of the rate in the first year, 0.43%, is significantly lower than the comparable value of 4.45% for Wilkie's model (Wilkie, 1995b, p904). This difference reflects the increased certainty gained from using the information contained in the initial yield curves.

3.2.3 The skewness coefficient is negative for the first 11 years and positive thereafter. This is inconsistent with the assumption made by Pentikäinen *et al.* (1995) that the price inflation distribution should be positively skewed. The price inflation distribution is only mildly leptokurtic. This illustrates that the jump-equilibrium model does not appear to allow for price inflation 'shocks' over annual intervals, such as those suggested by Clarkson (1991).

3.2.4 Price inflation rates are highly positively autocorrelated (see Section A.3). Using the parameters in Table 2.3.1, the correlation between the annual rate of price inflation in year 1 and year 2 is 0.75.

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3.3 *Averaged Total Returns*

3.3.1 Assuming that $\tau_b = 15$, $\tau_r = 10$, 'neutral' initial yield curves, and using the parameters in Table 2.3.1, Tables 3.3.1 to 3.3.4 report the mean, standard deviation and correlation of nominal and real averaged total returns over various terms (see Section A.4). These tables can be compared with the corresponding tables for Wilkie's model, which were calculated by simulation rather than analytically (see Wilkie, 1995, p904).

Table 3.3.1. Means and standard deviations of nominal averaged total returns

		Term						
Asset class	Statistic	1	2	5	10	20	50	100
Prices	Mean	3.97	3.97	3.97	3.98	3.99	4.01	4.06
	Std dev	0.43	0.60	0.96	1.35	1.91	3.02	4.28
Short-term	Mean	7.64	7.64	7.65	7.65	7.66	7.69	7.74
	Std dev	0.47	0.66	1.04	1.48	2.09	3.30	4.68
Long-term	Mean	10.37	10.04	9.83	9.77	9.75	9.76	9.80
	Std dev	11.51	8.00	4.60	2.70	1.40	2.05	3.74
Index-linked	Mean	8.87	8.74	8.66	8.64	8.63	8.66	8.70
	Std dev	7.42	5.16	3.11	2.20	2.03	3.00	4.42
Equity	Mean	16.72	15.71	15.09	14.89	14.79	14.76	14.79
	Std dev	21.24	15.11	9.43	6.51	4.55	3.69	4.60
Property	Mean	10.87	10.30	9.96	9.85	9.80	9.80	9.84
	Std dev	15.55	11.18	7.18	5.22	4.09	3.98	4.96

Table 3.3.2. Correlations between nominal averaged total returns

Asset class		Term						
j	l	1	2	5	10	20	50	100
Short-term	Prices	0.53	0.53	0.53	0.53	0.53	0.53	0.53
Long-term	Prices	−0.45	−0.45	−0.42	−0.37	−0.10	0.48	0.52
	Short-term	−0.85	−0.84	−0.80	−0.69	−0.19	0.90	0.99
Index-linked	Prices	0.43	0.47	0.60	0.78	0.88	0.73	0.63
	Short-term	−0.41	−0.35	−0.17	0.17	0.65	0.95	0.99
	Long-term	0.49	0.45	0.31	0.09	0.10	0.90	0.98
Equity	Prices	−0.14	−0.13	−0.09	−0.03	0.10	0.38	0.50
	Short-term	−0.37	−0.35	−0.28	−0.17	0.07	0.63	0.90
	Long-term	0.44	0.42	0.37	0.29	0.22	0.65	0.90
	Index-linked	0.28	0.25	0.19	0.11	0.16	0.63	0.90
Property	Prices	0.07	0.09	0.13	0.20	0.32	0.48	0.52
	Short-term	−0.02	0.01	0.10	0.24	0.48	0.82	0.95
	Long-term	0.03	0.00	−0.07	−0.15	−0.07	0.75	0.93
	Index-linked	0.11	0.10	0.09	0.15	0.39	0.80	0.94
	Equity	0.28	0.27	0.25	0.24	0.29	0.64	0.89

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Table 3.3.3. Means and standard deviations of real averaged total returns

		Term						
Asset class	Statistic	1	2	5	10	20	50	100
Short-term	Mean	3.53	3.53	3.53	3.54	3.55	3.57	3.62
	Std dev	0.43	0.60	0.95	1.35	1.90	3.01	4.26
Long-term	Mean	6.18	5.86	5.66	5.60	5.58	5.58	5.62
	Std dev	11.26	7.98	4.91	3.35	2.47	2.75	3.95
Index-linked	Mean	4.70	4.57	4.50	4.48	4.47	4.49	4.53
	Std dev	6.97	4.71	2.53	1.35	0.96	2.19	3.67
Equity	Mean	12.28	11.30	10.71	10.51	10.42	10.39	10.41
	Std dev	20.50	14.63	9.22	6.47	4.65	3.79	4.49
Property	Mean	6.63	6.09	5.76	5.65	5.60	5.60	5.63
	Std dev	14.93	10.72	6.84	4.93	3.79	3.59	4.46

Table 3.3.4. Correlations between real averaged total returns

Asset class		Term						
j	*l*	1	2	5	10	20	50	100
Short-term	*Prices*	−0.44	−0.44	−0.44	−0.44	−0.44	−0.44	−0.44
Long-term	Prices	−0.48	−0.51	−0.58	−0.69	−0.84	−0.78	−0.63
	Short-term	−0.42	−0.39	−0.29	−0.10	0.30	0.84	0.96
Index-linked	Prices	0.38	0.37	0.34	0.22	−0.22	−0.43	−0.44
	Short-term	−0.85	−0.83	−0.76	−0.50	0.50	0.97	0.99
	Long-term	0.50	0.48	0.41	0.30	0.39	0.85	0.97
Equity	Prices	−0.16	−0.17	−0.20	−0.25	−0.35	−0.50	−0.52
	Short-term	−0.24	−0.21	−0.15	−0.04	0.18	0.64	0.89
	Long-term	0.45	0.43	0.41	0.39	0.41	0.70	0.90
	Index-linked	0.28	0.26	0.21	0.15	0.22	0.65	0.89
Property	Prices	0.04	0.03	−0.01	−0.07	−0.18	−0.35	−0.41
	Short-term	−0.07	−0.04	0.04	0.18	0.42	0.79	0.94
	Long	0.04	0.03	0.00	−0.01	0.13	0.67	0.90
	Index-linked	0.09	0.07	0.01	−0.04	0.26	0.78	0.94
	Equity	0.28	0.27	0.26	0.26	0.31	0.64	0.88

3.3.2 Over increasing terms, the means and standard deviations of nominal and real averaged returns generally decrease initially and then increase, except for the means and standard deviations of price inflation and the returns on short-term fixed-interest securities, which increase constantly. The constant increase in the standard deviation of price inflation and the returns on short-term fixed-interest securities is related to their high autocorrelation coefficients (see ¶2.2.2). This contrasts with Wilkie's model, which is stationary, so that the corresponding standard deviations tend to zero over time.

3.3.3 The nominal returns on long-term fixed-interest securities have a higher mean and a lower standard deviation than the corresponding returns on short-term

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and index-linked securities over a 20-year term. This suggests that short-term fixed-interest securities and index-linked securities are less attractive long-term investments. However, for index-linked securities, this is not the case if real returns are considered. Property does not appear to be a particularly attractive long-term investment compared to long-term fixed-interest securities.

3.3.4 The correlations between the nominal returns generally decrease initially and then increase over increasing terms, except for the correlations with the returns on short-term fixed-interest securities, which increase constantly. Over a 50 and a 100-year term all the nominal returns are highly positively correlated, which confirms the observation made in ¶2.2.2. The correlations between price inflation and the nominal returns on long-term fixed-interest securities and on equity securities are initially negative. In addition, the correlations between the nominal returns on short-term fixed-interest securities and the nominal returns on all the other asset classes are initially negative. These correlations differ from the corresponding correlations for Wilkie's model in many respects. In particular, Wilkie's model assumes that there is little initial correlation between equity and property returns, and between short-term fixed-interest security returns and the returns on all the other asset classes, including price inflation.

3.3.5 The correlations between price inflation and the real returns on all the asset classes are generally negative. Over a 50 and a 100-year term all the real returns are highly positively correlated (see ¶2.2.2), which contrasts with Wilkie's model. The correlations between the asset class real returns, except for short-term fixed-interest securities, generally decrease and then increase over increasing terms. The correlations between the real returns on short-term fixed-interest securities and the real returns on all the other securities are initially negative and increase constantly.

4. Conclusion

4.1 This paper has attempted to further discuss the properties of the jump-equilibrium model. It does not attempt to provide a comprehensive review of the model, and, as a result, it should not be used to judge the model. Users of the model should assess the model on whether its strengths and weaknesses are relevant for their particular applications.

4.2 The properties discussed include that the model uses the current fixed-interest and index-linked yield curves to estimate expected future rates of return and price inflation. However, the yield curves that result in constant future rates of return and price inflation, or the 'neutral' initial yield curves, only have a reasonable shape for durations of less than roughly 35 years. After this, the 'neutral' yields decrease with increasing duration, which conflicts with the liquidity preference hypothesis. This is only of practical significance for very long-term actuarial applications, but it nevertheless suggests a failing of the model.

4.3 The distributions of rates of returns and price inflation were also

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examined over various terms and at various points in time. This showed that the model is non-stationary, as the variance of real and nominal rates of return and of price inflation tend to infinity over time. Moreover, the rates of return and price inflation implied by the model become increasingly cross-correlated and autocorrelated over time. Hence, the model assumes that the degree of uncertainty regarding future returns continues to increase over time, and that the modelled returns on different securities become increasingly similar over time.

4.4 Furthermore, the model implies that the distributions of annual price inflation and the annual returns on all the asset classes, except short-term fixed-interest securities, are initially negatively skewed. The distributions of the annual returns on equity securities and property are initially platykurtic. This suggests that, although the model produces negative 'shocks' over shorter time intervals, these shocks generally have little effect on annual returns.

ACKNOWLEDGEMENTS

Thanks are due to Philip Booth, Russell Gerrard, Steven Haberman, Andrew Smith, Richard Verrall, and to two anonymous referees for their helpful comments.

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https://doi.org/10.1017/S1357321700000143 Published online by Cambridge University Press

APPENDIX

PROOFS OF THE EQUATIONS

A.1 *Expected Bond Prices*

Equation (2.1.3) can be derived from equation (2.1.1) as follows:

$$\frac{\mathrm{E}_t[X_j(t+\tau)]}{X_j(t)} = \left(\frac{f_j(t+\tau)}{f_j(t)}\right)\cdot\prod_{k=1}^{n}\exp[-\lambda_{jk}\cdot\tau\cdot G_k(t)]$$

$$\cdot\prod_{k=1}^{n}\mathrm{E}_t\left[\exp\left[\beta_{jk}\cdot(G_k(t+\tau)-G_k(t))-\lambda_{jk}\cdot\int_t^{t+\tau}(G_k(w)-G_k(t))dw\right]\right]. \tag{A.1.1}$$

Now:

$$\mathrm{E}_t\left[\exp\left[\beta_{jk}\cdot(G_k(t+\tau)-G_k(t))-\lambda_{jk}\cdot\int_t^{t+\tau}(G_k(w)-G_k(t))dw\right]\right]$$

$$=\mathrm{E}_t\left[\lim_{h\to 0}\left\{\prod_{l=1}^{N=\frac{\tau}{h}}\exp[(\beta_{jk}-\lambda_{jk}\cdot h\cdot(N-l+1))\cdot(G_k(t+h\cdot l)-G_k(t+h\cdot(l-1)))]\right\}\right]$$

$$=\lim_{h\to 0}\left\{\prod_{l=1}^{N}(1-\beta_{jk}+\lambda_{jk}\cdot h\cdot(N-l+1))^{-\theta_k\cdot h}\right\}$$

$$=\exp\left[\lim_{h\to 0}\left\{-\theta_k\cdot\prod_{l=1}^{N}h\cdot\log_e(1-\beta_{jk}+\lambda_{jk}\cdot h\cdot(N-l+1))\right\}\right]$$

$$=\exp\left[-\theta_k\cdot\int_t^{t+\tau}\log_e(1-\beta_{jk}+\lambda_{jk}\cdot(t+\tau-w))dw\right]$$

$$=\exp[\theta_k\cdot\tau]\cdot\left(\frac{(1-\beta_{jk})^{1-\beta_{jk}}}{(1-\beta_{jk}+\lambda_{jk}\cdot\tau)^{1-\beta_{jk}+\lambda_{jk}\cdot\tau}}\right)^{\frac{\theta_k}{\lambda_{jk}}}.$$

Substituting this result into equation (A.1.1) gives equation (2.1.3).

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A.2 *Total Return Indices*

Equation (2.1.6) can be derived as follows:

$$\frac{RX_j(t,\tau)}{RX_j(0,\tau)} = \lim_{h\to 0}\left\{\prod_{l=1}^{N=\frac{t}{h}} \frac{\mathrm{E}_{l-h}[X_j((l-1)\cdot h+\tau)]}{\mathrm{E}_{(l-1)\cdot h}[X_j((l-1)\cdot h+\tau)]}\right\}. \tag{A.2.1}$$

Using equations (2.1.1) and (2.1.3):

$$\begin{aligned}
&\frac{\mathrm{E}_{l\cdot h}[X_j((l-1)\cdot h+\tau)]}{\mathrm{E}_{(l-1)\cdot h}[X_j((l-1)\cdot h+\tau)]} \\
&= \prod_{k=1}^{n} \exp[-h\cdot\theta_k + (\beta_{jk} - \lambda_{jk}\cdot(\tau-h))\cdot(G_k(l\cdot h) - G_k((l-1)\cdot h))] \\
&\quad \cdot\exp\left[-\lambda_{jk}\cdot\left(\int_{(l-1)\cdot h}^{l\cdot h} G_k(w)dw - h\cdot G_k((l-1)\cdot h)\right)\right] \\
&\quad \cdot(1-\beta_{jk}+\lambda_{jk}\cdot(\tau-h))^{\theta_k\cdot h}\cdot\left[1+\frac{-\lambda_{jk}\cdot h}{1-\beta_{jk}+\lambda_{jk}\cdot\tau}\right]^{\theta_k\cdot h\cdot\left(\frac{1-\beta_{jk}+\lambda_{jk}\cdot\tau}{-\lambda_{jk}\cdot h}\right)}.
\end{aligned} \tag{A.2.2}$$

Substituting equation (A.2.2) into (A.2.1) gives:

$$\begin{aligned}
\frac{RX_j(t,\tau)}{RX_j(0,\tau)} &= \lim_{h\to 0}\prod_{k=1}^{n}\exp[(\beta_{jk}-\lambda_{jk}\cdot(\tau-h))\cdot(G_k(t)-G_k(0))] \\
&\quad\cdot(1-\beta_{jk}+\lambda_{jk}\cdot(\tau-h))^{\theta_k\cdot t} \\
&\quad\cdot\exp\left[-\lambda_{jk}\cdot\left(\int_0^t G_k(w)dw - \sum_{l=1}^{N} h\cdot G_k((l-1)\cdot h)\right)\right] \\
&\quad\cdot\exp[-\theta_k\cdot t]\cdot\left[1+\frac{-\lambda_{jk}\cdot h}{1-\beta_{jk}+\lambda_{jk}\cdot\tau}\right]^{\theta_k\cdot t\cdot\left(\frac{1-\beta_{jk}+\lambda_{jk}\cdot\tau}{-\lambda_{jk}\cdot h}\right)} \\
&= \prod_{k=1}^{n}\exp[(\beta_{jk}-\lambda_{jk}\cdot\tau)\cdot G_k(t)]\cdot(1-\beta_{jk}+\lambda_{jk}\cdot\tau)^{\theta_k\cdot t}.
\end{aligned} \tag{A.2.3}$$

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A.3 *Annual Rates of Return*

A.3.1 The expected annual nominal rate of return on asset class j over the interval $t-u$ to t is given by (for $j=$ m, b, r, s, and p and $t>0$):

$$\mathrm{E}\left[\frac{R_j(t)}{R_j(t-u)}-1\right] = \left(\frac{f_{\mathrm{x}}(t-u)}{f_{\mathrm{x}}(t)}\right)\cdot \mathrm{M}_{\mathrm{x}j}(1,u)\cdot\prod_{k=1}^{n}(1-\beta_{jk}+\lambda_{jk}\cdot\tau_j)^{u\cdot\theta_k}\cdot(1-u\cdot\lambda_{\mathrm{x}k})^{-\theta_k\cdot(t-u)}-1 \tag{A.3.1}$$

where (for $1-t\cdot\beta_{jk}>0$ and $1-t\cdot(\beta_{jk}-\lambda_{jk}\cdot\tau)>0$):

$$\mathrm{M}_j(t,\tau)=\prod_{k=1}^{n}\exp[\theta\cdot\tau]\cdot\left(\frac{(1-t\cdot\beta_{jk})^{1-t\cdot\beta_{jk}}}{(1-t\cdot(\beta_{jk}-\lambda_{jk}\cdot\tau))^{1-t\cdot(\beta_{jk}-\lambda_{jk}\cdot\tau)}}\right)^{\frac{\theta_k}{t\cdot\lambda_{jk}}} \tag{A.3.2}$$

$$\beta_{\mathrm{x}jk}=\beta_{jk}-\lambda_{jk}\cdot\tau_j-\beta_{\mathrm{x}k} \quad\text{and}\quad \lambda_{\mathrm{x}jk}=-\lambda_{\mathrm{x}k}. \tag{A.3.3}$$

This result follows directly from the proof in ¶A.1.

A.3.2 The variance of the annual nominal rate of return on asset class j over the interval $t-u$ to t is given by (for $j=$ m, b, r, s, and p and $t>0$):

$$\begin{aligned}\mathrm{var}\left[\frac{R_j(t)}{R_j(t-u)}-1\right] &= \left(\frac{f_{\mathrm{x}}(t-u)}{f_{\mathrm{x}}(t)}\right)^2\cdot\left(\prod_{k=1}^{n}(1-\beta_{\mathrm{x}jk}+\lambda_{\mathrm{x}jk}\cdot\tau_j)^{2\cdot u\cdot\theta_k}\right)\\ &\cdot\left(\mathrm{M}_{\mathrm{x}j}(2,u)\cdot\prod_{k=1}^{n}(1+2\cdot u\cdot\lambda_{\mathrm{x}jk})^{-\theta_k\cdot(t-u)}\right.\\ &\left.-\left(\mathrm{M}_{\mathrm{x}j}(1,u)\cdot\prod_{k=1}^{n}(1+u\cdot\lambda_{\mathrm{x}jk})^{-\theta_k\cdot(t-u)}\right)^2\right).\end{aligned} \tag{A.3.4}$$

A.3.3 The autocovariance between the annual nominal rate of return on asset class j in year $t+u$ and year t is given by (for $j=$ m, b, r, s, and p and for $u,t>0$):

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$$\operatorname{cov}\left[\frac{R_j(t+u)}{R_j(t+u-1)}-1,\frac{R_j(t)}{R_j(t-1)}-1\right]=\left(\frac{f_x(t+u-1)}{f_x(t+u)}\right)\cdot\left(\frac{f_x(t-1)}{f_x(t)}\right)$$
$$\cdot\left(\prod_{k=1}^{n}(1-\beta_{xjk}+\lambda_{xjk}\cdot\tau_j)^{2\cdot\theta_k}\right)\cdot\left(-(\mathrm{M}_{xj}(1,1))^2\cdot\prod_{k=1}^{n}(1+\lambda_{xjk})^{-\theta_k\cdot(2\cdot t+u-2)}\right. \quad \text{(A.3.5)}$$
$$\left.+M_{xj}(2,1)\cdot\prod_{k=1}^{n}(1+\lambda_{xjk})^{-\theta_k\cdot(u-1)}\cdot(1+2\cdot\lambda_{xjk})^{-\theta_k\cdot(t-1)}\right).$$

A.3.4 The expected annual rate of price inflation over the interval $t-u$ to t is given by (for $t>0$):

$$\mathrm{E}\left[\frac{R_q(t)}{R_q(t-u)}-1\right]=\left(\frac{f_x(t-u)\cdot f_q(t)}{f_x(t)\cdot f_q(t-u)}\right)\cdot\mathrm{M}_{qx}(1,u)\cdot\prod_{k=1}^{n}(1+u\cdot\lambda_{qxk})^{-\theta_k\cdot(t-u)}-1 \quad \text{(A.3.6)}$$

where:

$$\beta_{qxk}=\beta_{qk}-\beta_{xk} \text{ and } \lambda_{qxk}=\lambda_{qk}-\lambda_{xk}. \quad \text{(A.3.7)}$$

A.3.5 The variance of the annual rate of price inflation over the interval $t-u$ to t is given by (for $t>0$):

$$\operatorname{var}\left[\frac{R_q(t)}{R_q(t-u)}-1\right]=\left(\frac{f_x(t-u)\cdot f_q(t)}{f_x(t)\cdot f_q(t-u)}\right)^2$$
$$\cdot\left(\mathrm{M}_{qx}(2,u)\cdot\prod_{k=1}^{n}(1+2\cdot u\cdot\lambda_{qxk})^{-\theta_k\cdot(t-u)}\right. \quad \text{(A.3.8)}$$
$$\left.-\left(\mathrm{M}_{qx}(1,u)\cdot\prod_{k=1}^{n}(1+u\cdot\lambda_{qxk})^{-\theta_k\cdot(t-u)}\right)^2\right).$$

A.3.6 The autocovariance between the annual rate of price inflation in year $t+u$ and year t is given by (for $u,t>0$):

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$$\operatorname{cov}\left[\frac{R_q(t+u)}{R_q(t+u-1)}-1,\frac{R_q(t)}{R_q(t-1)}-1\right]=\left(\frac{f_x(t+u-1)\cdot f_q(t+u)}{f_x(t+u)\cdot f_q(t+u-1)}\right)$$
$$\cdot\left(\frac{f_x(t-1)\cdot f_q(t)}{f_x(t)\cdot f_q(t-1)}\right)\cdot\left(M_{qx}(1,2)\cdot\prod_{k=1}^{n}(1+\lambda_{qxk})^{-\theta_k\cdot(u-1)}\right. \tag{A.3.9}$$
$$\left.\cdot(1+2\cdot\lambda_{qxk})^{-\theta_k\cdot(t-1)}-(M_{qx}(1,1))^2\cdot\prod_{k=1}^{n}(1+\lambda_{qxk})^{-\theta_k\cdot(2\cdot t+u-2)}\right).$$

A.3.7 The means, variances, autocovariances and covariances of the real rates of return are similar to the above equations and can generally be obtained by replacing the subscript x with the subscript q. Higher moments and moments calculated over other intervals can also be obtained using similar methods.

A.4 *Averaged Annual Returns*

A.4.1 Expected averaged annual nominal returns on asset class j over the interval 0 to t is given by (for $j=$ m, b, r, s, and p and $t>0$):

$$E[\overline{R}_j(t)]=\left(\frac{f_x(0)}{f_x(t)}\right)^{\frac{1}{t}}\cdot\left(\prod_{k=1}^{n}(1-\beta_{jk}+\lambda_{jk}\cdot\tau_j)^{\theta_k}\right)\cdot M_{xj}(\tfrac{1}{t},1)-1. \tag{A.4.1}$$

A.4.2 The covariance, or variance if $j=l$, between the averaged annual nominal returns on asset classes j and l over the interval 0 to t is given by (for $t>0$):

$$\operatorname{cov}[\overline{R}_j(t),\overline{R}_l(t)]=\left(\frac{f_x(0)}{f_x(t)}\right)^{\frac{2}{t}}\cdot\left(\prod_{k=1}^{n}(1-\beta_{jk}+\lambda_{jk}\cdot\tau_j)^{\theta_k}\right)$$
$$\cdot\left(\prod_{k=1}^{n}(1-\beta_{lk}+\lambda_{lk}\cdot\tau_l)^{\theta_k}\right)\cdot(M_{xjl}(\tfrac{1}{t},1)-M_{xj}(\tfrac{1}{t},1)\cdot M_{xl}(\tfrac{1}{t},1)) \tag{A.4.2}$$

where:

$$\beta_{xjlk}=\beta_{xjk}+\beta_{xlk}\quad\text{and}\quad\lambda_{xjlk}=\lambda_{xjk}+\lambda_{xlk}. \tag{A.4.3}$$

A.4.3 The means, variances, and covariances of the real rates of return are similar to the above equations and can generally be obtained by replacing the subscript x with q.

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A.4.4 Expected averaged annual price inflation over the interval 0 to t is given by (for $t>0$):

$$\mathrm{E}[\bar{R}_{\mathrm{q}}(t)] = \left(\frac{f_{\mathrm{q}}(t)\cdot f_{\mathrm{x}}(0)}{f_{\mathrm{x}}(t)\cdot f_{\mathrm{q}}(0)}\right)^{\frac{1}{t}} \cdot \prod_{k=1}^{n} \mathrm{E}\left[\exp\left[\frac{1}{t}\cdot\left(\beta_{\mathrm{qx}k}\cdot G_k(t) - \lambda_{\mathrm{qx}k}\cdot\int_0^t G_k(w)dw\right)\right]\right] - 1$$

$$= \left(\frac{f_{\mathrm{q}}(t)\cdot f_{\mathrm{x}}(0)}{f_{\mathrm{x}}(t)\cdot f_{\mathrm{q}}(0)}\right)^{\frac{1}{t}} \cdot \mathrm{M}_{\mathrm{qx}}(\tfrac{1}{t},1) - 1 \qquad \text{(A.4.4)}$$

where:

$$\bar{R}_j(t) = \left(\frac{R_j(t)}{R_j(0)}\right)^{\frac{1}{t}} - 1. \qquad \text{(A.4.5)}$$

A.4.5 The variance of averaged annual price inflation over the interval 0 to t is given by (for $t>0$):

$$\mathrm{var}[\bar{R}_{\mathrm{q}}(t)] = \left(\frac{f_{\mathrm{q}}(t)\cdot f_{\mathrm{x}}(0)}{f_{\mathrm{x}}(t)\cdot f_{\mathrm{q}}(0)}\right)^{\frac{2}{t}} \cdot (\mathrm{M}_{\mathrm{qx}}(\tfrac{2}{t},1) - (\mathrm{M}_{\mathrm{qx}}(\tfrac{1}{t},1))^2). \qquad \text{(A.4.6)}$$

A.4.6 The covariance between averaged annual price inflation and the annual nominal returns on asset class j over the interval 0 to t is given by (for $t>0$):

$$\mathrm{cov}[\bar{R}_{\mathrm{q}}(t), \bar{R}_j(t)] = \left(\frac{f_{\mathrm{q}}(t)\cdot(f_{\mathrm{x}}(0))^2}{(f_{\mathrm{x}}(t))^2\cdot f_{\mathrm{q}}(0)}\right)^{\frac{1}{t}} \cdot \left(\prod_{k=1}^{n}(1-\beta_{jk}+\lambda_{jk}\cdot\tau_j)^{\theta_k}\right) \qquad \text{(A.4.7)}$$

$$\cdot(\mathrm{M}_{\mathrm{qx}j}(\tfrac{1}{t},1) - \mathrm{M}_{\mathrm{qx}}(\tfrac{1}{t},1)\cdot\mathrm{M}_{\mathrm{x}j}(\tfrac{1}{t},1))$$

where:

$$\beta_{\mathrm{qx}jk} = \beta_{\mathrm{qx}k} + \beta_{\mathrm{x}jk} \quad \text{and} \quad \lambda_{\mathrm{qx}jk} = \lambda_{\mathrm{qx}k} + \lambda_{\mathrm{x}jk}. \qquad \text{(A.4.8)}$$

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