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An arithmetic Yau–Zaslow formula

Jesse Pajwani and Ambrus Pál

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Abstract

We prove an arithmetic refinement of the Yau–Zaslow formula by replacing the classical Euler characteristic in Beauville's argument by a motivic Euler characteristic, related to the work of Levine. Our result implies similar formulas for other related invariants, including a generalisation of a formula of Kharlamov and Răsdeaconu on counting real rational curves on real K3 surfaces, and Saito's determinant of cohomology.

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1. Introduction

Let X be a K3 surface over a field k that admits a complete linear system $\mathcal{C}$ consisting of curves of genus g. Assume that the curves in $\mathcal{C}$ are geometrically integral and belong to a primitive

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divisor class. It is well known that $\mathcal{C}$ contains a finite number n_g of rational curves. The starting point for this paper is the Yau–Zaslow formula from [Bea99, YZ96].

Theorem 1.1 (Complex Yau–Zaslow formula [Bea99, YZ96]). *Let $k=\mathbb{C}$. Then n_g depends only on g, not on the choice of K3 surface X or the choice of linear system of genus-g curves $\mathcal{C}$. Moreover, there is a generating series for n_g in $\mathbb{Z}[[t]]$:*

$$1+\sum_{g=1}^{\infty} n_g t^g = \prod_{n=1}^{\infty}(1-t^n)^{-24}.$$

Perhaps the historical significance of the above complex Yau–Zaslow formula is that it predicted certain genus-0 Gromov–Witten invariants; it counts the number of stable maps from $\mathbb{P}^1$ to a linear system $|L|$ with $c_1(L)^2=2g-2$. Curves in $|L|$ have arithmetic genus g so we can expect the number of rational curves to be finite, under the natural assumption that the rational curves are precisely those with g nodes, which is generically the case by Chen's theorem [Che02, Theorem 1.1]. Beauville, whose work we follow closely in this paper, noted that under good conditions these curves are precisely the ones whose compactified Jacobian has non-zero Euler characteristic, and that the Euler characteristic is 1, so he proposed taking the Euler characteristic of the relative compactified Jacobian over the linear system as another way to compute this genus-0 Gromov–Witten invariant. There has been subsequent work (see, for example, [BL00] and [FGvS99]), giving proofs in more general situations, while the state of the art is [KMP+10]. From the modern perspective, this equality between these genus-0 Gromov–Witten invariants and the Euler characteristic of the compactified Jacobian can be understood as a special case of the MNOP conjecture relating the Gromov–Witten and Donaldson–Thomas theories, with the latter simplifying to the Euler characteristic of the compactified Jacobian under good conditions. In this paper, we are concerned with giving arithmetic refinements of both sides of this equation with the aid of tools from motivic homotopy. It should be noted that another refinement is provided by [Göt14] in a different setting. While we take a different approach to these, recent papers such as [Vie25] and [EW23] compute refined Donaldson–Thomas invariants for $\mathbb{P}^3$ and $\mathbb{A}^3$, respectively, using motivic homotopy, providing evidence that it should be possible to refine this side in a general case. Similarly, in [KLS+23], the authors use a different approach to compute refinements of genus-0 Gromov–Witten invariants using tools from motivic homotopy.

For the Yau–Zaslow formula, Kharlamov and Răsdeaconu also proved a version of the Yau–Zaslow formula over $\mathbb{R}$ in [KR15].

Theorem 1.2 (Real Yau–Zaslow formula [KR15]). *Let $k=\mathbb{R}$. Let n_+ (respectively, n_-) denote the number of rational curves in $\mathcal{C}$ with Welschinger number $+$ (respectively, $-$), and let $\mathbb{W}(X,g):=n_+-n_-$. Then $\mathbb{W}(X,g)$ is given by the coefficient of t^g in the power series*

$$\prod_{i=1}^{\infty}(1+(-t^i))^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b}\in\mathbb{Z}[[t]],$$

where a is the compactly supported Euler characteristic of $X(\mathbb{R})$, and $b=\frac{1}{2}(24-a)$.

Recent papers show that for k any field of characteristic not 2, we can often refine results in enumerative geometry over $\mathbb{C}$ to obtain an element of the Grothendieck–Witt ring of our base field $\widehat{\mathrm{W}}(k)$, a ring that is widely studied for its number-theoretic properties. The element we obtain is an arithmetic refinement in the sense that we may recover the integer obtained by the classical count over $\mathbb{C}$ by applying a certain homomorphism out of this ring. Examples of

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such arithmetic refinements are an arithmetic refinement of the count of 27 lines on a smooth cubic surface in [KW21], and an arithmetic refinement of the count of lines meeting four lines in $\mathbb{P}^3$ in [SW21]. As a result of this paper, we obtain an arithmetic refinement of the Yau–Zaslow formula over any characteristic 0 field. To do this, we use a motivic Euler characteristic χ^{mot} taking values in $\widehat{\mathrm{W}}(k)$. The field of arithmetic curve counting has its origins in $\mathbb{A}^1$-homotopy theory and χ^{mot} is no different. The original definition of χ^{mot} comes from the categorical Euler characteristic in the stable motivic homotopy category over our base field. A theorem of Levine and Raksit in [LR20] allows us to compute this Euler characteristic geometrically when our varieties are smooth and projective. By [ABO+22, Theorem 2.13], this extends uniquely to a motivic measure on the ring of varieties, making it an excellent candidate for curve counting, and since the motivic Euler characteristic can be defined geometrically, we may use geometric techniques in our proofs.

To state our arithmetic Yau–Zaslow formula, we need some notation.

Notation 1.3. Let k be a characteristic 0 field, and let X and $\mathcal{C}$ be as above. Without loss of generality, we may assume that k is finitely generated over $\mathbb{Q}$, and let k^{const} denote the algebraic closure of $\mathbb{Q}$ in k. Suppose that the rational curves in $\mathcal{C}$ are nodal, which is generically the case by Chen's theorem [Che02, Theorem 1.1]. For C a curve, let $\mathcal{S}(C)$ be the set of singular points of C. Let $p \in \mathcal{S}(C)$ be a nodal point so that the tangents to C at p are defined either over $k(p)$ or over a quadratic extension, and write α_p for the element of $k(p)^\times/(k(p)^\times)^2$ such that the tangents are defined over $k(p)(\sqrt{\alpha_p})$. For q a closed point of $\mathbb{P}^g$, write C_q for the curve lying over q and $G(X)$ for the set of closed points of $\mathcal{C}$ such that C_q is rational. Let $X^{(m)}$ denote the mth symmetric power of X. Let $\Delta(g)$ be the integer from Theorem 6.18. Let $\Delta_g^{k^{\mathrm{const}}}$ be the group of order 2 characters of $\mathrm{Gal}_{k^{\mathrm{const}}}$ that are unramified at all primes of k^{const} whose residue characteristic p satisfies $p > \Delta(g) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1$, and let Δ_g^k denote the image of $\Delta_g^{k^{\mathrm{const}}}$ under the map $\mathrm{Hom}(\mathrm{Gal}_{k^{\mathrm{const}}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$. Let J_g denote the subset of elements q of $\widehat{\mathrm{W}}(k)$ such that $\mathrm{rank}(q) = 0$, $\mathrm{sign}(q) = 0$, and $\mathrm{disc}(q) \in \Delta_g^k$, which is an ideal by Definition 6.20. For L/k a finite field extension, let $\mathrm{Tr}_{L/k} : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(k)$ be the trace transfer map defined by taking a quadratic form q over L to $\mathrm{Tr}_{L/k} \circ q$, where $\mathrm{Tr}_{L/k}$ denote the Galois trace. Let $N_{L/k} : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(k)$ be Rost's norm transfer map from [Ros03, Corollary 5]. Let $\mathcal{J}$ be the ideal of $\widehat{\mathrm{W}}(k)[[t]]$ such that $t^g[q] \in \mathcal{J}$ if and only if $[q] \in J_g$. Finally, define

$$\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X) := \sum_{q \in G(X)} \mathrm{Tr}_{k(q)/k}\left(\prod_{p \in \mathcal{S}(C_q)} (1 + N_{k(p)/k(q)}([\langle -2 \rangle] - [\langle 2\alpha_p \rangle])) \right) \in \widehat{\mathrm{W}}(k)/J_g.$$

Theorem 1.4 (Arithmetic Yau–Zaslow formula). *The element* $\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)$ *is given by* $\langle (-1)^g \rangle$ *multiplied by the coefficient of* t^g *in the power series*

$$\prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \chi^{\mathrm{mot}}(X^{(m)}) \langle (-1)^d \rangle t^{md} \in \widehat{\mathrm{W}}(k)[[t]]/\mathcal{J}.$$

Because the ideal J_g is sufficiently small and because we compute $\chi^{\mathrm{mot}}(X^{(m)})$ modulo J in Corollary 8.21, we recover both the complex Yau–Zaslow formula of [Bea99] and the real Yau–Zaslow formula of [KR15] by taking the rank and signature of the above formula, respectively. We also obtain a Yau–Zaslow type formula by taking the discriminant of the above equation, modulo a finite error term in Δ_g^k. Our proof closely follows the steps laid out in Beauville's original proof (see [Bea99]) of the classical Yau–Zaslow formula, in that there are four main steps:

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(i) the computation of the motivic Euler characteristic of compactified Jacobians of curves, which captures both the number of rational curves on the surface as well as arithmetic information about their singularities;

(ii) applying a Fubini theorem, which says that the motivic Euler characteristic of the compactified Jacobian of the whole linear system is expressible in data that solely depend on the rational curves;

(iii) applying a Batyrev-Kontsevich type theorem to relate the motivic Euler characteristic of the compactified Jacobians to the motivic Euler characteristic of certain Hilbert schemes; and

(iv) applying a Göttsche formula to compute the motivic Euler characteristic of these Hilbert schemes in terms of the Euler characteristic of the surface.

Contents 1.5. The next three sections are dedicated to proving properties of the motivic Euler characteristic. In § 2, we define our motivic Euler characteristic, before showing its relationship to other invarants, including Levine's categorical Euler characteristic from [Lev20] and Saito's determinant of cohomology from [Sai94]. In § 3, we show a Galois-descent-type result for the motivic Euler characteristic. Section 4 is concerned with changing the base field, and showing that the motivic Euler characteristic is compatible with the natural transfer maps on the Grothendieck ring of varieties and the Grothendieck–Witt ring, respectively, by using the results from § 3. Sections 5–9 are concerned with the proof strategy laid out above. In § 5, we prove our version of a Fubini theorem by using an argument of Saito from [Sai94]. The Fubini theorem that we obtain is not perfect: as well as information directly related to the motivic Euler characteristic of the singular curves, we also need to take into account the monodromy action around these curves. The monodromy action gives rise to a Saito ϵ factor for computing the discriminant of the motivic Euler characteristic. We therefore compute these Saito factors, and show that for a given g, these ϵ factors lie in Δ_g^k, so in particular there are only finitely many possibilities. In § 6, we use a stratification of compactified Jacobians of curves in terms of their partial normalisations in order to directly compute their motivic Euler characteristics, and then we can apply the Fubini theorem to obtain an expression for the whole compactified Jacobian. Section 7 is concerned with a Batyrev–Kontsevich type theorem, allowing us to relate the motivic Euler characteristic of the compactified Jacobian of the linear system to the Hilbert scheme of points of the K3 surface. We prove this over real closed fields by using a model-theoretic argument to reduce to the real case, which follows by [KR15, Proposition 3.1]. For the rank and discriminant, we adapt Batyrev's original p-adic integration argument from [Bat99]. In § 8, we derive a motivic Göttsche formula, which allows us to compute the motivic Euler characteristic of these Hilbert schemes in terms of the motivic Euler characterstic of the underlying K3 surface by showing that we can apply a result of de Cataldo and Migliorini from [dCM02]. Finally, in § 9, we put these steps together to arrive at the arithmetic Yau–Zaslow formula, Theorem 1.4. We then show that Theorem 1.4 recovers the complex Yau–Zaslow formula of [Bea99] and the real Yau–Zaslow formula of [KR15], and we deduce a new formula for discriminants in Corollary 9.11.

1.1 Notation and conventions

We fix the following conventions for this paper.

– We let k denote the base field for our Yau–Zaslow formula. For this paper, we will often restrict to $\mathrm{char}(k) = 0$, and we will always restrict to $\mathrm{char}(k) \neq 2$.

– Fields denoted by $\mathbb{F}$ will be finite fields that normally arise through looking at the reduction of a model of our variety.

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- The symbol ℓ will always denote a prime number different from $\mathrm{char}(k)$ and $\mathrm{char}(\mathbb{F})$.
- If k is a field, a variety over k will mean a reduced separated scheme of finite type over $\mathrm{Spec}(k)$. We will not require varieties to be irreducible. We will consider the empty scheme $\emptyset = \mathrm{Spec}(\{0\})$ to be a variety over k.
- All orderings on fields will be assumed to be total orderings that are compatible with the operations on fields. A real closed field will mean a field equipped with an ordering such that no algebraic extension of the field can be equipped with an ordering.
- A Calabi–Yau variety will always mean a smooth projective variety with a trivial canonical bundle. In particular, a K3 surface is a two-dimensional, simply connected, Calabi–Yau variety.
- A morphism of varieties will always denote a finitely presented morphism $f : X \to Y$ that is compatible with the structure maps to the base.

We also fix the following notation for the whole paper.

$k^\times$	The multiplicative group of k.
$\overline{k}$	A fixed separable closure of k.
Gal_k	The absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k.
$\widehat{\mathrm{W}}(k)$	The Grothendieck–Witt ring of k (see Definition 2.1). This is written as $\mathrm{GW}(k)$ in some other literature in enriched enumerative geometry.
$K_0(\mathrm{Var}_k)$	The Grothendieck ring of varieties over k (see Definition 2.8).
g	A fixed positive integer.
$\mathcal{C}$	A linear system of curves of genus g lying on the K3 surface.
$\mathbb{G}_{m,k}$	The split one-dimensional affine torus, $\mathrm{Spec}(k[x, x^{-1}])$.

We should note that a choice of separable closure does not affect the results in this paper, since all results are isomorphism invariant.

2. The motivic Euler characteristic and related invariants

In this section, we first recall the definition of the motivic Euler characteristic from [ABO+22] that forms the basis for our arithmetic curve counting before showing how it relates to three different invariants, namely, the complex compactly supported Euler characteristic, the real compactly supported Euler characteristic, and the determinant of the ℓ-adic cohomology.

Definition 2.1. Let k be a field whose characteristic is different from 2. For every $a \in k$, let $\langle a \rangle$ denote the quadratic form of rank 1 such that $\langle a \rangle(\alpha) = a\alpha^2$ for any $\alpha \in k$. For every pair q, q' of quadratic forms, let $q \oplus q', q \otimes q'$ denote the orthogonal direct sum and the tensor product of q and q', respectively. The *Grothendieck–Witt ring* $\widehat{\mathrm{W}}(k)$ of quadratic forms over k is the ring generated by symbols $[q]$, where q is any quadratic form over k, subject to the relations $[q] = [q']$, when q and q' are isomorphic, the relations $[q \oplus q'] = [q] + [q']$ and $[q \otimes q'] = [q] \cdot [q']$, when q, q' are any pair of quadratic forms over k, and the relation $[\langle 0 \rangle] = 0$.

Define $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle$. The *Witt ring*, denoted by $\mathrm{W}(k)$, is the quotient $\widehat{\mathrm{W}}(k)/(\mathbb{H})$. We say that an element $q \in \widehat{\mathrm{W}}(k)$ is *hyperbolic* if it lies in the ideal generated by $\mathbb{H}$, which is the same as the abelian subgroup consisting of elements of the form $n\mathbb{H}$ for $n \in \mathbb{Z}$.

For a non-degenerate quadratic form q, the *rank* of q is the dimension of the underlying vector space, denoted by $\mathrm{rank}(q)$. It extends uniquely to a ring homomorphism $\mathrm{rank} : \widehat{\mathrm{W}}(k) \to \mathbb{Z}$. Write I for the kernel of this homomorphism. The subset $T \subset \widehat{\mathrm{W}}(k)$ of torsion elements is also an ideal, and we let $J = I^2 \cap T$.

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Definition 2.2. Let X be a smooth, irreducible projective variety over k of dimension n. Let $H^*_{\mathrm{dR}}(X/k)$ denote the de Rham cohomology of X over k. Then the composition of the cup-product square $a \mapsto a \cup a$ and the trace map $\mathrm{Tr}: H^*_{\mathrm{dR}}(X/k) \to k$ defines a quadratic form on the vector space $H^{2*}_{\mathrm{dR}}(X/k)$, which we may see is non-degenerate using Poincaré duality. Let $[H^{2*}_{\mathrm{dR}}(X/k)]$ denote the class of this quadratic form in $\widehat{\mathrm{W}}(k)$, and set

$$\chi^{\mathrm{dR}}(X) = [H^{2*}_{\mathrm{dR}}(X/k)] - \frac{1}{2}\left(\sum_{i\geq 0} \dim(H^{2i+1}_{\mathrm{dR}}(X/k))\right) \cdot \mathbb{H} \in \widehat{\mathrm{W}}(k),$$

which we will call the *de Rham Euler characteristic* of X.

In this paper, we will often work with the de Rham Euler characteristic due to its compatibility with results of Saito from [Sai94]. However, we may compare the de Rham Euler characteristic to other Euler characteristics taking values in $\widehat{\mathrm{W}}(k)$.

Definition 2.3. Fix X/k to be a smooth, irreducible projective variety of dimension n. Following [LR20, Construction 1.2], we define the *Hodge Euler characteristic* of X as follows.

Consider the trace map $\mathrm{Tr}: H^d(X, \Omega^d_{X/k}) \to k$. We can define a cohomologically bounded complex of k vector spaces $\mathrm{Hdg}(X/k)$ such that differentials are all zero and the vector space in degree n is given by $\bigoplus_{j=i+n} H^i(X, \Omega^j_{X/k})$. The cup product composed with the trace map then furnishes a non-degenerate symmetric bilinear form Tr on $\mathrm{Hdg}(X/k)$, as with the de Rham Euler characteristic. This therefore defines a class in the 0th Grothendieck–Witt group of the exact category of cohomologically bounded cochain complexes of k vector spaces in the sense of [Sch10, § 2.2]. Since cochain complexes of k vector spaces are always split, we can identify this group with $\widehat{\mathrm{W}}(k)$. We will write $\chi^{\mathrm{Hdg}}(X)$ to mean the element $(\mathrm{Hdg}(X/k), \mathrm{Tr}) \in \widehat{\mathrm{W}}(k)$ constructed above.

Lemma 2.4. *Let X/k be an irreducible smooth projective variety over k. Then we may identify $\chi^{\mathrm{dR}}(X) = \chi^{\mathrm{Hdg}}(X) \in \widehat{\mathrm{W}}(k)$.*

The following elegant proof was suggested to us by one of the referees of this paper. There is an alternative, more complicated, proof of this statement based on an argument by Speyer [Spe15], by closely studying the Hodge–de Rham spectral sequence.

Proof. We proceed by considering $X \times_k \mathbb{A}^1_k$, where we parameterise $\mathbb{A}^1_k$ by a coordinate t. Let $p_1 : X \times_k \mathbb{A}^1_k \to X$ denote projection onto the first factor and let $p_2 : X \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ denote projection onto the second factor. Consider the de Rham complex $(\Omega^*_{X/k}, d)$ on X. The fibrewise cup product and the trace map furnish a quadratic form on $Rp_{2*}(p_1^*\Omega^*_{X/k}, t \cdot d)$ in the category of cohomologically bounded cochain complexes of coherent sheaves on $\mathbb{A}^1_k$. This gives rise to an element of the 0th Grothendieck–Witt group of this abelian category in the sense of [Sch10, § 2.2], which we denote by $\widehat{\mathrm{W}}(\mathrm{Ch}^b\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^1_k}))$. Since $\mathbb{A}^1_k$ is a separated noetherian scheme, we may argue, as in [Wei13, Theorem 8.2], to identify $\widehat{\mathrm{W}}(\mathrm{Ch}^b\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^1_k})) = \widehat{\mathrm{W}}(\mathbb{A}^1_k) := KO^{[0]}_0(\mathbb{A}^1_k)$, where the Hermitian K-theory group $KO^{[0]}_0(\mathbb{A}^1_k)$ is as defined as in [PW19, § 5]. Similarly, by the same logic as the previous definition, we can identify $\widehat{\mathrm{W}}(\mathrm{Ch}^b\mathrm{Coh}(\mathcal{O}_{\mathrm{Spec}(k)})) = \widehat{\mathrm{W}}(k)$, which in turn we may identify with $KO^{[0]}_0(\mathrm{Spec}(k))$ by the definition of KO, as in [PW19, § 4].

Let $i_j : \mathrm{Spec}(k) \to \mathbb{A}^1_K$ be the inclusion at the points $j = 0, 1$. By the proper base-change theorem, we have an equality in the Grothendieck–Witt ring of k,

$$\widehat{\mathrm{W}}(\mathrm{Ch}^b\mathrm{Coh}(\mathcal{O}_{\mathrm{Spec}(k)})) \ni i_1^* Rp_{2*}(p_1^*\Omega^*_{X/k}, t \cdot d) = \bigoplus_n H^n_{\mathrm{dR}}(X)[-n] = \chi^{\mathrm{dR}}(X) \in \widehat{\mathrm{W}}(k),$$

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with cup product and trace form to k. Over $t=0$, the differential vanishes and so

$$\widehat{\mathrm{W}}(\mathrm{Ch}^b\mathrm{Coh}(\mathcal{O}_{\mathrm{Spec}(k)}))\ni i_0^*Rp_{2*}(p_1^*\Omega^*_{X/k},t\cdot d)=\bigoplus_{p,q}H^p(X,\Omega^q)[-p-q]=\chi^{\mathrm{Hdg}}(X)\in\widehat{\mathrm{W}}(k),$$

again with cup product and trace form to k, where we note that there is a potential even shift in the grading in the second complex, but this does not affect the class in $\widehat{\mathrm{W}}(k)$. Since the Hermitian K-theory groups are representable by a spectrum in the stable motivic homotopy category of k by [PW19, Theorem 1.1], they are $\mathbb{A}^1$-invariant. Therefore, $\widehat{\mathrm{W}}(k)=KO_0^{[0]}(\mathrm{Spec}(k))=KO_0^{[0]}(\mathbb{A}^1_k)$, and i_0^*, i_1^* induce identifications $KO_0^{[0]}(\mathbb{A}^1_k)=\widehat{\mathrm{W}}(k)$ with $i_0^*=i_1^*$. In particular, there is an equality in $\widehat{\mathrm{W}}(k)$:

$$\chi^{\mathrm{dR}}(X)=i_0^*(Rp_{2*}(p_1^*\Omega^*_{X/k},t\cdot d))=i_1^*(Rp_{2*}(p_1^*\Omega^*_{X/k},t\cdot d))=\chi^{\mathrm{Hdg}}(X),$$

as required. □

Definition 2.5. In [Lev20, Remark 2.1 (2)], the *categorical Euler characteristic* $\chi^{\mathrm{cat}}(X)$ is defined for an arbitrary algebraic variety X over k by using the infinite suspension spectrum $\Sigma^\infty_T X_+$ as an object of $\mathrm{SH}(k)$, the stable motivic homotopy category over k. Since $\Sigma^\infty_T X_+$ is dualisable, it gives us a canonical endomorphism of the unit object $\mathbb{S}_k$:

$$\chi^{\mathrm{cat}}(X)\in\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k).$$

By Morel's theorem [Mor02, Theorem 6.4.1], we can identify $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)\cong\widehat{\mathrm{W}}(k)$ when k has characteristic 0, and $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)\cong\widehat{\mathrm{W}}(k)[\frac{1}{p}]$ when k has positive odd characteristic.

One of the main theorems of [LR20] is the following.

Theorem 2.6 (Theorem 1.3 of [LR20]). *Let X be a smooth, projective variety over k. Then there is an equality of Euler characteristics $\chi^{\mathrm{cat}}(X)=\chi^{\mathrm{Hdg}}(X)$.*

Corollary 2.7. *Let X be a smooth, projective variety over k. Then there is an equality of Euler characteristics $\chi^{\mathrm{cat}}(X)=\chi^{\mathrm{dR}}(X)$.*

Proof. Combine the above theorem with Lemma 2.4. □

Assume that $\mathrm{char}(k)=0$ for the rest of the paper, unless otherwise stated.

Definition 2.8. Let k be any field. Let $K_0(\mathrm{Var}_k)$ denote the *Grothendieck ring of varieties over k*. This is the ring generated by symbols $[X]$, where X is any variety over k, subject to the relation $[X]=[X']$ when X and X' are isomorphic, the relation $[X]=[X-X']+[X']$ when X' is a closed subvariety in X, and $[X\times X']=[X]\cdot[X']$ when X, X' are any pair of varieties over k. The 0 element of this ring is $[\emptyset]$, and the multiplicative unit is $[\mathrm{Spec}(k)]$.

A *motivic measure* is a ring homomorphism out of $K_0(\mathrm{Var}_k)$.

The above Euler characteristic does not give us a motivic measure; see Remark 2.10. However, [ABO+22, Theorem 2.13] gives the following.

Theorem 2.9. *There is a unique ring homomorphism $\chi^{\mathrm{mot}}: K_0(\mathrm{Var}_k)\to\widehat{\mathrm{W}}(k)$ such that for every smooth, irreducible, projective variety X/k,*

$$\chi^{\mathrm{mot}}([X])=\chi^{\mathrm{Hdg}}(X)=\chi^{\mathrm{dR}}(X).$$

A key ingredient in the proof of the above is Bittner's presentation of the ring of varieties [Bit04, Theorem 3.1], which tells us that the underlying abelian group of $K_0(\mathrm{Var}_k)$ can be generated by smooth projective varieties, subject to only a smooth blow-up excision relation. In particular, the above theorem requires us to restrict to characteristic 0.

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Remark 2.10. We can show that the categorical Euler characteristic is not a motivic measure. For example, consider $\mathbb{P}^1$. Let $\{\infty\}$ denote the closed point at infinity. Then $X \setminus \{\infty\} \cong \mathbb{A}^1$, so in $K_0(\mathrm{Var}_k)$, $[\mathbb{P}^1] = [\mathbb{A}^1] + [\mathrm{Spec}(k)]$. In [Lev20, Proposition 2.5], $\chi^{\mathrm{cat}}(\mathbb{P}^1) = \langle 1\rangle + \langle -1\rangle$ is computed, but since $\mathbb{A}^1$ is clearly $\mathbb{A}^1$-contractible, we have $\chi^{\mathrm{cat}}(\mathbb{A}^1) = \chi^{\mathrm{cat}}(\mathrm{Spec}(k)) = \langle 1\rangle$. Therefore, $\chi^{\mathrm{cat}}(\mathbb{P}^1) \neq \chi^{\mathrm{cat}}(\mathbb{A}^1) + \chi^{\mathrm{cat}}(\mathrm{Spec}(k))$ whenever $\langle -1\rangle \neq \langle 1\rangle$, i.e. whenever -1 is not a square in k.

For the rest of the paper, we will write $\chi^{\mathrm{mot}}(X)$ for $\chi^{\mathrm{mot}}([X])$ for every variety X over k and call this the *motivic Euler characteristic* of X. This is a slight abuse of notation, since χ^{mot} depends on k. When the base field is ambiguous, we will write χ^{mot}_k; however, for the most part we will omit the subscript for readability.

Definition 2.11. Suppose that the dimension n of X is even. The composition of the cup product and the trace map $\mathrm{Tr} : H^{2n}_{\mathrm{dR}}(X/k) \to k$ defines a non-degenerate quadratic form on $H^n_{\mathrm{dR}}(X/k)$. Let $[H^n_{\mathrm{dR}}(X/k)] \in \widehat{\mathrm{W}}(k)$ denote the class of this quadratic form. Define integers b^+ and b^- by

$$b^- := \sum_{i<n} \dim_k H^i_{\mathrm{dR}}(X/k),$$
$$b^+ := \sum_{i<n} (-1)^i \dim_k H^i_{\mathrm{dR}}(X/k).$$

Lemma 2.12. *Suppose that X has even dimension n. Then*

$$\chi^{\mathrm{dR}}(X) = b^+ \cdot \mathbb{H} + [H^n_{\mathrm{dR}}(X/k)].$$

If X has odd dimension, then $\chi^{\mathrm{dR}}(X) = (b^+ - \frac{1}{2}\dim_k(H^n_{\mathrm{dR}}(X/k))) \cdot \mathbb{H}$.

Proof. Suppose that n is even, and consider the quadratic form on $H^{2*}_{\mathrm{dR}}(X/k)$. This decomposes as an orthogonal direct sum of the middle term $H^n_{\mathrm{dR}}(X/k)$ and of $H^{2i}_{\mathrm{dR}}(X/k) \oplus H^{2n-2i}_{\mathrm{dR}}(X/k)$, as the integer i runs between 0 and $n-1$. For every such i, the direct sum $H^{2i}_{\mathrm{dR}}(X/k) \oplus H^{2n-2i}_{\mathrm{dR}}(X/k)$ is a non-degenerate subspace of $H^{2*}_{\mathrm{dR}}(X/k)$ of rank $2\dim(H^{2i}_{\mathrm{dR}}(X/k))$ and $H^{2i}_{\mathrm{dR}}(X/k)$ is an isotropic subspace of dimension $\dim(H^{2i}_{\mathrm{dR}}(X/k))$. Therefore, the quadratic form $H^{2i}_{\mathrm{dR}}(X/k) \oplus H^{2n-2i}_{\mathrm{dR}}(X/k)$ is hyperbolic, and so is isomorphic to the orthogonal sum of $\dim(H^{2i}_{\mathrm{dR}}(X/k))$ copies of $\mathbb{H}$, so the claim is clear in the even case. The odd case follows by an identical argument. □

We are interested in recovering invariants related to quadratic forms from the Grothendieck–Witt ring. The motivic Euler characteristic encodes classical invariants on varieties, the first being the compactly supported Euler characteristic of the complex points.

Definition 2.13. For every variety X over k, let $e(X)$ denote the compactly supported Euler characteristic of the base change $X_{\overline{k}}$ of X to $\overline{k}$.

Theorem 2.14. *Let* $\mathrm{rank} : \widehat{\mathrm{W}}(k) \to \mathbb{Z}$ *be the rank homomorphism from Definition 2.1. We have* $e(X) = \mathrm{rank}(\chi^{\mathrm{mot}}(X))$ *for every algebraic variety X over k.*

Remark 2.15. We may use any Weil cohomology theory to compute the compactly supported Euler characteristic and we will get the same value; for example, we may work with ℓ-adic cohomology. Independence from the cohomology theory is underlined by Theorem 2.6, since $\chi^{\mathrm{mot}}(X)$ only depends on $\chi^{\mathrm{cat}}(X)$, an invariant that comes from the stable motivic homotopy category. For ℓ-adic cohomology, Laumon proved in [Lau81] that for any finite-type separated scheme over an algebraically closed field and any prime ℓ different from the characteristic, the ℓ-adic Euler characteristic and the compactly supported ℓ-adic Euler characteristic are equal.

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Proof of Theorem 2.14. For X a projective variety over $\mathbb{C}$, this is proven in [Lev20, Remark 2.3 (1)]. The general case can be reduced to this case as follows. The assignment $X \mapsto e(X)$ gives rise to a ring homomorphism $K_0(\mathrm{Var}_k) \to \mathbb{Z}$. Since $K_0(\mathrm{Var}_k)$ is generated by smooth projective varieties by [Bit04, Theorem 3.1], it is enough prove the claim for X smooth and projective. Let X be such a variety and let $F \subset k$ be a subfield that is finitely generated over $\mathbb{Q}$ such that X is already defined over F, i.e. there is a smooth projective variety Y over F whose base change to k is X. Fix an embedding of F into $\mathbb{C}$, which exists since F is countable. Using the naturality of the categorical Euler characteristic with respect to base change, and the invariance of the rank function on the Grothendieck–Witt ring under base change, we obtain that

$$\mathrm{rank}(\chi^{\mathrm{mot}}(X)) = \mathrm{rank}(\chi^{\mathrm{mot}}(Y \times_F k)) = \mathrm{rank}(\chi^{\mathrm{mot}}(Y)) = \mathrm{rank}(\chi^{\mathrm{mot}}(Y \times_F \mathbb{C})).$$

By proper base change for our choice of Weil cohomology theory, we have

$$e(X) = e(Y \times_F k) = e(Y) = e(Y \times_F \mathbb{C}).$$

Applying [Lev20, Remark 2.3 (1)] gives $\mathrm{rank}(\chi^{\mathrm{mot}}(Y \times_F \mathbb{C})) = e(Y \times_F \mathbb{C})$, as required. □

As well as the complex points of our variety, the motivic Euler characteristic allows us to recover information about the real points of our variety and natural generalisations of this.

Definition 2.16. Assume that k is a real closed field. Then, by Sylvester's law of inertia, every non-degenerate quadratic form q over k is isomorphic to a diagonal form

$$\underbrace{\langle 1\rangle \oplus \langle 1\rangle \oplus \cdots \oplus \langle 1\rangle}_{m} \oplus \underbrace{\langle -1\rangle \oplus \cdots \oplus \langle -1\rangle}_{n},$$

and $m - n$ only depends on the isomorphism class of q. This is called the *signature* of q and we write $\mathrm{sign}(q)$. It induces an isomorphism $\mathrm{sign} : \mathrm{W}(k) \to \mathbb{Z}$.

Now let k again be any field whose characteristic is not 2.

Definition 2.17. The *real spectrum* $\mathrm{Spr}(k)$ of k is the set of all orderings of the field k [Lam05, Definition 1.2, Chapter 8, § 1]. Note that we may have $\mathrm{Spr}(k) = \emptyset$; for example, if -1 is the sum of squares in k. In particular, $\mathrm{Spr}(k) = \emptyset$ if k has positive characteristic. Consider $\mathrm{Spr}(k)$ as a topological space by equipping it with the *Harrison topology*, which may be a slight abuse of notation if $\mathrm{Spr}(k) = \emptyset$. The definition can be found in [Lam05, Chapter 8, § 6], which we recall here for convenience. Let $S \subset k$ be a finite subset and let $\prec$ be a total ordering on the set S. Let $U(S, \prec) \subset \mathrm{Spr}(k)$ be the set consisting of all orderings $<$ on k such that $<$ restricted to S is $\prec$. The sets $U(S, \prec)$ form a sub-basis of the Harrison topology, which makes $\mathrm{Spr}(k)$ a compact and totally disconnected topological space.

Definition 2.18. For a topological space X and a ring A, let $\mathcal{C}(X, A)$ denote the ring of continuous A-valued functions, where we give A the discrete topology. For $<$ in $\mathrm{Spr}(k)$, let $k_<$ denote the real closure of the ordered field $(k, <)$. For every non-degenerate quadratic form q over k, let $\mathrm{sign}(q) : \mathrm{Spr}(k) \to \mathbb{Z}$ be the function sending each $<$ in $\mathrm{Spr}(k)$ to the signature of the base change of q to $k_<$. This is continuous with respect to the Harrison topology, so we obtain a ring homomorphism,

$$\mathrm{sign} : \mathrm{W}(k) \to \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}),$$

where we note that if $\mathrm{Spr}(k) = \emptyset$, then $\mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}) = \{0\}$, so sign is the trivial homomorphism. By slight abuse of notation, let $\mathrm{sign}: \widehat{\mathrm{W}}(k) \to \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z})$ denote the composition of the quotient map $\widehat{\mathrm{W}}(k) \to \mathrm{W}(k)$ with the map sign above.

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Definition 2.19. Again assume that k is a real closed field, and let X be an algebraic variety over k. Then $X(k)$ is a semi-algebraic set over k, so its *compactly supported Euler characteristic* $\chi^{\mathrm{rc}}(X(k)) \in \mathbb{Z}$ can be defined using real algebraic geometry (see, for example, [BCR98, § 11.2, pp. 266–67]). Now let k again be any field whose characteristic is not 2, and let X be as above. We will write $\epsilon(X) : \mathrm{Spr}(k) \to \mathbb{Z}$ for the function that assigns each $<$ in $\mathrm{Spr}(k)$ to the compactly supported real Euler characteristic of $X(k_<)$.

Theorem 2.20. *For every algebraic variety X over k, we have* $\mathrm{sign}(\chi^{\mathrm{mot}}(X)) = \epsilon(X)$.

Proof. If k has positive characteristic, this is trivial, since it has no real closures, so without loss of generality, assume that k is characteristic 0. When $k = \mathbb{R}$ and X is projective, this is [Lev20, Remark 2.3 (2)]. For k a general field, using the naturality of the motivic Euler characteristic with respect to base change, we may assume without loss of generality that k is real closed by passing to the real closure of $(k, <)$ for every point $<$ in $\mathrm{Spr}(k)$. Since the compactly supported Euler characteristic over real closed fields is a motivic measure, we may reduce to the case where X is smooth and projective, as in the proof of Theorem 2.14, so we can use the categorical Euler characteristic instead.

As we have already remarked, for every quasi-projective variety X over k, the set $X(k)$ has the structure of a semi-algebraic set over k. Let $S_\bullet(X(k))$ denote the *simplicial set of semi-algebraic singular simplexes* of the semi-algebraic space $X(k)$ of k-valued points of X as defined in [Pál22, Definition 2.5]. The assignment $X \mapsto S_\bullet(X(k))$ induces a functor $S_\bullet$ from the category of smooth quasi-projective varieties Sm/k to $s\mathbf{Set}$, the category of simplicial sets. Let SH denote the stable homotopy category of simplicial sets. The functor $S_\bullet$ gives rise to a realisation functor $\mathrm{Re}_B : \mathrm{SH}(k) \to \mathrm{SH}$ that for $X \in Sm/k$ sends the suspension spectrum $\Sigma^\infty_T X_+$ to the suspension spectrum $\Sigma^\infty(S_\bullet(X(k)))$. This induces a map $\sigma : \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \to \mathrm{End}_{\mathrm{SH}}(\mathbb{S}_k) \cong \mathbb{Z}$. We claim that σ is the signature map. It is enough to show that $\langle -1 \rangle$ goes to -1. Using the naturality of the constructions, we only need to check this for any real closed subfield $L \subseteq k$. By taking L to be the algebraic closure of $\mathbb{Q}$ in k, assume without loss of generality that L is the real closure $\mathbb{Q}^{\mathrm{real}}$ of $\mathbb{Q}$. Since $\mathbb{R}$ contains $\mathbb{Q}^{\mathrm{real}}$, we may use naturality again to reduce to the case $k = \mathbb{R}$. In this case, [Lev20, Remark 2.3 (2)] applies, since the forgetful map from the simplicial set of semi-algebraic singular simplexes to the usual simplicial set of continuous singular simplexes is a homotopy equivalence by [Pál22, Theorem 2.20(ii)]. □

Remark 2.21. By Pfister's local–global principle (see [Lam05, § VIII, Theorems 3.2, 4.9, and 6.9]), the induced map

$$\mathrm{sign} \otimes \mathrm{id}_{\mathbb{Z}[\frac{1}{2}]} : \mathrm{W}(k) \otimes \mathbb{Z}[\tfrac{1}{2}] \to \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}[\tfrac{1}{2}])$$

is an isomorphism. Therefore, the invariants rank and sign compute the Grothendieck–Witt ring up to torsion, so the motivic Euler characteristic recovers both the algebraically closed Euler characteristic in the form of the rank as well as the real closed Euler characteristic in the form of the signature. Conversely, the motivic Euler characteristic, up to torsion, can be computed from these invariants. The full motivic Euler characteristic contains interesting torsion information that cannot be recovered from these invariants. We turn our attention to some torsion information about χ^{mot} that also relates to a more classical invariant.

Definition 2.22. Assume that k is a field whose characteristic is not 2. Then every non-degenerate quadratic form q is isomorphic to a diagonal form

$$\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle.$$

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The *discriminant* $\mathrm{disc}(q)$ of the form q above is defined as the class of the product

$$a_1 \cdot a_2 \cdots a_n \text{ in } k^\times/k^{\times 2}.$$

This is independent of the diagonalisation of q, since it is the determinant of the symmetric matrix defining the bilinear form associated to q. It induces a homomorphism of the underlying abelian groups

$$\mathrm{disc} : \widehat{\mathrm{W}}(k) \to k^\times/k^{\times 2}.$$

We will freely use the identification $k^\times/k^{\times 2} = \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$.

For X a variety over k, we would like to relate $\mathrm{disc}(\chi^{\mathrm{mot}}(X))$ to other invariants on X.

Definition 2.23. Let ℓ be a prime number different from the characteristic of k, let U be a smooth scheme over k, and let $\mathcal{F}$ be a smooth ℓ-adic sheaf on U. Following Saito in [Sai94], we will consider the one-dimensional ℓ-adic representation of Gal_k given by the determinant

$$\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}) := \bigotimes_i \det H^i_c(U_{\overline{k}}, \mathcal{F})^{\otimes(-1)^i}.$$

The above representation is called the *determinant of cohomology* of $\mathcal{F}$, and for $\mathcal{F} = \mathbb{Q}_\ell$, we will call it *the ℓ-adic determinant of cohomology* of X. This is a one-dimensional ℓ-adic representation, so we will view it as an element of the abelian group $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$. For $\mathcal{F} = \mathbb{Q}_\ell$, we will write $\det_\ell(U)$ to mean $\det R\Gamma_c(U_{\overline{k}}, \mathbb{Q}_\ell)$.

Let X be a smooth proper variety over k of dimension n. As shown in [Sai94, Lemma 5], Poincaré duality allows us to obtain the identity

$$\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\tfrac{1}{2}ne(X)) \otimes \kappa(X),$$

where $\kappa(X)$ is a character of order at most 2 of Gal_k, which is independent of ℓ. Suppose that n is odd. As in [Sai94, Lemma 5], the existence of a non-degenerate alternating form on $H^n_{\acute{e}t}(X_{\overline{k}}, \mathbb{Q}_\ell)$ given by the cup product allows us to obtain that $\det(H^n_{\acute{e}t}(X_{\overline{k}}, \mathbb{Q}_\ell)) = \mathbb{Q}_\ell(-j)$ where $j = \frac{1}{2}\dim_{\mathbb{Q}_\ell}(H^n_{\acute{e}t}(X_{\overline{k}}, \mathbb{Q}_\ell))$. In particular, $\kappa(X) = 1$ if $\dim(X)$ is odd.

Theorem 2.24. *For every smooth projective variety X over k of dimension n, we have*

$$\mathrm{disc}(\chi^{\mathrm{cat}}(X)) = \mathrm{disc}(\chi^{\mathrm{dR}}(X)) = (-1)^{(1/2)n\cdot e(X)} \cdot \kappa(X).$$

If n is odd, then $e(X)$ is even, so $\frac{1}{2}n \cdot e(X)$ is always a positive integer. For X a K3 surface, this formula gives $\mathrm{disc}(\chi^{\mathrm{mot}}(X)) = \kappa(X)$. This result is essentially equivalent to a theorem of Saito from [Sai94]. First, we are going to formulate this as Saito did. The main result of Saito's paper [Sai94] linking the determinant of cohomology with de Rham cohomology is the following.

Theorem 2.25 (Saito's theorem [Sai94]). *Assume that X has even dimension n. Then*

$$\kappa(X) = (-1)^{(1/2)e(X)+b^-} \cdot \mathrm{disc}([H^n_{\mathrm{dR}}(X/k)]).$$

Proof. This is [Sai94, Theorem 2, p. 412]. □

Proof of Theorem 2.24. By Lemma 2.12, the result is trivial if $\dim(X)$ is odd, since we have $\chi^{\mathrm{dR}}(X) = \frac{1}{2}e(X) \cdot \mathbb{H}$. Therefore, suppose that $\dim(X)$ is even.

We can work with χ^{dR} instead of χ^{cat}, by Theorem 2.6. For all $q, q' \in \widehat{\mathrm{W}}(k)$, we have that the discriminant $\mathrm{disc}(q + q') = \mathrm{disc}(q) \cdot \mathrm{disc}(q')$. Applying the map disc to both sides of the equation

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in Lemma 2.12 therefore gives

$$\begin{aligned}\operatorname{disc}(\chi^{\mathrm{cat}}(X)) = \operatorname{disc}(\chi^{\mathrm{dR}}(X)) &= \operatorname{disc}(\mathbb{H})^{b^+} \cdot \operatorname{disc}([H^n_{\mathrm{dR}}(X/k)]) \\ &= (-1)^{b^-} \cdot \operatorname{disc}([H^n_{\mathrm{dR}}(X/k)]),\end{aligned}$$

where we use that $b^- \equiv b^+ \bmod 2$. The claim follows by Theorem 2.25. □

We want to extend Theorem 2.24 to all varieties by replacing χ^{dR} with χ^{mot}, and we do this by showing that the terms in Theorem 2.24 extend to motivic measures. Assume that $\operatorname{char}(k) = 0$.

Lemma 2.26. *The ℓ-adic determinant of cohomology $\det_\ell : K_0(\mathrm{Var}_k) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$ is a homomorphism of abelian groups.*

Proof. Let X be a variety over k, let $j : U \to X$ be an open immersion, and let $i : Z \to X$ be the closed immersion, which is the complement of j. We claim that $\det_\ell(X) = \det_\ell(U) \cdot \det_\ell(Z)$. Consider the short exact sequence of sheaves on X:

$$0 \to j_!(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell \to i_*(\mathbb{Q}_\ell) \to 0.$$

This short exact sequence gives rise to a distinguished triangle:

$$R_c\Gamma(X_{\overline{k}}, j_!(\mathbb{Q}_\ell)) \to R_c\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) \to R_c\Gamma(X_{\overline{k}}, i_*(\mathbb{Q}_\ell)) \to R_c\Gamma(X_{\overline{k}}, j_!(\mathbb{Q}_\ell))[1],$$

so taking the determinant gives

$$\det R_c\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) = \det R_c\Gamma(X_{\overline{k}}, j_!(\mathbb{Q}_\ell)) \cdot \det R_c\Gamma(X_{\overline{k}}, i_*(\mathbb{Q}_\ell)).$$

Since $j_!$ is exact, we have $\det R_c\Gamma(U_{\overline{k}}, \mathbb{Q}_\ell) = \det R_c\Gamma(X_{\overline{k}}, j_!(\mathbb{Q}_\ell))$, and the proper base change theorem gives $\det R_c\Gamma(Z_{\overline{k}}, \mathbb{Q}_\ell) = \det R_c\Gamma(X_{\overline{k}}, i_*(\mathbb{Q}_\ell))$, as required. □

Proposition 2.27. *For each non-negative integer i, there is a unique homomorphism of abelian groups $b_i : K_0(\mathrm{Var}_k) \to \mathbb{Z}$ that extends the ith Betti number of smooth projective varieties.*

Proof. Theorem 4.5 of [NS11] gives rise to a ring homomorphism $P(-,t) : K_0(\mathrm{Var}_k) \to \mathbb{Z}[t]$ such that if X is a smooth projective variety, $P(X,t) = \sum_{i=0}^{\infty}(-1)^i b_i(X) t^i$, where $b_i(X)$ denotes the ith Betti number of X. Let $c_i : \mathbb{Z}[t] \to \mathbb{Z}$ be the homomorphism of abelian groups given by taking $\sum_i \alpha_i t^i$ to $(-1)^i \alpha_i$. Defining b_i to be the composition of $P(-,t)$ with c_i gives the result. □

Definition 2.28. Since $b_i(X) = 0$ for $i > 2n$ for every smooth and projective of dimension n, every formal linear combination $\sum_{i\in\mathbb{N}} c_i b_i$ where $c_i \in \mathbb{R}$ is also well defined and gives rise to a unique homomorphism of abelian groups $K_0(\mathrm{Var}_k) \to \mathbb{R}$. When X is smooth and projective of dimension n, then

$$\begin{aligned}\sum_{i\in\mathbb{N}} \frac{(-1)^i i}{2} b_i &= \frac{(-1)^n n}{2} b_n + \sum_{i<n}\left(\frac{(-1)^i i}{2} b_i + \frac{(-1)^{2n-i}(2n-i)}{2} b_{2n-i}\right) \\ &= \frac{(-1)^n n}{2} b_n + \sum_{i<n}\left(\frac{(-1)^i i}{2} b_i + \frac{(-1)^i (2n-i)}{2} b_i\right) \\ &= \frac{n}{2}\left((-1)^n b_n + \sum_{i<n}(-1)^i 2 b_i\right) = \frac{e(X) n}{2},\end{aligned}$$

using Poincaré duality. This gives a unique homomorphism of abelian groups $w : K_0(\mathrm{Var}_k) \to \mathbb{R}$ extending the function $w(X) = [e(X)n]/2$ on smooth projective varieties over k, which we call the *total weight*. Since $[e(X)n]/2$ is an integer for X smooth and projective, [Bit04, Theorem 3.1] tells us that $w(X)$ is always an integer.

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Theorem 2.29. *For every algebraic variety X over k, we have*

$$\operatorname{disc}(\chi^{\mathrm{mot}}(X))(-1)^{w(X)} = \det_\ell(X) \cdot \mathbb{Q}_\ell(w(X)) \in \operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times).$$

Proof. The left-hand side lies in $k^\times/k^{\times 2}$, which we identify with $\operatorname{Hom}(\mathrm{Gal}_k, \{\pm 1\})$ so that we may view it as a subgroup of $\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$. By Theorem 2.24, the relation holds when X is smooth and projective. Since both sides of the equation are homomorphisms of abelian groups, the general case follows from the presentation of $K_0(\mathrm{Var}_k)$ from [Bit04, Theorem 3.1]. □

Remark 2.30. The above gives us an alternative way of characterising $w(X)$ when k is finitely generated over $\mathbb{Q}$. In this case, the cyclotomic character has infinite order, so $w(X)$ is the unique integer such that $\det_\ell(X) \cdot \mathbb{Q}_\ell(w(X))$ is torsion. Theorem 2.29 then guarantees that this character is 2-torsion, so can be identified with an element of $k^\times/k^{\times 2}$, and we may recover $\operatorname{disc}(\chi^{\mathrm{mot}}(X))$ as $(-1)^{w(X)}$ multiplied by this element. In particular, we may recover both $\operatorname{disc}(\chi^{\mathrm{mot}}(X))$ and $w(X)$ from $\det_\ell(X)$.

Corollary 2.31. *Assume that k is finitely generated over $\mathbb{Q}$. Then, for every variety X over k, we have that $\det_\ell(X)$ is trivial if and only if both $w(X)$ and $\operatorname{disc}(\chi^{\mathrm{cat}}(X))$ are trivial.*

Proof. By Theorem 2.29, it will be sufficient to show that $w(X)$ vanishes when $\det_\ell(X)$ vanishes. Now assume the latter. By the observation in Definition 2.23, the composition of $\det_\ell(X)$ and the map $\mathbb{Q}_\ell^\times \to \mathbb{Q}_\ell^\times$ given by squaring each element is the character $\mathbb{Q}_\ell(-2w(X))$ when X is smooth and projective. Therefore, since $\det_\ell$ and $w(-)$ are homomorphisms of abelian groups, this holds for general varieties by [Bit04, Theorem 3.1]. Since the cyclotomic character has infinite order when k is finitely generated over $\mathbb{Q}$, we obtain that $-2w(X) = 0$, and hence $w(X) = 0$. □

Remark 2.32. It is often easier to prove results about the ℓ-adic determinant of cohomology than the motivic Euler characteristic. The remainder of this section is dedicated to showing that we can deduce results about the motivic Euler characteristic from the ℓ-adic determinant of cohomology. To do this, we introduce the following terminology so that we obtain ring homomorphisms related to the discriminant, the ℓ-adic determinant of cohomology, and the total weight.

Definition 2.33. We say that a pair (R, ϵ) is an *augmented ring* if R is a commutative ring with unity and ϵ is a ring homomorphism $R \to \mathbb{Z}$ such that the composition of the canonical ring homomorphism $\mathbb{Z} \to R$ and ϵ is an isomorphism. If, in addition, $\operatorname{Ker}(\epsilon)^2 = 0$, then we say that (R, ϵ) is *elementary*.

Notation 2.34. Let $\mathbf{W}(k)$ denote the quotient of $\widehat{\mathrm{W}}(k)$ by I^2. Then $\operatorname{rank} : \widehat{\mathrm{W}}(k) \to \mathbb{Z}$ furnishes an augmentation $\epsilon : \mathbf{W}(k) \to \mathbb{Z}$. Since $\operatorname{Ker}(\epsilon)^2 = 0$, the pair $(\mathbf{W}(k), \epsilon)$ is elementary. Moreover, we can check that the discriminant function $\widehat{\mathrm{W}}(k) \to \operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$ factors through $\mathbf{W}(k)$, and when restricted to I/I^2, it furnishes a group isomorphism $\operatorname{Ker}(\epsilon) \cong I/I^2 \to \operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$.

Example 2.35. For every $\mathbb{Z}$-module M, let $E(M)$ denote the ring with underlying additive group $\mathbb{Z} \oplus M$ and multiplication $(a, m) \cdot (b, n) = (ab, an + bm)$. If we let $\epsilon : E(M) \to \mathbb{Z}$ be the projection onto the first factor, then $(E(M), \epsilon)$ is an elementary augmented ring. This construction is functorial; a group homomorphism $E(M) \to E(N)$ induces a ring homomorphism $M \to N$ respecting the augmentations.

Remark 2.36. For (R, ϵ) an augmented ring, we have a decomposition of the underlying additive group $R = \mathbb{Z} \oplus \operatorname{Ker}(\epsilon)$. When (R, ϵ) is elementary, the above decomposition is a natural isomorphism of augmented rings between R and $E(\operatorname{Ker}(\epsilon))$. In particular, we obtain a natural isomorphism $\mathbf{W}(k) \cong E(\operatorname{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$. Write E_{disc} for the quotient map

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$\widehat{\mathrm{W}}(k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$, and write $\chi^{\mathrm{disc}} : K_0(\mathrm{Var}_k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$ to denote the composition $E_{\mathrm{disc}} \circ \chi^{\mathrm{mot}}$.

Definition 2.37. Let χ_ℓ and χ_w denote the maps

$$\chi_\ell : K_0(\mathrm{Var}_k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)) \quad \text{and} \quad \chi_w : K_0(\mathrm{Var}_k) \to E(\mathbb{Z})$$

given by

$$[X] \mapsto (e(X), \det_\ell(X)) \quad \text{and} \quad [X] \mapsto (e(X), w(X)),$$

respectively. We will call χ_ℓ and χ_w the *augmented determinant of cohomology* and the *augmented total weight*, respectively.

Proposition 2.38. *The maps*

$$\chi_\ell : K_0(\mathrm{Var}_k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)) \quad \textit{and} \quad \chi_w : K_0(\mathrm{Var}_k) \to E(\mathbb{Z})$$

are ring homomorphisms.

Proof. We may assume without loss of generality that k is finitely generated over $\mathbb{Q}$. Define $s : E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times))$ to be the ring homomorphism induced by the group homomorphism $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$ induced by the squaring map $\mathbb{Q}_\ell^\times \to \mathbb{Q}_\ell^\times$. Let $c : E(\mathbb{Z}) \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times))$ be the ring homomorphism induced by the unique group homomorphism $\mathbb{Z} \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$ such that the image of -1 is the cyclotomic character. Then $c \circ \chi_w = s \circ \chi_\ell$. Since c is injective, we obtain that χ_w is a ring homomorphism if χ_ℓ is. Since χ_ℓ is a group homomorphism, we only need that for any X, Y, we have

$$\det_\ell(X \times Y) = \det_\ell(X)^{\otimes e(Y)} \otimes \det_\ell(Y)^{\otimes e(Y)}.$$

Using the Künneth formula,

$$\begin{aligned}
\det_\ell(X \times Y) &= \bigotimes_n \det H_c^n(X_{\overline{k}} \times Y_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^n} \\
&= \bigotimes_n \bigotimes_{i+j=n} (\det H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell) \otimes H_c^j(Y_{\overline{k}}, \mathbb{Q}_\ell))^{\otimes (-1)^n} \\
&= \bigotimes_{i,j} \det H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^{i+j} b_j(Y)} \otimes \det H_c^j(Y_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^{j+i} b_i(X)} \\
&= \bigotimes_i \det H_c^i(X_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^i e(Y)} \otimes \bigotimes_j \det H_c^j(Y_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^j e(X)},
\end{aligned}$$

so the claim is now clear. □

Proposition 2.39. *Assume that k is finitely generated over $\mathbb{Q}$. Let X, Y, Z be elements of $K_0(\mathrm{Var}_k)$ such that $\chi_\ell(X) = \chi_\ell(Y) \cdot \chi_\ell(Z)$. Then $\chi^{\mathrm{mot}}(x) = \chi^{\mathrm{mot}}(y) \cdot \chi^{\mathrm{mot}}(z) \mod I^2$ and $\chi_w(X) = \chi_w(Y)\chi_w(Z)$.*

Proof. Since $\chi_\ell(X) = \chi_\ell(Y)\chi_\ell(Z)$, we have $e(X) = e(Y)e(Z)$. This allows us to say the following:

$$\begin{aligned}
\chi_\ell(X) = \chi_\ell(Y) \cdot \chi_\ell(Z) &\iff \det_\ell(X) = \det_\ell(Y)^{\otimes e(Z)} \otimes \det_\ell(Z)^{\otimes e(Y)}, \\
\chi_w(X) = \chi_w(Y) \cdot \chi_w(Z) &\iff w(X) = e(Z)w(Y) + e(Y)w(Z), \\
\chi^{\mathrm{mot}}(X) = \chi^{\mathrm{mot}}(Y) \cdot \chi^{\mathrm{mot}}(Z) \bmod I^2 &\iff \\
&\mathrm{disc}(\chi^{\mathrm{mot}}(X)) = \mathrm{disc}(\chi^{\mathrm{mot}}(Y))^{e(Z)} \cdot \mathrm{disc}(\chi^{\mathrm{mot}}(Z))^{e(Y)},
\end{aligned}$$

using that all three augmented rings $E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times))$, $E(\mathbb{Z})$, and $\mathbf{W}(k)$ are elementary. By Theorem 2.29, it is enough to show that $\chi_\ell(X) = \chi_\ell(Y) \cdot \chi_\ell(Z)$ implies $\chi_w(X) = \chi_w(Y) \cdot \chi_w(Z)$.

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Let c and s be the homomorphisms from the proof of Proposition 2.38. Since $c \circ \chi_w = s \circ \chi_\ell$ and c is injective, the latter is clear. □

The main reason why it is easier to work with the ℓ-adic determinant of cohomology rather than the determinant is that we may relate the ℓ-adic determinant of cohomology to the zeta functions of reductions of the variety over finite fields. For the purposes of this paper, it is often helpful to have a formulation of the ℓ-adic determinant of cohomology of a variety in terms of ζ-functions of the reduction of the variety to finite fields.

Notation 2.40. Let F be a field finitely generated over $\mathbb{Q}$. Let X and Y be two quasi-projective varieties defined over F. Let U be a smooth connected scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ with function field F and choose models $\mathcal{X}, \mathcal{Y}$ for X, Y over U, respectively.

Since U is of finite type over $\mathrm{Spec}(\mathbb{Z})$, if s is a closed point of U, then the residue field $\mathbb{F}_s$ of s is finite. Write $\mathbb{F}_{s^n}$ to be the unique extension of $\mathbb{F}_s$ of degree n. Consider the variety $\mathcal{X}_s/\mathbb{F}_s$. Let $\zeta(\mathcal{X}_s, t) := \exp(\sum_{n\geq 1} \#\mathcal{X}_s(\mathbb{F}_{s^n})t^n)$.

By the proof of the Weil conjectures (see, for example, [Poo17, § 7.5.7]), we have

$$\zeta(\mathcal{X}_s, t) = \prod_i \det(\mathrm{Id} - t \cdot \mathrm{Frob}_s|_{H^i_c(\overline{\mathcal{X}_s}, \mathbb{Q}_\ell)})^{(-1)^i},$$

where Frob_s denotes the geometric Frobenius at s acting on the compactly supported ℓ-adic cohomology, and ℓ is any prime different from the characteristic of $\mathbb{F}_s$.

We claim that the ℓ-adic determinant of cohomology of X is determined by the ζ-functions of the reductions $\mathcal{X}_s$. In order to prove this, we first need some linear algebra.

Notation 2.41. Let $V = \bigoplus_i V_i$ be a finite-dimensional graded vector space over a field L of characteristic 0, let $\phi : V \to V$ be an invertible L-linear map respecting the grading, and define

$$\mathrm{Tr}(\phi) := \sum_i (-1)^i \mathrm{Tr}(\phi|_{V_i}), \quad \det(\phi) := \prod_i \det(\phi|_{V_i})^{(-1)^i}.$$

Define $e(V) := \mathrm{Tr}(\mathrm{Id}_V)$. Finally, define

$$\zeta(\phi, t) = \prod_i \det(\mathrm{Id}_{V_i} - t \cdot \phi|_{V_i})^{(-1)^i}.$$

Lemma 2.42. *We have*

$$\zeta(\phi, t) = \det(\phi)(-t)^{e(V)} \cdot u \quad \textit{where } u \in 1 + L[[t^{-1}]].$$

Proof. Since ϕ is the direct sum of the restrictions $\phi|_{V_i}$, and the terms $\zeta(\phi, t)$, $\det(\phi)$ and $(-t)^{e(V)}$ are all multiplicative with respect to direct sums, we may assume without loss of generality that only one graded piece of V is non-zero. Shifting the grading by j changes these terms by raising them to the $(-1)^j$th power, and the operation of raising to the $(-1)^j$th power preserves power series in $L[[t^{-1}]]$, so we may also assume that $V = V_0$. Note that

$$\det(\mathrm{Id} - t \cdot \phi) = (-t)^{\dim(V)} \cdot \det(\phi - t^{-1}\mathrm{Id}) = (-t)^{\dim(V)} c_\phi(t^{-1}),$$

where c_ϕ denotes the characteristic polynomial of ϕ, and so $c_\phi(t^{-1}) = \det(\phi) + t^{-1}u'$ for some polynomial $u' \in L[t^{-1}] \subseteq L[[t^{-1}]]$. Since ϕ is invertible, we may relabel $u'' = (1/\det\phi)u'$ and obtain

$$\det(\mathrm{Id} - t \cdot \phi) = (-t)^{\dim(V)} \cdot \det(\phi) \cdot (1 + t^{-1}u''),$$

which gives the result. □

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Proposition 2.43. *Let X be a quasi-projective variety over a finite field $\mathbb{F}$ and let ℓ be a prime different from* $\mathrm{char}(\mathbb{F})$. *Then*

$$\zeta(X,t) = \det_\ell(\mathrm{Frob}_{\mathbb{F}}) \cdot (-t)^{\chi(X)} \cdot u, \quad u \in 1 + \mathbb{Q}_\ell[[t^{-1}]],$$

where $\chi(X)$ is the compactly supported Euler characteristic of X, $\mathrm{Frob}_{\mathbb{F}} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ is the geometric Frobenius, and $\det_\ell(X)$ is the one-dimensional representation given by the determinant of cohomology of X as in Definition 2.23.

Proof. Let $V := \bigoplus_i H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)$ be the compactly supported ℓ-adic cohomology of X, which is a graded vector space over $\mathbb{Q}_\ell$, and let ϕ be the action of the geometric Frobenius $\mathrm{Frob}_{\mathbb{F}}$ on V. Then $e(V) = \chi(X)$, $\zeta(\phi, t) = \zeta(X, t)$, and $\det(\phi) = \det_\ell(\mathrm{Frob}_{\mathbb{F}})$, so the claim follows. □

Theorem 2.44. *Assume that for every closed point s of U, we have $\zeta(\mathcal{X}_s, t) = \zeta(\mathcal{Y}_s, t)$. Then there is an equality of Gal_F representations $\det_\ell(X) = \det_\ell(Y)$.*

Proof. Let s be a closed point of U. By the proper base change theorem for ℓ-adic cohomology and the previous lemma, if we pick $\mathrm{Frob}_s \in \mathrm{Gal}_F$ to be a lift of the Frobenius of $\mathbb{F}_s$, we may obtain $\det_\ell(X)(\mathrm{Frob}_{\mathbb{F}_s})$ from $\zeta(\mathcal{X}_s, t)$. Since $\zeta(\mathcal{X}_s, t) = \zeta(\mathcal{Y}_s, t)$, we see that $\det_\ell(X)(\mathrm{Frob}_s) = \det_\ell(Y)(\mathrm{Frob}_s)$. The Chebotarëv density theorem then shows that we have an equality of Gal_F representations $\det_\ell(X) = \det_\ell(Y)$. □

We will often prove facts about the motivic Euler characteristic by proving results separately for the rank, signature, and discriminant, then combining them to obtain an element in $\widehat{\mathrm{W}}(k)/J$. This is made possible by the following lemma.

Lemma 2.45. *Let $[q] \in \widehat{\mathrm{W}}(k)$. Then $[q] \in J$ if and only if all of the discriminant, signature, and rank vanish. In particular, if $[q]$ is an element of $\widehat{\mathrm{W}}(k)$, we can reconstruct the class of $[q]$ in $\widehat{\mathrm{W}}(k)/J$ from* $\mathrm{rank}([q]), \mathrm{sign}([q])$, *and* $\mathrm{disc}([q])$.

Proof. This is clear by definition of $J := I^2 \cap T$, since T is the intersection of $\ker(\mathrm{rank})$ and $\ker(\mathrm{sign})$, and $I^2 = \ker(\widehat{\mathrm{W}}(k) \to \mathbf{W}(k))$, and q lies in this kernel if and only if $\mathrm{rank}(q)$ and $\mathrm{disc}(q)$ both vanish. □

Corollary 2.46. *There is an injective ring homomorphism*

$$\mathrm{rank} \times \mathrm{sign} \times E_{\mathrm{disc}} : \widehat{\mathrm{W}}(k)/J \to \mathbb{Z} \times \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}) \times E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})).$$

Proof. This is essentially a restatement of the above lemma. □

3. Galois descent for the motivic Euler characteristic

In this section, we give a detailed account of a Galois-descent-type result for χ^{mot}. Fix K/k a finite Galois extension with Galois group $\mathrm{Gal}(K/k)$. If k has positive characteristic, assume further that $[K : k]$ is coprime to $\mathrm{char}(k)$.

Definition 3.1. Let $\mathrm{Var}_{K,\mathrm{desc}}$ denote the category of pairs (X, φ) of K-varieties X equipped with descent data φ. Explicitly, the descent data φ form a group homomorphism $\mathrm{Gal}(K/k) \to \mathrm{Aut}(X/k)$ such that for all $\tau \in \mathrm{Gal}(K/k)$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi(\tau)} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \xrightarrow{\tau} & \mathrm{Spec}(K) \end{array}$$

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where the vertical arrows denote the structure morphism $X \to \mathrm{Spec}(K)$. A morphism in $\mathrm{Var}_{K,\mathrm{desc}}$, $f : (X, \varphi) \to (X', \varphi')$ is a morphism of K-varieties $f : X \to X'$ that is compatible with the descent data in the sense that the obvious diagram commutes. We say that a morphism is an *open* (respectively, *closed*) *immersion* if the underlying morphism of varieties is an *open* (respectively, *closed*) *immersion*. This category also has a unit object with respect to the monoidal structure induced by the product: namely, the variety $\mathrm{Spec}(K)$ equipped with the identity as the structure morphism, with the only possible descent data. We will write $(\mathrm{Spec}(K), \mathrm{Id})$ for this unit object. This allows us to form the Grothendieck ring of K varieties with descent data, $K_0(\mathrm{Var}_{K,\mathrm{desc}})$, which is the ring generated by symbols $[(X, \varphi)]$, where (X, φ) is a K variety with descent data, subject to the relations:

(1) $[(X, \varphi)] = [(X', \varphi')]$ if (X, φ) and (X', φ') are isomorphic;
(2) $[(X, \varphi)] = [(U, \varphi|_U)] + [(X \setminus U, \varphi_{X\setminus U})]$ for $(U, \varphi_U) \hookrightarrow (X, \varphi)$ an open immersion, where the descent data on $X \setminus U$ are given by the restriction of the descent data on X; and
(3) $[(X, \varphi)][(X', \varphi')] = [(X \times_K X', \varphi \times \varphi')]$ for any pair of K varieties with descent data.

By Galois descent (see, for example, [Poo17, Corollary 4.4.6]), there is an equivalence of categories $\mathrm{Var}_{K,\mathrm{desc}} \cong \mathrm{Var}_k$. This equivalence of categories preserves the unit object in the category since it sends $(\mathrm{Spec}(K), \mathrm{Id})$ to $\mathrm{Spec}(k)$, where $(\mathrm{Spec}(K), \mathrm{Id})$. If (X, φ) is a K variety with descent data, write $\mathrm{des}_{K/k}(X, \varphi)$ to mean the associated k variety, and similarly if Y/k is a variety, write (Y_K, ϕ_Y) to mean the associated K variety with descent data.

This equivalence of categories preserves open and closed immersions (see [Poo17, Theorem 4.3.7]), so it induces an isomorphism $K_0(\mathrm{Var}_{K,\mathrm{desc}}) \cong K_0(\mathrm{Var}_k)$ given on the level of quasi-projective varieties by sending $[(X, \varphi)] \mapsto [\mathrm{des}_{K/k}(X, \varphi)]$, with inverse $[X] \mapsto [(X_K, \phi_X)]$.

We would like to apply a Galois-descent-style result for the motivic Euler characteristic. In order to do this, we need a Galois-descent result for quadratic forms.

Definition 3.2. Let K be a finite Galois extension of k, let V be a vector space over K with a semilinear $\mathrm{Gal}(K/k)$ action, and let $q : V \to K$ be a quadratic form. We say that q is *Galois compatible* if, for all $\tau \in \mathrm{Gal}(K/k)$ and $v \in V$, we have $q(\tau(v)) = \tau(q(v))$.

Write $\varphi : \mathrm{Gal}(K/k) \to \mathrm{Aut}_k(V)$ for the group homomorphism defining this semilinear action. We will write (q, φ) to denote the quadratic form on V along with the semilinear $\mathrm{Gal}(K/k)$ action on V induced by φ. Let $\widehat{\mathrm{W}}(K)^{\mathrm{desc}}$ denote the ring generated by the symbols $[(q, \varphi)]$, where $[(q, \varphi)]$ is a Galois-compatible quadratic form over K, subject to the relations:

(1) $[(q, \varphi)] = [(q', \varphi')]$ if there is an isomorphism between the underlying vector spaces of q, q' that is compatible with the quadratic forms and the semilinear $\mathrm{Gal}(K/k)$ actions;
(2) $[(q, \varphi)] + [(q', \varphi')] = [(q \oplus q', \varphi \oplus \varphi')]$, where $q \oplus q'$ (respectively, $\varphi \oplus \varphi'$) represents the natural quadratic form (respectively, semilinear $\mathrm{Gal}(K/k)$ action) on $V \oplus V'$; and
(3) $[(q, \varphi)] \cdot [(q', \varphi')] = [(q \otimes_K q', \varphi \otimes_K \varphi')]$, where $q \otimes_K q'$ (respectively, $\varphi \otimes_K \varphi'$) represents the natural quadratic form (respectively, semilinear $\mathrm{Gal}(K/k)$ action) on $V \otimes_K V'$.

For $a \in k^\times$, write $\langle a \rangle$ to mean the Galois-compatible quadratic form on K given by $x \mapsto ax^2$, and write $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle \in \widehat{\mathrm{W}}(K)^{\mathrm{desc}}$.

Theorem 3.3. *There is an equivalence of categories between the category of non-degenerate quadratic forms over k and the category of non-degenerate Galois-compatible quadratic forms over K.*

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Proof. This follows by Galois descent for vector spaces. If $q : V \to k$ is a quadratic form over k, we obtain a Galois-compatible quadratic form over K,

$$q \otimes Id_K : V \otimes_k K \to K,$$

where $\varphi : \mathrm{Gal}(K/k) \to \mathrm{Aut}_k(V \otimes_k K)$ is given by $\tau \mapsto \mathrm{Id} \otimes \tau$.

If $(q', \varphi) : V' \to K$ is a Galois-compatible quadratic form over K, and $v' \in V'^{\mathrm{Gal}(K/k)}$, then for all $\tau \in \mathrm{Gal}(K/k)$, we see that $\tau(q'(v')) = q'(\tau(v')) = q'(v')$, so $q'(v') \in k$. Restricting to the Galois-invariant subspace gives us a quadratic form over k:

$$q'|_{V'^{\mathrm{Gal}(K/k)}} : V'^{\mathrm{Gal}(K/k)} \to k.$$

We claim that $q'|_{V'^{\mathrm{Gal}(K/k)}}$ is non-degenerate. Let $B : V' \otimes_K V' \to K$ be the underlying bilinear form, and let $v \in V'^{\mathrm{Gal}(K/k)}$. Since B' is non-degenerate, pick $w \in V'$ such that $B(v, w) = 1$. Let $\hat{w} = \frac{1}{[K:k]} \sum_{\tau \in \mathrm{Gal}(K/k)} \tau(w)$. Then $\hat{w} \in V'^{\mathrm{Gal}(K/k)}$, and $B'(v, \hat{w}) = 1$, so $B'|_{V'^{\mathrm{Gal}(K/k)}}$ is non-degenerate. Finally, note that the functors $q \mapsto q \otimes Id_K$ and $q' \mapsto q'|_{V'^{\mathrm{Gal}(K/k)}}$ are quasi-inverse, which completes the proof. □

Corollary 3.4. *There is a canonical isomorphism* $\widehat{\mathrm{W}}(K)^{\mathrm{desc}} \cong \widehat{\mathrm{W}}(k)$, *given on the level of quadratic forms by sending* (q, φ) *to* $q|_{V^{\mathrm{Gal}(K/k)}}$.

Proof. Immediate by the above. □

We claim that Galois descent for varieties is compatible with a Galois-descent result for the Galois-compatible quadratic forms on their de Rham cohomology. This is made exact in the following.

Lemma 3.5. *Let* X *be a smooth projective* k *variety. Then there is an isomorphism of graded vector spaces with a semilinear* $\mathrm{Gal}(K/k)$ *action:*

$$H^*_{\mathrm{dR}}(X/k) \otimes_k K \cong H^*_{\mathrm{dR}}(X_K/K).$$

Let $\cup_k$ *denote the cup product on* $H^*_{\mathrm{dR}}(X/k)$, *and let* $\cup_K$ *denote the cup product on* $H^*_{\mathrm{dR}}(X_K/K)$. *Let* $\alpha, \alpha' \in H^*_{\mathrm{dR}}(X/k)$ *and* λ, λ' *in* K. *Then, under this isomorphism,*

$$(\alpha \otimes \lambda) \cup_K (\alpha' \otimes \lambda') = (\alpha \cup_k \alpha') \otimes \lambda\lambda'$$

and $Tr_{X_K}(\alpha \otimes \lambda) = \lambda Tr_X(\alpha)$.

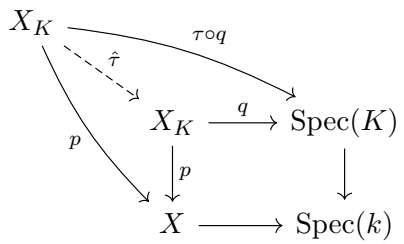
Proof. By the Künneth formula, there is an isomorphism of K vector spaces,

$$H^*_{\mathrm{dR}}(X/k) \otimes_k H^*_{\mathrm{dR}}(\mathrm{Spec}(K)/k) \cong H^*_{\mathrm{dR}}(X_K/K),$$

which respects the cup product and the trace map. Moreover, we see that we can identify $H^*_{\mathrm{dR}}(\mathrm{Spec}(K)/k) \cong K$. The left-hand side of the above isomorphism has a canonical semilinear action of $\mathrm{Gal}(K/k)$ on it, given by $\tau(x \otimes \alpha) = x \otimes \tau(\alpha)$. The right-hand side has a left $\mathrm{Gal}(K/k)$ action on it, induced by the right action on X_K. We claim that these two actions are the same, which is enough to prove the theorem.

First, consider $\mathrm{Spec}(K)$ as a k variety. The $\mathrm{Gal}(K/k)$ action on $\mathrm{Spec}(K)$ gives us a left action of $\mathrm{Gal}(K/k)$ on $H^*_{\mathrm{dR}}(K/k) \cong K$, which corresponds to the standard action of $\mathrm{Gal}(K/k)$ on K. For $\tau \in \mathrm{Gal}(K/k)$, write $\hat{\tau}$ for the automorphism of X_K given by the commutative diagram

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where $\hat{\tau}$ is the morphism induced by the universal property of the fibre product. Write p^*, q^*, τ^*, and $\hat{\tau}^*$ for the images of the morphisms p, q, τ and $\hat{\tau}$ after applying the contravariant functor, $H^*_{\mathrm{dR}}(-/k)$. For $x \in H^*_{\mathrm{dR}}(X)$ and $y \in K$, we have that

$$p^*(x) = x \otimes 1,$$
$$q^*(y) = 1 \otimes y,$$
$$\tau^*(y) = \tau(y).$$

Therefore, the map $\hat{\tau}^* : H^*_{\mathrm{dR}}(X_K/k) \to H^*_{\mathrm{dR}}(X_K/k) \otimes_k K$ is $x \otimes y \mapsto x \otimes \tau(y)$.

Note that $H^*_{\mathrm{dR}}(X_K/k)$ is the image of $H^*_{\mathrm{dR}}(X_K/K)$ under the forgetful functor from K vector spaces to k vector spaces, and the map $\hat{\tau}^*$ is the map on $H^*_{\mathrm{dR}}(X_K/K)$ induced by the semilinear $\mathrm{Gal}(K/k)$ action. This gives that $H^*_{\mathrm{dR}}(X_K/k) \cong H^*_{\mathrm{dR}}(X) \otimes_k K$, as required. □

Definition 3.6. For X/K a smooth projective variety over K of dimension n with descent data, there is a natural Galois-compatible quadratic form on $H^{2*+1}_{\mathrm{dR}}(X/K)$ defined as follows. Note that $H^{2i+1}_{\mathrm{dR}}(X/K)$ and $H^{2n-(2i+1)}_{\mathrm{dR}}(X/K)$ are canonically dual and both have a semilinear $\mathrm{Gal}(K/k)$ action on them. If $2i+1 \neq n$, then $2n-(2i+1) \neq 2i+1$. Therefore, the canonical isomorphism $H^{2n-(2i+1)}_{\mathrm{dR}}(X/K) \cong H^{2i+1}_{\mathrm{dR}}(X/K)^\vee$ gives rise to a canonical Galois-compatible quadratic form on $H^{2i+1}_{\mathrm{dR}}(X/K) \oplus H^{2n-(2i+1)}_{\mathrm{dR}}(X)$ given by taking a pair (α, f) to $f(\alpha)$. If $2i+1=n$, then $H^n_{\mathrm{dR}}(X/K)$ is canonically self-dual. Writing $\phi : H^n_{\mathrm{dR}}(X/K) \to H^n_{\mathrm{dR}}(X/K)^\vee$, we obtain a Galois-compatible quadratic form on $H^n_{\mathrm{dR}}(X/K)$ by taking α to $\phi(\alpha)(\alpha)$, which is hyperbolic since n is odd. This furnishes a Galois-compatible quadratic form on all of $H^{2*+1}_{\mathrm{dR}}(X/K)$, and we will write $[H^{2*+1}_{\mathrm{dR}}(X/K)]$ to be the corresponding element of $\widehat{\mathrm{W}}(K)^{\mathrm{desc}}$. Note that if we forget the descent data, the quadratic form is clearly hyperbolic.

Corollary 3.7. *Let (X, φ) be a K variety with descent data φ. Then there is a natural Galois-compatible non-degenerate quadratic form on $H^{2*}_{\mathrm{dR}}(X/K)$ with Galois action induced by φ given by the composition $x \mapsto \mathrm{Tr}_X(x \cup x)$.*

Proof. Immediate, since (X, φ) is given by the base change of $\mathrm{desc}_{K/k}(X, \varphi)$ to K. □

Theorem 3.8. *There is a unique ring homomorphism $\chi_k^{\mathrm{mot,desc}} : K_0(\mathrm{Var}_{K,\mathrm{desc}}) \to \widehat{\mathrm{W}}(K)^{\mathrm{desc}}$ such that if (X, φ) is a smooth projective variety, then*

$$\chi_k^{\mathrm{mot,desc}}([(X, \varphi)]) = [H^{2*}_{\mathrm{dR}}(X/K)] - [H^{2*+1}_{\mathrm{dR}}(X/K)].$$

Proof. Define $\chi^{\mathrm{mot,desc}}$ to be the composition $K_0(\mathrm{Var}_{K,\mathrm{desc}}) \cong K_0(\mathrm{Var}_k) \xrightarrow{\chi^{\mathrm{mot}}} \widehat{\mathrm{W}}(k) \to \widehat{\mathrm{W}}(K)^{\mathrm{desc}}$. The above corollary guarantees that this satisfies the desired property. Uniqueness is also immediate by the above corollary, since if φ is any other morphism as above, then the composition

$$K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_{K,\mathrm{desc}}) \xrightarrow{\varphi} \widehat{\mathrm{W}}(K)^{\mathrm{desc}} \cong \widehat{\mathrm{W}}(k)$$

is such that for a smooth projective variety X,

$$[X] \mapsto [H^{2*}_{\mathrm{dR}}(X/k)] - \tfrac{1}{2}\dim_k H^{2*+1}_{\mathrm{dR}}(X/k) \cdot \mathbb{H} = \chi^{\mathrm{dR}}(X),$$

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so the composition is equal to χ^{mot}. As noted in Theorem 2.9, the morphism χ^{mot} is unique with respect to this property, so $\chi^{\mathrm{mot,desc}}$ will also be unique. □

Remark 3.9. An argument similar to the one above has been noted in [LR20, § 8E].

4. The motivic Euler characteristic and transfer maps

Let L/k be a finite separable field extension. We can define various transfer maps between the rings $K_0(\mathrm{Var}_L)$ and $K_0(\mathrm{Var}_k)$, given by base change, forgetting the base, and Weil restriction, and analogously for $\widehat{\mathrm{W}}(L)$ and $\widehat{\mathrm{W}}(k)$ by base change, the trace transfer, and Rost's norm transfer from [Ros03]. In this section, we show that the motivic Euler characteristic is compatible with these transfer maps.

Definition 4.1. Let $\iota : k \hookrightarrow L$ be a field extension. Then there is a natural ring homomorphism $\iota_* : \widehat{\mathrm{W}}(k) \to \widehat{\mathrm{W}}(L)$ given on the level of quadratic forms by sending a quadratic form $q : V \to k$ to $q \otimes \mathrm{Id} : V \otimes_k L \to L$. Similarly, there is a ring homomorphism $(\mathrm{Spec}(\iota))^* : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_L)$, given by sending a variety X to X_L.

Lemma 4.2. *The following diagram is commutative.*

$$\begin{array}{ccc} K_0(\mathrm{Var}_k) & \xrightarrow{\chi_k^{\mathrm{mot}}} & \widehat{\mathrm{W}}(k) \\ \downarrow{\scriptstyle \mathrm{Spec}(\iota)^*} & & \downarrow{\scriptstyle \iota_*} \\ K_0(\mathrm{Var}_L) & \xrightarrow{\chi_L^{\mathrm{mot}}} & \widehat{\mathrm{W}}(L) \end{array}$$

Proof. By [Bit04, Theorem 3.1], it is enough to prove this result for smooth projective varieties, so we may use de Rham cohomology. Flat base change for de Rham cohomology gives us an isomorphism $H^*_{\mathrm{dR}}(X_L/L) \cong H^*_{\mathrm{dR}}(X/k) \otimes_k L$. Under this isomorphism, the cup product on $H^*_{\mathrm{dR}}(X_L)$ is the extension of the cup product on $H^*_{\mathrm{dR}}(X/k)$ via L-linearity, so it only remains to examine what happens to the trace map.

Since X/k is smooth and projective, there is a canonical isomorphism $H^{2n}_{\mathrm{dR}}(X/k) \cong H^n(X, \Omega^n_{X/k})$ via the cycle class map. The trace map $H^n(X, \Omega^n_{X/k}) \to k$ is defined to be the map given by the image of the identity under the isomorphism

$$Rp_{X/k,*} R\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_{X/k}, \Omega_{X/k})^0 \cong H^n(X, \Omega^n_{X/k})^\vee$$

induced by Grothendieck–Verdier duality (see [Har66, Chapter III, Corollary 11.2(f)]). In particular, we see that the trace map commutes with arbitrary base extension by [Har66, Corollary 11.2(c)], so the trace map $\mathrm{Tr}_{X/L}$ is given by $\mathrm{Tr}_{X/k} \otimes_k \mathrm{Id}_L$, which gives us the result. □

We now focus on two other natural transfer maps, which are maps $\widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(k)$ for quadratic forms and maps $K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_k)$ for varieties.

Definition 4.3. Let L/k be a finite separable field extension. Write $N_{L/k}$ to be the norm map $L^\times \to k^\times$ and $\mathrm{Tr}_{L/k}$ to mean the trace map $L \to k$. We have a natural homomorphism of abelian groups $\mathrm{Tr}_{L/k} : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(k)$, given on the level of quadratic forms by sending the quadratic form $q : V \to L$ to the quadratic form $\mathrm{Tr}_{L/k} \circ q$. This is a widely studied homomorphism in quadratically enriched enumerative geometry, appearing, for example, in [KW21, p. 19].

Write $\mathrm{disc}(L/k) := \mathrm{disc}(\mathrm{Tr}_{L/k}(\langle 1 \rangle))$. This is not misleading: for $k = \mathbb{Q}$, this agrees with the discriminant of the number field L. For $w \in \widehat{\mathrm{W}}(k)$, we have $\mathrm{rank}(\mathrm{Tr}_{L/k}(w)) = [L : k]\mathrm{rank}(w)$, so $\mathrm{Tr}_{L/k}$ is not a ring homomorphism.

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Analogously to the above, if we have a variety $X \xrightarrow{p} \mathrm{Spec}(L)$, we can consider X as a variety over k with structure map $X \xrightarrow{p} \mathrm{Spec}(L) \to \mathrm{Spec}(k)$. This induces a homomorphism of abelian groups $K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_k)$, though not a ring homomorphism.

As in Remark 2.36, identify $\widehat{\mathrm{W}}(k)/I^2 = \mathbf{W}(k)$ with $E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$. Let E_{disc} denote the quotient map

$$\begin{aligned} E_{\mathrm{disc}} : \widehat{\mathrm{W}}(k) &\to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})), \\ w &\mapsto (\mathrm{rank}(w), \mathrm{disc}(w)). \end{aligned}$$

Lemma 4.4. *Let $w \in \widehat{\mathrm{W}}(L)$, such that $E_{\mathrm{disc}}(w) = (\alpha, \beta)$. Then $\mathrm{Tr}_{L/k}(w)$ satisfies*

$$E_{\mathrm{disc}}(\mathrm{Tr}_{L/k}(w)) = ([L:k]\alpha, \mathrm{disc}(L/k)^{\alpha} N_{L/k}(\beta)).$$

Proof. The statement about the rank is immediate, so we only need to show this for the discriminant. Since both disc and $\mathrm{Tr}_{L/k}$ are abelian group homomorphisms, it is enough to show the statement on generators of $\widehat{\mathrm{W}}(L)$, so assume that $w = \langle\beta\rangle$. Let m_β be the matrix defining the k linear map on L given by multiplication by β, so that $\det(m_\beta) = N_{L/k}(\beta)$. Let A be a symmetric matrix representing the symmetric k-bilinear form on L given by $(x, y) \mapsto \mathrm{Tr}_{L/k}(xy)$ for all $x, y \in L$. Then A is a symmetric matrix such that $\det(A) = \mathrm{disc}(L/k)$. We obtain a matrix representing the quadratic form $\mathrm{Tr}_{L/k}(\langle\beta\rangle)$, given by Am_β. The result follows since

$$\mathrm{disc}(\mathrm{Tr}_{L/k}(\langle\beta\rangle)) = \det(Am_\beta) = \det(A)\det(m_\beta) = \mathrm{disc}(L/k)N_{L/k}(\beta). \qquad \square$$

Lemma 4.5. *The following diagram is commutative.*

$$\begin{array}{ccc} K_0(\mathrm{Var}_L) & \longrightarrow & K_0(\mathrm{Var}_k) \\ \downarrow{\scriptstyle \chi_L^{\mathrm{mot}}} & & \downarrow{\scriptstyle \chi_k^{\mathrm{mot}}} \\ \widehat{\mathrm{W}}(L) & \xrightarrow{\mathrm{Tr}_{L/k}} & \widehat{\mathrm{W}}(k) \end{array}$$

Proof. We first note that we can reduce to the case where X/L is smooth, projective, and connected, so that we may work with the de Rham Euler characteristic. Let X/L be a smooth projective variety of dimension n. There is an isomorphism of k-vector spaces $H^*_{\mathrm{dR}}(X/L) \cong H^*_{\mathrm{dR}}(X/k)$, and this isomorphism is compatible with cup products, so it only remains to check what happens to the trace map $H^{2n}_{\mathrm{dR}}(X/k) \to k$. From here, the proof is largely an application of [Har66, Chapter 3, Corollary 11.2], which we present here for the convenience of the reader.

After applying the cycle class isomorphism $H^{2n}_{\mathrm{dR}}(X/k) \cong H^n(X, \Omega^n_{X/k})$, we see that the trace map $H^n(X, \Omega^n_{X/k}) \to k$ is the image of the identity under the isomorphism

$$Rp_{X/k,*}\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_{X/k}, \Omega_{X/k})^0 \cong H^n(X, \Omega^n_{X/k})^\vee$$

induced by Grothendieck–Verdier duality, and similarly if we replace k by L as above. By [Har66, Chapter 3, Corollary 11.2(e)], it is implied that the image of the identity under the isomorphism

$$Rp_{L/k,*}\mathcal{H}\mathrm{om}_L(\Omega_{L/k}, \Omega_{L/k})^0 \cong H^n(\mathrm{Spec}(L), \Omega^n_{L/k})^\vee$$

is given by the standard trace map $\mathrm{Tr}_{L/k}$. Since $p_{L/k}$ is étale, $p^!_{L/k} = p^*_{L/k}$ is the left adjoint of $Rp_{L/k,*}$, and since it is finite, $Rp_{L/k,*} = p_{L/k,*}$. Moreover, since $p^!$ is right adjoint to p_*, we have

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canonical isomorphisms $p^!_{X/k}(\mathcal{O}_k) = \Omega^n_{X/k}[-n]$, and similarly when replacing k with L. Similarly, we see that

$$p^!_{X/k}(k) = p^!_{X/L}p^!_{L/k}(k) = p^!_{X/L}(p^*_{L/k}(k)) = p^!_{X/L}(L).$$

Using this, we have the following commutative diagram, in which all of the arrows are isomorphisms.

$$\begin{array}{ccc}
Rp_{X/k,*}R\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(p^!_{X/k}(k), p^!_{X/k}(k))^0 & \longrightarrow & p_{L/k,*}Rp_{X/L,*}R\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(p^!_{X/L}(L), p^!_{X/L}p^!_{L/k}(k))^0 \\
\downarrow & & \downarrow \\
R\mathcal{H}\mathrm{om}_k(Rp_{X/k,*}p^!_{X/k}(k), k)^0 & & p_{L/k,*}R\mathcal{H}\mathrm{om}_L(Rp_{X/L,*}p^!_{X/L}(L), p^!_{L/k}(k))^0 \\
\downarrow & & \downarrow \\
 & & p_{L/k,*}\,\mathrm{Hom}_L(H^n(X, \Omega^n_{X/L}), p^!_{L/k}k) \\
 & & \downarrow \Phi \\
\mathrm{Hom}_k(H^n(X, \Omega^n_{X/k}), k) & \xrightarrow{\quad = \quad} & \mathrm{Hom}_k(p_{L/k,*}H^n(X, \Omega^n_{X/L}), k)
\end{array}$$

Again by [Har66, Chapter 3, Corollary 11.2(e) and (f)], the isomorphism Φ takes a morphism $f \in \mathrm{Hom}_L(H^n(X, \Omega^n_{X/L}), L)$ to $\mathrm{Tr}_{L/k} \circ f \in \mathrm{Hom}_k(H^n(X, \Omega^n_{X/L}), k)$. Consider the identity element $\mathrm{Id} \in Rp_{X/k,*}R\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(p^!_{X/k}(k), p^!_{X/k}(k))^0$. Going clockwise around the above diagram maps the element Id to $\mathrm{Tr}_{L/k} \circ \mathrm{Tr}_{X/L}$. However, going counterclockwise maps Id to $\mathrm{Tr}_{X/k}$, giving the desired equality. □

We now turn our attention to the norm and Weil restriction transfer maps. This first requires us to use the machinery of polynomial maps: while the transfer maps defined by the trace and forgetful functor are group homomorphisms, the transfer maps that we obtain by looking at Weil restrictions and norms have a different structure.

Definition 4.6. Let B be an abelian group, let A an abelian monoid, and let $P : A \to B$ be a function on the underlying sets. We say that P is a *polynomial map of degree* ≤ -1 if $P = 0$. For $n \geq 0$, we say that P is a *polynomial map of degree* $\leq n$ if, for each $x \in A$, the map

$$\Delta_x P : A \to B,$$
$$\Delta_x P(y) = P(y + x) - P(y)$$

is a polynomial map of degree $\leq n - 1$. We say that P is a *polynomial map* if it is a polynomial map of degree $\leq n$ for some n. Note that P is a polynomial map of degree ≤ 0 if and only if P is constant.

Now suppose that A and B are commutative rings. A polynomial map is called *multiplicative* if $P(1) = 1$ and if for every $x, y \in A$, we have $P(xy) = P(x)P(y)$. Note that a multiplicative polynomial map of degree ≤ 1 is either identically 1 or a ring homomorphism.

Polynomial maps can be extended from monoids to rings by the following.

Lemma 4.7 (Proposition 1.6 of [Jou00]). *Let $\tilde{A}$ be an abelian monoid, let $\tilde{f} : \tilde{A} \to B$ be a polynomial map of degree $\leq n$, and let A be the group completion of $\tilde{A}$. Then there exists a unique polynomial map of degree $\leq n$ $f : A \to B$ that extends $\tilde{f}$.*

The statement about the degree of the polynomial map is not in the statement of [Jou00, Proposition 1.6]; however, it follows from the proof.

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For every finite separable extension K/F of fields, let $\mathrm{N}_{K/F}: K \to F$ denote the norm map. The following proposition also serves as a definition of the norm map.

Proposition 4.8 (Corollary 5 of [Ros03]). *Let F be a field of characteristic $\neq 2$, and let K/F be a finite separable field extension of degree n. Then there exists a unique multiplicative polynomial map $\mathrm{N}_{K/F}: \widehat{\mathrm{W}}(K) \to \widehat{\mathrm{W}}(F)$ of degree $\leq n$ such that $\mathrm{N}_{K/F}(\langle b \rangle) = \langle \mathrm{N}_{K/F}(b) \rangle$ for all $b \in k^\times$.*

This rest of this section is dedicated to proving Theorem 4.22, which says that this norm polynomial map computes the motivic Euler characteristic of the Weil restriction of a variety. In order to prove this, we first need some smaller lemmas.

Lemma 4.9. *Suppose that $f, g: A \to B$ are polynomial maps of degree $\leq n$, $\leq m$, respectively. Then $f+g$ is a polynomial map of degree $\leq \max(n,m)$. Similarly, if $m=1$ and f, g are multiplicative, then fg is a multiplicative polynomial map of degree $\leq n+1$.*

Proof. We prove the first statement by induction on the degree of f. If $\deg(f)=0$, then f is constant, so $\Delta_x(f+g) = \Delta_x(g)$. Suppose that $\deg(f)=n$. Then consider $\Delta_x(f+g): A \to B$. We compute $\Delta_x(f+g)(y) = \Delta_x(f)(y) + \Delta_x(g)(y)$. Since $\Delta_x(f)$ and $\Delta_x(g)$ are polynomial maps of degree $\leq n-1$ and $\leq m-1$, respectively, the result holds by induction.

For the second part of the lemma, we proceed by induction on the degree of the polynomial map, with the case that f has degree ≤ 0 being trivial, since f is then identically 1. We compute

$$\begin{aligned}\Delta_x(fg)(y) &= g(x+y)f(x+y) - g(y)f(y)\\ &= g(x)f(x+y) + g(y)f(x+y) - g(y)f(y)\\ &= g(x)f(x+y) + g(y)(\Delta_x f(y)).\end{aligned}$$

By induction, we have that $g \cdot (\Delta_x f)$ is a polynomial map of degree $\leq n$. Note that we have a polynomial map $y \mapsto g(x)f(x+y)$ of degree $\leq n$, since $g(x)$ is constant, and $f(x+y) = \Delta_x f + f$ is a polynomial map of degree $\leq n$ by the first part of the lemma. Therefore, $\Delta_x(fg)$ is polynomial of degree $\leq n$, so fg is polynomial of degree $\leq n+1$. The fact that fg is multiplicative is immediate. □

Corollary 4.10. *Let $g_1, \ldots, g_n: A \to B$ be ring homomorphisms. Then $g = \prod_{i=1}^n g_i$ is a polynomial function of degree $\leq n$.*

Proof. Proceed by induction on n, with the case $n=1$ being trivial, and the induction step following by the second part of the above lemma. □

The next few results are dedicated to showing that the Weil restriction functor from L to k gives rise to a polynomial map $K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_k)$.

Definition 4.11. Let L/k be a finite separable field extension. We let $I_L := \{\iota: L \to \overline{k}\}$ be the set of embeddings of L into $\overline{k}$ with $\iota|_k = \mathrm{id}_k$. This set has a natural right action of the absolute Galois group Gal_k, given by $\iota \cdot \sigma = \sigma^{-1} \circ \iota$.

If K/k is a finite separable field extension with $L \subseteq K$, we have a Gal_k equivariant map $I_K \to I_L$ given by restriction, and the fibres of this map have cardinality $[K:L]$. From here, fix L/k a finite field extension, let K be the Galois closure of L in $\overline{k}$, and let $G = \mathrm{Gal}(K/k)$. If H is the subgroup of G such that $L = K^H$, then $I_L = G/H$ as a G set.

Definition 4.12. Let X be a quasi-projective variety over L. Define a K variety $\mathrm{Ind}_L^K X$ by setting $\mathrm{Ind}_L^K X := \prod_{\iota \in I_L} X_\iota$, where X_ι is the pullback of $X \to \mathrm{Spec}(L)$ along the inclusion

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$\mathrm{Spec}(K) \xrightarrow{\iota} \mathrm{Spec}(L)$. We give $\mathrm{Ind}_L^K X$ a canonical semilinear G action. Let $g \in G$. Define a right G action on $\mathrm{Ind}_L^K X$ by $g : X_{g\circ\iota} \to X_\iota$, where this morphism is the pullback.

$$\begin{array}{ccc} X_{g\circ\iota} & \longrightarrow & X_\iota \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \xrightarrow{g} & \mathrm{Spec}(K) \end{array}$$

This defines a semilinear action of G on $\mathrm{Ind}_L^K X$, which we call φ_G. This in turn gives us a variety $\mathrm{des}_{K/k}(\mathrm{Ind}_L^K X, \varphi_G)$ by Galois descent. By [Wei12, Theorem 1.3.2], the variety $\mathrm{des}_{K/k}(\mathrm{Ind}_L^K X, \varphi_G)$ is isomorphic to $\mathrm{Res}_{L/k}(X)$.

Lemma 4.13. *The assignment $X \to \mathrm{Ind}_L^K X$ extends to a polynomial map of degree $\leq n$, denoted $\mathrm{Ind}_L^K : K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_K)$.*

Proof. For each $\iota \in I_L$, note that $g_\iota : [X] \mapsto [X \times_{L,\iota} K]$ is a ring homomorphism. Applying Corollary 4.10 gives us the result. □

In order to show that maps into $K_0(\mathrm{Var}_k)$ are polynomial maps, it is helpful to note the following, which is probably well known.

Lemma 4.14. *Let Y/k be a variety. Then the class $[Y]$ in $K_0(\mathrm{Var}_k)$ is trivial if and only if $Y = \emptyset$.*

Proof. By naturality of $K_0(\mathrm{Var}_k)$, it is enough to prove this in the case where k is algebraically closed. Suppose that Y is a non-empty variety, and let $\dim(Y) = d$. As in Proposition 2.27, we have abelian group homomorphisms $b_i : K_0(\mathrm{Var}_k) \to \mathbb{Z}$, which, if X/k is a smooth projective variety, has that $b_i(X)$ is the ith Betti number of X. The proof of [Bit04, Theorem 3.1] shows that we may write $[Y] = [\overline{X}] + S$, where $S \in K_0(\mathrm{Var}_k)$ can be written in terms of classes of smooth projective varieties of dimension $< d$. In particular, $b_{2d}(Y) = b_{2d}(\overline{X}) + b_{2d}(S)$. Since S can be written in terms of smooth projective varieties of dimension $< d$, we see that $b_{2d}(S) = 0$. Moreover, since $\overline{X}$ is a smooth projective variety of dimension d, we see that $b_{2d}(\overline{X}) = \#\pi_0(\overline{X}) \neq 0$. Therefore, $b_{2d}(Y) \neq 0$, so $[Y] \neq 0$. □

Lemma 4.15. *There is a multiplicative polynomial map $K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_k)$, given on smooth quasi-projective varieties by $[X] \mapsto [\mathrm{Res}_{L/k}(X)]$.*

Proof. By Lemma 4.7, we only need to define the polynomial map on the subset of $K_0(\mathrm{Var}_L)$ consisting of elements that are given by $[X]$, where X is a smooth projective variety. As in Definition 4.12, we note that $\mathrm{des}_{K/k}(\mathrm{Ind}_L^K X, \varphi_G)$. Therefore, as noted in Definition 3.1, it is enough to define a map $K_0(\mathrm{Var}_L) \to K_0(\mathrm{Var}_{K,\mathrm{desc}})$ such that for X/L a quasi-projective variety, we have that $[X] \mapsto [(\mathrm{Ind}_L^K X, \varphi_G)]$.

By Lemma 4.7, it is enough to show that the assignment $\mathrm{Res}_{L/k} : [X] \mapsto [(\mathrm{Ind}_L^K X, \varphi_G)]$ satisfies the conditions to be a polynomial map whenever X is projective. That is, we need to show that for any projective varieties $Y_0, \ldots, Y_n$, we have $\Delta_{Y_0}\Delta_{Y_1} \ldots \Delta_{Y_n}\mathrm{Res}_{L/k}([X]) = 0$. Consider $\Delta_Y \mathrm{Res}_{L/k} : [X] \mapsto [(\mathrm{Ind}_L^K (X \amalg Y), \varphi_G)] - [(\mathrm{Ind}_L^K X, \varphi_G)]$. Note that $\mathrm{Ind}_L^K X$ embeds as a closed subvariety into $\mathrm{Ind}_L^K (X \amalg Y)$ in a way that is compatible with the descent data on both varieties. Therefore, we can write $\Delta_Y \mathrm{Res}_{L/k}([X]) = [(X_Y, \varphi_G)]$, where (X_Y, φ_G) is the K variety with descent data given by the complement of $(\mathrm{Ind}_L^K (X), \varphi_G)$ in $(\mathrm{Ind}_L^K (X \amalg Y), \varphi_G)$.

Therefore, $\Delta_Y \mathrm{Res}_{L/k}([X])$ is represented by the class of an actual K variety with descent data (X_Y, φ_G) and so gives us a k variety $\mathrm{des}_{K/k}(X_Y, \varphi_G)$. Iterating this process, if $Y_0, \ldots, Y_n$

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are L varieties, we can write

$$\Delta_{Y_0}\Delta_{Y_1}\dots\Delta_{Y_n}\mathrm{Res}_{L/k}([X]) = [(X_{Y_0,\dots,Y_n}, \varphi_G)],$$

where $(X_{Y_0,\dots,Y_n}, \varphi_G)$ is a K variety with descent data. We claim that $[(X_{Y_0,\dots,Y_n}, \varphi_G)] = 0$. Note that $[(X_{Y_0,\dots,Y_n}, \varphi_G)]$ is identified with $[\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)]$ under the isomorphism $K_0(\mathrm{Var}_{K,\mathrm{desc}}) \cong K_0(\mathrm{Var}_k)$, so it is enough to show that $[\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)] = 0$. Since this is the class of a k variety, $\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)$, this class vanishes if and only if the variety $\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)$ is the empty variety by Lemma 4.14.

We see that $\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G) = \emptyset$ if and only if $\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)_K$ is trivial as a K variety. However, $[\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)_K] = [\Delta_{Y_0}\Delta_{Y_1}\dots\Delta_{Y_n}\mathrm{Ind}_L^K(X)]$, which vanishes since Ind_L^K is a polynomial map of degree $\leq n$. In particular, $[\mathrm{des}_{K/k}((X_{Y_0,\dots,Y_n}), \varphi_G)] = 0$, and so $\Delta_{Y_0}\Delta_{Y_1}\dots\Delta_{Y_n}\mathrm{Res}_{L/k}([X]) = 0$, as required. □

Definition 4.16. Let q be a quadratic form over L, on a vector space, V. For $\iota \in I_L$, define q_ι to be the quadratic form on $V \otimes_{L,\iota} K$ given by $q \otimes_{L,\iota} \langle 1\rangle$, where we embed L into K by ι. It is clear that $f_\iota : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(K)$ given by $[q] \mapsto [q_\iota]$ is a homomorphism. Define $\mathrm{Ind}_L^K(q) := \otimes_\iota q_\iota$ as a quadratic form on K. Note that $\mathrm{Ind}_L^K(q)$ has underlying vector space $\bigotimes_{\iota\in I_L}(V \otimes_\iota K)$ and that we may give this space a semilinear G action, where $g \in G$ acts by

$$g(v \otimes \lambda)_\iota = (v \otimes g(\lambda))_{g\iota},$$

where $(v \otimes \lambda)_\iota$ means $v \otimes \lambda \in V \otimes_\iota K$. We see that $\mathrm{Ind}_L^K(q)$ is a Galois-compatible quadratic form on $\bigotimes_{\iota\in I_L}(V \otimes_\iota K)$ with respect to this Galois action. Write $\mathrm{Ind}_L^K(q)$ for the Galois-compatible quadratic form on $\bigotimes_{\iota\in I_L}(V \otimes_\iota K)$, and let $N_{L/k}(q)$ be the associated quadratic form over k that results from applying the equivalence of categories in Theorem 3.3 to $\mathrm{Ind}_L^K(q)$.

By the above work, it is clear that $\mathrm{rank}(N_{L/k}(q)) = \mathrm{rank}(q)^{[L:k]}$.

Remark 4.17. *A priori*, it is not clear that $N_{L/k}(q)$, as defined above, is equivalent to the norm as in Proposition 4.8. The following lemma shows that this is not something to be concerned about, and that the maps are the same.

Lemma 4.18. *Let $q = \langle\alpha\rangle$ for $\alpha \in L$. Then $[N_{L/k}(q)] = \langle N_{L/k}(\alpha)\rangle \in \widehat{\mathrm{W}}(k)$.*

Proof. Note that $g_\iota(\langle\alpha\rangle) = \langle\iota(\alpha)\rangle$. Similarly, we see that $\bigotimes_{\iota\in I_L} k$ is the subspace of $\bigotimes_{\iota\in I_L} L \otimes_\iota K$ fixed by G. In particular, $N_{L/k}(\langle\alpha\rangle)(1 \otimes \dots \otimes 1) = \prod_{\iota\in I_L}\iota(\alpha) = N_{L/k}(\alpha)$, which gives us that $N_{L/k}(\langle\alpha\rangle) = \langle N_{L/k}(\alpha)\rangle$, as required. □

Corollary 4.19. *The map $N_{L/k}(q)$ from Proposition 4.8 and the map $N_{L/k}(q)$ from Definition 4.16 agree.*

Proof. This is immediate by the above lemma and Proposition 4.8. □

Remark 4.20. Since $N_{L/k} : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(k)$ is a polynomial map, we obtain a polynomial map $\mathrm{Ind}_L^K : \widehat{\mathrm{W}}(L) \to \widehat{\mathrm{W}}(K)^{\mathrm{desc}}$ by composing with the isomorphism in Corollary 3.4.

Corollary 4.21. *Let $w \in \widehat{\mathrm{W}}(L)$ with $E_{\mathrm{disc}}(w) = (\alpha, \beta) \in E((L^\times)/(L^\times)^2)$. Then,*

$$E_{\mathrm{disc}}(N_{L/k}(w)) = (\alpha^{[L:k]}, N_{L/k}(\beta)).$$

Proof. We know that $E_{\mathrm{disc}} \circ N_{L/k}$ is a polynomial map of degree $\leq n$, so we can reduce to the case where w is a quadratic form, and the result will follow by Lemma 4.7. This is then a straightforward computation, similar to the lemma above. □

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Theorem 4.22. *The following diagram is commutative.*

$$\begin{array}{ccc} K_0(\mathrm{Var}_L) & \xrightarrow{\chi_L^{\mathrm{mot}}} & \widehat{\mathrm{W}}(L) \\ \downarrow{\scriptstyle \mathrm{Res}_{L/k}} & & \downarrow{\scriptstyle N_{L/k}} \\ K_0(\mathrm{Var}_k) & \xrightarrow{\chi_k^{\mathrm{mot}}} & \widehat{\mathrm{W}}(k) \end{array}$$

In particular, if X/L is a variety such that $\mathrm{Res}_{L/k}(X)$ exists, then

$$\chi_k^{\mathrm{mot}}(\mathrm{Res}_{L/k}(X)) = N_{L/k}(\chi_L^{\mathrm{mot}}(X)).$$

Proof. The composition of two multiplicative polynomial functions is again a multiplicative polynomial function. Therefore, if for X a smooth projective variety over L, we have

$$\chi^{\mathrm{mot}}(\mathrm{Res}_{L/k}(X)) = N_{L/k}(\chi^{\mathrm{mot}}(X)),$$

then $\chi^{\mathrm{mot}} \circ \mathrm{Res}_{L/k}$ and $N_{L/k} \circ \chi^{\mathrm{mot}}$ define the same polynomial function from $K_0(\mathrm{Var}_L)$ to $\widehat{\mathrm{W}}(k)$ by Lemma 4.7. Since $\mathrm{Res}_{L/k}(X) = \mathrm{des}_{K/k}((\mathrm{Ind}_L^K(X), \varphi_G))$, and $N_{L/k}(\chi_L^{\mathrm{mot}}(X))$ is the image of $\mathrm{Ind}_L^K(\chi_L^{\mathrm{mot}}(X)^{\mathrm{desc}})$ under the isomorphism $\widehat{\mathrm{W}}(k) \to \widehat{\mathrm{W}}(K)^{\mathrm{desc}}$, the theorem follows if we can show that the following diagram is commutative.

$$\begin{array}{ccc} K_0(\mathrm{Var}_L) & \xrightarrow{\chi_L^{\mathrm{mot}}} & \widehat{\mathrm{W}}(L) \\ \downarrow{\scriptstyle \mathrm{Ind}_L^K} & & \downarrow{\scriptstyle Ind_L^K} \\ K_0(\mathrm{Var}_{K,\mathrm{desc}}) & \xrightarrow{\chi_k^{\mathrm{mot,desc}}} & \widehat{\mathrm{W}}(K)^{\mathrm{desc}} \end{array}$$

Therefore, we need to show that

$$\chi^{\mathrm{mot,desc}}(\mathrm{Ind}_L^K X) = \prod_\iota g_\iota(\chi^{\mathrm{mot}}(X)) \in \widehat{\mathrm{W}}(k),$$

which is clear by naturality of the motivic Euler characteristic, the semilinear actions on both sides, and the fact that Galois descent is compatible with the motivic Euler characteristic as in Corollary 3.7. □

5. Fubini theorems

Historically, since the motivic Euler characteristic is a motivic measure, we might think of it as a volume. Indeed, the motivic Gauss–Bonnet formula of [LR20] says that we may obtain the motivic Euler characteristic as the integral of an Euler class across our space. Let X, Y be varieties over k, and suppose that we have a morphism $f : X \to Y$. For the purposes of this paper, a Fubini theorem is a theorem that allows us to compute properties of $\chi^{\mathrm{mot}}(X)$ from the motivic Euler characteristic $\chi^{\mathrm{mot}}(Y)$ as well as the motivic Euler characteristics of the fibres $\chi^{\mathrm{mot}}(X_y)$. For example, suppose that our base field is $\mathbb{C}$, so that χ^{mot} is the compactly supported Euler characteristic applied to the complex points. By the standard Fubini theorem, $\chi_c(X_y(\mathbb{C})) = 0$ for all y implies that $\chi_c(X(\mathbb{C})) = 0$. This section is concerned with proving certain Fubini theorems for the motivic Euler characteristic modulo J. We first look at the situation where $f : X \to Y$ satisfies that $\chi^{\mathrm{mot}}(X_y) = 0$ for every closed point y of Y. The principle that 'if an invariant is zero on the fibres, then it is zero on the total space' will be referred to as a naïve Fubini theorem. It does not tend to hold in our context, and we obtain a Fubini theorem for the ℓ-adic determinant of cohomology in Corollary 5.19, which does not satisfy this naïve property.

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However, in Corollary 5.27, we show that we may bound the ramification of the character that we obtain by looking at the ℓ-adic determinant of cohomology of the total space, which allows us to restrict the possibilities of this character to a finite set. We then turn to looking at a different Fubini theorem, one for maps whose fibres are algebraic groups, where we can use the structure of algebraic groups in order to prove a naïve Fubini theorem for torsors in Corollary 5.31.

5.1 The Fubini theorem over real closed fields

The goal of this subsection is to prove the first part of our Fubini theorem.

Theorem 5.1. *Suppose that X and Y are varieties over k, and let $f : X \to Y$ be a morphism such that for every y a closed point of Y, we have that $\chi^{\mathrm{mot}}(X_y)$ is torsion. Then $\chi^{\mathrm{mot}}(X)$ is torsion.*

We call this the real closed Fubini theorem as, by Pfister's local–global principle, we have that $\chi^{\mathrm{mot}}(X)$ is torsion if and only if $\mathrm{rank}(\chi^{\mathrm{mot}}(X)) = 0$ and $\mathrm{sign}(\chi^{\mathrm{mot}}(X)) = 0$. This is therefore is a true naïve Fubini theorem. This holds if we can show the corresponding statement for the rank and sign of motivic Euler characteristics. We first prove this statement for real closed fields and their algebraic closures. Until specified otherwise, let k be either a real closed field or the algebraic closure of a real closed field.

Definition 5.2. Let $\mathcal{SAS}_k$ be the category of semi-algebraic sets over k. Define $K_0(\mathcal{SAS}_k)$ to be the ring generated by isomorphism classes of semi-algebraic sets over k subject to the relations:

(1) $[\emptyset] = 0$;
(2) for U a closed semi-algebraic subset of X, we have $[X] = [X \setminus U] + [U]$; and
(3) $[X][Y] = [X \times Y]$.

There are canonical homomorphisms $K_0(\mathrm{Var}_k) \to K_0(\mathcal{SAS}_k)$, given by sending a variety X to $X(k)$ or to $X(\overline{k})$.

Let X be a semi-algebraic set. Write $H^i_c(X; k)$ to mean the ith real closed singular cohomology with compact support, and coefficients in k. Note that the semi-algebraic simplicial cohomology is isomorphic to the sheaf cohomology with coefficients in the constant sheaf $\underline{k}$. This is proven in the non-compactly supported case in [KD82, Theorem 3.3]. This isomorphism extends to the compactly supported case, using that the compactly supported cohomology is given by

$$H^*_c(X; k) = \varinjlim H^*(X, X \setminus V; k),$$

where V runs over all of the compact subsets of X. Since we have canonical isomorphisms

$$H^*_{\mathrm{sheaf}}(X, X \setminus V; \underline{k}) \cong H^*_{\mathrm{sing}}(X, X \setminus V; k),$$

this isomorphism carries over to the limit.

Definition 5.3. Let X be a semi-algebraic set over k. Define the *real closed Euler characteristic* $\chi^{\mathrm{rc}}(X) := \sum_{i=0}^{\infty}(-1)^i \dim_k(H^i_c(X; k)) \in \mathbb{Z}$.

Lemma 5.4. *The real closed Euler characteristic χ^{rc}, as defined above, furnishes a ring homomorphism $\chi^{\mathrm{rc}} : K_0(\mathcal{SAS}_k) \to \mathbb{Z}$.*

Proof. We only need to show that the three relations defining $K_0(\mathcal{SAS}_k)$ are satisfied. The relation that $\chi^{\mathrm{rc}}([\emptyset]) = 0$ is trivial. By [EP15, Theorem 4.1], we see that the sheaf cohomology with compact support and coefficients in $\underline{R}$ satisfies the exactness and excision axioms. Therefore,

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for U an open subset of X, we have a long exact sequence

$$\ldots \to H_c^n(U;k) \to H_c^n(X;k) \to H_c^n(X \setminus U;k) \to H_c^{n+1}(U;k) \to \ldots.$$

This immediately tells us that $\chi^{\mathrm{rc}}(X) = \chi^{\mathrm{rc}}(U) + \chi^{\mathrm{rc}}(X \setminus U)$. Multiplicativity of χ^{rc} follows from the Künneth formula [EP20, Theorem 3.40]. □

Theorem 5.5. *Let $f : W \to Z$ be a morphism of semi-algebraic sets over k. Suppose that for each $p \in Z$, the real closed Euler characteristic of the fibre $\chi^{\mathrm{rc}}(W_p) = q$. Then $\chi^{\mathrm{rc}}(W) = q\chi^{\mathrm{rc}}(Z)$.*

Proof. By [BCR98, Lemma 9.3.2], there exists a stratification of Z into constructible subsets $Z_1, \ldots, Z_n$, such that for each i, we have an isomorphism $f^{-1}(Z_i) \cong F_i \times W_i$ for some F_i fitting into the commutative diagram

$$\begin{array}{ccc} f^{-1}(Z_i) & \longrightarrow & F_i \times Z_i \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & Z_i & \end{array}$$

where the morphism $F_i \times Z_i \to Z_i$ is the canonical projection. This means that for $p \in Z_i$, there is an isomorphism $W_p \cong F_i$, so $\chi^{\mathrm{rc}}(F_i) = \chi^{\mathrm{rc}}(W_p) = q$. In $K_0(\mathcal{SAS}_k)$ we have $[W] = \sum_{i=1}^n [F_i][Z_i]$, so applying χ^{rc} and noting that $\chi^{\mathrm{rc}}(F_i) = q$, we obtain that $\chi^{\mathrm{rc}}(W) = q\chi^{\mathrm{rc}}(Z)$. □

Lemma 5.6. *For X a k variety, we have an equality in $\widehat{\mathrm{W}}(k)$:*

$$\chi^{\mathrm{mot}}(X) = \tfrac{1}{2}(\chi^{\mathrm{rc}}(X(\overline{k})) + \chi^{\mathrm{rc}}(X(k)))\langle 1\rangle + \tfrac{1}{2}(\chi^{\mathrm{rc}}(X(\overline{k})) - \chi^{\mathrm{rc}}(X(k)))\langle -1\rangle.$$

Proof. By Theorem 2.14, we have that $\mathrm{rank}(\chi^{\mathrm{mot}}(X)) = e(X) = \chi^{\mathrm{rc}}(X(\overline{k}))$. Moreover, we have that $\mathrm{sign}(\chi^{\mathrm{mot}}(X)) = \epsilon(X) = \chi^{\mathrm{rc}}(X(k))$ by Theorem 2.20. The result follows since if q is any element of $\widehat{\mathrm{W}}(k)$, we have that

$$q = \tfrac{1}{2}(\mathrm{rank}(q) + \mathrm{sign}(q))\langle 1\rangle + \tfrac{1}{2}(\mathrm{rank}(q) - \mathrm{sign}(q))\langle -1\rangle.$$ □

Theorem 5.7. *Let X and Y be k varieties, and let $f : X \to Y$ be a morphism. For p a closed point of Y let X_p be the fibre of f over p. Then, if $\chi_k^{\mathrm{mot}}(X_p)$ is trivial for all p, then so is $\chi^{\mathrm{mot}}(X)$.*

Proof. By Lemma 5.6,

$$\chi^{\mathrm{mot}}(X) = \tfrac{1}{2}(\chi^{\mathrm{rc}}(X(\overline{k})) + \chi^{\mathrm{rc}}(X(k)))\langle 1\rangle + \tfrac{1}{2}(\chi^{\mathrm{rc}}(X(\overline{k})) - \chi^{\mathrm{rc}}(X(k)))\langle -1\rangle,$$

so it is enough to prove the result for $\chi^{\mathrm{rc}}(X(\overline{k}))$ and $\chi^{\mathrm{rc}}(X)$. First, suppose that k is a real closed field. If p has residue field k, then $\chi^{\mathrm{rc}}(X_p(k)) = \mathrm{sign}(\chi^{\mathrm{mot}}(X_p)) = 0$ by assumption. If p has residue field $\overline{k}$, then $X_p(k) = \emptyset$, so $\chi^{\mathrm{rc}}(X_p(k)) = 0$, and therefore $\chi^{\mathrm{rc}}(X_p(k)) = 0$ for all p. Theorem 5.5 gives that $\chi^{\mathrm{rc}}(X(k)) = 0$. Similarly, $\chi^{\mathrm{rc}}(X_p(\overline{k})) = \mathrm{rank}(\chi^{\mathrm{mot}}(X_p))$, so Theorem 5.5 gives $\chi^{\mathrm{rc}}(X(\overline{k})) = 0$, as required.

If k is the algebraic closure of a real closed field, then $\chi^{\mathrm{rc}}(X(k)) = \chi^{\mathrm{mot}}(X)$, so we again apply Theorem 5.5 to $X(k)$ and obtain $\chi^{\mathrm{mot}}(X) = 0$. □

We wish to extend the above result to our real closed Fubini theorem for any field of characteristic 0. We first need the following lemma.

Lemma 5.8. *Let n be a non-negative integer. Then* $\mathrm{Spr}(\mathbb{Q}(t_1, \ldots, t_n)) \neq \emptyset$.

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Proof. We prove this by induction, with the case $n=0$ being trivial. Suppose that it is true for $n'<n$, let $K=\mathbb{Q}(t_1,\dots,t_{n-1})$, and let $\prec\in \mathrm{Spr}(K)$. As in [BCR98, Example 1.1.2], we may define an order $<$ on $K(t_n)=\mathbb{Q}(t_1,\dots,t_n)$ as follows. Let $p(t_n)\in K[t_n]$ be non-zero. Then $p(t_n)=a_kt_n^k+\ldots+a_mt_n^m$ for some non-negative integers $k\le m$ with $a_k\neq 0$. Then say $0<p(t_n)$ if $0\prec a_k$. For a general element $[p(t_n)]/[q(t_n)]$ of $K(t_n)$, we say that $0<[p(t_n)]/[q(t_n)]$ if and only if $0<p(t_n)q(t_n)$. As in [BCR98, Example 1.1.2], this defines a total order on $K(t_n)$ extending $\prec$, as required. □

Now suppose that k is an arbitrary field of characteristic 0.

Lemma 5.9. *Let $f:X\to Y$ be a morphism such that for every y a closed point of Y, we have $\mathrm{rank}(\chi_k^{\mathrm{mot}}(X_y))=0$. Then $\mathrm{rank}(\chi^{\mathrm{mot}}(X))=0$.*

Proof. This lemma follows by reducing to the case of Theorem 5.7, and so we need to reduce to the case where k is the algebraic closure of a real closed field. We first reduce to the case where our base field is the algebraic closure of a finitely generated extension of $\mathbb{Q}$.

Since X,Y are varieties over k, we may assume that X,Y are defined over a field F that is finitely generated over $\mathbb{Q}$. Let X_0 and Y_0 be varieties over F such that $X=X_0\times_F\mathrm{Spec}(k)$ and $Y=Y_0\times_F\mathrm{Spec}(k)$. Pick an algebraic closure $\overline{F}$ of F, and for every finite extension F'/F, pick compatible embeddings $F'\hookrightarrow\overline{F}$. Let $\overline{X}:=X_0\times_F\mathrm{Spec}(\overline{F})$, and similarly for $\overline{Y}$. Then, by Lemma 4.2, we see that $\mathrm{rank}(\chi_k^{\mathrm{mot}}(X))=\mathrm{rank}(\chi_F^{\mathrm{mot}}(X_0))=\mathrm{rank}(\chi_{\overline{F}}^{\mathrm{mot}}(\overline{X}))$. We claim that for every $\overline{y}$ a closed point of $\overline{Y}$, we have $\mathrm{rank}(\chi_{\overline{F}}^{\mathrm{mot}}(\overline{X}_{\overline{y}}))=0$. Let $y\in Y$ be the closed point of Y lying under $\overline{y}$ under the morphism $\overline{Y}\to Y$. Let F_y denote the residue field at y, and let $n:=[F_y:F]$. Note that $\overline{X}_{\overline{y}}$ is defined over F_y, and $\overline{X}_{\overline{y}}=X_y\times_{\mathrm{Spec}(F_y)}\mathrm{Spec}(\overline{F})$. By Lemma 4.2, we see that $\mathrm{rank}(\chi_{\overline{F}}^{\mathrm{mot}}(\overline{X}_{\overline{y}}))=\mathrm{rank}(\chi_{F_y}^{\mathrm{mot}}(X_y))$. Lemma 4.5 then gives $\chi_F^{\mathrm{mot}}(X_y)=\mathrm{Tr}_{F_y/F}(\chi_{F_y}^{\mathrm{mot}}(X_y))$, so taking ranks, we see that $0=n_y\cdot\mathrm{rank}(\chi_{F_y}^{\mathrm{mot}}(X_y))$. Therefore, $\mathrm{rank}(\chi_{F_y}^{\mathrm{mot}}(X_y))=0$, as required.

Therefore, it suffices to prove the statement for $\overline{X}\to\overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are varieties over $\overline{F}$ such that the fibres of the morphism have trivial motivic Euler characteristic. Let n denote the transcendence degree of $F/\mathbb{Q}$. By Noether's normalisation lemma (see, for example, [Lan02, Chapter 8, Theorem 2.1]), the field F is an finite extension of $\mathbb{Q}(t_1,\dots,t_n)$. By Lemma 5.8, we have $\mathrm{Spr}(\mathbb{Q}(t_1,\dots,t_n))\neq\emptyset$. Pick $<\in\mathrm{Spr}(\mathbb{Q}(t_1,\dots,t_n))$, and let F^{real} be the real closure of $\mathbb{Q}(t_1,\dots,t_n)$ with respect to this ordering. Then $\overline{F}$ is the algebraic closure of F^{real}, so the extension $\overline{F}/F^{\mathrm{real}}$ is a degree-2 field extension. We may therefore think of $\overline{X},\overline{Y}$ and the fibres $\overline{X}_y$ as varieties over F^{real}. The result follows by Theorem 5.7. □

Corollary 5.10. *Let k now be any characteristic 0 field. Let X and Y be varieties over k, and let $f:X\to Y$ be a morphism. Let p be a closed point of Y, and let X_p be the fibre of f over p. If $\chi^{\mathrm{mot}}(X_p)$ is torsion in $\widehat{\mathrm{W}}(k)$ for all p, then $\chi^{\mathrm{mot}}(X)$ is torsion.*

Proof. By Pfister's local–global principle, we see that $\chi^{\mathrm{mot}}(X_p)$ is torsion if and only if we have $\mathrm{rank}(\chi^{\mathrm{mot}}(X_p))=0$ and $\chi^{\mathrm{mot}}(X_{k_<,p})=0$ for all $<\in\mathrm{Spr}(k)$, and similarly for $\chi^{\mathrm{mot}}(X)$. If $\mathrm{Spr}(k)=\emptyset$, then the only conditions are on the rank, so the result follows by Lemma 5.9. Suppose then that $\mathrm{Spr}(k)\neq\emptyset$. Let $<\in\mathrm{Spr}(k)$. Since χ^{mot} commutes with base change, we have $\chi^{\mathrm{mot}}(X_{k_<,y})=0$ for all closed points y of Y. Theorem 5.7 gives $\chi^{\mathrm{mot}}(X_{k_<})=0$, as required. □

5.2 The determinant of cohomology

Now suppose that k is an arbitrary field of characteristic 0. In this subsection, we show a Fubini theorem for the ℓ-adic determinant of cohomology that is not a naïve Fubini theorem. Let U be a smooth scheme over k and $\mathcal{F}$ be a smooth ℓ-adic sheaf for $\ell\neq\mathrm{char}(k)$. Consider the

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one-dimensional ℓ-adic representation of Gal_k:

$$\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}) = \bigotimes_i \det H^i_c(U_{\overline{k}}, \mathcal{F})^{\otimes(-1)^i}.$$

In order to prove our Fubini theorem, we make use of the following theorem, which is the main theorem of [Sai94].

Theorem 5.11. *Assume that there is a smooth projective variety X/k that contains U such that $D = X \setminus U$ is a divisor with simple normal crossings. Moreover, assume that the ramification of $\mathcal{F}$ on D is tame. Finally, assume that $\mathcal{F}$ is defined on a model of U that is defined over a ring of finite type over $\mathbb{Z}$. Then we have*

$$\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}) \otimes \det R\Gamma_c(U_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes -\mathrm{rank}\mathcal{F}} = c^*_{X,U/k}(\det \mathcal{F}) \otimes J_{D,\mathcal{F}}^{\otimes(-1)}$$

as one-dimensional ℓ-adic representations of Gal_k.

The first term on the right-hand side is the pullback of the determinant character $\det \mathcal{F}$ of $\pi_1^{\acute{e}t}(U)^{\mathrm{ab,tame}}$ to Gal_k by the pairing with the relative canonical class $c_{X,U/k} \in CH_0(X, D)$. In this paper, we only need that $c^*_{X,U/k}$ respects tensor products, and we will only ever consider this in the case when $\det(\mathcal{F})$ is trivial, so this term will always vanish. We refer to the bottom of [Sai94, p. 414] for the precise construction of $c^*_{X,U/k}$. We will, however, require the definition of $J_{D,\mathcal{F}}$, as set out in the paragraph before [Sai94, Theorem 1, p. 424], as well as the definition on [Sai94, p. 416], and also [Sai94, Proposition 2, p. 417]. We will recall these below for the convenience of the reader.

Definition 5.12. Let A denote a regular noetherian ring, let $S = \mathrm{Spec}(A)$, and suppose that S is connected with function field k. Following [Sai94], define a *Jacobi datum* on S to be a triple (T, χ, n) such that:

(1) $T = (T_i)_{i\in I}$ is a finite family of finite étale covers $T_i \to S$;
(2) $\chi = (\chi_i)_{i\in I}$ is a family of characters $\chi_i : \mu_{d_i}(T_i) \to \overline{\mathbb{Q}_l}^\times$ of the group of d_ith roots of unity for some integer d_i invertible in A such that $\mu_{d_i} \simeq \mathbb{Z}/d_i\mathbb{Z}$ on T_i; and
(3) $n = (n_i)_{i\in I}$ is a family of integers satisfying the norm condition below.

Write $T_i = \coprod_{j\in J_i} T_{ij}$, where each T_{ij} is connected. Let the norm $N_{T_{ij}/S}(\chi_i)$ denote the product of the conjugates, and let the norm $N_{T_i/S}$ be given by

$$\begin{aligned} N_{T_i/S}(\chi_i) : \mu_{d_i}(T_i) &\to \overline{\mathbb{Q}_\ell}^\times \\ \zeta &\mapsto \prod_j \chi_i(N_{T_{ij}/S}(\zeta)). \end{aligned}$$

By construction, each $N_{T_i/S}(\chi_i)$ factors through some $\mu_{d_i'}(T_i)$ such that $d_i' | d_i$ and $\mu_{d_i'} = \mathbb{Z}/d_i'\mathbb{Z}$ on S. Let d be a common multiple of all of the d_i', and regard $N_{T_i/S}(\chi_i)$ as a character on μ_d via the composition

$$\begin{aligned} \mu_d &\to \mu_{d_i'}, \\ \zeta &\mapsto \zeta^{\frac{d}{d_i'}}. \end{aligned}$$

The norm condition for our Jacobi datum is that we require that $\prod_{i\in I} N_{T_i/S}(\chi_i)^{n_i} = 1$, where this product is considered as a product of characters of μ_d on S.

If A' is another ring equipped with a morphism $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$, we may pull back the Jacobi datum (T, χ, n) on $\mathrm{Spec}(A)$ to $\mathrm{Spec}(A')$ to obtain $f^*(T, \chi, n) = (f^*T, f^*\chi, n)$, where

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$f^*(T) = (f^*T_i)$ is given by the pullback of the finite étale covers $T_i \to \mathrm{Spec}(A)$ along the morphism f, and $f^*\chi_i$ is the composition $\chi_i : \mu_{d_i}(f^*T_i) \to \mu_{d_i}(T_i) \to \mathbb{Q}_\ell^\times$. We say that the Jacobi datum (T', χ', n') on A' is *defined on* A if there exists a Jacobi datum (T, χ, n) on A such that we may obtain (T', χ', n') by pulling back the Jacobi datum (T, χ, n) along the morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. If s is a closed point of A with residue field $\kappa(s)$, we will say that the *reduction of* (T, χ, n) *at* s is the Jacobi datum on $\kappa(s)$ given by pulling back (T, χ, n) along the closed immersion corresponding to s.

Definition 5.13. Let $\mathbb{F}$ be a finite field of order q. Let χ be a multiplicative character, i.e. a group homomorphism $\chi : \mathbb{F}^\times \to \overline{\mathbb{Q}_l}^\times$. Let ψ be an additive character, i.e. a group homomorphism $\psi : \mathbb{F} \to \overline{\mathbb{Q}_\ell}^\times$. Define the *Gauss sum*

$$G_{\mathbb{F}}(\chi, \psi) := -\sum_{a \in \mathbb{F}^\times} \chi^{-1}(a)\psi(a).$$

Definition 5.14. Let $\mathbb{F}$ be a finite field of order q. Then, if we have a Jacobi datum (T, χ, n) on $\mathrm{Spec}(\mathbb{F})$, we see that each $T_i = \coprod_{j \in \overline{I}_i} \mathrm{Spec}(\mathbb{F}_{ij})$ for some $\mathbb{F}_{ij}/\mathbb{F}$ a finite extension and $\overline{I}_i$ a finite index set. Define the *Jacobi sum* $j_\chi = j_{T,\chi,n}$ for a Jacobi datum (T, χ, n) on $\mathrm{Spec}(\mathbb{F})$ by

$$j_\chi := \prod_{i \in I} \prod_{j \in \overline{I}_i} (G_{\mathbb{F}_{ij}}(\overline{\chi}_{ij}, \psi_0 \circ \mathrm{Tr}_{\mathbb{F}_{ij}/\mathbb{F}}))^{n_i}.$$

Here, ψ_0 is a non-trivial additive character of k, and if $\mathbb{F}_{ij}$ is of order q_{ij} then $\overline{\chi}_{ij}$ is a multiplicative character of $\mathbb{F}_i$ defined by

$$\overline{\chi}_i(a) := \chi_i(a^{(q_{ij}-1)/d_i}).$$

As mentioned on [Sai94, p. 417], we can consider characters on $\mathbb{F}^\times$ as characters of $\mu_{q-1}(\mathbb{F})$. Moreover, as characters of $\mu_{q-1}(\mathbb{F})$, we have $\prod_{j \in \overline{I}_i} \overline{\chi}_i|_{\mathbb{F}^\times} = N_{T_i/\mathrm{Spec}(S)}(\chi_i)$. As mentioned on [Sai94, p. 417], [Del77, Proposition 4.15] tells us that the product j_χ is independent of our choice of ψ_0, since $\prod_{i \in I} N_{\mathbb{F}_i/\mathbb{F}}(\chi_i)^{n_i} = 1$.

Definition 5.15. As in [Sai94, Proposition 2], we can associate an ℓ-adic representation $J_{(T,\chi,n)}$ of Gal_k to each Jacobi datum (T, χ, n) on a field k as follows. A Jacobi datum on k is defined on a regular ring A of finite type over $\mathbb{Z}$, since each of the T_i are varieties over k. The representation $J_{(T,\chi,n)}$ is the pullback of the representation of $\pi_1^{\acute{e}t}(\mathrm{Spec}(A))^{ab}$ characterised by the following condition: for each closed point s of $\mathrm{Spec}(A)$, the action of the geometric Frobenius Frob_s at s is given by the multiplication by the Jacobi sum $j_\chi(s)$ defined by the reduction of the Jacobi datum (T, χ, n) at s. By [Sai94, Proposition 2], this character both exists and is unique. Say that $J_{(T,\chi,n)}$ is the *associated ℓ-adic character* of the Jacobi datum (T, χ, n).

Definition 5.16 (Saito [Sai94]). Let U and $\mathcal{F}$ be as in Theorem 5.11. We can define a Jacobi datum on k associated to the ramification of $\mathcal{F}$ along D. Let D_i be the irreducible components of D with constant field k_i. Let ρ be the ℓ-adic representation of $\pi_1(U, \overline{x})^{\mathrm{tame}}$ corresponding to $\mathcal{F}$. The kernel $\pi_1^{\acute{e}t}(U, \overline{x})^{\mathrm{tame}} \to \pi_1^{\acute{e}t}(X, \overline{x})$ is the normal subgroup topologically generated by the local monodromy groups $\widehat{\mathbb{Z}}'(1)_{D_i}$ along the D_i where $\widehat{\mathbb{Z}}'(1) = \varprojlim \mu_d$ with d invertible in k. Let ρ_i be the restriction of ρ to $\widehat{\mathbb{Z}}'(1)_{D_1} \simeq \widehat{\mathbb{Z}}'(1)_{k_i}$.

Since we have a model of finite type over $\mathbb{Z}$, Grothendieck's monodromy theorem [Del72, Théorème de monodromie 1.2] tells us that the ρ_i are quasi-unipotent. Therefore, the action of

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$\widehat{\mathbb{Z}}'(1)$ on the semi-simplification ρ_i^{ss} factors through a finite quotient. This means that we can decompose

$$\rho_i^{\mathrm{ss}} \simeq \bigoplus_{j\in I_i} \mathrm{Tr}_{k_{ij}/k_i}(\chi_{ij}).$$

Here, k_{ij} is the finite extension of k_i obtained by adjoining the d_{ij}th roots of unity, χ_{ij} is a character of $\mu_{d_{ij}}(k_{ij})$ of order d_{ij}, and we define Tr_{k_{ij}/k_i} to be the $[k_{ij}:k_i]$ dimensional $\widehat{\mathbb{Z}}'(1)$ representation

$$\mathrm{Tr}_{k_{ij}/k}(\chi_{ij}) := \bigoplus_{\sigma\in \mathrm{Gal}(k_{ij}/k_i)} \chi_{ij}\circ\sigma.$$

For $i\in I$, let $D_i^* = D_i - \bigcup_{J\neq i} D_j$ and let c_i be the compactly supported Euler characteristic of $(D_i^* \times_{k_i} \mathrm{Spec}(\overline{k_i}))$. Define our Jacobi datum (T,χ,n) by $\overline{I} = \coprod_i I_i$, $T=(k_{ij})$, $\chi=(\chi_{ij})$ and $n=(n_{ij})$, with $n_{ij}=c_i$ for $i\in I$ and $j\in I_i$. Define the term $J_{D,\mathcal{F}}$ to be the ℓ-adic representation $J_{(T,\chi,n)}$.

Definition 5.17. Let $f: W\to Y$ be a projective morphism of positive relative dimension. Assume that there is a smooth and projective variety Z over k containing Y such that the complement $D=Z-Y$ is a divisor with simple normal crossings, that $R^if_*\mathbb{Q}_\ell$ is smooth over Y, and that the ramification along D is tame for all i. We define

$$\prod_{i=0}^{2(\dim(W)-\dim(Y))} J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}$$

to be the *Saito factor* of f at D.

Theorem 5.18. *Let $f: W\to Y$ be a projective morphism of positive relative dimension. Assume that there is a projective and smooth variety Z over k containing Y such that the complement $D=Z-Y$ is a divisor with simple normal crossings, that $R^if_*\mathbb{Q}_\ell$ is smooth, and that the ramification is tame for all i. Moreover, assume that $\sum_i(-1)^i\mathrm{rank}(R^if_*\mathbb{Q}_\ell)=0$, $\bigotimes_i \det R^if_*\mathbb{Q}_\ell^{(-1)^i}$ is trivial, and f is defined on a model over a ring of finite type over $\mathbb{Z}$. Then,*

$$\det_\ell(W) = \prod_{i=0}^{2(\dim(W)-\dim(Y))} J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}.$$

Proof. By the Leray spectral sequence,

$$\det_\ell(W) = \prod_{i=0}^{2(\dim(W)-\dim(Y))} \det R\Gamma_c(Y_{\overline{k}}, R^if_*\mathbb{Q}_\ell)^{(-1)^i}.$$

Since f is defined on a model over a ring of finite type over $\mathbb{Z}$, the same is true for the sheaves $R^if_*\mathbb{Q}_\ell$, so we apply Saito's theorem (Theorem 5.11) to obtain

$$\prod_i (\det R\Gamma_c(Y_{\overline{k}}, R^if_*\mathbb{Q}_\ell)^{(-1)^i}) \otimes \det_\ell(Y)^{\otimes -\sum_i(-1)^i \mathrm{rank} R^if_*\mathbb{Q}_\ell} = c_{Y,D/k}^*\Big(\bigotimes_i (\det R^if_*\mathbb{Q}_\ell)^{(-1)^i}\Big) \otimes \prod_i J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}},$$

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using the bilinearity of the pairing $CH_0(X,D)\times\pi_1^{ét}(U)^{\mathrm{ab,tame}}\to \mathrm{Gal}_k$. By our assumptions on the rank and determinants of $R^if_*\mathbb{Q}_\ell$, we see that

$$\det_\ell(Y)^{\otimes -\sum_i(-1)^i \mathrm{rank} R^if_*\mathbb{Q}_\ell}=1,$$
$$c^*_{Y,D/k}\Big(\bigotimes_i(\det R^if_*\mathbb{Q}_\ell)^{(-1)^i}\Big)=1,$$

where 1 denotes the trivial representation of Gal_k. Since the sheaves $R^if_*\mathbb{Q}_\ell$ vanish whenever $i>2(\dim(W)-\dim(Y))$, this gives the result. □

Corollary 5.19. *Let $f:W\to Y$ be a projective morphism of positive relative dimension g. Assume that there is a projective and smooth variety Z over k containing Y such that the complement $D=Z-Y$ is a divisor with simple normal crossings, that $R^if_*\mathbb{Q}_\ell$ is smooth, and that the ramification along D is tame for all i. Moreover, assume that for every closed point of Y the fibre, W_y has $\chi^{\mathrm{mot}}(W_y)\equiv 0 \pmod J$. Then the motivic Euler characteristic modulo J satisfies $\chi^{\mathrm{mot}}(W)\equiv\kappa_W \pmod J$, where κ_W is an element of $\widehat{\mathrm{W}}(k)/J$ such that $\mathrm{rank}(\kappa_W)=0$, $\mathrm{sign}(\kappa_W)=0$, and $\mathrm{disc}(\kappa_W)=\det_\ell(W)=\prod_{i=0}^{2g}J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}$.*

Proof. The fact that $\mathrm{rank}(\kappa_{\mathcal{C}})$ and $\mathrm{sign}(\kappa_{\mathcal{C}})$ both vanish is due to our real closed Fubini theorem, Corollary 5.10. Consider the condition $\chi^{\mathrm{mot}}(W_y)\equiv 0 \pmod J$ for all y. Taking ranks tells us that $\mathrm{rank}(\chi^{\mathrm{mot}}(W_y))=0$ for all y, and applying proper base change tells us that $\sum_i(-1)^i\mathrm{rank}(R^if_*\mathbb{Q}_\ell)=0$. Similarly, we see that $\det_\ell(W_y)=0$ for all y, which is equivalent to saying that $\bigotimes_i\det(R^if_*\mathbb{Q}_\ell)^{(-1)^i}$ is trivial as well. The above theorem gives us the result. □

Remark 5.20. This Fubini theorem reduces our computation of $\chi^{\mathrm{mot}} \pmod J$ to Saito factors associated to the boundary of our morphism, thought of as characters in $\mathrm{Hom}(\mathrm{Gal}_k,\overline{\mathbb{Q}_\ell}^\times)$. Since $\mathrm{disc}(\kappa_W)=\det_\ell(W)=\prod_{i=0}^{2g}J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}$, and $\mathrm{disc}(\kappa_W)$ is a character of order 2 by construction, the Saito factor $\prod_{i=0}^{2g}J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}$ will also be a character of order 2. For the rest of this section, we show that these characters of Gal_k naturally factor through the absolute Galois group of a number field, and we can control the ramification of these characters.

Definition 5.21. For k a field, let k^{const} denote the algebraic closure of its prime subfield in k. In particular, if k is a field that is finitely generated over $\mathbb{Q}$, the field k^{const} is a number field.

Lemma 5.22. *Let (T,χ,n) be a Jacobi datum on k, a field of characteristic 0. Let d denote the lowest common multiple of the d_j, and let $A:=\mathrm{Spec}(\mathcal{O}_{k^{\mathrm{const}}}[\frac{1}{d}])$. Then there exists a Jacobi datum (T',χ',n') on $\mathrm{Spec}(A)$ such that the character $J_{(T,\chi,n)}$ is given by the composition*

$$\mathrm{Gal}_k\to\pi_1^{ét}(\mathrm{Spec}(A))\xrightarrow{J_{(T',\chi',n')}}\overline{\mathbb{Q}_\ell}^\times.$$

Proof. This follows instantly by the second half of the corollary on [Sai94, p. 418]. □

Corollary 5.23. *The character $J_{(T,\chi,n)}$ factors as a composition $\mathrm{Gal}_k\twoheadrightarrow\mathrm{Gal}_{k^{\mathrm{const}}}\to\overline{\mathbb{Q}_\ell}^\times$.*

Proof. First, note that the character $J_{(T,\chi,n)}$ factors through the map $\mathrm{Gal}_k\to\pi_1^{ét}(\mathrm{Spec}(A))$, which in turn factors as $\mathrm{Gal}_k\to\mathrm{Gal}_{k^{\mathrm{const}}}\to\pi_1^{ét}(\mathrm{Spec}(A))$. The map $\mathrm{Gal}_k\to\mathrm{Gal}_{k^{\mathrm{const}}}$ is surjective since k^{const} is algebraically closed in k by construction. □

Definition 5.24. Since $\mathrm{Gal}_k\to\mathrm{Gal}_{k^{\mathrm{const}}}$ is surjective, the character $\mathrm{Gal}_{k^{\mathrm{const}}}\to\overline{\mathbb{Q}_\ell}^\times$ appearing in the above corollary is unique. Call this character $J^{\mathrm{const}}_{(T,\chi,n)}$.

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Corollary 5.25. *Let p be a prime number coprime to d_j for every j, and let $\mathfrak{p}$ be a prime of k^{const} with residue characteristic p. Then the character $J^{\mathrm{const}}_{(T,\chi,n)}$ is unramified at $\mathfrak{p}$.*

Proof. By construction, $J^{\mathrm{const}}_{(T,\chi,n)}$ factors as $\mathrm{Gal}_{k^{\mathrm{const}}} \to \pi_1^{\acute{e}t}(\mathrm{Spec}(A)) \to \overline{\mathbb{Q}_\ell}^{\times}$. For $\mathfrak{p}$ as above, we see that $\mathfrak{p} \in \mathrm{Spec}(A)$. Therefore, the inertia group of $\mathfrak{p}$ is in the kernel of the morphism $\mathrm{Gal}_{k^{\mathrm{const}}} \to \pi_1^{\acute{e}t}(\mathrm{Spec}(A))$. In particular, it is in the kernel of $J^{\mathrm{const}}_{(T,\chi,n)}$, as required. □

Corollary 5.26. *Suppose that we are now in the situation of Theorem 5.11, and let $J_{D,\mathcal{F}}$ be the ℓ-adic character appearing in the theorem. Suppose that p is a prime number satisfying*

$$p > \mathrm{rank}(\mathcal{F}) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1,$$

and let $\mathfrak{p}$ be a prime of k^{const} with residue characteristic p. Then $J^{\mathrm{const}}_{D,\mathcal{F}}$ is unramified at $\mathfrak{p}$.

Proof. By the above corollary, we only need to show that p does not divide any d_j. The d_j appear through the decomposition of the semi-simplification of the monodromy representation associated to $\mathcal{F}$,

$$\rho^{\mathrm{ss}} = \bigoplus_j \mathrm{Tr}_{k_j/k}(\chi_j),$$

where χ_j is a representation on $\mu_{d_j}(k_j)$ and, moreover, $\dim(\rho^{\mathrm{ss}}) = \mathrm{rank}(\mathcal{F})$. Note that

$$\dim \mathrm{Tr}_{k_j/k}(\chi_j) = [k_j : k] \geq [k(\zeta_{d_j}) : k] = [k^{\mathrm{const}}(\zeta_{d_j}) : k^{\mathrm{const}}],$$

where ζ_{d_j} denotes a primitive d_jth root of unity. We also see that

$$\begin{aligned} [k^{\mathrm{const}}(\zeta_p) : \mathbb{Q}] &= [k^{\mathrm{const}}(\zeta_p) : k^{\mathrm{const}}] \cdot [k^{\mathrm{const}} : \mathbb{Q}], \\ [k^{\mathrm{const}}(\zeta_p) : \mathbb{Q}] &\geq [\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1. \end{aligned}$$

Putting this together gives that $[k^{\mathrm{const}}(\zeta_p) : k^{\mathrm{const}}] \geq \frac{p-1}{[k^{\mathrm{const}} : \mathbb{Q}]}$, and by the assumption on p, this gives $[k^{\mathrm{const}}(\zeta_p) : k^{\mathrm{const}}] > \mathrm{rank}(\mathcal{F})$. Suppose that p divides d_j for some d_j, seeking a contradiction. Then,

$$\dim(\mathrm{Tr}_{k_j/k}(\chi_j)) \geq [k^{\mathrm{const}}(\zeta_{d_j}) : k^{\mathrm{const}}] \geq [k^{\mathrm{const}}(\zeta_p) : k^{\mathrm{const}}] > \mathrm{rank}(\mathcal{F}) = \dim(\rho^{\mathrm{ss}}),$$

which is a contradiction, since $\mathrm{Tr}_{k_j/k}(\chi_j)$ is a factor in the direct-sum decomposition of ρ^{ss}. □

Corollary 5.27. *Suppose that we are in the situation of Corollary 5.19. Then the character $\mathrm{disc}(\kappa_W)$ factors as*

$$\mathrm{Gal}_k \twoheadrightarrow \mathrm{Gal}_{k^{\mathrm{const}}} \xrightarrow{\kappa_W^{\mathrm{const}}} \{\pm 1\} \subseteq \mathbb{Q}_\ell^{\times},$$

and the character $\mathrm{disc}(\kappa_W^{\mathrm{const}})$ is unramified at all primes of k^{const} whose residue characteristic is larger than $\mathrm{rank}(R^i f_* \mathbb{Q}_\ell) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1$ for all i.

Proof. We see that $\mathrm{disc}(\kappa_W^{\mathrm{const}})$ is given by $\prod_i J_{D,R^i f_* \mathbb{Q}_\ell}^{(-1)^i}$. Each $J_{D,R^i f_*,\mathbb{Q}_\ell}$ is unramified at primes whose residue characteristic is larger than $\mathrm{rank}(R^i f_* \mathbb{Q}_\ell) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1$, so if a prime has residue characteristic larger than $\mathrm{rank}(R^i f_* \mathbb{Q}_\ell) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1$ for all i, it will be unramified for all of the characters $J_{D,R^i f_* \mathbb{Q}_\ell}$, and so will be unramified for $\mathrm{disc}(\kappa_W)$. □

5.3 A naïve Fubini theorem for algebraic groups

In the case that $f : X \to Y$ is a torsor under an algebraic group, we can achieve a stronger Fubini theorem by using Lang's theorem.

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Theorem 5.28. *Let G a connected algebraic group over k and let $f : X \to Y$ be such that for every closed point $y \in Y$ with residue field $\kappa(y)$, the fibre X_y over y is a torsor under $G_{\kappa(y)}$. Then,*

$$\chi^{\mathrm{mot}}(X) \equiv \chi^{\mathrm{mot}}(G) \cdot \chi^{\mathrm{mot}}(Y) \pmod{J}.$$

Proof. We prove this statement for the rank, signature, and discriminant. For the rank and signature, this follows by Theorem 5.7, so it only remains to show this for the discriminant. Without loss of generality, assume that k is finitely generated over $\mathbb{Q}$. By Proposition 2.39, it is enough to show a similar claim for the augmented determinant of cohomology. We can assume that $X \to Y$ and G are defined on a ring of finite type over $\mathbb{Z}$, denoted by A, and let $\mathcal{X}, \mathcal{Y}$ denote smooth projective models for X, Y over $\mathrm{Spec}(A)$. By Chebotarëv's density theorem, it is enough to prove that for every s a closed point of $\mathrm{Spec}(A)$ such that the fibre of X over s is smooth,

$$\det_\ell(X)(\mathrm{Frob}_s) = \det_\ell(G)(\mathrm{Frob}_s) \cdot \det_\ell(Y)(\mathrm{Frob}_s).$$

Theorem 2.44 tells us that $\det_\ell(X)(\mathrm{Frob}_s)$ is determined by the ζ-function of $\mathcal{X}_s$. Write X_s, Y_s, and G_s for the reductions of $\mathcal{X}, \mathcal{Y}$, and G at s, respectively, and let $\mathbb{F}_s$ denote the residue field of s. The result reduces to showing that $\zeta(X_s, t) = \zeta(G_s \times Y_s, t)$. Lang's theorem [Lan56, Theorem 2] gives us $H^1(\mathbb{F}_s, G) = 0$, since $\mathbb{F}_s$ is finite. Therefore, the fibres of $f_s : X_s \to Y_s$ are all isomorphic to a suitable base change of G. Let p be a closed point of Y_s with residue field $\mathbb{F}_p$, and let X_p be the fibre of X_s over p. Then $X_p \cong G_{\mathbb{F}_p}$. Counting the points of X gives an equality of zeta functions $\zeta(X_s, t) = \zeta(Y_s \times_{\mathbb{F}_s} G, t)$, as required. □

Theorem 5.29. *Let X be a positive-dimensional abelian variety over k. Then $\chi^{\mathrm{mot}}(X) = 0$.*

Proof. Since X is smooth and proper, we have $\chi^{\mathrm{mot}}(X) = \chi^{\mathrm{dR}}(X)$, so we may work with de Rham cohomology. By [Mil86, Lemma 15.2], we see that $H^*_{\mathrm{dR}}(X) \cong \bigoplus_{i=0}^{2n} \bigwedge^i H^1_{\mathrm{dR}}(X)$. Note that $\mathrm{rank}(\chi^{\mathrm{dR}}(X)) = \sum_{i=0}^{2n} (-1)^i \binom{n}{i} = 0$. Choosing a basis of $H^1_{\mathrm{dR}}(X)$ and taking the wedge product gives rise to a basis of $H^n_{\mathrm{dR}}(X) = \bigwedge^n H^1_{\mathrm{dR}}(X)$. With respect to this basis, the quadratic form on $H^n_{\mathrm{dR}}(X)$ is clearly hyperbolic, so there is some m such that $[H^n_{\mathrm{dR}}(X)] = m \cdot \mathbb{H}$. Applying the previous lemma tells us that $\chi^{\mathrm{dR}}(X) = (b^+ + m) \cdot \mathbb{H}$. Finally, note that $\mathrm{rank}(\chi^{\mathrm{dR}}(X)) = 2(b^+ + m)$. Since $\mathrm{rank}(\chi^{\mathrm{dR}}(X)) = 0$, this gives the result. □

Remark 5.30. Readers familiar with motivic homotopy theory should note that we may prove the above theorem directly using the categorical Euler characteristic and the motivic Gauss–Bonnet formula of [LR20]. Let p_X denote the structure morphism $p_X : X \to \mathrm{Spec}(k)$. Since X is smooth and projective, we have $\chi^{\mathrm{mot}}(X) = \chi^{\mathrm{cat}}(X)$ by Theorem 2.14. By [LR20, p. 3], we can define an Euler class of the tangent bundle of X valued in the Hermitian K-theory spectrum of X, which we call $e^{\mathrm{BO}}(T_{X/k})$. Since X/k is an abelian variety, $T_{X/k}$ is trivial, so this Euler class vanishes. The motivic Gauss–Bonnet formula [LR20, Theorem 1.5] says that $\chi^{\mathrm{cat}}(X) = p_{X,*}(e^{\mathrm{BO}}(T_{X/k}))$. This is akin to the classical Gauss–Bonnet formula in that it says that the Euler characteristic of a variety may be obtained by integrating the Euler class of the tangent bundle of the variety. The theorem follows since $e^{\mathrm{BO}}(T_{X/k})$ is trivial.

Corollary 5.31. *Let A be a connected commutative algebraic group over k whose maximal proper quotient group variety has positive dimension. Then $\chi^{\mathrm{mot}}(A) \in J$.*

Proof. There is a short exact sequence of algebraic groups,

$$0 \to G \to A \xrightarrow{\pi} B \to 1,$$

where G is a connected linear algebraic group and B is the maximal proper quotient group variety of X. The map $\pi : A \to B$ is a principal G-bundle, so by Theorem 5.28, we see that

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$\chi^{\mathrm{mot}}(A) \equiv \chi^{\mathrm{mot}}(G) \cdot \chi^{\mathrm{mot}}(B) \pmod{J}$. Since B is a positive-dimensional abelian variety, Theorem 5.29 gives $\chi^{\mathrm{mot}}(B) = 0$, as required. □

6. The motivic Euler characteristic of compactified Jacobians of curves

6.1 The motivic Euler characteristics of compactified Jacobians of nodal curves

Definition 6.1. Let G be a one-dimensional torus. The group of cocharacters is a Gal_k-module that is isomorphic to $\mathbb{Z}$ as a group. Let $\chi_G : \mathrm{Gal}_k \to \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ denote the homomorphism corresponding to this group action. Since we can identify $k^\times/k^{\times 2}$ with $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$, there exists some $\alpha \in k^\times/k^{\times 2}$ corresponding to χ_G. We will write $\mathbb{G}_{m,k}^{[\alpha]}$ for the one-dimensional torus whose cocharacter lattice has Gal_k action corresponding to the class $\alpha \in k^\times/k^{\times 2}$. A model for $\mathbb{G}_{m,k}^{[\alpha]}$ is given by the subscheme $x^2 - \alpha y^2 = 1$ in $\mathbb{A}_k^2$.

Lemma 6.2. *We have* $\chi^{\mathrm{mot}}(\mathbb{G}_{m,k}^{[\alpha]}) = \mathbb{H} - [\langle 2\rangle \oplus \langle 2\alpha\rangle] = [\langle -2\rangle] - [\langle 2\alpha\rangle]$.

Proof. Note that we have a smooth compactification of $\mathbb{G}_{m,k}^{[\alpha]}$ sitting inside $\mathbb{P}_k^1$ with complement $\mathrm{Spec}(k(\sqrt{\alpha}))$. Using the additivity of the motivic Euler characteristic, we obtain

$$\chi^{\mathrm{mot}}(\mathbb{G}_{m,k}^{[\alpha]}) = \chi^{\mathrm{mot}}(\mathbb{P}_k^1) - \chi^{\mathrm{mot}}(\mathrm{Spec}(k(\sqrt{\alpha}))) = \mathbb{H} - [\mathrm{Tr}_{k(\sqrt{\alpha})/k}].$$

Computing the trace form in the basis $1, \sqrt{\alpha}$ gives $[\mathrm{Tr}_{k(\sqrt{\alpha})/k}] = [\langle 2\rangle \oplus \langle 2\alpha\rangle]$. The final part of the claim follows since $\mathbb{H} = \langle 1\rangle + \langle -1\rangle = \langle 2\rangle + \langle -2\rangle$. □

Definition 6.3. Following [AK80, Definition 5.9], let $f : Y \to S$ be a finitely presented morphism with integral geometric fibres. An $\mathcal{O}_Y$ module I is called *relatively torsion free and rank 1 over* S if I is locally finitely presented, S-flat, and the fibre I_s is a rank-1 torsion-free $\mathcal{O}_{Y_s}$ module for every geometric point s of S.

For the rest of this subsection, let C denote a projective, geometrically integral curve over k. As in [AK80, Definition 5.11 and Theorem 8.1], rank-1 torsion-free sheaves on C of degree n are parameterised by the *compactified Jacobian* $\overline{\mathrm{Pic}}^n(C)$, which by [AK80, Theorem 8.1] is a representable as an étale sheaf by a k scheme that we will also call $\overline{\mathrm{Pic}}^n(C)$. Similarly, by [AK80, Theorem 8.1], if $\mathcal{C} \to \mathbb{P}^g$ is a complete linear system of genus-g curves, then we may form the relative compactified Jacobian $\overline{\mathrm{Pic}}^g(\mathcal{C}) \to \mathbb{P}^g$, whose fibres are the compactified Jacobians of the fibres of $\mathcal{C} \to \mathbb{P}^g$.

Let C be a curve of genus g. The insight to prove the complex Yau–Zaslow formula is that $e(\overline{\mathrm{Pic}}^g(C)) = 0$ unless C is rational, in which case, if C is nodal this counts the appropriate multiplicity of rational curves. Therefore, counting the number of rational curves can be done by looking at the Euler characteristic of the compactified Jacobians of the linear system. Here, we compute $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C)) \pmod{J}$ instead. We do this by first showing that if C is not rational, then $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C)) = 0 \pmod{J}$. For C a nodal and rational curve, we compute $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C))$ by successively expressing $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C)) \pmod{J}$ in terms of the motivic Euler characteristic of the compactified Jacobian of the normalisation of C at a singular point.

Definition 6.4. A *partial normalisation* of C is a surjective map $f : C' \to C$ of integral projective curves that induces isomorphism between the function fields of C' and C. Two such partial normalisations $f_1 : C_1 \to C$ and $f_2 : C_2 \to C$ are isomorphic if there is an isomorphism $g : C_1 \to C_2$ such that $f_1 = f_2 \circ g$. Since every partial normalisation is dominated by the normalisation of C, the set $\mathcal{N}(C)$ of isomorphism classes of partial normalisations is finite.

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There is a stratification of $\overline{\mathrm{Pic}}^n(C)$ by locally closed subvarieties

$$\overline{\mathrm{Pic}}^n(C) = \coprod_{f \in \mathcal{N}(C)} \overline{\mathrm{Pic}}^n_f(C).$$

Here, the morphism $f : C' \to C$ runs over all partial normalisations of C, and $\overline{\mathrm{Pic}}^n_f(C)$ is such that the $\overline{k}$-valued points of $\overline{\mathrm{Pic}}^n_f(C)$ correspond exactly to those rank-1 sheaves $\mathcal{L}$ on $C_{\overline{k}}$ such that the endomorphism ring of $\mathcal{L}$, as an $\mathcal{O}_{C_{\overline{k}}}$-subalgebra of the sheaf of rational functions on C, is $f_*(\mathcal{O}_{C'_{\overline{k}}})$. Note that in the case where $C' = C$ and f is the identity, then $\overline{\mathrm{Pic}}^n_{\mathrm{Id}}(C) = \mathrm{Pic}^n(C)$. More generally, if $f : C' \to C$ is a partial normalisation of the point p where the residue field $\kappa(p)$ is a degree-d extension of k, then $\overline{\mathrm{Pic}}^n_f(C)$ is the image of $\mathrm{Pic}^{n-d}(C')$ under f_*.

Lemma 6.5. *For every partial normalisation $f : C' \to C$, the stratum $\overline{\mathrm{Pic}}^n_f(C)$ is equipped with the free action of the quotient group variety $\mathrm{Pic}^0(C)/\mathrm{Ker}(f^*)$ over k, where $f^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C')$ is the map induced by functoriality.*

Proof. There is an action

$$\mathrm{Pic}^0(C) \times \overline{\mathrm{Pic}}^n(C) \to \overline{\mathrm{Pic}}^n(C)$$

of $\mathrm{Pic}^0(C)$ on $\overline{\mathrm{Pic}}^n(C)$, which at the level of represented functors is an action given by the rule $(\mathcal{L}, \mathcal{G}) \mapsto \mathcal{L} \otimes \mathcal{G}$. This action of $\mathrm{Pic}^0(C)$ leaves the stratification invariant. By [Bea99, Lemma 2.1], the stabiliser of any geometric point of the stratum $\overline{\mathrm{Pic}}^n_f(C)$ is $\mathrm{Pic}^0(C)/\mathrm{Ker}(f^*)$. The claim is now clear. □

Proposition 6.6. *Assume that the geometric genus of C is non-zero. Then $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C))$ lies in J for any n.*

Proof. Using the additivity of χ^{mot}, it is enough to show that $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n_f(C))$ lies in J for every partial normalisation $f : C' \to C$. Let $\widetilde{f} : \widetilde{C} \to C$ denote the normalisation of C. There is a short exact sequence of algebraic groups,

$$0 \to G \xrightarrow{\iota} \mathrm{Pic}^0(C) \xrightarrow{\widetilde{f}^*} \mathrm{Pic}^0(\widetilde{C}) \to 1,$$

where G is a connected linear algebraic group and $\widetilde{f}^*$ is induced by the pullback with respect to $\widetilde{f}$. By Lemma 6.5, the algebraic group $\mathrm{Pic}^0(C)/\mathrm{Ker}(f^*)$ acts freely on $\overline{\mathrm{Pic}}^n_f(C)$. Since the genus of $\widetilde{C}$, which is the geometric genus of C, is positive, the dimension of the abelian variety $\mathrm{Pic}^0(\widetilde{C})$ is also positive. Moreover, $\mathrm{Ker}(f^*) \subseteq G$ as $\widetilde{f}$ factors through f. Therefore, $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n_f(C)) \in J$ by Corollary 5.31. □

Corollary 6.7. *Assume that the geometric genus of C is at least 1. Then both $\det_\ell(\overline{\mathrm{Pic}}^n(C))$ and $e(\overline{\mathrm{Pic}}^n(C))$ are trivial.*

Proof. This follows by showing that $\chi_\ell(C) = 0$, where χ_ℓ is the ring homomorphism from Definition 2.38. Since $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) \in J$, this gives the result. □

We now turn to computing the motivic Euler characteristics of the compactified Jacobian of nodal rational curves.

Lemma 6.8. *Let $f : C' \to C$ be a partial normalisation of C at the nodal point p of degree d. The morphism $f_* : \overline{\mathrm{Pic}}^{n-d}(C') \to \overline{\mathrm{Pic}}^n(C)$ is a closed embedding.*

Proof. See [Bea99, Lemma 3.1]. □

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Let $\overline{\mathrm{Pic}}_b^n(C)$ denote the image of the morphism f_* above, and let $\overline{\mathrm{Pic}}_o^n(C)$ be the complement of $\overline{\mathrm{Pic}}_b^n(C)$ in $\overline{\mathrm{Pic}}^n(C)$.

Notation 6.9. In the remainder of this section, we assume that C is semi-stable. Let $\mathcal{S}(C)$ denote the set of nodes of C. For every $p \in \mathcal{S}(C)$, let $k(p)$ denote the residue field of p and let $\deg(p)$ denote the degree of $k(p)$ over k. Now let p be a node of C and pick a $k(p)$-rational point $\widetilde{p}$ of C whose set-theoretic image is exactly p. The formal completion of the base change of C to $k(p)$ at $\widetilde{p}$ is isomorphic to the formal completion of $x^2 = \alpha_p y^2$ at $(0,0)$ for some $\alpha_p \in k(p)^\times$. The element α_p is unique up to multiplication by an element of $(k(p)^\times)^2$, so the isomorphism class of the quadratic form $\langle 2\alpha_p \rangle$ over $k(p)$ is unique and only depends on p.

We have the following computation of the compactified Jacobians of normalisations, which serves as an arithmetic generalisation of [KR15, Proposition 2.8].

Proposition 6.10. *Let $f : C' \to C$ be the partial normalisation of C at the nodal point p. The pullback with respect to f induces a map*

$$\overline{\mathrm{Pic}}_o^n(C) \to \overline{\mathrm{Pic}}^n(C'),$$

which exhibits $\overline{\mathrm{Pic}}_o^n$ as a torsor under $\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})$.

Proof. By naturality, we obtain $\overline{\mathrm{Pic}}_o^n(C) \times_k \overline{k} = \overline{\mathrm{Pic}}_o^n(C_{\overline{k}})$. Therefore, $\overline{\mathrm{Pic}}_o^n(C)$ is a variety, and so is determined by $\overline{\mathrm{Pic}}_o^n(C)(\overline{k})$ along with its (right) action of Gal_k.

Let $d := [k(p) : k]$, and let $q, q' \in C'$ be the preimages of p in C'. Let $p_1, \dots, p_d$ be the points of $C(\overline{k})$ that lie over p and, similarly, let $q_1, \dots, q_d$ be the points of $C'(\overline{k})$ lying over q and let $q'_1, \dots, q'_d$ be the points of $C'(\overline{k})$ lying over q', so that the map $C'(\overline{k}) \to C(\overline{k})$ sends q_i and q'_i to p_i. We also see that $\{p_1, \dots, p_d\}$ is isomorphic as a right Gal_k set to $I_{k(p)}$.

We explicitly construct an action of $\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})$ on $\overline{\mathrm{Pic}}_o^n(C)$. That is, we construct a map $\phi : \overline{\mathrm{Pic}}_o^n(C) \times_k \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]}) \to \overline{\mathrm{Pic}}_o^n(C)$ that fits into the commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{Pic}}_o^n(C) \times_k \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]}) & \xrightarrow{\phi} & \overline{\mathrm{Pic}}_o^n(C) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \overline{\mathrm{Pic}}_o^n(C) & \longrightarrow & \overline{\mathrm{Pic}}^n(C') \end{array}$$

where pr_1 is the projection onto the first factor and the unlabelled maps are those from the statement. Our map ϕ is therefore given by a Gal_k equivariant map

$$\overline{\mathrm{Pic}}_o^n(C)(\overline{k}) \times \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k}) \to \overline{\mathrm{Pic}}_o^n(C)(\overline{k}).$$

Identify $\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k}) \cong (\overline{k}^\times)^{I_{k(p)}}$, where the right action of Gal_k on $(\overline{k}^\times)^{I_{k(p)}}$ is given by

$$(\lambda_i)_{\iota_i} \cdot \sigma = (\sigma^{-1}(\lambda_i))_{\sigma^{-1} \circ \iota_i} \quad \text{if } q_i \cdot \sigma = q_j$$

and

$$(\lambda_i)_{\iota_i} \cdot \sigma = (\sigma^{-1}(\lambda_i)^{-1})_{\sigma^{-1} \circ \iota_i} \quad \text{if } q_i \cdot \sigma = q'_j.$$

Let $\mathcal{F} \in \overline{\mathrm{Pic}}_o^n(C)(\overline{k})$, and let $\mathcal{G}$ be the image of $\mathcal{F}$ in $\overline{\mathrm{Pic}}^n(C')(\overline{k})$. The data of $\mathcal{F}$ is given by $\mathcal{G}$, along with isomorphisms $\psi_i : \mathcal{G}_{q_i} \cong \mathcal{G}_{q'_i}$ for every i. By the proof of [KR15, Lemma 2.5], we can canonically identify $\mathrm{Isom}(\mathcal{G}_{q_i}, \mathcal{G}_{q'_i}) \cong \overline{k}^\times$, so that $\psi_i, \psi_j \in \overline{k}^\times$. There is a canonical Gal_k action on $\prod_i \mathrm{Isom}(\mathcal{G}_{q_i}, \mathcal{G}_{q'_i})$, which we can describe.

Let $\sigma \in \mathrm{Gal}_k$ be such that $p_i \cdot \sigma = p_j$, and let $\psi_i \in \mathrm{Isom}(\mathcal{G}_{q_i}, \mathcal{G}_{q'_i})$, be such that $\psi_i \cong \lambda_i \in \overline{k}^\times$. Then $\psi_i \cdot \sigma = \sigma^{-1}(\lambda_i) \in \mathrm{Isom}(\mathcal{G}_{q_i \cdot \sigma}, \mathcal{G}_{q'_i \cdot \sigma})$. There are now two cases.

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In the case when $q_i \cdot \sigma = q_j$, we have $\mathrm{Isom}(\mathcal{G}_{q_i \cdot \sigma}, \mathcal{G}_{q_i' \cdot \sigma}) = \mathrm{Isom}(\mathcal{G}_{q_j}, \mathcal{G}_{q_j'})$.

In the case when $q_i \cdot \sigma = q_j'$, we have $\mathrm{Isom}(\mathcal{G}_{q_i \cdot \sigma}, \mathcal{G}_{q_i' \cdot \sigma}) = \mathrm{Isom}(\mathcal{G}_{q_j'}, \mathcal{G}_{q_j})$. We can canonically identify this with $\mathrm{Isom}(\mathcal{G}_{q_j}, \mathcal{G}_{q_j'})$, under the isomorphism $\lambda \mapsto \lambda^{-1}$.

Putting this together, we see that there is an isomorphism

$$\prod_i \mathrm{Isom}(\mathcal{G}_{q_i}, \mathcal{G}_{q_i'}) \cong \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k})$$

that is compatible with the Gal_k actions on each side. Explicitly, if we identify

$$\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k}) \cong (\overline{k}^{\times})^{I_{k(p)}},$$

and each $\mathrm{Isom}(\mathcal{G}_{q_i}, \mathcal{G}_{q_i'}) \cong \overline{k}^{\times}$, this isomorphism is given by $(\lambda_i)_i \mapsto (\lambda_i)_{\iota_i}$ and the action of Gal_k is given by $(\lambda_i)_i \cdot \sigma = (\sigma^{-1}(\lambda_i))_{\sigma^{-1} \circ \iota_i}$ if $q_i \cdot \sigma = q_j$ for some j, and $(\lambda_i)_i \cdot \sigma = (\sigma^{-1}(\lambda_i)^{-1})_{\sigma^{-1} \circ \iota_i}$ if $q_i \cdot \sigma = q_j'$ for some j, which is exactly to say that $\prod_i \mathrm{Isom}(\mathcal{G}_{q_j}, \mathcal{G}_{q_j'}) \cong (\overline{k}^{\times})^{I_{k(p)}}$. Putting this together, we can write every $\mathcal{F} \in \overline{\mathrm{Pic}}_o^n(C)(\overline{k})$ as a pair $(\mathcal{G}, (\psi_i))$, where $\mathcal{G} \in \overline{\mathrm{Pic}}^n(C')(\overline{k})$ and $(\psi_i) \in (\overline{k}^{\times})^{I_{k(p)}}$.

Define the morphism

$$\phi : \overline{\mathrm{Pic}}_o^n(C)(\overline{k}) \times \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k}) \to \overline{\mathrm{Pic}}_o^n(C)(\overline{k})$$

as follows. Let $(\mathcal{G}, \{\psi_i\}) \in \overline{\mathrm{Pic}}_o^n(C)(\overline{k})$, and let $\{\lambda_i\} \in \mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})(\overline{k})$. Define

$$\phi((\mathcal{G}, \{\psi_i\}), \{\lambda_i\}) := (\mathcal{G}, \{\lambda_i \psi_i\}).$$

We can check that ϕ is Galois equivariant and so gives us a map of the underlying k varieties, i.e. a $\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})$ action on $\overline{\mathrm{Pic}}_o^n(C)$ over $\overline{\mathrm{Pic}}^n(C')$. This action is Galois equivariant and acts freely and transitively on the fibres and so gives the result. □

Corollary 6.11. *Let $f : C' \to C$ be a partial normalisation of C at the nodal point p of degree d. Then,*

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) \equiv \overline{\mathrm{Pic}}^{n-d}(C') + (N_{k(p)/k}(\langle -2 \rangle - \langle 2\alpha_p \rangle))\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C')) \pmod{J}.$$

Proof. Lemma 6.8 tells us that

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) = \chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^{n-d}(C')) + \chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}_o^n(C)).$$

The above proposition, along with Theorem 5.28 allows us to compute

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}_o^n(C)) \equiv \chi^{\mathrm{mot}}(\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]}))\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C')) \pmod{J}.$$

Finally, Theorem 4.22 and Lemma 6.2 allow us to see that

$$\chi^{\mathrm{mot}}(\mathrm{Res}_{k(p)/k}(\mathbb{G}_{m,k(p)}^{[\alpha_p]})) = N_{k(p)/k}(\langle -2 \rangle - \langle 2\alpha_p \rangle),$$

as required. □

Notation 6.12. The above corollary motivates the following definition. Let C be a nodal curve defined over k, and let $p \in \mathcal{S}(C)$. Define

$$\Phi_p := 1 + N_{k(p)/k}(\langle -2 \rangle - \langle 2\alpha_p \rangle) \in \widehat{\mathrm{W}}(k).$$

Corollary 6.13. *Assume that C has geometric genus 0 and arithmetic genus g. Then, for any integer n, there is an equality in $\widehat{\mathrm{W}}(k)/J$:*

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) = \prod_{p \in \mathcal{S}(C)} \Phi_p.$$

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Proof. We prove the claim by induction on the cardinality of $\mathcal{S}(C)$. When $\mathcal{S}(C)$ is empty, then C is smooth and genus 0 and hence $\overline{\mathrm{Pic}}^n(C)$ is isomorphic to $\mathrm{Spec}(k)$ for every n. Therefore, the claim holds in this case, since the product over an empty index is 1. Assume that the claim holds for every nodal curve C' of geometric genus 0 such that $|\mathcal{S}(C')| < |\mathcal{S}(C)|$. Let $p' \in \mathcal{S}(C)$ be a point of degree d, and let C' be the normalisation of C at p', so that $\mathcal{S}(C) = \mathcal{S}(C') \cup \{p'\}$. Then, by the previous corollary,

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) \equiv \chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^{n-d}(C')) + (N_{k(p')/k}(\langle -2\rangle - \langle 2\alpha_{p'}\rangle)\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) \pmod{J}.$$

The induction hypothesis gives

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^{n-d}(C')) \equiv \chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C')) \equiv \prod_{p \in \mathcal{S}(C')} \Phi_p \pmod{J}.$$

Therefore,

$$\begin{aligned}\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^n(C)) &\equiv \left(\prod_{p \in \mathcal{S}(C')} \Phi_p\right)(1 + (N_{k(p)/k}(\langle -2\rangle - \langle 2\alpha_p\rangle)) \pmod{J} \\ &\equiv \left(\prod_{p \in \mathcal{S}(C')} \Phi_p\right) \cdot \Phi_{p'} \pmod{J} \\ &\equiv \prod_{p \in \mathcal{S}(C)} \Phi_p \pmod{J},\end{aligned}$$

as required. □

Corollary 6.14. *Suppose that $k = \mathbb{C}$ in the previous corollary. Then $e(\overline{\mathrm{Pic}}^g(C)) = 1$.*

Proof. This follows immediately, as $\mathrm{rank}(\Phi_p) = 1$ for every p. □

The above recovers [Bea99, Corollary 3.4]. Another corollary is the real closed case, which generalises [KR15, Proposition 1.8] to all real closed fields. Now suppose that k is a real closed field, and identify $k^\times/k^{\times 2} \cong \{\pm 1\}$.

Corollary 6.15. *There is an equality $\mathrm{sign}(\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C))) = (-1)^r$, where r is the number of points $p \in \mathcal{S}(C)$ such that $k(p) = k$ and $\alpha_p = 1$, i.e. the number of singular points that are both defined over k and have their tangent spaces defined over k.*

Proof. Let $p \in \mathcal{S}(C)$ be one of the singular points. If $k(p) = \overline{k}$, then we see that the corresponding factor at p is given by $\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle)$, which is zero. Therefore, $\mathrm{sign}(\Phi_p) = 1$. If $k(p) = k$, then the singular point p is a k point. If $\alpha_p = -1$, then the tangent spaces at p are not defined over k, and $\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle) = 0$, so $\mathrm{sign}(\Phi_p) = 1$. If $\alpha_p = 1$, then $\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle) = \langle -1\rangle - \langle 1\rangle$, and we obtain

$$N_{k/k}(\langle -1\rangle - \langle 1\rangle) = \langle -1\rangle - \langle 1\rangle,$$

which means that $\Phi_p = \langle -1\rangle$ and so has sign -1. Putting this together gives us

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C)) = (-1)^r,$$

where r is the number of singular points on C such that p and the tangent spaces to p are both defined over k. □

Corollary 6.16. *Let k be any field of characteristic 0. Then,*

$$\mathrm{disc}(\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C))) = (-1)^g \prod_{p \in \mathcal{S}(C)} N_{k(p)/k}(\alpha_p) \in k^\times/k^{\times 2}.$$

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Proof. We obtain this by applying disc to Corollary 6.13. By definition,

$$\operatorname{disc}(\Phi_p) = \operatorname{disc}(1 - N_{k(p)/k}(\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle))) = \operatorname{disc}(N_{k(p)/k}(\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle))).$$

Note that $\operatorname{disc}(\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle)) = -\alpha_p$, so Corollary 4.21 gives us

$$\operatorname{disc}(N_{k(p)/k}(\mathbb{H} - (\langle 2\rangle + \langle 2\alpha_p\rangle))) = (-1)^{[k(p):k]} N_{k(p)/k}(\alpha_p).$$

Multiplying all the Φ_p terms gives us

$$\operatorname{disc}\left(\prod_{p\in\mathcal{S}(C)} \Phi_p\right) = \prod_{p\in\mathcal{S}(C)} (-1)^{[k(p):k]} N_{k(p)/k}(\alpha_p) = (-1)^g \prod_{p\in\mathcal{S}(C)} N_{k(p)/k}(\alpha_p),$$

as required. □

6.2 Motivic Euler characteristics of the compactified Jacobian of the linear system

Let X be a polarised K3 surface equipped with a degree-$2g$ polarisation and let $\mathcal{C}$ be the associated g-dimensional linear system of genus-g curves on X, as in § 1. Let $\pi : \mathcal{C} \to \mathbb{P}^g$ be the associated morphism given by the linear system. We now combine the above results with our Fubini theorem to compute $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C}))$.

Definition 6.17. For q a closed point of $\mathbb{P}^g_k$, write $k(q)$ to mean the residue field at q. As a slight abuse of notation, we also write $q : \operatorname{Spec}(k(q)) \to \mathbb{P}^g_k$ to be the corresponding closed immersion, and $\mathcal{C}_q$ to mean the fibre of $\mathcal{C} \xrightarrow{\pi} \mathbb{P}^g_k$ over q. Assume that the curves in $\mathcal{C}$ belong to a primitive divisor class and are geometrically integral, which will hold if $g \geq 2$ and X is of Picard rank 1 and the curves in the linear system belong to a primitive divisor class. Assume that the rational curves in $\mathcal{C}$ are all nodal, which is generically the case by Chen's theorem [Che02, Theorem 1.1]. Let $G(X)$ be the set of closed points q of $\mathbb{P}^g_k$ such that C_q has geometric genus 0. We see that $G(X)$ is finite, since the set of geometric points lying above $G(X)$ is finite by the argument presented in [Bea99, Corollary 2.3], which holds over any algebraically closed field of characteristic 0. By slight abuse of notation, let $G(X)$ also denote the reduced closed subscheme of $\mathbb{P}^g_k$ whose underlying set of closed points is $G(X)$. Let $\pi_g : \overline{\mathrm{Pic}}^g(\mathcal{C}) \to \mathbb{P}^g_k$ be the relative compactified Picard variety. Let $Y := \mathbb{P}^g_k \setminus G(X)$ and define $W := \pi_g^{-1}(Y)$. Write $\pi_{g,W} : W \to Y$ for the restriction of π_g to W.

Theorem 6.18. *There exists an integer $\Delta(g)$ such that for any K3 surface as above,*

$$\dim_{\mathbb{Q}_\ell} H^i(\overline{\mathrm{Pic}}^g(C_q)_{\overline{k}}, \mathbb{Q}_\ell) \leq \Delta(g),$$

for all closed points q of $\mathbb{P}^g$ and for all i.

Proof. Since this statement is only about the dimension of the cohomology groups, it is enough to prove this statement after base changing to the algebraic closure of the field, so that $k = \overline{k}$. By [Huy16, Chapter 5, § 1], let $\mathcal{M}_g$ denote the moduli stack of K3 surfaces equipped with a polarisation of degree $2g$ (note that the construction in [Huy16] makes sense over any algebraically closed field of characteristic 0). This is a Deligne–Mumford stack by [Huy16, Chapter 5, Proposition 4.10], so there exists a $\overline{k}$ variety $\mathbf{M}_g$, equipped with an étale surjective morphism $\mathbf{M}_g \to \mathcal{M}_g$. Since $\mathcal{M}_g$ is the moduli stack representing polarised K3 surfaces over $\overline{k}$, pulling back the universal family along this morphism equips $\mathbf{M}_g$ with a flat family $\mathcal{X} \to \mathbf{M}_g$ of polarised K3 surfaces. Moreover, since $\mathbf{M}_g \to \mathcal{M}_g$ is surjective, every K3 surface equipped with a polarisation of degree $2g$ appears as a fibre over some point in the family $\mathcal{X} \to \mathbf{M}_g$.

Consider $\mathbf{L} \in \mathrm{Pic}_{\mathcal{X}/\mathbf{M}_g}(\mathbf{M}_g)$. This determines a linear system of divisors $\mathbf{C}_g$ on $\mathcal{X}$ regarded as an $\mathbf{M}_g$ variety, which in turn comes with a map $\mathbf{C}_g \to \mathbb{P}^g_{\mathbf{M}_g} \cong \mathbb{P}^g_{\overline{k}} \times_{\overline{k}} \mathbf{M}_g$. Let $\mathbf{q} \in \mathbb{P}^g_{\mathbf{M}_g}(\overline{k})$, so that $\mathbf{q}$ is given by a pair (α, q), where $\alpha \in \mathbf{M}_g(\overline{k})$ and $q \in \mathbb{P}^g(\overline{k})$. Then the fibre of $\mathbf{C}_g \to \mathbb{P}^g_{\mathbf{M}_g}$ over $\mathbf{q}$

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is given by the fibre of $\mathcal{C}_\alpha \to \mathbb{P}^g$ over q, and so in particular is an integral curve of arithmetic genus g. Therefore, [AK80, Theorem 8.1] allows us to form the relative compactified Jacobian $\Pi_g : \overline{\mathrm{Pic}}^g(\mathbf{C}_g) \to \mathbb{P}^g_{\mathbf{M}_g}$. The fibres of this morphism have dimension $2g$, since the fibres are the relative compactified Jacobians of the curves as above.

Consider the map Π_g, which is proper, and let i be a positive integer. Since the ℓ-adic sheaf $\mathbb{Q}_\ell$ is locally constant, the derived pushforward $R^i\Pi_{g,*}\mathbb{Q}_\ell$ is constructible by [Del77, § XIV, Theorem 1]. If $i \leq 2g$, then since $R^i\Pi_{g,*}\mathbb{Q}_\ell$ is constructible, there exists a stratification $U^i_1, \ldots, U^i_n$ of $\mathbb{P}^g_{\mathbf{M}_g}$ such that $R^i\Pi_{g,*}\mathbb{Q}_\ell$ is locally constant on U^i_j. Let $\Delta^{i,j}(g)$ denote the rank of $R^i\Pi_{g,*}\mathbb{Q}_\ell$ on U^i_j, and let $\Delta^i(g) := \max_j \Delta^{i,j}(g)$, which is finite since the stratification is finite. If $i > 2g$, then $R^i\Pi_{g,*}\mathbb{Q}_\ell = 0$, so set $\Delta^i(g) = 0$. Let $\Delta(g) := \max_i \Delta^i(g)$. Precisely by construction, if $f : U \to \mathbb{P}^g_{\mathbf{M}_g}$ is such that $f^*R^i\Pi_{g,*}\mathbb{Q}_\ell$ is locally constant, then $\mathrm{rank}(f^*R^i\Pi_{g,*}\mathbb{Q}_\ell) \leq \Delta(g)$.

Now suppose that X is our polarised K3 surface over $\overline{k}$. Let $\mathcal{C} \to \mathbb{P}^g$ denote the linear system on X, let $q \in \mathbb{P}^g(\overline{k})$, and let C_q denote the curve over q. We claim that $\dim_{\mathbb{Q}_\ell} H^i(\overline{\mathrm{Pic}}^g(C_q), \mathbb{Q}_\ell) \leq \Delta(g)$.

By the definition of the moduli stack, X determines a point $[X] \in \mathcal{M}_g(\overline{k})$. Pick $\alpha \in \mathbf{M}_g(\overline{k})$ lying over $[X]$ under the morphism $\mathbf{M}_g(\overline{k}) \to \mathcal{M}_g(\overline{k})$, which exists since $\mathbf{M}_g \to \mathcal{M}_g$ is étale and surjective. Then the fibre of $\overline{\mathrm{Pic}}^g(\mathbf{C}_g) \to \mathbb{P}^g_{\mathbf{M_g}}$ over the point (α, q) is isomorphic to $\overline{\mathrm{Pic}}^g(C_q)$ by construction.

Fix an integer i and consider $H^i(\overline{\mathrm{Pic}}^g(C_q), \mathbb{Q}_\ell)$. This vanishes for $i > 2g$, so assume that $i \leq 2g$. Consider $(\alpha, q) \in \mathbb{P}^g_{\mathbf{M}_g}(\overline{k})$. By construction, the fibre of Π_g over (α, q) is $\overline{\mathrm{Pic}}^g(C_q)$. Therefore, by the proper base change theorem (see [AGV72, § XII, Theorem 5.1(iii)]), we see that

$$\dim_{\mathbb{Q}_\ell}(H^i(\overline{\mathrm{Pic}}^g(C_q), \mathbb{Q}_\ell)) = \mathrm{rank}((\alpha, q)^* R^i\Pi_{g,*}\mathbb{Q}_\ell) \leq \Delta(g),$$

as required. □

Remark 6.19. In the case that C_q is a nodal curve, we may use Proposition 6.10 and the Leray–Serre spectral sequence to see that $\dim_{\mathbb{Q}_\ell}(H^i(\overline{\mathrm{Pic}}^g(C_q), \mathbb{Q}_\ell)) \leq \binom{2g}{g}$, and if C_q is smooth, $H^i(\overline{\mathrm{Pic}}^g(C_q), \mathbb{Q}_\ell) = \binom{2i}{i}$. Therefore, $\Delta(g) \geq \binom{2g}{g}$.

Definition 6.20. Let $\Delta(g)$ be the constant from Theorem 6.18. For k^{const} a number field, let $\Delta_g^{k^{\mathrm{const}}}$ be the finite subgroup of $\mathrm{Hom}(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$ consisting of characters χ such that χ is unramified at all primes $\mathfrak{p}$ whose residue characteristic p satisfies $p > \Delta(g) \cdot [k^{\mathrm{const}} : \mathbb{Q}] + 1$. For k a finitely generated field over $\mathbb{Q}$, let Δ_g^k be the finite subgroup of $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$ consisting of elements in the image of $\Delta_g^{k^{\mathrm{const}}}$ under the map $\mathrm{Hom}(\mathrm{Gal}_{k^{\mathrm{const}}}, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})$. We will also think of Δ_g^k as a subgroup of $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$ using the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell^\times$.

The map $\widehat{\mathrm{W}}(k) \to E(\mathrm{Hom}(\mathrm{Gal}_k, (\mathbb{Z}/2\mathbb{Z})/\Delta_g^k)$ given by $[q] \mapsto (\mathrm{rank}(q), \mathrm{disc}(q)) \pmod{\Delta_g^k}$ is a ring homomorphism with kernel I_g containing I^2. Let $J_g := I_g \cap T$. The homomorphisms rank and sign clearly both factor through $\widehat{\mathrm{W}}(k)/J_g$, so we can still recover these invariants after passing to this quotient.

Remark 6.21. Let k_g be the abelian extension of k corresponding to the fixed field of the subgroup $\bigcap_{\chi \in \Delta_g^k} \ker(\chi)$. The canonical map $\widehat{\mathrm{W}}(k)/J \to \widehat{\mathrm{W}}(k_g)/J$ factors through J_g. We can therefore view J_g as recovering a discriminant in k_g. However, we may lose information about sign by passing to $\widehat{\mathrm{W}}(k_g)/J$. For example, if -1 is not a square in k, and if the character corresponding to $k(\sqrt{-1})$ lies in Δ_g^k, then $\mathrm{Spr}(k_g) = \emptyset$, but this may not be the case for $\mathrm{Spr}(k)$. However, we do not lose information about sign in the case that k is a real closed field.

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Lemma 6.22. *We have* $\chi^{\mathrm{mot}}(W) \equiv 0 \pmod{J_g}$.

Proof. This proof is an application of Corollary 5.19 to this situation. We first show that this situation satisfies the properties for this result.

Let $i \geq 0$, and consider $R^i\pi_{g,*}\mathbb{Q}_\ell$. Since π_g is proper, [AGV72, § XIV, Theorem 1] gives us that $R^i\pi_{g,*}\mathbb{Q}_\ell$ is constructible, so there exists a stratification $Y_1^i, \dots, Y_{n_i}^i$ of Y into locally closed subvarieties such that $R^i\pi_*\mathbb{Q}_\ell$ is locally constant on Y_j^i for all j. Moreover, since π_g has relative dimension $2g$, this is the trivial sheaf whenever $i > 2g$. Since the restriction of a locally constant sheaf is locally constant and each stratification is finite, we may find a common stratification $Y_1, \dots, Y_n$ such that $R^i\pi_{g,W,*}\mathbb{Q}_\ell$ is locally constant on Y_j for all i and for all j. Define a partition of W by setting $W_j := \pi_{g,W}^{-1}(Y_j)$, so $\chi^{\mathrm{mot}}(W) = \sum_j \chi^{\mathrm{mot}}(W_j)$. It is therefore enough to show that $\chi^{\mathrm{mot}}(W_j) \equiv 0 \pmod{J_g}$ for each W_j.

By construction, for every i, $R^i\pi_{g,W_j,*}\mathbb{Q}_\ell$ is locally constant on each Y_j, so they are smooth. We would like to apply Corollary 5.19 to each morphism $W_j \to Y_j$. In order for this to hold, we first need Y_j to have a compactification $\overline{Y_j}$ such that $\overline{Y_j} \setminus Y_j$ is a divisor with simple normal crossings and the ramification of $R^i\pi_{g,W_j,*}\mathbb{Q}_\ell$ on $\overline{Y_j} \setminus Y_j$ is tame. This follows immediately by taking $\overline{Y_j}'$ to be the closure of Y_j in $\mathbb{P}^g$, and blowing up $\overline{Y_j}'$ at $\overline{Y_j}' \setminus Y_j$ in order to obtain that the boundary is a divisor.

The fibres of $\chi^{\mathrm{mot}}(W_j)$ are given by $\overline{\mathrm{Pic}}^g(C)$ for C a non-rational curve, and so their motivic Euler characteristic lying in J by Proposition 6.6. Moreover, for all i, $\mathrm{rank}(R^i\pi_{g,W_j,*}\mathbb{Q}_\ell) \leq \Delta(g)$ by Theorem 6.18. Corollary 5.19 then gives that $\chi^{\mathrm{mot}}(W_j)$ has rank 0, signature 0, and discriminant given by $\prod_{i=0}^{2g} J_{D,R^if_*\mathbb{Q}_\ell}^{\otimes(-1)^{i+1}}$. Corollary 5.27 gives us the required bound on the ramification of this character, and so we see that $\chi^{\mathrm{mot}}(W) \in J_g$. □

Definition 6.23. Define the following element of $\widehat{\mathrm{W}}(k)$:

$$\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X) := \sum_{q \in G(X)} \mathrm{Tr}_{k(q)/k}\left(\prod_{p \in \mathcal{S}(C_q)} \Phi_p\right) \in \widehat{\mathrm{W}}(k).$$

This element makes an appearance in our final arithmetic Yau–Zaslow formula. Note that $\mathrm{rank}(\Phi_p) = 1$, so $\mathrm{rank}(\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)) = \#G(X)(\overline{k})$. In particular, we see that $\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)$ 'counts' the number of rational curves in $\mathcal{C}$, as well as providing arithmetic information about nodes of the rational curves.

Lemma 6.24. *We have an equality modulo* J_g*:*

$$\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X) \equiv \chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C})) \pmod{J_g}.$$

Proof. Note that $\overline{\mathrm{Pic}}^g(\mathcal{C}) = W \amalg \pi_g^{-1}(G(X))$, so by the additivity of χ^{mot}, we obtain

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C})) = \chi_k^{\mathrm{mot}}(W) + \chi_k^{\mathrm{mot}}(\pi_g^{-1}(G(X))),$$

and Lemma 6.22 gives $\chi_k^{\mathrm{mot}}(W) = 0 \pmod{J_g}$. By additivity of the motivic Euler characteristic,

$$\chi^{\mathrm{mot}}(\pi_g^{-1}(G(X))) = \sum_{q \in G(X)} \chi_k^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C_q)).$$

Note that $C_q \to \mathrm{Spec}(k(q))$ is a nodal curve of the geometric genus-0 curve over $k(q)$, so applying Corollary 6.13 gives

$$\chi_{k(q)}^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C_q)) = \prod_{p \in \mathcal{S}(C_q)} \Phi_p \in \widehat{\mathrm{W}}(k(q))/J.$$

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Applying Lemma 4.5 gives

$$\chi_k^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(C_q)) = \mathrm{Tr}_{k(q)/k}\left(\prod_{p\in\mathcal{S}(C_q)} \Phi_p\right) \in \widehat{\mathrm{W}}(k)/J,$$

so summing over $q \in G(X)$ gives the result as required. □

7. The motivic Euler characteristic of Calabi–Yau varieties

Recall that X is a K3 surface admitting a g-dimensional linear system $\mathcal{C}$ of geometrically integral curves of genus g and that the curves in this linear system correspond to a primitive class in the Néron–Severi group. The assumption that the curves in $\mathcal{C}$ are geometrically integral allows us to apply [Muk84, Example 0.5] to show that the relative compactified Picard varieties $\overline{\mathrm{Pic}}^g(\mathcal{C})$ can be identified with connected components of the moduli space of simple sheaves on X. In particular, $\overline{\mathrm{Pic}}^g(\mathcal{C})$ is nonsingular and has a trivial canonical bundle, by [Muk84, Theorem 0.1]. Note that if X has Picard rank 1 and $g \geq 2$, the assumption that the curves in $\mathcal{C}$ are geometrically integral is automatically satisfied and $\mathcal{C}$ will be unique with these properties.

Proposition 7.1 (Beauville, Proposition 1.3 of [Bea99]). *There exists a birational morphism from the relative compactified Picard variety $\overline{\mathrm{Pic}}^g(\mathcal{C})$ to $X^{[g]}$.*

Remark 7.2. In [Bea99], the result is stated over $\mathbb{C}$; however, the construction holds over any field.

In [Bat99], it is shown that birational Calabi–Yau varieties have the same Betti numbers, and so also the same complex Euler characteristic. This section is dedicated to proving the required analogue of this theorem for the motivic Euler characteristic of the Calabi–Yau varieties that we are considering.

Theorem 7.3. *In the situation as above, there is an equality of the motivic Euler characteristics modulo J:*

$$\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g\mathcal{C}) \equiv \chi^{\mathrm{mot}}(X^{[g]}) \pmod{J}.$$

It is natural to conjecture that the above equality holds for the whole motivic Euler characteristic, rather than just modulo J. Note that if this theorem holds for the whole motivic Euler characteristic, then Theorems 2.14, 2.20, and 2.24 would imply Lemmas 7.4, 7.5, and 7.6, respectively. We begin with a lemma that allows us to use the result of Batyrev and Kontsevich in our context.

Lemma 7.4. *Let X and Y be two birational Calabi–Yau varieties over k. Then,*

$$\mathrm{rank}(\chi^{\mathrm{mot}}(X)) = \mathrm{rank}(\chi^{\mathrm{mot}}(Y)).$$

Proof. Over $\mathbb{C}$, the motivic Euler characteristic χ^{mot} is just the compactly supported topological Euler characteristic. The result follows by [Bat99, Theorem 1.1], which says that two birational Calabi–Yau varieties over $\mathbb{C}$ have the same Betti numbers. Now suppose that $k \neq \mathbb{C}$. As in Theorem 2.14, there exists a finitely generated field L such that X is defined over L, and therefore by enlarging L, the varieties X and Y are both defined over some L. Since L has finite transcendence degree over $\mathbb{Q}$, we can embed L into $\mathbb{C}$. The result holds for $X_{\mathbb{C}}$, i.e. $\mathrm{rank}(\chi^{\mathrm{mot}}(X)) = \mathrm{rank}(\chi^{\mathrm{mot}}(Y))$. Since the motivic Euler characteristic commutes with base change and base changing preserves the rank, this gives the result. □

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We now wish to prove a lemma that allows us to use [KR15, Proposition 3.1] over any real closed field.

Lemma 7.5. *For every real closure $k_<$ of k, we have an equality of real closed Euler characteristics*

$$\chi^{\mathrm{rc}}((\overline{\mathrm{Pic}}^g\mathcal{C})_{k_<}(k_<)) = \chi^{\mathrm{rc}}((X^{[g]})_{k_<}(k_<)).$$

Proof. First, since $\overline{\mathrm{Pic}}^g(-)$ and $(-)^{[g]}$ are representable functors, they commute with base change, so $\overline{\mathrm{Pic}}^g(\mathcal{C}_{k_<}) \cong (\overline{\mathrm{Pic}}^g\mathcal{C})_{k_<}$, and similarly $(X^{[g]})_{k_<} \cong (X_{k_<})^{[g]}$. It therefore suffices to show this statement in the case that k is a real closed field. For $k = \mathbb{R}$, the first equality follows by [KR15, Proposition 3.1], which says that our birational map $\phi : \overline{\mathrm{Pic}^g\mathcal{C}}(\mathbb{R}) \dashrightarrow X^{[g]}(\mathbb{R})$ defined in [Bea99, Proposition 1.3] gives rise to a diffeomorphism $\overline{\phi}$ between these two real manifolds, which in turn tells us that they have the same Euler characteristic. We claim that we can choose $\overline{\phi}$ to be semi-algebraic, so that $\overline{\phi}$ is a Nash diffeomorphism in the sense of [BCR98, Definition 2.9.9].

We first claim that if $Y/\mathbb{R}$ is a smooth projective variety, then there exists a Nash diffeomorphism from $Y(\mathbb{R})$ to a closed, bounded Nash submanifold of $\mathbb{R}^N$ for some N. It is enough to show this in the special case when X is a projective space. In this case, this is [BCR98, Theorem 3.4.4, p. 72]. In particular, $\overline{\mathrm{Pic}^g\mathcal{C}}(\mathbb{R})$ (respectively, $X^{[g]}(\mathbb{R})$) is Nash diffeomorphic to closed, bounded Nash submanifolds of $\mathbb{R}^N$, which we will call M (respectively, M'). By [Shi83, Theorem 2], since M, M' are closed bounded Nash submanifolds of $\mathbb{R}^N$, there is a diffeomorphism between them if and only if there is a Nash diffeomorphism between them by [Shi83, Theorem 2]. In particular, since $\overline{\mathrm{Pic}^g\mathcal{C}}(\mathbb{R})$ and $X^{[g]}$ are diffeomorphic, M and M' are diffeomorphic, and so are Nash diffeomorphic. This in turn gives that $\overline{\mathrm{Pic}^g\mathcal{C}}(\mathbb{R})$ and $X^{[g]}$ are Nash diffeomorphic, which proves the claim.

To prove the statement in general, let k be a general real closed field. Note that if K is a real closed field containing k and X/k is a variety, then the motivic Euler characteristic of the base change $\chi^{\mathrm{mot}}(X_K) = \chi^{\mathrm{mot}}(X) \in \mathbb{Z} \oplus \mathbb{Z}$, where we identify $\widehat{\mathrm{W}}(k) \cong \mathbb{Z} \oplus \mathbb{Z}$ by sending $n\langle 1\rangle + m\langle -1\rangle$ to (n, m). Therefore, by base changing to an extension of k containing $\mathbb{R}$, it suffices to prove the statement when k is a real closed field containing $\mathbb{R}$. We proceed by parameterising the K3 surfaces using moduli spaces over k. First, let $\mathbb{Q}^{\mathrm{real}}$ denote the real closure of $\mathbb{Q}$. For $d \geq 0$, define $\mathrm{Hilb}^d := \mathrm{Hilb}^{9dt^2+2}_{\mathbb{P}^{9d+1}_{\mathbb{Q}^{\mathrm{real}}}}$. Let H^d denote the open subvariety of Hilb^d from [Huy16, Proposition 2.1]. It is defined by the universal property that a morphism $T \to \mathrm{Hilb}^d$ factors through H^d if and only if the pullback of the universal family $f : \mathcal{Z}_T \to T$ satisfies the following.

(1) The morphism f is a smooth family and all the fibres are K3 surfaces.
(2) If we have $p : \mathcal{Z}_T \to \mathbb{P}^{9d+1}$ is a projection, then $p^*\mathcal{O}(1) \cong L^3 \otimes f^*L_0$ for $L \in Pic(\mathcal{Z}_T)$ and $L_0 \in \mathrm{Pic}(T)$.
(3) The line bundle L is primitive on each geometric fibre.
(4) For all fibres $\mathcal{Z}_s$ of $f : \mathcal{Z}_T \to T$, the restriction map gives isomorphisms:

$$H^0(\mathbb{P}^N_{k(s)}, \mathcal{O}(1)) \cong H^0(\mathcal{Z}_s, L_s^3).$$

Now let X be a K3 surface over k with a polarisation given by an ample line bundle L such that $\deg(L) = d$. Then the discussion before [Huy16, Proposition 2.1] tells us that we can view X as a projective subvariety of $\mathbb{P}^{9d+1}_k$, with Hilbert polynomial $9dt^2 + 2$. By the definition of the Hilbert scheme, this corresponds to a morphism $\mathrm{Spec}(k) \to \mathrm{Hilb}^d$ such that the pullback of the universal family is isomorphic to X. By the properties that H^d has, this morphism factors through H, so we obtain a morphism $\mathrm{Spec}(k) \to H^d$. Therefore, every K3 surface over k with an ample line bundle L is the pullback of the universal family of a morphism $f : \mathrm{Spec}(k) \to H^d$ for some d.

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We will abuse notation slightly and write $(X, L) \in \mathrm{Hom}(\mathrm{Spec}(k), H^d)$ to mean the K3 surface and line bundle specified by the morphism. This choice of (X, L) specifies a birational map $\phi_{X,L} : \overline{\mathrm{Pic}}^g(\mathcal{C}) \to X^{[g]}$, as defined in [Bea99, Proposition 1.3]. Consider the following statement.

For $d \in \mathbb{Z}_{>0}$ and $(X, L) \in \mathrm{Hom}(\mathrm{Spec}(k), H^d)$, there exists a Nash diffeomorphism

$$\overline{\phi}_{(X,L)} : \overline{\mathrm{Pic}}^g(\mathcal{C})(k) \to X^{[g]}(k).$$

This can be written as a first-order formula, and we have seen that this formula is true for $k = \mathbb{R}$. Therefore, by the model completeness of real closed fields, it is true for every real closed field k containing $\mathbb{R}$. The existence of such a Nash diffeomorphism means that these semi-algebraic sets have the same real closed Euler characteristic, as required. □

Lemma 7.6. *Let X and Y be two birational Calabi–Yau varieties. Then $\det_\ell(X) = \det_\ell(Y)$.*

Proof. This lemma essentially follows from the proof of [Bat99, Theorem 1.1]. We recall the argument here for convenience. Assume without loss of generality that k is finitely generated over $\mathbb{Q}$, since we can take k to be the smallest field over which both X and Y are defined. Let S be an integral scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ whose function field is k. We may also assume that both X and Y have smooth projective models over S, denoted by $\mathcal{X}$ and $\mathcal{Y}$, respectively. For every closed point s of S, let $\mathbb{F}_s$, $\mathcal{X}_s$, and $\mathcal{Y}_s$ denote the residue field of s, and the fibre of $\mathcal{X}$ and $\mathcal{Y}$ over s, respectively. Let $\mathbb{F}_{s^n}$ denote the unique extension of $\mathbb{F}_s$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. As noted on p. 7 in the proof of [Bat99, Theorem 1.1], since $\mathcal{X}_s$ and $\mathcal{Y}_s$ are birational and Calabi–Yau, we have $|\mathcal{X}_s(\mathbb{F}_{s^n})| = |\mathcal{Y}_s(\mathbb{F}_{s^n})|$ for every s and n. This in turn gives us an equality of zeta functions $\zeta(\mathcal{X}_s, t) = \zeta(\mathcal{Y}_s, t)$ for every s. Theorem 2.44 then gives the result. □

Proof of Theorem 7.3. Example 0.5 of [Muk84] says that $\overline{\mathrm{Pic}}^g(\mathcal{C})$ is Calabi–Yau, Example 0.4 of [Muk84] says that $X^{[g]}$ is also Calabi–Yau, and Proposition 7.1 says that these two varieties are birational. We can therefore apply Lemmas 7.4 and 7.6 to obtain that $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g\mathcal{C})$ and $\chi^{\mathrm{mot}}(X^{[g]})$ have the same rank and ℓ-adic determinant of cohomology. Moreover, since they are birational, they also have the same dimension and hence the same total weight, so by Proposition 2.39, we have $\mathrm{disc}(\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g\mathcal{C})) = \mathrm{disc}(\chi^{\mathrm{mot}}(X^{[g]}))$. Finally, by Lemma 7.5, we see that $\overline{\mathrm{Pic}}^g\mathcal{C}$ and $X^{[g]}$ have the same signature. Therefore, by Lemma 2.45, they differ only by an element of J. □

8. The motivic Göttsche formula

In this section, we prove a refinement of the Göttsche formula for the classical Euler characteristic, which is [Göt90, Theorem 0.1, p. 193]. This formula allows us to compute the motivic Euler characteristic of $X^{[g]}$ modulo J.

Notation 8.1. Let X be a smooth geometrically irreducible projective surface over k, let $X^{(n)}$ denote its nth symmetric product, and let $X^{[n]}$ be the Hilbert scheme of zero-dimensional subschemes of X of length n. Let $\mathfrak{P}(n)$ denote the set of partitions of n. The elements of $\mathfrak{P}(n)$ are sequences $\underline{m} = (m_1, m_2, \ldots, m_r)$ of non-negative integers such that $1m_1 + 2m_2 + \cdots + rm_r = n$. The length $l(\underline{m})$ of the partition $\underline{m}$ is defined as $l(\underline{m}) = m_1 + m_2 + \cdots + m_r$.

Notation 8.2. Let $\mathrm{Chow}(k)$ denote the category of rational Chow motives over k, and for every smooth projective variety X and idempotent correspondence α from X to X over k, let (X, α) denote the object of $\mathrm{Chow}(k)$ corresponding to this pair. For every object A of $\mathrm{Chow}(k)$ and integer n, let $A(n)$ denote the nth Tate twist of A, and denote direct sums and tensor products in the category $\mathrm{Chow}(k)$ by the usual symbols for sums and products, respectively. For X as above, let $M(X)$ denote (X, Δ_X), where Δ_X is the diagonal cycle on $X \times X$.

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Theorem 8.3 (de Cataldo–Migliorini). *There is an isomorphism of Chow motives with rational coefficients:*

$$M(X^{[n]})(-n) \cong \sum_{\underline{m}=(m_1,m_2,\ldots,m_r)\in\mathfrak{P}(n)} M(X^{(m_1)})M(X^{(m_2)})\cdots M(X^{(m_r)})(-l(\underline{m})) \text{ in } \mathrm{Chow}(k).$$

Proof. The claim follows immediately from [dCM02, Theorem 6.2.5], and from [dCM02, Remark 6.2.2], where the authors have mentioned that their result holds over any base field. □

Definition 8.4. Define $K_0(\mathrm{Chow}(k))$ to be the Grothendieck ring of rational Chow motives, i.e. the ring generated by isomorphism classes of Chow motives over k subject to the obvious addition and multiplication relations coming from the direct sum and tensor product. Let $K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$ denote the subring of $K_0(\mathrm{Chow}(k))$ generated by classes of the form $[M(X)(j)]$, where X/k is a smooth projective variety and j is an integer.

Corollary 8.5. *There is an equality in* $K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$*:*

$$[M(X^{[n]})(-n)] = \sum_{\underline{m}=(m_1,m_2,\ldots,m_r)\in\mathfrak{P}(n)} \prod_{i=1}^{r}([M(X^{(m_i)})(-m_i)]).$$

Proof. As noted in [dCM02, Remark 6.2.4], this follows immediately by Theorem 8.3 and the definition of $K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$, where we absorb the Tate twists into the appropriate terms to obtain the formula in the form shown above. □

Lemma 8.6. *There is a ring homomorphism* $M : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \subseteq K_0(\mathrm{Chow}(k))$ *such that if* X/k *is a smooth projective variety,* $M([X]) = [M(X)]$.

Proof. We first check that this is a well-defined homomorphism of abelian groups. By [Bit04, Theorem 3.1], the ring $K_0(\mathrm{Var}_k)$ is generated as an abelian group by the classes of smooth projective varieties, subject to a single blow-up relation, given by

$$[\mathrm{Bl}_Z(X)] + [Z] = [X] + [E],$$

for any smooth projective variety X and smooth closed subvariety Z, where $\mathrm{Bl}_Z(X)$ is the blow up of X along Z with exceptional divisor E. This same blow-up relation is satisfied in the category of Chow motives by the corollary on [Man68, p. 463]. This assignment therefore gives us a well-defined homomorphism of abelian groups, and the fact that it is a ring homomorphism follows instantly by the Künneth formula. □

In this section, we construct a ring homomorphism $\chi^{\mathrm{Chow}} : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to \widehat{\mathrm{W}}(k)/J$ such that $\chi^{\mathrm{Chow}} \circ M(X) = \chi^{\mathrm{mot}}(X)$ in $\widehat{\mathrm{W}}(k)/J$ and examine how it applies to Tate twists. We construct χ^{Chow} by constructing versions of the complex Euler characteristic, the real Euler characteristic, and the discriminant for elements of $K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$ and then examining how they behave under Tate twists. We then appeal to Lemma 2.45 to recover an element of $\widehat{\mathrm{W}}(k)$. In Corollary 8.18, we apply χ^{Chow} to Theorem 8.3 in order to obtain a formula for the motivic Euler characteristic of $X^{[n]}$ modulo J.

Definition 8.7. Let M be a Chow motive over a field k. We will abuse notation slightly and use the same notation as for varieties. Let $e : K_0(\mathrm{Chow}(k)) \to \mathbb{Z}$ denote the Euler characteristic of M, i.e. $e(M) := \sum_i (-1)^i \dim(H^i_{\ell-\mathrm{adic}}(M))$.

When k is a real closed field, we may define its real Euler characteristic as

$$\epsilon(M) := \mathrm{Tr}(c^M),$$

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where $c^M : H^*_{\text{Betti}}(M) \to H^*_{\text{Betti}}(M)$ is complex conjugation, and Tr denotes graded trace as in Notation 2.41. Similarly to in Definition 2.18, the above data furnish a ring homomorphism $\chi^{\text{real}} : K_0(\text{Chow}(k)) \to \mathcal{C}(\text{Spr}(k), \mathbb{Z})$.

We can also define the ℓ-adic determinant of cohomology $\det_\ell(M)$ to be the one-dimensional ℓ-adic representation $\prod_i (\det(H^i_{\ell-\text{adic}}(M))^{(-1)^i})$. Similarly to Definition 2.37, there is a ring homomorphism $\chi_l : K_0(\text{Chow}(k)) \to E(\text{Hom}(\text{Gal}_k, \mathbb{Q}_\ell^\times))$ given by $M \mapsto (e(M), \det_\ell(M))$.

Lemma 8.8. *The compactly supported Euler characteristic $e : K_0(\text{Var}_k) \to \mathbb{Z}$, the real closed compactly supported Euler characteristic $\chi^{\text{real}} : K_0(\text{Var}_k) \to \mathcal{C}(\text{Spr}(k), \mathbb{Z})$, and the ℓ-adic determinant of cohomology $\det_\ell : K_0(\text{Var}_k) \to \text{Hom}(\text{Gal}_k, \mathbb{Q}_\ell^\times)$ all factor through the map $M : K_0(\text{Var}_k) \to K_0(\text{Chow}(k))$ from the previous lemma.*

Proof. This is clear since all of these invariants for smooth projective varieties can be read off from the Galois representation on the underlying cohomology ring for our Weil cohomology theories. This means that $e(M(X)) = e(X)$, $\chi^{\text{real}}(M(X)) = \chi^{\text{real}}(X)$, and $\det_\ell(M(X)) = \det_\ell(X)$. □

Remark 8.9. The reason why the above lemma only applies to these invariants, rather than the whole motivic Euler characteristic, is that the definition of the motivic Euler characteristic using de Rham cohomology in Definition 2.2 makes essential use of Poincaré duality as well as the cohomology groups.

We now study how these homomorphisms out of $K_0(\text{Chow}(k))$ behave under Tate twists.

Notation 8.10. Let G be a group, and let $\rho : G \to GL_L(V)$ an L-linear representation on a finite-dimensional graded vector space over the field L. Moreover, suppose that $V = \bigoplus_i V_i$ is graded, and that ρ respects the grading. Let $\chi : G \to L^\times$ be a one-dimensional representation. Define the twisted representation $\rho(\chi) : G \to GL_L(V)$ by the formula

$$(\forall g \in G), \rho(\chi)(g) = \chi(g)\text{Id}_V \cdot \rho(g).$$

We can compute the character and the determinant of the representation $\rho(\chi)$ in terms of these invariants of ρ and χ.

Lemma 8.11. *We have*

$$\text{Tr}(\rho(\chi)(g)) = \chi(g) \cdot \text{Tr}(\rho(g)) \quad \text{and} \quad \det((\rho(\chi)(g))) = \chi(g)^{e(V)} \cdot \det(\rho(g)),$$

where Tr *and* det *denote the graded versions as in Notation 2.41.*

Proof. We may view $\chi(g) \in L^\times$ as a scalar, and $\rho(\chi)(g) = \chi(g) \cdot \rho(g)$. Shifting the grading multiplies the trace by -1 and raises the determinant to the power of -1, so it is enough to prove an ungraded version for $V = V_0$. This then follows immediately. □

Proposition 8.12. *The functions e, χ^{real} and $\det_\ell$ out of $K_0(\text{Chow}(k))$ satisfy the following formulae under Tate twists.*

(i) *We have that $e(M(j)) = e(M)$.*
(ii) *When k is real closed $\epsilon(M(j)) = (-1)^j \epsilon(M)$ and so $\chi^{\text{real}}(M(j)) = (-1)^j \chi^{\text{real}}(M)$.*
(iii) *We have $\det_\ell(M(j)) = \det_\ell(M) \cdot \mathbb{Q}_\ell(j \cdot e(M))$.*

Proof. The first part is immediate, since the dimension of the Betti realisation does not change under the Tate twist.

For the second part, it suffices to prove the statement when k is real closed and we are looking at the real closed Euler characteristic ϵ. Define $\chi : \text{Gal}_k \to \overline{k}^\times$ by sending the non-trivial element

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of Gal_k to $-1 \in \overline{k}^\times$. We may identify the vector spaces $H^*_{\mathrm{Betti}}(M(j)) \cong H^*_{\mathrm{Betti}}(M)$, and under this identification, we see that $c^{M(j)} = (-1)^j c^M$. In particular,

$$\epsilon(M(j)) = \mathrm{Tr}(c^{M(j)}) = (-1)^j \mathrm{Tr}(c^M) = (-1)^j \epsilon(M),$$

as required.

For the statement about $\det_\ell(M(j))$, let $g \in \mathrm{Gal}_k$. Write ρ^M for the representation of Gal_k on $H^*_{\ell-\mathrm{adic}}(M)$. We may identify the vector spaces $H^*_{\ell-\mathrm{adic}}(M(j)) \cong H^*_{\ell-\mathrm{adic}}(M)$, and under this identification, we see that $\rho^{M(j)}(g) = \chi(j) \cdot \rho^M(g)$, where χ denotes the ℓ-adic cyclotomic character on Gal_k. By definition, $\det_\ell(M)(g) = \det(\rho^M(g))$, so Lemma 8.11 gives the result. □

The goal is to use these invariants to define a map χ^{Chow} that extends the motivic Euler characteristic to Chow motives. While we can define the ℓ-adic determinant of cohomology for a general Chow motive, it is unclear how to define the discriminant in an analogous way. However, for elements of $K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$, we can define the discriminant as follows.

Definition 8.13. Suppose that $k/\mathbb{Q}$ is finitely generated, and let M be a rational Chow motive over k such that $M = M(X)(j)$ for some variety X/k. Then $H^i_{\ell-\mathrm{adic}}(M) \cong H^i_{\acute{e}t}(X_{\overline{k}}, \mathbb{Q}_\ell) \cdot \mathbb{Q}_\ell(j)$ for each i. Putting this together gives

$$\det_\ell(M) = \det_\ell(X) \cdot \mathbb{Q}_\ell(j \cdot e(X)).$$

Multiplying again by $\mathbb{Q}_\ell(w(X) - j \cdot e(X))$ gives

$$\det_\ell(M) \cdot \mathbb{Q}_\ell(w(X) - j \cdot e(X)) = \det_\ell(X) \cdot \mathbb{Q}_\ell(w(X)).$$

Theorem 2.29 implies that the above representation is a 2-torsion character whose class in $k^\times/k^{\times 2}$ is $(-1)^{w(X)}\mathrm{disc}(\chi^{\mathrm{mot}}(X))$. As in Remark 2.30, define $w(M) := w(X) - j \cdot e(X)$, so that $w(M)$ is the unique integer such that $\det_\ell(M) \cdot \mathbb{Q}_\ell(w(M))$ is torsion. For such Chow motives, define $\mathrm{disc}(M)$ to be the element of $k^\times/k^{\times 2}$ such that

$$\mathrm{disc}(M) \cdot (-1)^{w(M)} := \det_\ell(M) \cdot \mathbb{Q}_\ell(w(M)).$$

We may also argue as in Remark 2.36 to obtain a ring homomorphism

$$\chi^{\mathrm{disc}}_k : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$$

given by $M \mapsto (e(M), \mathrm{disc}(M))$.

If k is any field of characteristic 0, since k is a union of its finitely generated subfields, we have that $K_0(\mathrm{Var}_k)^{\mathrm{Chow}} = \varinjlim_{L' \subseteq k} K_0(\mathrm{Var}_L)^{\mathrm{Chow}}$.

For k an arbitrary field of characteristic 0, define $\chi^{\mathrm{disc}}_k : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$ to be the inductive limit of all of the $\chi^{\mathrm{disc}}_L : K_0(\mathrm{Var}_L)^{\mathrm{Chow}} \to E(\mathrm{Hom}(\mathrm{Gal}_L, \mathbb{Z}/2\mathbb{Z}))$, where the fields L run over all finitely generated subfields of k. When the base field is clear, we will simply write χ^{disc}.

Lemma 8.14. *For X/k a smooth projective variety, the discriminant* $\mathrm{disc}(M(X)(j)) = \langle (-1)^{j \cdot e(M)} \rangle \cdot \mathrm{disc}(\chi^{\mathrm{mot}}(X))$.

Proof. This follows by the above definition, noting that $(-1)^{w(M(X)(j))} = (-1)^{w(X)} \cdot (-1)^{j \cdot e(M)}$. □

Theorem 8.15. *The homomorphism*

$$e \times \chi^{\mathrm{real}} \times \chi^{\mathrm{disc}} : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to \mathbb{Z} \times \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}) \times E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$$

factors through the inclusion $\widehat{\mathrm{W}}(k)/J \hookrightarrow \mathbb{Z} \times \mathcal{C}(\mathrm{Spr}(k), \mathbb{Z}) \times E(\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z}))$ *from Lemma 2.45.*

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Proof. It is enough to show that for X a variety and j an integer, there exists some $q \in \widehat{\mathrm{W}}(k)/J$ such that $e(M(X)(j)) = \mathrm{rank}(q)$, $\chi^{\mathrm{real}}(M(X)(j)) = \mathrm{sign}(q)$ and $\chi^{\mathrm{disc}}(M(X)(j)) = E_{\mathrm{disc}}(q)$. Proposition 8.12 and Lemma 8.14 tell us that taking $q = \langle(-1)^j\rangle \cdot \chi^{\mathrm{mot}}(X)$ gives us the result. □

Definition 8.16. In light of the above, define $\chi^{\mathrm{Chow}} : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to \widehat{\mathrm{W}}(k)/J$ to be the unique homomorphism such that the following diagram is commutative.

$$\begin{array}{ccc} K_0(\mathrm{Var}_k)^{\mathrm{Chow}} & & \\ \downarrow{\scriptstyle \chi^{\mathrm{Chow}}} & \searrow^{e\times\chi^{\mathrm{real}}\times\chi^{\mathrm{disc}}} & \\ \widehat{\mathrm{W}}(k)/J & \xrightarrow[\mathrm{rank}\times\mathrm{sign}\times E_{\mathrm{disc}}]{} & \mathbb{Z}\times\mathcal{C}(\mathrm{Spr}(k),\mathbb{Z})\times E(\mathrm{Hom}(\mathrm{Gal}_k,\mathbb{Z}/2\mathbb{Z})) \end{array}$$

Corollary 8.17. *For X/k a variety and j an integer, we have*

$$\chi^{\mathrm{Chow}}(M(X)(j)) = \langle(-1)^j\rangle\chi^{\mathrm{mot}}(X) \in \widehat{\mathrm{W}}(k)/J.$$

Proof. Note that the twist by j map $(j) : K_0(\mathrm{Var}_k)^{\mathrm{Chow}} \to K_0(\mathrm{Var}_k)^{\mathrm{Chow}}$ given by sending $M(X)(i)$ to $M(X)(i+j)$ is a homomorphism of the underlying additive abelian group. Similarly, the map $\widehat{\mathrm{W}}(k)/J \to \widehat{\mathrm{W}}(k)/J$ given by multiplication by $\langle(-1)^j\rangle$ is a homomorphism on the underlying abelian group. Therefore, the compositions

$$\chi^{\mathrm{Chow}} \circ (j) \circ M : K_0(\mathrm{Var}_k) \to \widehat{\mathrm{W}}(k)/J$$

and

$$(\langle(-1)^j\rangle\cdot) \circ \chi^{\mathrm{mot}} : K_0(\mathrm{Var}_k) \to \widehat{\mathrm{W}}(k)/J$$

are both homomorphisms of abelian groups. It therefore suffices that these homomorphisms are equal on generators of the abelian group $K_0(\mathrm{Var}_k)$. Through [Bit04, Theorem 3.1], we can check this only for X/k, a smooth projective variety. This is then immediate by the proof of Theorem 8.15. □

Corollary 8.18 (Motivic Göttsche formula). *We have*

$$1 + \sum_{n=1}^{\infty} \chi^{\mathrm{mot}}(X^{[n]})\langle(-1)^n\rangle t^n = \prod_{d=1}^{\infty}\left(\sum_{m=0}^{\infty} \chi^{\mathrm{mot}}(X^{(m)})\langle-1\rangle^m t^{md}\right) \text{ in } (\widehat{\mathrm{W}}(k)/J)[[t]].$$

Proof. Fix n to be a positive integer. Applying the ring homomorphism χ^{Chow} to the formula in Corollary 8.5 gives

$$\chi^{\mathrm{Chow}}(M(X^{[n]})(-n)) = \sum_{(m_1,m_2,\ldots,m_r)\in\mathfrak{P}(n)} \prod_{i=1}^{r} (\chi^{\mathrm{Chow}}(M(X^{(m_i)})(-m_i))) \in \widehat{\mathrm{W}}(k)/J.$$

By Corollary 8.17, we may rewrite this as

$$\chi^{\mathrm{mot}}(X^{[n]})\langle(-1)^n\rangle = \sum_{(m_1,m_2,\ldots,m_r)\in\mathfrak{P}(n)} \left(\prod_{i=1}^{r} \chi^{\mathrm{mot}}(X^{(m_i)})\langle(-1)^{m_i}\rangle\right).$$

We claim that the right-hand side of this is the coefficient of t^n in the power series

$$\prod_{d=1}^{\infty}\left(\sum_{m=0}^{\infty} \chi^{\mathrm{mot}}(X^{(m)})\langle(-1)^m\rangle t^{md}\right).$$

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Relabeling the indices m by m_d in order to keep track of where they appear in the product, the above series becomes

$$\prod_{d=1}^{\infty}\left(\sum_{m_d=0}^{\infty}\chi^{\mathrm{mot}}(X^{(m_d)})\langle(-1)^{m_d}\rangle t^{m_d d}\right).$$

Expanding the product and looking at the coefficient of t^n in the above power series, we see that the t^n term is given by

$$\sum_{\{(m_1,\ldots,m_r):\sum dm_d=n\}}\left(\prod_{d=1}^{r}\chi^{\mathrm{mot}}(X^{(m_d)})\langle(-1)^{m_d}\rangle t^{dm_d}\right).$$

As noted in Notation 8.2, we may identify $\mathfrak{P}(n)$ with $\{(m_1,\ldots,m_r):\sum dm_d=n\}$, which gives us that the coefficient of t^n is exactly given by

$$\sum_{(m_1,m_2,\ldots,m_r)\in\mathfrak{P}(n)}\left(\prod_{d=1}^{r}\chi^{\mathrm{mot}}(X^{(m_d)})\langle(-1)^{m_d}\rangle\right).$$

Putting all of this together, $\chi^{\mathrm{mot}}(X^{[n]})\langle(-1)^n\rangle\in\widehat{\mathrm{W}}(k)/J$ is equal to the coefficient of t^n in the generating series

$$\prod_{d=1}^{\infty}\left(\sum_{m=0}^{\infty}\chi^{\mathrm{mot}}(X^{(m)})\langle(-1)^m\rangle t^{md}\right)\in(\widehat{\mathrm{W}}(k)/J)[[t]],$$

which gives the theorem. □

We now show that we can recover the rank, sign, and discriminant of $\chi^{\mathrm{mot}}(X^{(m)})$ to extract other Göttsche formulae based on this.

Lemma 8.19. *Let X be a K3 surface. Then*

$$e(X^{(m)})=\binom{e(X)+m-1}{m}.$$

If k is a real closed field, then we may write $\chi^{\mathrm{mot}}(X)=b\mathbb{H}+a\langle\pm1\rangle$ for some non-negative integers a,b. Moreover,

$$\epsilon(X)=\mathrm{sign}(\chi^{\mathrm{mot}}(X^{(m)}))=\pm\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{a+m-2j-1}{m-2j}\binom{b+j-1}{j},$$

where the sign is $(-1)^m$ if $\mathrm{sign}(\chi^{\mathrm{mot}}(X))$ is negative, and 1 if $\mathrm{sign}(\chi^{\mathrm{mot}}(X))$ is positive.

Proof. Since X is a K3 surface, the odd Betti numbers vanish, so $H^i_{\mathrm{dR}}(X/k)=0$ if i is odd. Therefore, $\chi^{\mathrm{mot}}(X)=[H^{2*}_{\mathrm{dR}}(X/k)]$ is an actual quadratic form, so $\chi^{\mathrm{mot}}(X)=b\mathbb{H}+a\langle\pm1\rangle$ for some $a,b\geq0$. For the rank, we may use Theorem 2.14 to reduce this to a statement about symmetric powers of CW complexes. The main result of [Mac62] then gives the result.

Suppose that k is a real closed field, and consider the complex $H^*_{\mathrm{sing}}(X^{(m)}(\overline{k}),k)$, along with the action of σ, the non-trivial element of Gal_k. Moreover, σ is diagonalisable, so pick a basis of eigenvectors for $H^*_{\mathrm{sing}}(X^{(m)}(\overline{k}),k)$, denoted by $v_1,\ldots,v_{e(X)}$. As noted in the second part of [Lev20, Remark 1.3], we apply the Lefschetz trace formula to compute $\mathrm{Trace}(\sigma)=\epsilon(X)=e(X(k))$.

Let q be a quadratic form on $H^*_{\mathrm{sing}}(X(\overline{k}),k)$ such that q is diagonal with respect to $v_1,\ldots,v_{e(X)}$, and such that $\sigma(v_i)=q(v_i)v_i$. This means that we obtain $e(X)=\mathrm{rank}(q)$ and $\epsilon(X)=\mathrm{sign}(q)$. By the Künneth formula, we can identify $H^*_{\mathrm{sing}}(X^{(m)}(\overline{k}),k)=S^m(H^*_{\mathrm{sign}}(X(\overline{k}),k))$,

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where S^m denotes the symmetric product, and where the action of σ on the left-hand side is given by $S^m(\sigma)$ on the right-hand side. Therefore, the quadratic form we obtain from looking at the action of σ on both sides is the same and is given by $S^m(q)$, the symmetric power of the form q. Recall that $q = b\mathbb{H} + a\langle\pm 1\rangle$, where $a, b \geq 0$ and the $\pm$ comes from whether the 1 or -1 eigenspace of σ on $H^*_{\text{sign}}(X(\overline{k}), k)$ is larger. For now, suppose that the -1 eigenspace is larger, as this is the more difficult case.

We see that $S^m(q) = \bigoplus_{i=0}^m S^i(b\mathbb{H})S^{m-i}(a\langle -1\rangle)$. By [McG05, Proposition 4.12], we see that if i is odd, then $\text{sign}(S^{m-i}(b\mathbb{H})) = 0$, and $\text{sign}(S^i(b\mathbb{H})) = \binom{b+j-1}{j}$ if $i = 2j$ is even. We calculate

$$S^{m-2j}(a\langle -1\rangle) = \langle (-1)^{m-2j}\rangle \binom{a+m-2j-1}{m-2j-1}.$$

Applying sign then gives

$$\text{sign}(S^m(q)) = (-1)^m \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} \binom{a+m-2j-1}{m-2j}\binom{b+j-1}{j}.$$

The argument for the case when the 1 eigenspace is larger is similar. □

Lemma 8.20. *Let X be a K3 surface. Then,*

$$\det_\ell(X^{(m)}) = \det_\ell(X)^{\binom{m+e(X)-1}{m}}.$$

Proof. Since X is a K3 surface, we have $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell) = 0$ for all i odd. Therefore,

$$\det_\ell(X) = \det(R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell)) = \det(H^*(X_{\overline{k}}, \mathbb{Q}_\ell)).$$

Note that $H^*(X^{(m)}_{\overline{k}}, \mathbb{Q}_\ell) = S^m(H^*(X_{\overline{k}}, \mathbb{Q}_\ell))$. Since $X^{(m)}$ is proper, the compactly supported cohomology agrees with the normal cohomology, so $\det_\ell(X)$ is the determinant of the Galois representation $H^*(X^{(m)}_{\overline{k}}, \mathbb{Q}_\ell)$. Therefore, for all $\sigma \in \text{Gal}_k$, we have

$$\det_\ell(X^{(m)})(\sigma) = \det_\ell(X)^{\binom{m+e(X)-1}{m}},$$

since for $T : V \to V$, the determinant of the symmetric power,

$$\det(S^m(T)) = \det(T)^{\binom{\dim(V)+m-1}{m}}.$$ □

Corollary 8.21. *Let X be a K3 surface, and let*

$$w_m = \binom{e(X)+m-1}{m}.$$

Then $\chi^{\text{mot}}(X^{(m)})$ is the unique element of $\widehat{\text{W}}(k)/J$ satisfying the following properties.

(1) *The rank is given by* $\text{rank}(\chi^{\text{mot}}(X^{(m)})) = w_m$.

(2) *The signature is given by*

$$\text{sign}(\chi^{\text{mot}}(X^{(m)}_{k_<})) = \pm \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} \binom{a+m-2j-1}{m-2j}\binom{b+j-1}{j},$$

for any choice of $k_< \in \text{Spr}(k)$, where a is given by $|\text{sign}(\chi^{\text{mot}}(X)_{k_<})|$, $b = \frac{1}{2}(w_m - a)$ and the sign at the front is equal to 1 if $\text{sign}(\chi^{\text{mot}}(X)_{k_<})$ is positive, and $(-1)^m$ if $\text{sign}(\chi^{\text{mot}}(X)_{k_<})$ is negative.

(3) *The discriminant is* $\text{disc}(\chi^{\text{mot}}(X^{(m)})) = \text{disc}(\chi^{\text{mot}}(X))^{w_m}$.

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Proof. Uniqueness is immediate due to the definition of J. The first two equalities follow by the previous two lemmas. For the final equality, Lemma 8.20 applied to the ℓ-adic cohomology gives $\det_\ell(X^{(m)}) = \det_\ell(X)^{w_m}$, and the result follows by Remark 2.30 and Theorem 2.29. □

Remark 8.22. Applying the rank homomorphism to the equation in Corollary 8.18 and using Theorem 2.14 recovers Göttsche's formula for Euler characteristics:

$$1+\sum_{n=1}^{\infty} e(X^{[n]})t^n = \prod_{d=1}^{\infty}\left(1+\sum_{m=1}^{\infty}\binom{m+e(X)-1}{m}t^{md}\right) = \prod_{d=1}^{\infty}(1-t^d)^{-e(X)} \text{ in } \mathbb{Z}[[t]].$$

Assume that k is real closed, so that $\mathrm{Spr}(k)$ is a single point and $\mathcal{C}(\mathrm{Spr}(k),\mathbb{Z}) = \mathbb{Z}$.

Corollary 8.23 (Real closed Göttsche formula). *We have*

$$1+\sum_{n=1}^{\infty}\epsilon(X^{[n]})t^n = \prod_{i=1}^{\infty}(1+(-t)^i))^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b} \text{ in } \mathbb{Z}[[t]],$$

where $a=\epsilon(X)$ *and* $b=\frac{1}{2}(e(X)-\epsilon(X))$.

Proof. First, suppose that $a \geq 0$, so that $\chi^{\mathrm{dR}}(X) = b\mathbb{H} + a\langle 1\rangle$, where both $a, b \geq 0$. Then, taking sign of both sides of the equation in Corollary 8.18 and using Corollary 8.21, we obtain

$$\begin{aligned}
1+\sum_{n=1}^{\infty}\epsilon(X^{[n]})(-1)^n t^n &= \prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{a+m-2j-1}{m-2j}\binom{b+j-1}{j}\right)(-1)^m t^{md}\\
&= \prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{a+m-2j-1}{m-2j}(-1)^{m-2j}t^{(m-2j)d}\binom{b+j-1}{j}t^{2jd}\right)\\
&= \prod_{d=1}^{\infty}\left(\left(\sum_{i=0}^{\infty}\binom{a+i-1}{i}(-1)^i t^{id}\right)\cdot\left(\sum_{j=0}^{\infty}\binom{b+j-1}{j}t^{2jd}\right)\right)\\
&= \prod_{d=1}^{\infty}((1+t^d)^{-a}(1-t^{2d})^{-b})\\
&= \prod_{i=1}^{\infty}(1+t^i)^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b}.
\end{aligned}$$

Relabelling $t=-t$, we obtain

$$1+\sum_{n=1}^{\infty}\epsilon(X^{[n]})t^n = \prod_{i=1}^{\infty}(1+(-t)^i))^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b},$$

as required. It remains to show the case when $a<0$. Let $\tilde{a}=-a$ and $\tilde{b}=\frac{1}{2}(e(X)-\tilde{a})$, so that $\tilde{b}=b-a$. Taking sign gives

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$$\sum_{n=1}^{\infty}\epsilon(X^{[n]})(-1)^n t^n = \prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{\tilde{a}+m-2j-1}{m-2j}\binom{\tilde{b}+j-1}{j}\right)t^{md}$$
$$= \prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{\tilde{a}+m-2j-1}{m-2j}t^{(m-2j)d}\binom{\tilde{b}+j-1}{j}t^{2jd}\right)$$
$$= \prod_{d=1}^{\infty}((1-t^d)^{-\tilde{a}}(1-t^{2d})^{-\tilde{b}}).$$

Using that $(1-t^d) = (1-t^{2d})\cdot(1+t^d)^{-1}$, we can rewrite

$$\prod_{d=1}^{\infty}((1-t^d)^{-\tilde{a}}(1-t^{2d})^{-\tilde{b}}) = \prod_{d=1}^{\infty}((1+t^d)^{\tilde{a}}\cdot(1-t^{2d})^{-\tilde{b}-\tilde{a}}).$$

Rewriting $\tilde{a} = -a$ and $\tilde{b} = b-a$ gives

$$\prod_{d=1}^{\infty}((1+t^d)^{\tilde{a}}\cdot(1-t^{2d})^{-\tilde{b}-\tilde{a}}) = \prod_{i=1}^{\infty}(1+t^i)^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b}.$$

Putting everything together and relabelling t for $-t$ gives the result. □

In the case that $k=\mathbb{R}$, this exactly recovers the real Göttsche formula of [KR15, Theorem 1.2]. Let k again be an arbitrary field of characteristic 0. Identify $\widehat{\mathrm{W}}(k)/I^2 = E(k^\times/k^{\times 2})$ as in Remark 2.36. Write χ^{disc} for the composition $K_0(\mathrm{Var}_k)\xrightarrow{\chi^{\mathrm{mot}}}\widehat{\mathrm{W}}(k)\to\widehat{\mathrm{W}}(k)/I^2$, so that after this identification $\chi^{\mathrm{disc}}(X) = (e(X), \mathrm{disc}(\chi^{\mathrm{mot}}(X)))$.

Theorem 8.24. *There is an equality in the ring $E(k^\times/k^{\times 2})[[t]]$:*

$$1+\sum_{n=1}^{\infty}(\chi^{\mathrm{disc}}(X^{[n]}))t^n = \prod_{d=1}^{\infty}(1-(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X)))t^m)^{-e(X)}.$$

Proof. Note that by Corollary 8.21,

$$\chi^{\mathrm{disc}}(X^{(m)}) = \left(\binom{m+e(X)-1}{m}, \mathrm{disc}(\chi^{\mathrm{mot}}(X))^{\binom{m+e(X)-1}{m}}\right) = \binom{m+e(X)-1}{m}(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X))).$$

Applying the homomorphism χ^{disc} to Corollary 8.18 gives

$$1+\sum_{n=1}^{\infty}\chi^{\mathrm{disc}}(X^{[n]})(1,(-1))^n t^n = \prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\binom{m+e(X)-1}{m}(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X))\cdot(1,-1)^m t^{md}$$
$$= \prod_{d=1}^{\infty}(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X)))\cdot\sum_{m=0}^{\infty}\binom{m+e(X)-1}{m}((1,-1)t^d)^m$$
$$= \prod_{d=1}^{\infty}(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X)))(1-(1,-1)t^d)^{-e(X)}.$$

Relabelling $t = (1,-1)t$ gives

$$1+\sum_{n=1}^{\infty}\chi^{\mathrm{disc}}(X^{[n]})t^n = \prod_{d=1}^{\infty}(1,\mathrm{disc}(\chi^{\mathrm{mot}}(X)))(1-t^d)^{-e(X)},$$

as required. □

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9. The arithmetic Yau–Zaslow formula

Recall that X is a K3 surface admitting a complete g-dimensional linear system $\mathcal{C}$ of geometrically integral curves of genus g and that the curves in this linear system correspond to a primitive divisor class. Suppose that the rational curves in $\mathcal{C}$ are all nodal, which is generically the case by Chen's theorem [Che02, Theorem 1.1].

Lemma 9.1. *The motivic Euler characteristic* $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C}))\pmod J$ *is given by* $\langle(-1)^g\rangle$ *multiplied by the coefficient of* t^g *in the generating series*

$$\prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\chi^{\mathrm{mot}}(X^{(m)})\langle(-1)^m\rangle t^{md}\in(\widehat{\mathrm{W}}(k)/J)[[t]].$$

Proof. By Theorem 7.3, we have that $\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C}))\equiv\chi^{\mathrm{mot}}(X^{[g]})\pmod J$, so applying Corollary 8.18 gives us the result. □

Corollary 9.2. *The element* $\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)$ *from Definition 6.23 only depends on* X *and* g *modulo* J_g*, and not on* $\mathcal{C}$ *itself.*

Proof. Lemma 6.24 gives that $\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)\equiv\chi^{\mathrm{mot}}(\overline{\mathrm{Pic}}^g(\mathcal{C}))\pmod{J_g}$, so the result holds by the above lemma. □

The above corollary allows us to write $\mathbf{B}_g^{\mathrm{mot}}(X):=\mathbf{B}_{\mathcal{C}}^{\mathrm{mot}}(X)$. Our calculation of $\overline{\mathrm{Pic}}^g(\mathcal{C})$ only holds modulo J_g, so we need the following definition.

Definition 9.3. Let $\mathcal{J}$ denote the subset of $\widehat{\mathrm{W}}(k)[[t]]$ given by power series of the form $\sum_{i=0}^{\infty}a_it^i$, where each $a_i\in J_i$, where J_i is the ideal from Definition 6.20. Since each J_i is an ideal of $\widehat{\mathrm{W}}(k)$ with $J_i\subseteq J_{i+1}$, we see that $\mathcal{J}$ is an ideal of $\widehat{\mathrm{W}}(k)[[t]]$.

Theorem 9.4 (Arithmetic Yau–Zaslow formula). *Let* $X,\mathcal{C}$ *be as above. Then* $\langle(-1)^g\rangle\mathbf{B}_g^{\mathrm{mot}}(X)$ *is given by the coefficient of* t^g *in the power series*

$$\prod_{d=1}^{\infty}\sum_{m=0}^{\infty}\chi^{\mathrm{mot}}(X^{(m)})\langle(-1)^m\rangle t^{md}\in\widehat{\mathrm{W}}(k)[[t]]/\mathcal{J}.$$

Proof. Apply Lemma 6.24 to the result of Theorem 9.1. □

Remark 9.5. Consider the rank of $\mathbf{B}_g^{\mathrm{mot}}(X)$:

$$\mathrm{rank}(\mathbf{B}_g^{\mathrm{mot}}(X))=\mathrm{rank}\sum_{q\in G(X)}\mathrm{Tr}_{k(q)/k}\Big(\prod_{p\in\mathcal{S}(C_q)}\Phi_p\Big).$$

Note that $\mathrm{rank}(\mathrm{Tr}_{k(q)/k}(w))=[k(q):k]\mathrm{rank}(q)$, so since Φ_p has rank 1, this gives

$$\mathrm{rank}(\mathbf{B}_g^{\mathrm{mot}}(X))=\sum_{q\in G(X)}[k(q):k]=\#G(X)(\overline{k}).$$

That is, applying rank to $\mathbf{B}_g^{\mathrm{mot}}(X)$ recovers the number of rational curves in $\mathcal{C}$ after base changing to the algebraic closure. Applying Remark 8.22 to the right-hand side of the above theorem therefore recovers the complex Yau–Zaslow formula of [Bea99] when working over $\mathbb{C}$.

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Definition 9.6. Assume that k is real closed. Let C be a semi-stable curve over k. Following [KR15], we call $(-1)^s$ the *Welschinger number* $W(C)$ of C, where s is the number of zero-dimensional semi-algebraic connected components of $C(k)$. We set

$$\mathbb{W}(X,\mathcal{C}) := \sum_{q\in G(X)(k)} W(C_q) \in \mathbb{Z}.$$

Lemma 9.7. *For g the dimension of $\mathcal{C}$, we have*

$$\mathbb{W}(X,\mathcal{C}) = (-1)^g \mathrm{sign}(\mathbf{B}^{\mathrm{mot}}_{\mathcal{C}}(X)).$$

Proof. By Corollary 6.15, the signature $\mathrm{sign}(\prod_{p\in\mathcal{S}(C_q)}\Phi_p) = (-1)^{c_{C_q}}$, where c_{C_q} denotes the number of points in $\mathcal{S}(C_q)$ such that both p and the tangent spaces at p are defined over k. Let s_{C_q} denote the number of points in $\mathcal{S}(C_q)$ which are defined over k, whose tangent spaces are not defined over k, and let n_{C_q} denote the number of complex points in $\mathcal{S}(C_q)$. Note that $s_{C_q} + c_{C_q} + 2n_{C_q} = g$, so $(-1)^{c_{C_q}} = (-1)^g W(C_q)$. Therefore,

$$\mathrm{sign}(\mathbf{B}^{\mathrm{mot}}_{\mathcal{C}}(X)) = \sum_{q\in G(X)(k)} (-1)^g W(C_q) = (-1)^g \mathbb{W}(X,\mathcal{C}),$$

as required. □

Corollary 9.8 (Real closed arithmetic Yau–Zaslow formula). *The number $\mathbb{W}(X,\mathcal{C})$ depends only on g and X so we can write $\mathbb{W}(X,g)$. Moreover, $\mathbb{W}(X,g)$ is given by the coefficient of t^g in the power series*

$$\prod_{i=1}^{\infty}(1+(-t^i))^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b} \in \mathbb{Z}[[t]],$$

where $a = e(X(k))$ and $b = e(X(\overline{k})) - \frac{1}{2}e(X(k))$.

Proof. Applying sign to the power series in Theorem 9.4, and using the Göttsche formula 8.23, gives that $(-1)^g \mathrm{sign}(\mathbf{B}^{\mathrm{mot}}_{\mathcal{C}}(X)) = \mathbb{W}(X,\mathcal{C})$ is the coefficient of t^g in the power series

$$\prod_{i=1}^{\infty}(1+(-t)^i))^{-a}\cdot\prod_{j=1}^{\infty}(1-t^{2j})^{-b}.$$ □

Remark 9.9. The above corollary generalises the main theorem of Kharlamov–Răsdeaconu from [KR15] to all real closed fields.

We now turn to the discriminant. Let k be an arbitrary field of characteristic 0. Define the integer $r_g := \mathrm{rank}(\mathbf{B}^{\mathrm{mot}}_g(X))$, and let $\mathbb{D}^{\mathrm{mot}}_g(X) := \mathrm{disc}(\mathbf{B}^{\mathrm{mot}}_g(X)) \in k^\times/k^{\times 2}$.

Lemma 9.10. *There is an equality in* $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Z}/2\mathbb{Z})/\Delta^k_g$*:*

$$\mathbb{D}^{\mathrm{mot}}_g(X) = (-1)^{r_g}\prod_{q\in G(X)}\mathrm{disc}(k(q)/k)\prod_{p\in\mathcal{S}(\pi_g^{-1}(q))}\mathrm{N}_{k(p)/k}(\alpha_p).$$

Proof. By Lemma 4.4 and Corollary 4.21, we obtain

$$\mathrm{disc}(\mathbf{B}^{\mathrm{mot}}_g(X)) = \prod_{q\in G(X)}\mathrm{disc}(k(q)/k)\cdot(-1)^{[k(q):k]}\prod_{p\in\mathcal{S}(\pi_g^{-1}(q))}\mathrm{N}_{k(p)/k}(\alpha_p),$$

which gives the result since $\sum_{q\in G(X)}[k(q):k] = r_g$. □

This allows us to obtain the following Yau–Zaslow formula for discriminants.

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Corollary 9.11. *Suppose that k is finitely generated over $\mathbb{Q}$. Then the pair $(r_g, \mathbb{D}_g^{\mathrm{mot}}(X))$, viewed as an element of $E((k^\times/k^{\times 2})/\Delta_g^k)$, is given by the coefficient of t^g in the generating series*

$$\prod_{d=1}^{\infty}(1, \mathrm{disc}(\chi^{\mathrm{mot}}(X)))(1-t^d)^{-e(X)} \in E((k^\times/k^{\times 2})/\Delta_g^k)[[t]].$$

Proof. Theorem 7.3 gives $\chi^{\mathrm{disc}}(\overline{\mathrm{Pic}}^g(\mathcal{C})) = \chi^{\mathrm{disc}}(X^{[g]})$. By the previous lemma, this is $(r_g, \mathbb{D}_g^{\mathrm{mot}}(X))$. Applying Theorem 8.24 gives the result. □

Remark 9.12. In particular, the above formula implies that $\mathbb{D}_g^{\mathrm{mot}}(X) \in k^\times/k^{\times 2} \pmod{\Delta_g^k}$ is always either 1 or $\mathrm{disc}(\chi^{\mathrm{mot}}(X))$.

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Conflicts of interest

None.

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