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RESEARCH ARTICLE

Selberg, Ihara and Berkovich

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Abstract

We use the Selberg zeta function to analyze the limiting behavior of resonances in a degenerating family of Kleinian Schottky groups. We prove that, after a suitable rescaling, the Selberg zeta functions converge to the Ihara zeta function of a limiting finite graph associated with the relevant non-Archimedean Schottky group acting on the Berkovich projective line.

To reveal more interesting and intricate structures of resonances, we introduce *intermediate zeta functions* that encapsulate both Archimedean and non-Archimedean information – whereas the classical Selberg and Ihara zeta functions reflect only Archimedean and non-Archimedean properties, respectively. As an application of our techniques, we derive an exponential error term in a result of McMullen (recently extended by Dang–Mehmeti) concerning the asymptotic vanishing rate of the Hausdorff dimension of limit sets of certain degenerating Schottky groups generating symmetric three-funnel surfaces.

Contents

1 Introduction **2**
1.1 Example: symmetric three-funnel surfaces . . . 3
1.2 Main results: degenerating Schottky families . . . 3
1.3 On the proof of Theorem 1.3 . . . 6
1.4 Notations . . . 7
2 Preliminaries on Schottky groups **8**
2.1 Schottky groups of $\mathrm{PGL}_2(\mathbb{C})$ . . . 8
2.2 Kleinian Schottky spaces . . . 8
2.3 Limit sets of Kleinian Schottky groups . . . 8
2.4 Degenerations of Kleinian Schottky groups . . . 9
2.5 Non-Archimedean Schottky groups . . . 10
2.6 Hybrid space . . . 13
3 Asymptotic convergence of zeta function **16**
3.1 Convergence on the half-plane: Euler product . . . 16
3.2 Boundedness of zeta function through transfer operator . . . 20
3.2.1 Preliminaries on trace-class operators . . . 20
3.2.2 Transfer operator: $\mathrm{SL}_2(\mathbb{R})$ case . . . 21
3.2.3 Transfer operator: $\mathrm{SL}_2(\mathbb{C})$ case . . . 25

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3.3 Convergence on the whole plane . . . 28
3.4 Uniform separation . . . 29
4 Effective convergence of zeta function **33**
4.1 Expansion of lengths . . . 33
4.2 Convergence to intermediate zeta functions: statement of results . . . 35
4.3 Convergence to intermediate zeta functions: proofs . . . 36
5 Analytic extension of intermediate zeta functions **42**
5.1 Weighted Ihara zeta function . . . 42
5.2 M-th expansion of lengths and derivative cocycles . . . 43
6 Examples **49**
6.1 Two-generator case . . . 49
6.1.1 Figure Θ graph . . . 53
6.1.2 Dumbbell graph . . . 54
6.1.3 Figure eight graph . . . 54
6.2 Symmetric three-funnel surface . . . 55
A Appendix **57**
A.1 Critical exponent of the non-Archimedean Poincaré series . . . 57
A.2 Proof of Proposition 5.10 . . . 57
References **61**

1. Introduction

A *Schottky group* of rank g *over* $\mathbb{C}$ (also called a Kleinian Schottky group) is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ that is purely loxodromic and isomorphic to the free group of rank g. Schottky introduced these groups in 1877, and they were classically studied in connection with the famous uniformization theorem for Riemann surfaces; indeed, Koebe proved in 1910 that any closed Riemann surface is uniformized by some Kleinian Schottky group.

Partly motivated by the great success of studying degenerating families of Riemann surfaces – culminating in the celebrated Deligne–Mumford compactification of moduli spaces and its many applications – we investigate, in this paper, *degenerating* families of Schottky groups over $\mathbb{C}$ through the lens of spectral theory, a perspective of independent interest.

More concretely, given a family Γ_z of Schottky groups over $\mathbb{C}$ generated by 2×2 matrices whose entries are meromorphic functions of $z \in \mathbb{D}^*$, it is sometimes possible to introduce a Schottky group Γ^{na} over the non-Archimedean field $\mathbb{C}((t))$ (of Laurent series in t) whose features can be used to extract asymptotic information on Γ_z as $|z| \to 0$. In fact, this philosophy of studying degenerating families of complex objects via non-Archimedean limiting objects is nowadays widespread in the literature (cf. [20], [4], [10], [13], and [21] for some recent works). In the context of Schottky groups, this idea was exploited by Dang and Mehmeti [9, Theorem 3] to prove that certain degenerating families Γ_z of Schottky groups over $\mathbb{C}$ possess *limit sets* $L(\Gamma_z) \subset \mathbb{P}^1_{\mathbb{C}}$ whose Hausdorff dimensions satisfy

$$\dim(L(\Gamma_z)) \sim \frac{d_0}{\log(1/|z|)} \tag{1.1}$$

as $|z| \to 0$, where d_0 is the Hausdorff dimension of the limit set of a non-Archimedean Schottky group Γ^{na} acting on the *Berkovich projective line* $\mathbb{P}^{1,an}_{\mathbb{C}((t))}$. (We also recommend that the reader look at the very recent paper [8] by Courtois and Guilloux, which contains a partial extension of the work of Dang and Mehmeti to *higher* dimensions via an *alternative method*, namely, the study of the infinite-dimensional hyperbolic space.)

In the present paper, we pursue this line of inquiry to describe the Selberg zeta functions of certain degenerating families of Schottky groups over $\mathbb{C}$ in terms of Ihara zeta functions of finite graphs naturally attached to non-Archimedean Schottky groups (see Subsection 1.2 for precise statements). To reveal

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richer and more intricate structures of resonances, we introduce *intermediate zeta functions* that integrate both Archimedean and non-Archimedean data – whereas the classical Selberg and Ihara zeta functions capture only one of these aspects, respectively. Our approach combines tools from spectral theory with the analytic and geometric study of degenerating families of Schottky groups over $\mathbb{C}$. As a direct application, we significantly refine the result of Dang and Mehmeti mentioned above. In particular, we improve upon McMullen's work [24] on the Hausdorff dimension of limit sets of Schottky reflection groups associated with symmetric three-funnel hyperbolic surfaces and enhance the theoretical and numerical works of Weich [37], Borthwick [5], and Pollicott–Vytnova [34] on the convergence of rescaled Selberg zeta functions of symmetric three-funnel hyperbolic surfaces (see Subsection 1.1 for detailed statements). Furthermore, the techniques developed in this paper to enable fine-scale analysis of families of Selberg zeta functions may also be useful for future investigations into other dynamical properties of degenerating families of Schottky groups (see, for example, Remark 1.5).

1.1. Example: symmetric three-funnel surfaces

Let $X(\ell)$ be the symmetric three-funnel hyperbolic surface with parameter ℓ, that is, the noncompact hyperbolic surface of genus zero that has three funnels of width $\ell > 0$. This gives an example of a family of degenerating surfaces when ℓ tends to infinity. Denote by $\Gamma(\ell)$ the Schottky group uniformizing $X(\ell)$, and let $L(\Gamma(\ell))$ be the limit set of $\Gamma(\ell)$ on the visual boundary $\partial\mathbb{H}^2$. McMullen [24] studied the asymptotics of the Hausdorff dimension of $L(\Gamma(\ell))$ as $\ell \to \infty$, and he obtained $\dim(L(\Gamma(\ell))) \approx \frac{\log 4}{\ell}$.

For this particular example, as will be explained in Example 6.7 below, the main results of this paper imply an exponential error term in McMullen's asymptotic formula above:

Theorem 1.1. *For the case of symmetric three-funnel hyperbolic surfaces, we have for any $\epsilon > 0$,*

$$\dim(L(\Gamma(\ell))) = \frac{\log 4}{\ell} + O_\epsilon(e^{-(1-\epsilon)\ell/4}).$$

Moreover, for the Selberg zeta function (see equation (1.2) for the definition), we have

$$|Z(\Gamma(\ell), s/\ell) - (1 - 6e^{-s} + 9e^{-2s} - 4e^{-3s})| = O_{K,\epsilon}(e^{-(1-\epsilon)\ell/4}),$$

for any s in a fixed compact set K and any $\epsilon > 0$.

The convergence of the Selberg zeta function without speed is due to Weich [37]. Pollicott–Vytnova [34] proved a polynomial error term $O(1/\sqrt{\ell})$ in the convergence of the Selberg zeta functions and $O(1/\sqrt{\ell^3})$ in the convergence of Hausdorff dimensions (and, in fact, Example 6.8 below shows how to recover their results from our techniques). The exponential error term above also explains that the numerics of resonances in Borthwick [5] behave well even for small ℓ such as $\ell = 10$: see Example 6.5 below for more details.

Our proof differs from previous work on symmetric three-funnel surfaces. Indeed, this family serves merely as a representative example within a broader class of degenerating Schottky families. As explained below, our analysis encompasses the limiting behavior of all such families.

1.2. Main results: degenerating Schottky families

The study of the degenerations at the origin $z = 0$ of meromorphic families Γ_z of Kleinian Schottky (and, more generally, nonelementary) groups depending on a complex parameter $z \in \mathbb{D}^*$ in the punctured unit disc of $\mathbb{C}$ is a topic of intense investigation, with a vast literature surrounding it. For instance, such families are useful for building compactifications of representation varieties, character varieties, or moduli spaces (see, e.g., [7], [27], and [32]), their Lyapunov exponents for the associated random walks satisfy fascinating asymptotics (see, e.g., [2] and [10]), and, as we already mentioned above, Hausdorff dimensions of their limit sets have attracted the attention of many authors.

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In the present paper, a central object of discussion will be the Selberg zeta functions. Recall that the Selberg zeta function for a Schottky group Γ of $\mathrm{SL}_2(\mathbb{C})$ is defined as follows. Letting $\mathcal{P}$ be the set of conjugacy classes in Γ containing a nontrivial primitive element, we have

$$Z(\Gamma, s) = \prod_{\gamma \in \mathcal{P}} \prod_{k_1, k_2 \geq 0} (1 - e^{-(s+k_1+k_2)\ell(\gamma)} e^{-i\theta_\gamma (k_1 - k_2)}). \tag{1.2}$$

Here, by a slight abuse of notation, we also denote by γ an oriented closed geodesic representing a loxodromic element $\gamma \in \mathrm{SL}_2(\mathbb{C})$, so that $\ell(\gamma)$ is the length of the periodic geodesic γ in $\Gamma\backslash\mathbb{H}^3$, and $e^{i\theta_\gamma}$ is the holonomy given by $\gamma'(\gamma_+) = e^{-\ell(\gamma) - i\theta_\gamma}$, with γ_+ being the attracting fixed point of γ. Similarly, for $\Gamma < \mathrm{SL}_2(\mathbb{R})$, the classical Selberg zeta function is defined as

$$Z(\Gamma, s) = \prod_{\gamma \in \mathcal{P}} \prod_{k \geq 0} (1 - e^{-(s+k)\ell(\gamma)}).$$

Note that the definitions for $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ do not coincide, and for this reason, we will deal with the two cases separately.

The function $Z(\Gamma, s)$ is convergent and analytic in the region $\Re s > \delta_\Gamma$, where δ_Γ is the so-called critical exponent. It admits an analytic continuation to the whole complex plane $\mathbb{C}$, and hence we obtain an entire function $Z(\Gamma, s)$. Due to [31], we know that the poles of the resolvent of the Laplacian on $\Gamma\backslash\mathbb{H}^3$ (also called resonances) correspond to a subset of the zeros of the Selberg zeta function $Z(\Gamma, s)$.

For a degenerating family of Schottky groups, we introduce *intermediate zeta functions*, which serve as a core object in our analysis.

Definition 1.2 (Intermediate zeta function). Let $\gamma \in \mathrm{SL}_2(\mathbb{C}((t)))$ be a loxodromic element and let $M \in \mathbb{Z}_{\geq 0}$. We define the *intermediate length* as

$$\ell_M(\gamma, z) = \ell^{na}(\gamma) + \Re \sum_{j=0}^{M} a_j(\gamma) z^j / \log(1/|z|),$$

where $\ell^{na}(\gamma)$ is the non-Archimedean length defined in Definition 2.16. The coefficients $a_j(\gamma)$ are determined by the expansion

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + \Re \sum_{j=0}^{\infty} a_j(\gamma) z^j / \log(1/|z|).$$

Here, $\gamma_z \in \mathrm{SL}_2(\mathbb{C})$ denotes the evaluation of γ at z. The term $\ell(\gamma_z)$ is given by $2\log|\lambda_1(\gamma_z)|$, where $\lambda_1(\gamma_z)$ and $\lambda_2(\gamma_z)$ are the two eigenvalues of γ_z such that $|\lambda_1(\gamma_z)| > |\lambda_2(\gamma_z)|$. The *intermediate M-zeta function* is then defined for $\Re s$ sufficiently large by

$$Z_M(\Gamma, z, s) = \prod_{[\gamma] \in \mathcal{P}} (1 - e^{-s\ell_M(\gamma, z)}).$$

In Proposition 4.2 and Corollary 5.8, we establish that this function is well-defined and admits a holomorphic extension to all $s \in \mathbb{C}$.[1]

Here is our main result:

Theorem 1.3. *Let Γ_z be a degenerating Schottky family satisfying condition ($\bigstar$) in Subsection 2.4, and denote by $\Gamma < \mathrm{SL}_2(\mathbb{C}((t)))$ the corresponding non-Archimedean Schottky group. Then for any $s \in \mathbb{C}$, as $|z| \to 0$, we have*

$$Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s), \tag{1.3}$$

[1] See Example 6.5 for an explicit but nontrivial intermediate zeta function for a hyperbolic funneled torus.

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where $Z_I(\Gamma, s)$ is the Ihara zeta function associated with Γ (defined in Proposition 2.19) and the convergence on any compact set is uniform.

Moreover, for any compact set K and $\epsilon > 0$, we have for all $0 < |z| < 1/e$ and for any $s \in K$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_0(\Gamma, z, s)| \lesssim_{K,\epsilon} |z|^{1-\epsilon}. \tag{1.4}$$

Furthermore, for any $C, \epsilon > 0$ and $M \in \mathbb{Z}_{\geq 0}$, we have for $s \in [-C, C] + i[-C|z|^{-M}, C|z|^{-M}]$ and $0 < |z| < 1/e$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon}. \tag{1.5}$$

As a direct corollary, we obtain the following.

Corollary 1.4. *Let Γ_z be a degenerating Schottky family satisfying condition (★) in Subsection 2.4. Let $P(|z|)$ be the first zero of the intermediate zeta function $Z_0(\Gamma, z, s)$ in the variable s. We have for any $\epsilon > 0$ and $0 < |z| < 1/e$,*

$$\dim(L(\Gamma_z)) = P(|z|) + O_\epsilon(|z|^{1-\epsilon}).$$

For special cases, such as symmetric three-funnel surfaces and Example 6.4, the main term has the form $P(|z|) = d_0/\log(1/|z|)$, where d_0 is the Hausdorff dimension of the limit set of the corresponding non-Archimedean Schottky group $\Gamma < \mathrm{SL}_2(\mathbb{C}((t)))$ acting on $\mathbb{P}^{1,an}_{\mathbb{C}((t))}$. In general settings, the main term $P(|z|)$ is a zero of a polynomial in exponential functions of the variable s, which is of the form $\sum a_j e^{-s(b_j + c_j/\log(1/|z|))}$. It would be interesting to investigate the Taylor expansion of $P(|z|)$ in terms of $1/\log(1/|z|)$.

Remark 1.5. Let us expand the above discussion. Due to the definition of Z_0 (Definition 1.2) and Proposition 4.2, where we establish the approximate length function, we know that both $P(|z|)$ and the coefficients of the expansion depend only on the coefficients of $\mathrm{tr}(\gamma)$ with $\gamma \in \Gamma$. If we write $Z_0(\Gamma, z, s) = Z_0(s, u)$ with $u = 1/\log(1/|z|)$, then by the implicit function theorem, we obtain

$$P(|z|) = \frac{d_0}{\log(1/|z|)} - \frac{\partial_2 Z_0}{\partial_1 Z_0}(d_0, 0)\frac{1}{(\log(1/|z|))^2} + O\left(\frac{1}{(\log(1/|z|))^3}\right).$$

An intriguing question is whether one can find a dynamical interpretation of the term $\frac{\partial_2 Z_0}{\partial_1 Z_0}(d_0, 0)$, similar to the known interpretation in terms of variations of Hausdorff dimension [25] or the expansion of the Lyapunov exponent [2]. Another possibility is to explore connections with invariant distributions, as discussed in [12]. Understanding this term could shed light on deeper geometric and dynamical properties of the underlying moduli space of Schottky groups over $\mathbb{C}$.

The zeros of the Selberg zeta function correspond to the resonances of the Laplacian on the quotient manifold (see [5, Chapter 10]). By definition, these resonances are the poles of the resolvent of the Laplacian and appear in the long-time asymptotics of the wave equation on the quotient manifold. Despite extensive efforts in spectral theory to understand these resonances, many questions remain open – particularly concerning their high-frequency distribution (see Remark 1.7). Our main theorem yields a direct corollary regarding the distribution of resonances in the region where $|\Im s| < |z|^{-N}$ as $|z| \to 0$. Additionally, by employing intermediate zeta functions, Corollary 4.6 provides a more refined approximation of such resonances. As an application, we recover the result in [34] in Example 6.8, and our method extends to a broad class of hyperbolic surfaces.

Corollary 1.6. *Under the assumption of Theorem 1.3, we have that the zeros of $Z_I(\Gamma, s)$ have the structure $Res_I(\Gamma) = \{\mu_j\} + 2\pi i\mathbb{Z}$, where $\{\mu_j\}$ is a finite set. Outside $s = 0$, the resonances of Γ_z*

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converge to $Res_I(\Gamma)$ *(counted with multiplicity) after rescaling, that is,*

$$\log(1/|z|)\, Res(\Gamma_z) \setminus \{0\} \to Res_I(\Gamma) \setminus \{0\}, \quad |z| \to 0 \tag{1.6}$$

on compact sets. In particular, the spectral gap of Γ_z *converges to* 0 *after rescaling by* $\log(1/|z|)$.

Proof. We use Rouché's theorem to deduce from Theorem 1.3 that the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ converge to the zeros of $Z_I(\Gamma, s)$ in any compact set. By Proposition 2.19, the Ihara zeta function $Z_I(\Gamma, s)$ associated with Γ is the Ihara zeta function of a finite graph which comes from some Berkovich space quotient by Γ, and hence $Z_I(\Gamma, s)$ is a polynomial in e^{-s} (see Proposition 5.1), which yields the statement on the structure of $\mathrm{Res}_I(\Gamma)$. The convergence in equation (1.6) follows from the fact that the resonances of the Laplacian coincide with the zeros of $Z(\Gamma_z, s)$ outside $s \in \mathbb{Z}_{\leq 0}$ (see [5, Chapter 10] for the two-dimensional case and [31] for the general case). □

Remark 1.7.

1. A classical result states that the Hausdorff dimension of the limit set of a Kleinian Schottky group equals the first zero of its Selberg zeta function. For the non-Archimedean case, we prove that the Hausdorff dimension of the limit set also equals the first zero of the Ihara zeta function (see Section A.1). Hence, we recover the convergence theorem of the Hausdorff dimension of the limit sets of [9] stated in equation (1.1) from Corollary 1.6.
2. The work [26] establishes that the family of congruence Schottky surfaces exhibits a uniform spectral gap across both low and high frequencies. [22] and [6] investigate the optimal size of the low-frequency spectral gap for random covers. Our results demonstrate that this gap becomes smaller for degenerating families. A natural question is to ask if the gap also shrinks in high frequencies, that is, when $\Im s$ is big. The conjecture of Jakobson–Naud [19] suggests that this does not happen.
3. The fractal Weyl law conjecture [38, Conjecture 5] states that the number of resonances in a strip grows according to the dimension of the limit set at $T \to \infty$:

$$\#\mathrm{Res}(\Gamma) \cap \{\Re s > -C,\ |\Im s| \leq \mathrm{T}\} \sim \mathrm{T}^{1+\delta}, \quad \delta = \dim(L(\Gamma)).$$

   Moreover, it is conjectured in [38, Conjecture 7] that the resonances concentrate near the axis $\Re s = \delta/2$. The upper bound is known from [14], but the sharp lower bound is not known in any nontrivial example of Schottky groups. Our work provides a perspective on this conjecture via the intermediate zeta functions, which we conjecture to have a growth pattern similar to the fractal Weyl law; see the discussion after Example 6.8.
4. When the corresponding Mumford curve Σ_X of the non-Archimedean Schottky group Γ is a q-regular graph with each edge of length 1 (see Section 2.5 for the definition), the Ihara zeta function is given by $Z_I(\Gamma, s) = (1 - e^{-2s})^{r-1} \det(I - A_X e^{-s} + (q-1)e^{-2s})$, where r is the number of generators of Γ and A_X is the adjacency matrix of Σ_X (see, for example, [16]). Let $\lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 = q$ be the eigenvalues of A_X. Therefore, the set of resonances $\mathrm{Res}_I(\Gamma)$ is determined by the μ_j's, which satisfy $e^{\mu_j} = (\lambda_i \pm \sqrt{\lambda_i^2 - 4(q-1)})/2(q-1)$ or $e^{\mu_j} = \pm 1$. This is similar to the relation between $\mathrm{Res}(\Gamma_z)$ and eigenvalues/resonances of the Laplacian operator on the quotient manifold.
   Moreover, if the graph Σ_X is a Ramanujan graph, that is to say, we have $-2\sqrt{q-1} \leq \lambda_n \leq \cdots \leq \lambda_2 \leq 2\sqrt{q-1}$, then μ_j has three possibilities: $\Re\mu_j = \log(q-1)/2$; $\mu_j = \log(q-1)$; $\mu_j = 0$ or πi.

1.3. On the proof of Theorem 1.3

The ideas behind the proof of Theorem 1.3 can be outlined as follows. To establish the asymptotic convergence in equation (1.3), we divide the complex plane into two regions. In the region where $\Re s \gg 1$, convergence follows from a detailed analysis of how hyperbolic lengths converge to their non-Archimedean counterparts (see Section 3.1). In the complementary region, we employ tools from

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spectral theory, studying the relevant transfer operators $\mathcal{L}_{z,s}$ for Γ_z, which we show are trace-class operators. We relate their Fredholm determinants $\det(1-\mathcal{L}_{z,s})$ to the Selberg zeta function $Z(\Gamma_z, s)$ (Lemma 3.10). A key step in proving convergence is to obtain a uniform bound on $\det(1-\mathcal{L}_{z,s})$ for small $|z|$, which we achieve by bounding the singular values uniformly. This uniform control is made possible by establishing a uniform separation property for the limit sets of Γ_z with respect to z, as shown in Proposition 3.11 and Proposition 3.12 for $\mathrm{SL}_2(\mathbb{R})$, and in Proposition 3.14 and Proposition 3.15 for $\mathrm{SL}_2(\mathbb{C})$. Here, it is worth pointing out that the Berkovich techniques are mainly used to analyze length convergence and prove this uniform separation. The effective convergence in equations (1.4) and (1.5) is then obtained through the use of intermediate zeta functions, which allow for a more sophisticated integration of analytic and geometric information from degenerating Schottky families. An interesting aspect of our approach is a new perspective that connects the asymptotic expansion of lengths with the derivative cocycle – highlighting a rich interplay between analytic data and geometric structure (see Section 5.2).

1.4. Notations

Throughout the paper, we adopt the following conventions.

- We write $A \lesssim B$ if there exists a positive constant $C > 0$ (that may differ from expression to expression) such that $A \leq CB$. We specify the dependencies of C by a subscript, for example, $A \lesssim_\Gamma B$. We use $A \ll B$ to mean that A is much smaller than B.
- $D(a,r) := \{w \in \mathbb{C} : |w-a| < r\}$ is the disc of radius r centered at $a \in \mathbb{C}$, and let

$$|D(a,r)| = \pi r^2 \tag{1.7}$$

  denote its Lebesgue measure.
- $\mathbb{D} = D(0,1)$ is the unit disc and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Similarly, $\mathbb{D}_r = D(0,r)$ and $\mathbb{D}_r^* = \mathbb{D}_r \setminus \{0\}$.
- The variable t usually means the formal variable in the field of Laurent series $\mathbb{C}((t))$. This field is equipped with its usual t-adic norm $|f|_{na} = \exp(-\mathrm{ord}_{t=0}(f)) = e^{-n}$ for $f = t^n \sum_{m=0} f_m t^m \in \mathbb{C}((t))$ with $f_0 \neq 0$. The variable z usually means a complex number that parametrizes a degenerating family of Schottky groups over $\mathbb{C}$ or $\mathbb{R}$.
- The Banach ring $A_r \subset \mathbb{C}((t))$ is defined by $f = \sum a_n t^n \in \mathbb{C}((t))$ such that the hybrid norm

$$\|f\|_{A_r} = \sum \max\{|a_n|_\infty, |a_n|_0\} r^n$$

  is finite ($|\cdot|_0$ is the trivial norm). We will usually take $r = 1/e$ and define $\|\cdot\|_{\mathrm{hyb}} := \|\cdot\|_{A_{1/e}}$.
- Let $M \in \mathbb{Z}_{\geq 0}$ be a nonnegative integer and $a = t^m \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}((t))$ with $a_0 \neq 0$. We define the leading M terms

$$\mathrm{lt}_M(a) := t^m \sum_{n=0}^{M} a_n t^n.$$

- We use Fraktur letters $\mathfrak{a}, \mathfrak{b}$ to mean elements of the (fixed) set of generators of Γ. We use bold letters $\mathbf{a}, \mathbf{b}$ to mean words of the generators.
- For an element $\gamma \in \mathrm{SL}(2, \mathbb{C}((t)))$, we use γ_z to denote an element in $\mathrm{SL}(2,\mathbb{C})$ by evaluating at $t = z$ if all the coefficients of γ converge at z.
- $\ell(\gamma) \in \mathbb{R}_{\geq 0}$ means the length of a closed geodesic associated with $\gamma \in \Gamma$. $\ell^{na}(\gamma) \in \mathbb{Z}_{\geq 0}$ is the non-Archimedean length defined in Definition 2.16. $l(\gamma) \in \mathbb{Z}_{\geq 0}$ is the word length of γ for a fixed set of generators $\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}$ with $\mathfrak{a}_{i+g} = \mathfrak{a}_i^{-1}$, that is, $l(\gamma) = n$ whenever $\gamma = \mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}$ with $\mathfrak{a}_{i_{j+1}} \neq \mathfrak{a}_{i_j}^{-1}$.

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2. Preliminaries on Schottky groups

We recall some preliminaries on Schottky groups.

2.1. Schottky groups of $PGL_2(\mathbb{C})$

An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ is called *loxodromic* if it induces a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ on $\mathbb{C}$ which is conjugate to $z \mapsto \lambda z$ for a complex number λ with $0 < |\lambda| < 1$.

Definition 2.1. We say that a free subgroup $\Gamma = \langle M_1, \ldots, M_g \rangle \subset \mathrm{PGL}_2(\mathbb{C})$ with $g \geq 2$ generators is a *Schottky group of rank g* if

- every $\gamma \in \Gamma \setminus \{id\}$ is loxodromic;
- Γ admits a nonempty domain of discontinuity, that is, it acts freely and properly on a nonempty connected invariant open subset of $\mathbb{P}^1(\mathbb{C})$.

Given a Jordan curve C in $\mathbb{P}^1(\mathbb{C})$ and a reference point $o \notin C$, we say that the exterior (resp. interior) of C is the connected component of $\mathbb{P}^1(\mathbb{C}) \backslash C$ containing (resp. not containing) o. After Maskit [23], a Schottky group has the form $\langle T_1, \ldots, T_g \rangle$, where the T_j's induce Möbius transformations with the following property: there are a reference point o and Jordan curves $C_1, C_1', \ldots, C_g, C_g'$ with disjoint interiors such that each T_j maps the exterior of C_j to the interior of C_j'. If all the Jordan curves C_j, C_j' can be chosen to be circles, then the corresponding Schottky group is called a *classical Schottky group*.

2.2. Kleinian Schottky spaces

A loxodromic element $\gamma \in \mathrm{PGL}_2(\mathbb{C})$ is determined by three parameters: its attracting fixed point $\alpha \in \mathbb{P}^1(\mathbb{C})$, its repelling fixed point $\beta \in \mathbb{P}^1(\mathbb{C})$, and its contraction rate $\lambda \in \mathbb{D}^*$. In the sequel, we shall denote by $M(\alpha, \beta, \lambda)$ the loxodromic matrix parametrized by α, β, and λ.

The *Schottky space* $S_g(\mathbb{C})$ *of rank* $g \geq 2$ is the space of g elements $M_1, \ldots, M_g$ of $\mathrm{PGL}_2(\mathbb{C})$ that generate a Schottky group up to the natural equivalence given by simultaneous conjugation by Möbius transformations (cf. Bers [3]). These elements are parametrized by $3g$ quantities $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \lambda_1, \ldots, \lambda_g$. Since the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is 3-point transitive, we can set $\alpha_1 = 0$, $\beta_1 = 2$, $\alpha_2 = 1$. In this way, we can regard $S_g(\mathbb{C})$ as a subset of $\mathbb{C}^{3g-3}$: each $p \in S_g(\mathbb{C})$ is parametrized by the so-called *Koebe coordinates*

$$(\alpha_3, \ldots, \alpha_g, \beta_2, \ldots, \beta_g, \lambda_1, \ldots, \lambda_g) = (\underline{\alpha}, \underline{\beta}, \underline{\lambda}) \in (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, 2\})^{2g-3} \times (\mathbb{D}^*)^g.$$

As was shown by Hejhal [15] (see also [3]), $S_g(\mathbb{C})$ is a connected complex manifold of dimension $3g-3$.

2.3. Limit sets of Kleinian Schottky groups

Given a point $p \in S_g(\mathbb{C})$, let $L(\Gamma_p)$ be the limit set of the Schottky group $\Gamma = \Gamma_p$ associated with p. Sullivan [36] showed that the Hausdorff dimension $\dim(L(\Gamma_p))$ coincides with the critical exponent δ_{Γ_p} of Γ, which is given by

$$\delta_{\Gamma_p} = \inf\{s > 0 : \mathcal{P}(s) < \infty\}.$$

Here, $\mathcal{P}(s) = \sum_{\gamma \in \Gamma_p} |\gamma'(o)|_\infty^s$ is the usual Poincaré series, o is a point outside $L(\Gamma_p)$, and $|.|_\infty$ is the norm with respect to the Euclidean metric on $\mathbb{C}$. Due to the work of Patterson [28, 29, 30], we know that the critical exponent is also equal to the first zero of the Selberg zeta function.

As established by Anderson and Rocha [1], $\dim(L(\Gamma_p))$ is a real analytic function of $p \in S_g(\mathbb{C})$.

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2.4. Degenerations of Kleinian Schottky groups

As in the case of the moduli space of complex algebraic curves, one can try to approach the boundary of $S_g(\mathbb{C})$ by studying degenerations of one-parameter families of Schottky groups. More concretely, we always consider a meromorphic function $p : \mathbb{D} \to \mathbb{C}^{2g-3} \times \mathbb{D}^g, z \mapsto p(z) = (\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))$ such that $p(z) \in S_g(\mathbb{C})$ for all $z \in \mathbb{D}^*$. From now on, when talking about a Schottky family, we mean a family given by this meromorphic map $p(z)$.

Since it is not easy to handle *general* degenerations of Schottky groups, we shall focus on the degenerations $p(z)$ leading to nice (Schottky) non-Archimedean limits (in the sense of Poineau and Turchetti [33]) as $z \to 0$. For this purpose, one follows Dang and Mehmeti [9] by considering the quantity

$$K_{\min} = \min_{\substack{i\neq j, i\neq k \\ u\in\{\alpha,\beta\}}} \operatorname{ord}_{z=0}\big(\lambda_i(z) \cdot [u_j(z) : u_k(z); \alpha_i(z) : \beta_i(z)]\big)$$

(based on the order of vanishing of certain holomorphic functions) measuring how close the attracting and repelling fixed points can get in terms of cross-ratios versus the contraction rates.[2]

Definition 2.2. We say that a one-parameter family of Schottky groups satisfies condition (★) if

- $\lambda_j(z)$ vanishes at $z = 0$ for all $j = 1, \dots, g$ and
- $K_{\min} > 0$.

For an explanation of this condition using the non-Archimedean norm on $\mathbb{C}((t))$, please see Remark 2.14.

In this setting, Dang and Mehmeti showed the following.

Theorem 2.3. *There exists $d_0 > 0$ with*

$$\delta_{\Gamma_z} = dim(L(\Gamma_z)) \sim d_0/\log(1/|z|)$$

as $z \to 0$ whenever condition (★) holds.

Remark 2.4. Similar results for Lyapunov exponents in place of the Hausdorff dimension were obtained by Favre [13] (more recently, an error term was derived by Ingram et al. [18] in arithmetic situations).

Let $\mathbb{M}(\mathbb{D})$ be the ring of meromorphic functions on $\mathbb{D}$ with a possible pole at 0. From a Koebe coordinate $(\alpha(z), \beta(z), \lambda(z))$, we get a matrix $\gamma_z = \begin{pmatrix}\alpha(z) & \beta(z)\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \lambda(z)\end{pmatrix}\begin{pmatrix}\alpha(z) & \beta(z)\\ 1 & 1\end{pmatrix}^{-1}$ in $\mathrm{GL}_2(\mathbb{M}(\mathbb{D}))$. To simplify later computations, we want a matrix in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. The only difficulty is that $\sqrt{\lambda(z)} = \sqrt{t^{-n}f(z)}$, for some $n \in \mathbb{Z}$ and holomorphic $f(z)$ nonzero at 0, may not be meromorphic. We can make a change of variable. Let $z = w^2$ for $w \in \mathbb{D}$; then, due to $\mathbb{D}$ being simply connected, the square root $\sqrt{\lambda(w^2)} = w^{-n}\sqrt{f(w^2)}$ is still meromorphic. Let $\beta_w = \begin{pmatrix}\alpha(w^2) & \beta(w^2)\\ 1 & 1\end{pmatrix}\begin{pmatrix}w^n(f(w^2))^{-1/2} & 0\\ 0 & w^{-n}(f(w^2))^{1/2}\end{pmatrix}\begin{pmatrix}\alpha(w^2) & \beta(w^2)\\ 1 & 1\end{pmatrix}^{-1}$, which is in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. For $z = w^2$, from the definition, we have $\ell(\gamma_z) = \ell(\beta_w)$ and $2\ell^{na}(\gamma) = \ell^{na}(\beta)$, where ℓ^{na} will be defined in Definition 2.16 and is taken with respect to $\mathbb{C}((t))$ and $\mathbb{C}((w))$, respectively. Denote the Schottky group generated by β_w as Δ_w. Then the corresponding Selberg zeta functions and Ihara zeta functions satisfy

$$Z(\Gamma_z, s/\log(1/|z|)) = Z(\Delta_w, s/2\log(1/|w|)) \text{ and } Z_I(\Gamma, s) = Z_I(\Delta, s/2).$$

We can obtain the result for Γ_z from that of Δ_w. Hence, up to a possible change of variable $z = w^2$, the family of Schottky groups in $\mathrm{PGL}_2(\mathbb{M}(\mathbb{D}))$ can be replaced by a family in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. We do not give the detailed results in PGL_2. The readers are encouraged to write for themselves.

[2]For example, by taking $i = 1$, $j = k = 2$, one gets $\lambda_1(z) \cdot [1 : \beta_2(z); 0 : 2] = \lambda_1(z)(2 - \beta_2(z))/\beta_2(z)$.

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2.5. Non-Archimedean Schottky groups

Let $(k, |\cdot|_k)$ be a complete normed non-Archimedean field. See [33] for more details of this part.

Definition 2.5. For $n \in \mathbb{N}$, the *Berkovich analytification* $\mathbb{A}_k^{n,an}$ of the n-dimensional affine space $\mathbb{A}_k^n$ is the set of multiplicative seminorms on the polynomial ring $k[T_1, \cdots, T_n]$ which extend the norm on k. The topology on $\mathbb{A}_k^{n,an}$ is the coarsest topology such that for any polynomial P in $k[T_1, \cdots, T_n]$, the evaluation map from $\mathbb{A}_k^{n,an}$ to $\mathbb{R}$ given by $|\cdot|_x \mapsto |P|_x$ is continuous.

The *Berkovich analytification* $\mathbb{P}_k^{1,an}$ of the projective line $\mathbb{P}_k^1$ is constructed by gluing two copies of $\mathbb{A}_k^{1,an}$. More precisely, let $|\cdot|_x$ and $|\cdot|_y$ be two seminorms of the two copies. They are equivalent if the map $T \to S^{-1}$ maps one seminorm on $k[T, T^{-1}]$ to the other on $k[S, S^{-1}]$.

Remark 2.6. When $k = \mathbb{C}$, $\mathbb{A}_{\mathbb{C}}^{n,an}$ is homeomorphic to the usual analytic $\mathbb{A}_{\mathbb{C}}^n$, and hence $\mathbb{P}_{\mathbb{C}}^{1,an}$ is homeomorphic to $\mathbb{P}_{\mathbb{C}}^1$. When k is non-Archimedean, $\mathbb{P}_k^{1,an}$ has the structure of a real tree.

Example 2.7. Set $n = 1$. When k is non-Archimedean, for $a \in k$ and $r \in \mathbb{R}_{\geq 0}$, we can define the point $\eta_{a,r} \in \mathbb{A}_k^{1,an}$ by $\sum_n a_n(T-a)^n \mapsto \max_n |a_n|_k r^n$. In this setting, $a \in k$ corresponds to $\eta_{a,0} \in \mathbb{A}_k^{1,an}$, and the element $\eta_{0,1} \in \mathbb{A}_k^{1,an}$ is called the *Gauss point*.

Definition 2.8. An open disc (resp. a closed disc) in $\mathbb{P}_k^{1,an}$ is an open subset (resp. a closed subset) isomorphic to a set

$$D^-(a,r) = \{x \in \mathbb{A}_k^{1,an} || T-a|_x < r\}$$
$$(\text{resp. } D^+(a,r) = \{x \in \mathbb{A}_k^{1,an} || T-a|_x \leq r\}),$$

where $a \in k$ and $r \in \mathbb{R}_{>0}$, and $\mathbb{A}_k^{1,an}$ is an affine chart of $\mathbb{P}_k^{1,an}$.

The *Shilov boundaries* of both $D^-(a,r)$ and $D^+(a,r)$ are defined to be the singleton $\{\eta_{a,r}\}$.

The action of an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{PGL}_2(k)$ on $\mathbb{A}_k^{1,an} - \{-d/c\}$ is given by

$$|P(T)|_{\gamma(x)} = \left| P\left(\frac{aT+b}{cT+d}\right) \right|_x = \frac{|P_1(T)|_x}{|P_2(T)|_x}$$

for $P, P_1, P_2 \in k[T]$ and $P_1(T)/P_2(T) = P(\frac{aT+b}{cT+d})$. Moreover, it is possible to naturally extend this formula to obtain an action of γ on $\mathbb{P}_k^{1,an}$: see [32, II.1.3] for more details.

An element $\gamma \in \mathrm{PGL}_2(k)$ is called *loxodromic* if, given a representative in $\mathrm{GL}_2(k)$, its eigenvalues in an algebraic closure k^{alg} of k have different absolute values (by [32, II.1.4], the eigenvalues are in k). Schottky groups of $\mathrm{PGL}_2(k)$ ([33, Definition 3.5.1]) are defined similarly to Schottky groups of $\mathrm{PGL}_2(\mathbb{C})$ (Definition 2.1) with the projective space $\mathbb{P}^1(\mathbb{C})$ replaced by $\mathbb{P}_k^{1,an}$.

Definition 2.9. For a loxodromic element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$ with $c \neq 0$ and $\lambda \in \mathbb{R}_{>0}$, let

$$D^-_{\gamma,\lambda} = \{x \in \mathbb{A}_k^{1,an} |\ |(cZ+d)(x)|^2 < \lambda |ad-bc|\},$$
$$D^+_{\gamma,\lambda} = \{x \in \mathbb{A}_k^{1,an} |\ |(cZ+d)(x)|^2 \leq \lambda |ad-bc|\},$$

where we only consider an affine chart $\mathbb{A}_k^{1,an}$ of $\mathbb{P}_k^{1,an}$ and Z is the variable of $k[Z]$. They are called open and closed *twisted Ford discs*, respectively.

Remark 2.10. The original definition in [33, Definition 3.5.5] contains a typo. We need to define for all $x \in \mathbb{A}_k^{1,an}$ to have the desired property $\gamma D^-_{\gamma,\lambda} = \mathbb{P}_k^{1,an} - D^+_{\gamma^{-1},\lambda^{-1}}$. The definition relies on the choice of the affine chart $\mathbb{A}_k^{1,an}$. In the entire argument, we fix the affine chart once and for all, such that ∞ is not in the limit set.

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Definition 2.11 (Schottky figure). Let Γ be a Schottky group in $\mathrm{PGL}_2(k)$. Let $\{\gamma_1, \cdots, \gamma_g\}$ be a set of free generators of Γ. Let $\mathcal{B} = \{D^+_{\gamma_i}, D^+_{\gamma_i^{-1}} : i = 1, \cdots g\}$ be a collection of disjoint closed discs in $\mathbb{P}^{1,an}_k$. Then $\mathcal{B}$ is called a *Schottky figure* adapted to $\{\gamma_1, \cdots, \gamma_g\}$ if each $i \in \{1, \cdots, g\}$ and $\epsilon \in \{1, -1\}$,

$$D^-_{\gamma_i^\epsilon} := \gamma_i^\epsilon (\mathbb{P}^{1,an}_k - D^+_{\gamma_i^{-\epsilon}})$$

is the maximal open disc inside $D^+_{\gamma_i^\epsilon}$.

The following theorem is due to Gerritzen and is stated in the notation of Poineau–Turchetti ([33, Theorem 3.5.9]).

Theorem 2.12 (Gerritzen). *Let Γ be a Schottky group of $\mathrm{PGL}_2(k)$. Then there exists a set of free generators $\{\gamma_1, \cdots, \gamma_g\}$ of Γ and $\lambda_i \in (0, 1)$ for $i = 1, \cdots, g$ such that the collection of twisted Ford discs*

$$\mathcal{B} = \{D^+_{\gamma_i^{-1}, \lambda_i^{-1}}, D^+_{\gamma_i, \lambda_i} : i = 1, \cdots, g\}$$

is a Schottky figure adapted to $\{\gamma_1, \cdots, \gamma_g\}$.

Remark 2.13. There is no analog of this theorem in the Archimedean setting: indeed, it is known that some Kleinian Schottky groups are not classical, that is, they cannot be described in terms of Schottky figures involving only round discs.

Remark 2.14. A meromorphic family $(\Gamma_z)_{z \in \mathbb{D}^*}$ of Kleinian Schottky groups can be seen as a subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}((t)))$. In this context, as shown in [33, Proposition 4.4.2], condition ($\bigstar$) is *equivalent* to Γ being a Schottky subgroup of $\mathrm{PGL}_2(\mathbb{C}((t)))$ (with $\mathbb{C}((t))$ equipped with the non-Archimedean norm $|\sum_{j \geq m} a_j t^j|_{na} = e^{-m}$ for $a_m \neq 0$).

Mumford curve and its skeleton

For an analog of periodic geodesics in the non-Archimedean case, we need to use the description of the limit set in $\mathbb{P}^{1,an}_k$. The advantage is that $\mathbb{P}^{1,an}_k$ is a compact real tree, while $\mathbb{P}^1_k$ is discrete.

Let Γ be a Schottky group of rank g in $\mathrm{PGL}_2(k)$. As in the special case where $k = \mathbb{C}$, a point $x \in \mathbb{P}^{1,an}_k$ is a *limit point* if there exist $x_o \in \mathbb{P}^{1,an}_k$ and a sequence $\{\gamma_n\}_n \subset \Gamma$ of distinct elements such that $\lim_{n\to\infty} \gamma_n x_0 = x$. Let $L(\Gamma)$ be the limit set of Γ on $\mathbb{P}^{1,an}_k$, that is, the set of limit points. We have that $L(\Gamma) \subset \mathbb{P}^1_k$ [32, Corollary II.3.14].

Moreover, let $O = \mathbb{P}^{1,an}_k - L(\Gamma)$. It is shown that O is a domain of discontinuity of Γ, that is, it is Γ-invariant, and the action of Γ on O is free and proper. The quotient space $X = \Gamma\backslash O$ is a Mumford curve. (See [33, Theorem II.3.18].) Let $\Sigma = O \cap \cup_{x,y \in L}[x, y]$, where $\mathbb{P}^{1,an}_k$ has the structure of a real tree and $[x, y]$ is the unique injective path from x to y. See [33, Section 2.5] and [32, Proposition I.6.12].[3] Let $\Sigma_X = \Gamma\backslash\Sigma$, which is called the skeleton of the Mumford curve and is a finite graph of genus g. (See [33, Notation 4.2.2].) The skeleton Σ_X can also be obtained by identifying the Shilov boundary points p_i, q_i of $D^+_{\gamma_i}, D^+_{\gamma_i^{-1}}$, which is a Schottky figure adapted to $\{\gamma_1, \cdots, \gamma_g\}$.

Remark 2.15. This skeleton Σ_X is the analog of the convex core of a hyperbolic manifold for the Archimedean case. This can be seen from the fact that $\mathbb{P}^{1,an}_k$ contains the Bruhat–Tits building (Bruhat–Tits tree in the $\mathbb{P}^1_k$ case), which is the analog of the simply connected hyperbolic manifold in the Archimedean case.

Metric structure

Recall from [32, Section I.8] and [33, Section 2.5] the definition of multiplicative length. For $\alpha, \beta \in k$ and $0 < r \leq s$, we define the *additive* length of the elements $\eta_{\alpha,r}, \eta_{\beta,s}$ by:

[3]The path is the union of two intervals given by $\{\eta_{x,r},\ r \in [0, |x - y|_k]\}$ and $\{\eta_{y,r},\ r \in [0, |x - y|_k]\}$ with $\eta_{x,|x-y|_k} = \eta_{y,|x-y|_k}$.

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- if $\max\{r,s\} \geq |\alpha-\beta|_k$, $d_a(\eta_{\alpha,r},\eta_{\beta,s}) := \log(s/r)$;
- if $\max\{r,s\} < |\alpha-\beta|_k$, $d_a(\eta_{\alpha,r},\eta_{\beta,s}) := \log(|\alpha-\beta|_k/r) + \log(|\alpha-\beta|_k/s)$.

This length is invariant under the action of Möbius transformations. The tree Σ consists of points of $\eta_{a,r}$ with $a \in k$ and $r \in \mathbb{R}_{>0}$.[4] Using this length, we can define a length on Σ. The length of an edge of the graph $\Sigma_X = \Gamma\backslash\Sigma$ comes from the quotient of this additive length.

Definition 2.16. For a loxodromic element γ, we define the non-Archimedean length by

$$\ell^{na}(\gamma) = \log(|\mathrm{tr}\tilde{\gamma}|_k^2/|\det(\tilde{\gamma})|_k),$$

where $\tilde{\gamma}$ is a representative of γ in $\mathrm{GL}_2(k)$.

This definition also coincides with the logarithm of the attraction rate at the attracting fixed point of the loxodromic element. Hence, it serves as a suitable analog of the length of periodic geodesics in the non-Archimedean case.

The following lemma connects the length of loops in the graph with the non-Archimedean length of loxodromic elements in Γ.

Lemma 2.17. *Let γ be a nontrivial loxodromic element in Γ and $\alpha,\alpha' \in \mathbb{P}^1_k$ be its two fixed points. Then for any $x \in [\alpha,\alpha'] - \{\alpha,\alpha'\}$, we have*

$$d_a(x,\gamma x) = \ell^{na}(\gamma),$$

where $[x,\gamma x] \subset [\alpha,\alpha'] - \{\alpha,\alpha'\} \subset \Sigma$.

For any $y = \eta_{\beta,s} \notin [\alpha,\alpha']$ with $0 < s < \infty$, we have

$$d_a(y,\gamma y) > \ell^{na}(\gamma).$$

Proof. Due to invariance under conjugation, it is sufficient to consider $\gamma = \mathrm{diag}\{\lambda,1\}$ with $|\lambda|_k > 1$. Then the line $[\alpha,\alpha'] = \{\eta_{0,r}, 0 \leq r \leq \infty\}$. We have $d_a(x,\gamma x) = d_a(\eta_{0,r},\eta_{0,|\lambda|_k r}) = \log|\lambda|_k = \ell^{na}(\gamma)$.

The second statement follows from the fact that $\mathbb{P}^{1,an}_k$ is a tree and γ preserves the distance d_a. □

Ihara zeta function

The non-Archimedean case of the Selberg zeta function is well studied. The analogous zeta function is called the Ihara zeta function for finite graphs. See [17] and [16].

Suppose $G = (V,E)$ is a finite graph. Let $E = \{e_1,\cdots,e_{2J}\}$ be the set of oriented edges of G with e_j and e_{j+J} in opposite directions. Each edge has a positive length. A loop P in G is a finite sequence $P = (e_{i_1},\cdots,e_{i_n})$ with the endpoint of e_{i_j} equal to the starting point of $e_{i_{j+1}}$ (we use the convention $e_{i_{n+1}} = e_{i_1}$). The length $\ell(P)$ of a path P is the sum of the lengths of the e_{i_j}. A path is nonbacktracking if there are no consecutive edges that are inverse to each other; that is, there is no j such that $e_{i_{j+1}} = e_{i_j+J}$. A loop is primitive if it cannot be written as a multiple of another loop. Two loops that differ only by starting point are defined to be in the same equivalence class of loops, denoted by $[P]$. Let $\mathcal{P}$ be the set of equivalence classes of primitive nonbacktracking loops.

Theorem 2.18 (Ihara, Hashimoto, Bass). *Let G be a finite graph. Define the Ihara zeta function by*

$$Z_I(G,s) = \prod_{[P]\in\mathcal{P}} (1 - e^{-s\ell(P)}).$$

Then the Ihara zeta function extends analytically to the whole complex plane.

See Proposition 5.1 for a proof and a way to compute the Ihara zeta function.

[4] Points like $\eta_{x,0} = x$ are not in $\Sigma = (\mathbb{P}^{1,an}_k - L) \cap \cup_{x,y\in L}[x,y]$.

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Let Γ be a Schottky group in $\mathrm{PGL}_2(k)$. Recall that the skeleton of the Mumford curve associated with Γ is the quotient space $\Sigma_X = \Gamma\backslash\Sigma$. Since Σ is a tree, which is simply connected, each nonbacktracking loop P of Σ_X corresponds to a conjugacy class $[\gamma_P]$ in Γ; this follows from the standard correspondence between free homotopy classes of loops and conjugacy classes in the fundamental group. The length of each edge in Σ_X is given by the length of its lift in Σ. Moreover, from Lemma 2.17, we obtain that for the nonbacktracking loop P, we have $\ell(P) = \ell^{na}(\gamma_P)$. Therefore, we obtain the following.

Proposition 2.19. *Let $Z_I(\Gamma, s)$ be the Ihara zeta function for $\Gamma < PGL_2(k)$,*[5] *that is,*

$$Z_I(\Gamma, s) = \prod_{[\gamma]\in\mathcal{P}} (1 - e^{-s\ell^{na}(\gamma)}).$$

Then we have

$$Z_I(\Gamma, s) = Z_I(\Sigma_X, s). \tag{2.1}$$

Remark 2.20. If all the edges of G are of length 1, then

$$Z_I(s) = (1 - e^{-2s})^{g-1} \det(1 - A_G e^{-s} + Q_G e^{-2s}),$$

where A is the adjacent matrix of the graph G, g is the rank of the fundamental group of G, and Q_G is a diagonal matrix with the i-th element equal to the degree minus 1 of the i-th vertex.

Therefore, if $|k^*| = a^{\mathbb{Q}}$ for some $a \in (0, 1)$ (in particular, in the case of the algebraic closure of $\mathbb{C}((t))$), then the edge lengths of Σ_X are rational multiples of $|\log a|$. By rescaling the metric, we obtain a graph with integer edge lengths. Subdividing these edges with additional vertices yields a graph where every edge has unit length. Consequently, we can apply this formula to compute the Ihara zeta function.

2.6. Hybrid space

After Poineau–Turchetti [33], one can build a Schottky space S^{an}_{g,A_r} over a Banach ring A_r.

Set $r = 1/e$. The Banach ring A_r can be described as follows. We say that $f = \sum a_n t^n \in \mathbb{C}((t))$ belongs to A_r if its *hybrid norm*

$$\|f\|_{\mathrm{hyb}} := \sum \max\{|a_n|_\infty, |a_n|_0\}(1/e)^n \tag{2.2}$$

is finite. (Here, $|.|_0$ is the trivial norm.) Due to $\max\{|a_n|_\infty, |a_n|_0\} \geq |a_n|_\infty$, the elements $f \in A_r$ can be realized as meromorphic functions on $\mathbb{D}_r$ with a possible pole at 0.

Definition 2.21. The Berkovich affine space $\mathbb{A}^{3g-3,an}_{A_r}$ consists of elements x which are bounded multiplicative seminorms $|\cdot|_x$ on the ring $R = A_r[\alpha_3, \ldots, \alpha_g, \beta_2, \ldots, \beta_g, \lambda_1, \ldots, \lambda_g]$, that is, $|\cdot|_x$ satisfies

- $|0|_x = 0, |1|_x = 1$;
- $|P + Q|_x \leq |P|_x + |Q|_x$ and $|P \cdot Q|_x = |P|_x|Q|_x$ for all $P, Q \in R$;
- $|f|_x \leq \|f\|_{\mathrm{hyb}}$ for all $f \in A_r$.

The topology is the weakest topology such that for all elements $P \in R$, the map from $\mathbb{A}^{3g-3,an}_{A_r}$ to $\mathbb{R}$ given by $|P|_x$ is continuous.

When $n = 0$, we denote by $\mathcal{M}(A_r)$ the space $\mathbb{A}^{0,an}_{A_r}$, and call it the *Berkovich spectrum* of A_r. It is known that $\mathcal{M}(A_r)$ is naturally homeomorphic to the closed disc $\overline{\mathbb{D}}_r$ and, by restricting the seminorms to the base ring, one obtains a continuous morphism $\Pi : \mathbb{A}^{3g-3,an}_{A_r} \to \mathcal{M}(A_r)$.

[5] This function is also called the Ruelle zeta function.

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For each $x \in \mathbb{A}^{3g-3,an}_{A_r}$, we denote by $\mathcal{H}(x)$ the completion of the fraction field of $R/\mathrm{Ker}|\cdot|_x$. We call x *Archimedean* if $\mathcal{H}(x)$ is Archimedean, and *non-Archimedean* otherwise.

Continuous map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$

Let $p : \mathbb{D} \to \mathbb{C}^{3g-3}$, $z \mapsto p(z) = (\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))$, be a continuous function in $\overline{\mathbb{D}}^*_r$, holomorphic in $\mathbb{D}^*_r$, and meromorphic at 0. By abusing notation, we construct a map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$ as follows: for any $F = \sum f_n(t) P_n(\underline{\alpha}, \underline{\beta}, \underline{\lambda}) \in A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$, we set

$$|F|_{p(z)} = \begin{cases} \left|\sum f_n(z) P_n(\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))\right|_\infty^{1/\log(1/|z|)}, & \text{if } z \neq 0 \\ \exp(-\mathrm{ord}_{z=0}(\sum f_n(z) P_n(\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z)))), & \text{if } z = 0, \end{cases} \tag{2.3}$$

where $P_n(\underline{\alpha}, \underline{\beta}, \underline{\lambda})$ is a monomial of total degree n. The key advantage of this construction is that, from [4, Proposition A.4], the map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$ is continuous. In the beginning, we deal with meromorphic functions. Using this hybrid construction, we do not have a singularity at $z = 0$ and give a sense of the limiting behavior at $z = 0$.

For $z \in \overline{\mathbb{D}}^*_r$, we have $\mathcal{H}(|\cdot|_{p(z)}) = (\mathbb{C}, |\cdot|_\infty^{1/\log(1/|z|)})$, and therefore $p(z)$ is an Archimedean point. For $z = 0$, we have $\mathcal{H}(|\cdot|_{p(0)}) = (\mathbb{C}((t)), |\cdot|_{na})$, and therefore $p(0)$ is a non-Archimedean point.

Now, we build projective spaces where the Schottky groups attached to the points in the Berkovich affine space act in a natural way. Given $x \in \mathbb{A}^{3g-3,an}_{A_r}$, let

$$\mathbb{P}^{1,an}_{\mathcal{H}(x)} \simeq \mathbb{P}^1(\mathbb{C})$$

if the residue field $\mathcal{H}(x)$ is Archimedean, and $\mathbb{P}^{1,an}_{\mathcal{H}(x)} \simeq$ a tree otherwise. In this context, a point $x \in \mathbb{A}^{3g-3,an}_{A_r}$ belongs to the Schottky space S^{an}_{g,A_r} if $0 < |\lambda_j|_x < 1$ for all $1 \leq j \leq g$, the points $0, 1, 2, \alpha_3, \ldots, \alpha_g, \beta_2, \ldots, \beta_g$ are mutually distinct, and the matrices $\Psi(x) \subset \mathrm{PGL}_2(\mathcal{H}(x))$ associated with x induce a Schottky group over $\mathbb{P}^{1,an}_{\mathcal{H}(x)}$. Interestingly enough, Poineau and Turchetti proved an analog of Hejhal's theorem by establishing that S^{an}_{g,A_r} is an open path-connected subset of $\mathbb{A}^{3g-3,an}_{A_r}$.

For the family of Schottky groups satisfying condition (★), due to Remark 2.14, we know $p(0) \in S^{an}_{g,A_r}$. Due to the openness of S^{an}_{g,A_r}, we recover [9, Proposition 6.1].

Proposition 2.22. *For the family of Schottky groups satisfying condition (★), there exists $0 < \eta \leq r$ such that the continuous map p restricted to $\mathbb{D}_\eta$ has its image in S^{an}_{g,A_r}.*

Uniform Schottky basis

For an element $\gamma \in \mathrm{PGL}_2(\mathbb{C}((t)))$, whose entries are meromorphic on $\mathbb{D}^*_r$ with a possible pole at 0, we define a family of twisted Ford discs:

$$D^+_{\gamma,\lambda}(z) = \begin{cases} \{x \in \mathbb{C} \mid |c(z)x + d(z)|_\infty^{2/\log(1/|z|)} \leq \lambda |(ad - bc)(z)|_\infty^{1/\log(1/|z|)}\} \text{ if } z \neq 0, \\ \{x \in \mathbb{A}^{1,an}_{\mathbb{C}((t))} \mid |(cZ + d)(x)|^2 \leq \lambda |(ad - bc)(x)|\} \text{ if } z = 0, \end{cases} \tag{2.4}$$

where $c(z)$ means the complex value of the meromorphic function c at t.

We state a result on uniform Schottky figures for all small parameters. This is a translation of this special case of [32, Theorem 4.3.2 and Corollary 4.3.3].

Lemma 2.23. *Let $\Gamma = \langle M_1, \cdots, M_g \rangle$ be a family of Schottky groups satisfying condition (★). There exist $s \in (0, r)$ and an automorphism $\tau \in Aut(F_g)$ such that the following holds. Let*

$$N_1 = \tau M_1, \cdots, N_g = \tau M_g \in \mathrm{PGL}_2(\mathbb{C}((t))).$$

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There exist positive real numbers $\lambda_1, \cdots, \lambda_g$ such that the family of twisted Ford discs

$$(D^+_{N_1,\lambda_1}(z), \cdots, D^+_{N_g,\lambda_g}(z), D^+_{N_1^{-1},\lambda_1^{-1}}(z), \cdots, D^+_{N_g^{-1},\lambda_g^{-1}}(z))$$

is a Schottky figure adapted to the basis $(N_1, \cdots, N_g)$ for each $z \in \mathbb{C}$ with $|z| < s$.

Proof. We explain the language used in [33] and how to obtain the lemma.

Recall that $S = S^{an}_{g,A_r}$ is the Schottky space. Let $\mathcal{O}(S)$ be the ring of analytic functions, which means that locally the function can be uniformly approximated by rational functions.[6] Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}(S))$ be a loxodromic element with $c \neq 0$ and $0 < \lambda$. Let $\mathbb{A}^{1,an}_S$ be the affine line over S with variable Z. In [33], relative twisted Ford discs are defined by

$$D^+_{\gamma,\lambda} = \{x \in \mathbb{A}^{1,an}_S | \ |(cZ+d)(x)|^2 \leq \lambda |(ad-bc)(x)|\}.$$

We will prove that this gives the same family of twisted Ford discs as in equation (2.4).

The projection map π from $\mathbb{A}^{1,an}_S$ to S is the restriction of the seminorm to the ring $A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$. For any $y \in S$, the fiber $\pi^{-1}(y)$ is the affine line $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$.[7] Relative Ford discs are the union of twisted Ford discs on these fibers.

For the continuous map p, we know $p(0)$ is a non-Archimedean point in S. Then we can apply [32, Theorem 4.3.2 and Corollary 4.3.3] to find relative twisted Ford discs on an open neighborhood U of $p(0)$ in S, which is a Schottky figure adapted to a chosen basis. The new basis is denoted by $N_1 = \tau M_1, \cdots, N_g = \tau M_g$ for some $\tau \in Aut(F_g)$ and with parameters $\lambda_1, \cdots, \lambda_g$. Moreover, for any $y \in U$, the restriction of the relative twisted Ford discs to the fiber $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$ gives a Schottky figure on that fiber. Then we only need to prove that this gives the twisted Ford discs in the statement of the lemma.

For $\mathcal{H}(y)$ Archimedean, the norm $(\mathcal{H}(y), |\cdot|_y)$ is always given by $(\mathbb{C}, |\cdot|_\infty^{\epsilon(y)})$ with some $\epsilon(y) \leq 1$. Due to our choice of $p(z)$, we know that $\epsilon(p(z)) = 1/\log(1/|z|)$ for $z \neq 0$. Over $(\mathbb{C}, |\cdot|_\infty^{\epsilon})$, the analytification gives no new point. Hence, a seminorm $|\cdot|_x \in \mathbb{A}^{1,an}_{\mathcal{H}(p(z))} = \mathbb{A}^{1,an}_{\mathbb{C},|\cdot|_\infty^{\epsilon}}$ reads as

$$|P(Z)(x)| = \left|(\sum a_n Z^n)(x)\right| = \left|\sum a_n Z(x)^n\right|_\infty^{1/\log(1/|z|)},$$

for some unique $Z(x) \in \mathbb{C}$ determined by x and for a polynomial $P \in \mathbb{C}[Z]$. Due to the openness of U and the continuity of the map p, we can find $0 < s < r$ such that $p(\mathbb{D}_s)$ is contained in U. Therefore, the twisted Ford discs at the fiber over $p(z)$ with $z \neq 0$ and $|z| < s$ read as

$$\begin{aligned} D^+_{\gamma,\lambda}(z) &= \{x \in \mathbb{A}^{1,an}_{\mathcal{H}(p(z))} | \ |(cZ+d)(x)|^2 \leq \lambda |(ad-bc)(x)|\} \\ &= \{x \in \mathbb{C} | \ |c(z)x + d(z)|_\infty^{2/\log(1/|z|)} \leq \lambda |(ad-bc)(z)|_\infty^{1/\log(1/|z|)}\}. \end{aligned}$$

The proof is complete. □

We define the discs $D_{\mathfrak{a},z}$ to be the Ford discs

$$D_{\mathfrak{a},z} := D^-_{\mathfrak{a}^{-1},\lambda_{\mathfrak{a}^{-1}}}(z) \tag{2.5}$$

[6] We will only use meromorphic functions on $\mathbb{D}$, so we do not provide a detailed definition.

[7] The fiber $\pi^{-1}(y)$ is the set of seminorms on $R[Z]$ whose restriction to $R = A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$ coincides with the seminorm y. Recall that $\mathcal{H}(y)$ is the completion of the fraction field of $R/\ker(y)$. Hence, this set of seminorms, $\pi^{-1}(y)$, is isomorphic to the set of multiplicative seminorms on $\mathcal{H}(y)[Z]$ whose restriction to $\mathcal{H}(y)$ coincides with y, which is exactly the affine line $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$.

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for any generator $\mathfrak{a}$ in the generator set $\{N_1, \cdots, N_g, N_1^{-1}, \cdots, N_g^{-1}\}$, where the inverse is used because we want the attracting fixed point $\mathfrak{a}^+$ of $\mathfrak{a}$ to satisfy $\mathfrak{a}^+ \in D_{\mathfrak{a}}$. For any element $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \Gamma$, we define

$$D_{\gamma,z} = \mathfrak{a}_{i_1,z} \cdots \mathfrak{a}_{i_{n-1},z} D_{\mathfrak{a}_{i_n},z}.$$

We can further suppose the Schottky basis is uniformly bounded for $|z| \ll 1$. Indeed, we may assume that the limit set and the Schottky figure of Γ are in the unit ball $D(0,1)$ of the Berkovich affine line $\mathbb{A}_k^{1,an}$, which can be realized by suitably conjugating the Schottky group. Then, for small $|z|$, the Schottky figures of Γ_z are in a uniform compact set in $\mathbb{C}$ due to the centers of Schottky discs being meromorphic functions and their radii being functions of the form $\lambda^{\log(1/|z|)}/|c(z)|$. Here, λ and $c(z)$ are determined by the disc $D^-_{\mathfrak{a}^{-1},\lambda_{\mathfrak{a}^{-1}}}(z)$ corresponding to $\mathfrak{a}_z = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$ and $\lambda_{\mathfrak{a}^{-1}} = \lambda$.

3. Asymptotic convergence of zeta function

Let $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ be a family of Schottky groups satisfying (★).

3.1. Convergence on the half-plane: Euler product

Theorem 3.1. *For $\Re s \gg 1$, we have that*

$$\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_I(\Gamma, s)} \to 1 \quad \text{as } z \to 0.$$

The following lemma gives the convergence of the length of each conjugacy class.

Lemma 3.2. *Let $[\gamma]$ be a nontrivial conjugacy class in Γ. Then*

$$\lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{\ell^{na}(\gamma)} = 1.$$

Proof. Using the definition of non-Archimedean length (Definition 2.16) and the assumption that $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$, we have

$$|\mathrm{tr}(\gamma)|_{na} = e^{\ell^{na}(\gamma)/2}.$$

Every nontrivial element γ in Γ is loxodromic in $\mathrm{SL}_2(\mathbb{C}((t)))$, and hence $\ell^{na}(\gamma) > 0$. By the definition of the hybrid norm (2.2), $\mathrm{tr}(\gamma_z)$ is meromorphic, so that $\lim_{z\to 0} |\mathrm{tr}(\gamma_z)|^{1/\log(1/|z|)} = |\mathrm{tr}(\gamma)|_{na}$. Additionally, the assumption that Γ satisfies (★) implies that $\ell(\gamma_z) \to \infty$ as $z \to 0$, which follows from the standard relationship between contraction rates and translation lengths combined with the coarse additivity of the latter. We obtain

$$|\mathrm{tr}(\gamma_z)|/e^{\ell(\gamma_z)/2} = (e^{\ell(\gamma_z)/2} + e^{-\ell(\gamma_z)/2})/e^{\ell(\gamma_z)/2} \to 1.$$

We obtain

$$\lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{\ell^{na}(\gamma)} = \lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{2\log|\mathrm{tr}(\gamma_z)|/\log(1/|z|)} \frac{\log|\mathrm{tr}(\gamma_z)|/\log(1/|z|)}{\log|\mathrm{tr}(\gamma)|_{na}} \frac{2\log|\mathrm{tr}(\gamma)|_{na}}{\ell^{na}(\gamma)} = 1. \qquad \square$$

We also need a uniform bound independent of z for the lengths of geodesics. We first recall the uniform distortion estimates from Lemma 5.6 and Corollary 5.9 in [9]. Meanwhile, Proposition 2.22 and Lemma 2.23 state that there exists $\eta \in (0, r)$ such that the map $p : \mathbb{D}_\eta \to \mathcal{S}^{an}_{g,A_r}$ is continuous,

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and $\{D_{\mathfrak{a},z}\}$ is a uniform Schottky basis for any $z \in \mathbb{D}_\eta$. Hence, we can apply the results to obtain the following.

Lemma 3.3. *There exist $\eta_0 \in (0,r)$ and constants $R>0$, $c \in (0,1)$ such that for any $|z| \le \eta_0$ and any nontrivial word γ, the radius $r_{\gamma,z}$ of $D_{\gamma,z}$ satisfies*[8]

$$r_{\gamma,z} \le Rc^{l(\gamma)-1}. \tag{3.1}$$

Moreover, there exists a constant $N>0$ such that for any $\gamma = \mathfrak{a}\mathbf{b}$ with $\mathfrak{a}\mathbf{b}$ reduced, and $l(\mathfrak{a})=1$ and $l(\mathbf{b})>N$,

$$\frac{\sup_{x\in D_{\mathbf{b},z}} |\mathfrak{a}'(x)|_z}{\inf_{x\in D_{\mathbf{b},z}} |\mathfrak{a}'(x)|_z} \le \exp\left(\frac{1}{4} r_{\mathbf{b},z}\right). \tag{3.2}$$

Here, the radius and the derivative are computed with respect to the norm $|\cdot|_z$. More precisely, for $z\neq 0$, $|\cdot|_z$ is a norm on $\mathbb{C}$ given by $|\cdot|_z := |\cdot|_\infty^{1/\log(1/|z|)}$, and for $z=0$, $|\cdot|_z$ is a norm on $\mathbb{C}((t))$ defined by $|\cdot|_z = |\cdot|_{na}$.

Lemma 3.4. *Let $\eta_0 \in (0,r)$, $R>0$, $c\in(0,1)$ be the constants given in Lemma 3.3. Then there exists $C>0$ such that for any word $\gamma = \mathbf{a}\mathfrak{b}$ with $\mathbf{a}\mathfrak{b}$ reduced and $l(\mathfrak{b})=1$ and $\mathbf{a}$ of arbitrary length, we have*

$$\frac{\sup_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z}{\inf_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z} \le C\exp(R/(1-c)) \quad \textit{for any} \quad z\in\mathbb{D}_{\eta_0}.$$

Proof. Assume $l(\mathbf{a})=n$ and write $\mathbf{a} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$. Recall that $\mathfrak{a}_{i_j}\cdots\mathfrak{a}_{i_n} D_{\mathfrak{b},z} = D_{\mathfrak{a}_{i_j}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}$ for any $z\in\mathbb{D}_{\eta_0}$. We estimate the distortion of $\mathbf{a}$:

$$\begin{aligned}\frac{\sup_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z}{\inf_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z} &\le \frac{\sup_{x\in D_{\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}} |\mathfrak{a}_{i_1}'(x)|_z}{\inf_{x\in D_{\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}} |\mathfrak{a}_{i_1}'(x)|_z} \cdot \frac{\sup_{x\in D_{\mathfrak{a}_{i_3}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}} |\mathfrak{a}_{i_2}'(x)|_z}{\inf_{x\in D_{\mathfrak{a}_{i_3}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}} |\mathfrak{a}_{i_2}'(x)|_z} \cdots \frac{\sup_{x\in D_{\mathfrak{b},z}} |\mathfrak{a}_{i_n}'(x)|_z}{\inf_{x\in D_{\mathfrak{b},z}} |\mathfrak{a}_{i_n}'(x)|_z} \\ &\lesssim \exp\left(\sum_{j<n-N} \frac{1}{4} r_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\mathfrak{b},z}\right) \lesssim \exp\left(\frac{1}{4}R/(1-c)\right).\end{aligned}$$

Here, to obtain the second and third inequalities, for j satisfying $n-j>N$, we estimate the distortion of $\mathfrak{a}_{i_j}$ by applying equation (3.2) and then equation (3.1); and for j satisfying $n-j\le N$, we bound the distortion of $\mathfrak{a}_j$ by a constant that depends only on Γ and N. □

Lemma 3.5. *There exists $c\in(0,1)$, depending on the Schottky family, such that for any conjugacy class $[\gamma]$ with $\ell^{na}(\gamma) > 1/c$ and $|z|<c$, we have*

$$\ell(\gamma_z) \ge c\log(1/|z|)\ell^{na}(\gamma).$$

Proof. Let $l(\gamma)$ be the word length of γ with the fixed generating set. Choose a γ in the conjugacy class that is cyclically reduced, meaning that the first and last letters of the word γ are not inverse to each other.

Note that γ has a unique attracting fixed point, say, in the disc $D_{\mathfrak{a}_{i_1},z}$. Using $\gamma D_{\mathfrak{a}_{i_1},z} = D_{\gamma\mathfrak{a}_{i_1},z}$, combined with Lemma 3.4, we obtain

$$-\log|\gamma'(\gamma\text{-attracting fixed point})|_z \ge \left|\log\left(\frac{r_{\gamma\mathfrak{a}_{i_1},z}}{r_{\mathfrak{a}_{i_1},z}}\right)\right| - \frac{R}{(1-c)}. \tag{3.3}$$

[8] Recall that $l(\gamma)$ stands for the word length: cf. Subsection 1.4.

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As we require γ to be cyclically reduced, we have $l(\gamma\mathfrak{a}_{i_1}) = l(\gamma) + 1$. Hence, by equation (3.1),

$$\begin{aligned}\ell(\gamma_z) &= -\log(1/|z|)\log|\gamma'(\gamma\text{-attracting fixed point})|_z \geq \log(1/|z|)\left|\left|\log\left(\frac{r_{\gamma\mathfrak{a}_{i_1},z}}{r_{\mathfrak{a}_{i_1},z}}\right)\right| - \frac{R}{(1-c)}\right| \\ &\geq \log(1/|z|)\left(l(\gamma)|\log c| - |\log r_{\mathfrak{a}_{i_1},z}| - \log R - \frac{R}{(1-c)}\right) \gtrsim \log(1/|z|)l(\gamma),\end{aligned}$$

where the last inequality is true if $l(\gamma)$ is large enough compared to these constants. For the finitely many γ with $l(\gamma)$ small, we use Lemma 3.2 to get a uniform bound.

Lemma 3.6. *Let γ be a cyclically reduced word. For the word length $l(\gamma)$ and the non-Archimedean length $\ell^{na}(\gamma)$, they satisfy $\ell^{na}(\gamma) \lesssim l(\gamma)$, where the implicit constant is independent of γ.*

Proof. Since $\ell^{na}(\gamma)$ is the least translation length for the type of points $\eta_{\alpha,r}$ (Lemma 2.17), we have for the Gauss point $o = \eta_{0,1}$,

$$\begin{aligned}\ell^{na}(\gamma) &\leq d_a(o,\gamma o) \leq d_a(o,\mathfrak{a}_{i_1}o) + d_a(\mathfrak{a}_{i_1}o,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}o) + \cdots + d_a(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n-1}}o,\gamma o) \\ &\leq \sum_j d_a(o,\mathfrak{a}_{i_j}o) \lesssim l(\gamma).\end{aligned}$$

□

The proof is completed by combining the above lemma. □

Since equation (3.3) also works for the non-Archimedean point $p(0)$, the same proof implies the following.

Lemma 3.7. *Let Γ be a non-Archimedean Schottky group. For any cyclically reduced word γ, we have*

$$\ell^{na}(\gamma) \gtrsim l(\gamma).$$

Proof of Theorem 3.1. Suppose $s \in \mathbb{C}$ with $\Re s \gg 1$. We have

$$\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_I(\Gamma,s)} = \prod_{[\gamma]\in\mathcal{P}} \frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}} \cdot \mathcal{R}(\gamma), \tag{3.4}$$

where

$$\mathcal{R}(\gamma) = \begin{cases} \prod_{k\geq 1}(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)}) & \text{if } \Gamma_z < \mathrm{SL}_2(\mathbb{R}), \\ \prod_{k_1+k_2\geq 1}(1-e^{-(s/\log(1/|z|)+k_1+k_2)\ell(\gamma_z)}e^{-i\theta_\gamma(k_1+k_2)}) & \text{if } \Gamma_z < \mathrm{SL}_2(\mathbb{C}). \end{cases}$$

We will use the inequality

$$1/(1-x) \leq \exp(2x) \ \text{ for } \ 0 \leq x \leq 1/2. \tag{3.5}$$

We estimate $|\mathcal{R}(\gamma)|^{-1}$. Using Lemma 3.5, we have for $|z|$ small, if $\Gamma_z < \mathrm{SL}_2(\mathbb{R})$, then

$$\begin{aligned}|\mathcal{R}(\gamma)|^{-1} &= \prod_{k\geq 1}|1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)}|^{-1} \leq \exp\left(2\sum_{k\geq 1}e^{-kc\log(1/|z|)\ell^{na}(\gamma)}\right) \\ &= \exp\left(\frac{2e^{-c\log(1/|z|)\ell^{na}(\gamma)}}{1-e^{-c\log(1/|z|)\ell^{na}(\gamma)}}\right).\end{aligned}$$

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If $\Gamma_z < \mathrm{SL}_2(\mathbb{C})$, since $\#\{(k_1,k_2)\in\mathbb{N}^2 : k_1+k_2=k\} = k+1$, we have

$$|\mathcal{R}(\gamma)|^{-1} \le \prod_{k\ge 1} |1-e^{-(\Re s/\log(1/|z|)+k)\ell(\gamma_z)}|^{-(k+1)} \le \exp\left(2\sum_{k\ge 1}(k+1)e^{-kc\log(1/|z|)\ell^{na}(\gamma)}\right)$$
$$= \exp\left(\frac{2}{(1-e^{-c\log(1/|z|)\ell^{na}(\gamma)})^2} - 2\right) \le \exp\left(\frac{4e^{-c\log(1/|z|)\ell^{na}(\gamma)}}{(1-e^{-c\log(1/|z|)\ell^{na}(\gamma)})^2}\right).$$

Combining these two cases, we have for $|z|$ small,

$$\prod_{[\gamma]\in\mathcal{P}} |\mathcal{R}(\gamma)|^{-1} \le \exp\left(8\sum_{[\gamma]\in\mathcal{P}} e^{-c\log(1/|z|)\ell^{na}(\gamma)}\right). \tag{3.6}$$

We first sum over $\{[\gamma]\in\mathcal{P} : \ell^{na}(\gamma)=n\}$ and then sum over $n\in\mathbb{N}$. Due to Lemma 3.7, there exists a constant C depending only on the family such that

$$\#\{[\gamma]\in\mathcal{P} : \ell^{na}(\gamma)=n\} \le \#\{\gamma \text{ cyclically reduced} : l(\gamma)\le C'n\} \le e^{Cn}. \tag{3.7}$$

Hence,

$$Eq.\ (3.6) \le \exp\left(\frac{8e^{-(c\log(1/|z|)-C)}}{1-e^{-(c\log(1/|z|)-C)}}\right),$$

which converges to 1 as $|z|$ tends to zero.

Now, we consider the other factor in equation (3.4). Take $N \gg 1$. We separate into two parts: $\ell^{na}(\gamma)\le N$ and $\ell^{na}(\gamma) > N$. For the first part, which is a product of finite terms, due to Lemma 3.2,

$$\prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)\le N}} \frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}} \to 1.$$

For the second part, due to Lemma 3.5, we have

$$\left|\log \prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}} \left|\frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}}\right|\right| \le 2\sum_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}} e^{-\Re s/\log(1/|z|)\ell(\gamma_z)} + e^{-\Re s\ell^{na}(\gamma)}$$
$$\le 2\sum_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}} e^{-\Re sc\ell^{na}(\gamma)} + e^{-\Re s\ell^{na}(\gamma)} \le 2\sum_{n>N} e^{Cn}(e^{-n\Re sc} + e^{-\Re sn}) = \frac{4e^{-N(\Re sc-C)}}{1-e^{-N(\Re sc-C)}},$$

where the first inequality is due to equation (3.5) and the last inequality is due to equation (3.7). Therefore, given $\Re s \gg 1$, we have that for N sufficiently large, the product

$$\prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}} \frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}}$$

is close to 1. The proof is complete. □

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3.2. Boundedness of zeta function through transfer operator

3.2.1. Preliminaries on trace-class operators

We recall some preliminaries on trace-class operators. See [11, Appendix B] for more details.

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator between two Hilbert spaces. The singular values μ_ℓ of $\mathcal{A}$ are defined to be the eigenvalues λ_ℓ of $(\mathcal{A}^*\mathcal{A})^{1/2}$ in decreasing order, that is,

$$\mu_\ell(\mathcal{A}) := \lambda_\ell((\mathcal{A}^*\mathcal{A})^{1/2}), \quad \ell = 0, 1, 2, \cdots.$$

We recall the minimax property of singular values:

$$\mu_\ell(\mathcal{A}) = \min_{\operatorname{codim}(V,\mathcal{H}_1)=\ell} \max_{v\in V\setminus\{0\}} \frac{\|\mathcal{A}v\|_{\mathcal{H}_2}}{\|v\|_{\mathcal{H}_1}}. \tag{3.8}$$

This follows directly from the minimax property for self-adjoint operators since $\|\mathcal{A}v\|^2_{\mathcal{H}_2} = (\mathcal{A}^*\mathcal{A}v, v)_{\mathcal{H}_2}$. As a corollary, for any compact operators $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_2$, we have the inequality

$$\mu_\ell(\mathcal{A}+\mathcal{B}) \le \mu_{\ell_1}(\mathcal{A}) + \mu_{\ell_2}(\mathcal{B}), \quad \forall\, \ell = \ell_1 + \ell_2. \tag{3.9}$$

Definition 3.8. A compact operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert spaces is a trace-class operator if

$$\|\mathcal{A}\|_1 := \sum_\ell \mu_\ell(\mathcal{A}) < \infty.$$

$\|\cdot\|_1$ is called the trace norm.

For a trace-class operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ on a single space, one can define the trace by

$$\operatorname{tr}(\mathcal{A}) := \sum_j \langle \mathcal{A}e_j, e_j\rangle, \tag{3.10}$$

where $\{e_j\}$ is an orthonormal basis of $\mathcal{H}$. One can verify that the definition does not depend on the choice of $\{e_j\}$. By Lidskii's theorem [11, Proposition B.31], we have

$$\operatorname{tr}(\mathcal{A}) = \sum_\ell \lambda_\ell(\mathcal{A}).$$

The Fredholm determinant is defined by

$$\det(1-\mathcal{A}) := \prod_\ell (1-\lambda_\ell(\mathcal{A})).$$

We have the estimates

$$|\operatorname{tr}(\mathcal{A})| \le \sum_\ell |\lambda_\ell(\mathcal{A})| \le \|\mathcal{A}\|_1, \quad |\det(1-\mathcal{A})| \le \prod_\ell (1+\mu_\ell(\mathcal{A})) \le e^{\|\mathcal{A}\|_1}. \tag{3.11}$$

For a trace-class operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and a bounded operator $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$, it follows from the minimax property that $\mu_\ell(\mathcal{B}\mathcal{A}) \le \|\mathcal{B}\|_{\mathcal{H}_2\to\mathcal{H}_3}\mu_\ell(\mathcal{A})$ and thus

$$\|\mathcal{B}\mathcal{A}\|_1 \le \|\mathcal{B}\|_{\mathcal{H}_2\to\mathcal{H}_3}\|\mathcal{A}\|_1. \tag{3.12}$$

Moreover, it is shown in [11, Eq (B.4.9)] that for a trace-class operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and a bounded operator $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_1$,

$$\operatorname{tr}(\mathcal{A}\mathcal{B}) = \operatorname{tr}(\mathcal{B}\mathcal{A}). \tag{3.13}$$

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3.2.2. Transfer operator: $\mathrm{SL}_2(\mathbb{R})$ case

To estimate the Selberg zeta function in the region where the infinite product expression does not converge, we employ the transfer operator $\mathcal{L}_s$. The observation is that the transfer operator $\mathcal{L}_s$ is a trace-class operator acting on the space of holomorphic functions with bounded L^2 norm, and its Fredholm determinant gives the Selberg zeta function.

We first consider the case of a Schottky group $\Gamma < \mathrm{SL}_2(\mathbb{R})$ of rank $g \geq 2$. Write $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{2g} \rangle$ with $\mathfrak{a}_i \mathfrak{a}_{i+g} = id$ for $i = 1, 2, \cdots, g$. By conjugating Γ if necessary, there exist open discs $D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \cdots, D_{\mathfrak{a}_{2g}}$ in $\mathbb{C}$ centered in $\mathbb{R}$ such that their closures are disjoint, and for each $\mathfrak{a} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$, we have $\mathfrak{a}(\hat{\mathbb{C}} - \overline{D_{\mathfrak{a}^{-1}}}) = D_{\mathfrak{a}}$ and $\mathfrak{a}(\hat{\mathbb{R}} - I_{\mathfrak{a}^{-1}}) = I_{\mathfrak{a}}^{\circ}$, where $I_{\mathfrak{a}} := \overline{D_{\mathfrak{a}}} \cap \mathbb{R}$.

We define the Hilbert space $\mathcal{H} := \oplus_{i=1}^{2g} \mathcal{H}(D_{\mathfrak{a}_i})$, where $\mathcal{H}(D_{\mathfrak{a}}) = \{u \in L^2(D_{\mathfrak{a}}) : u \text{ is holomorphic}\}$ is equipped with the norm $\|u\|^2_{\mathcal{H}(D_{\mathfrak{a}})} := |D_{\mathfrak{a}}|^{-1} \|u\|^2_{L^2(D_{\mathfrak{a}})}$, where $|D_{\mathfrak{a}}|$ is the Lebesgue measure of the disc $D_{\mathfrak{a}}$ given as in equation (1.7). The transfer operator $\mathcal{L}_s : \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{L}_s u(w) := \sum_{\mathfrak{a}_i \neq \mathfrak{b}^{-1}} \mathfrak{a}_i'(w)^s u(\mathfrak{a}_i(w)) \quad \text{for} \quad w \in D_{\mathfrak{b}} \quad \text{with} \quad \mathfrak{b} \in \{\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}\}. \tag{3.14}$$

Here, note that the derivative satisfies $\mathfrak{a}_i'(w) > 0$ for $w \in D_{\mathfrak{b}} \cap \mathbb{R}$, and hence we define $\mathfrak{a}_i'(w)^s := e^{s \log \mathfrak{a}_i'(w)}$, where we choose the branch of the complex logarithm so that if $s \in \mathbb{R}$, then $\mathfrak{a}_i'(w)^s$ is the unique holomorphic function on $D_{\mathfrak{b}}$ satisfying $\mathfrak{a}_i'(w)^s > 0$ for $w \in D_{\mathfrak{b}} \cap \mathbb{R}$.

We introduce a few more notations. We use $\mathcal{W}^n$ to denote words of length n with respect to the generators $\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}$, that is, words of the form $\mathfrak{a}_{i_1} \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_n}$ with $\mathfrak{a}_{i_{j+1}} \neq \mathfrak{a}_{i_j}^{-1}$. For $\mathbf{a} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \mathcal{W}^n$, we denote

$$\mathbf{a}(w) = \mathfrak{a}_{i_1}(\mathfrak{a}_{i_2}(\cdots \mathfrak{a}_{i_n}(w)))$$

and $\hat{\mathbf{a}} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{n-1}}$.

For a disc D of radius R, and a real number $\rho > 0$, we use ρD to denote the disc with the same center and of radius ρR.

The following lemma shows that $\mathcal{L}_s$ is a trace-class operator (see also [5, Lemma 15.7]).

Lemma 3.9. *Suppose $\gamma : D_{\mathfrak{b}} \to \rho D_{\mathfrak{a}}$ is a holomorphic function between two discs with $0 < \rho < 1$ and $\mathfrak{a}, \mathfrak{b} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$. Then the pullback operator*

$$\gamma^* u(w) = u(\gamma(w)) : \mathcal{H}(D_{\mathfrak{a}}) \to \mathcal{H}(D_{\mathfrak{b}})$$

is a trace-class operator satisfying

$$\mu_\ell(\gamma^*) \leq C(1-\rho)^{-2} \rho^\ell \sqrt{\ell + 1}.$$

Consequently, for any $s \in \mathbb{C}$, $\mathcal{L}_s : \mathcal{H} \to \mathcal{H}$ is a trace-class operator.

Proof. We may assume $D_{\mathfrak{a}}$ is centered at 0 and take an orthonormal basis $\phi_n(w) := \sqrt{n+1}\left(\frac{w}{R}\right)^n$ of $\mathcal{H}(D_{\mathfrak{a}})$, where R is the radius of $D_{\mathfrak{a}}$. Then

$$\|\gamma^* \phi_n\|^2_{\mathcal{H}(D_{\mathfrak{b}})} = |D_{\mathfrak{b}}|^{-1} \int_{D_{\mathfrak{b}}} (n+1) \frac{|\gamma(w)|^{2n}}{R^{2n}} dm(w) \leq (n+1)\rho^{2n}.$$

By the minimax property equation (3.8), we conclude

$$\mu_\ell(\gamma^*) \leq \sum_{n \geq \ell} \|\gamma^* \phi_n\|_{\mathcal{H}(D_{\mathfrak{b}})} \leq \sum_{n \geq \ell} \sqrt{n+1} \rho^n \leq C(1-\rho)^{-2} \rho^\ell \sqrt{\ell+1}.$$

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Thus,

$$\sum_{\ell} \mu_\ell(\gamma^*) \le C(1-\rho)^{-2} \sum_{\ell} \rho^\ell \sqrt{\ell+1} < \infty,$$

which implies that γ^* is a trace-class operator. Since $\mathcal{L}_s$ is a finite direct sum of such pullback operators composed with multiplication by holomorphic functions, we conclude that $\mathcal{L}_s$ is also a trace-class operator using equation (3.9) and equation (3.12). □

Since $\mathcal{L}_s$ is a trace-class operator, we can define its Fredholm determinant. We claim $\det(1-\mathcal{L}_s)$ is exactly the Selberg zeta function; see [5, Lemma 15.10]. Here, we include a proof for completeness, but we will omit the proof later for similar cases. Note that since $\mathcal{L}_s$ is of trace class and depends holomorphically on $s \in \mathbb{C}$, $\det(1-\mathcal{L}_s)$ is an entire function. So, this provides an alternative proof of the holomorphic extension of the Selberg zeta function.

Lemma 3.10. *For $s \in \mathbb{C}$, we have*

$$\det(1-\mathcal{L}_s) = Z(\Gamma, s). \tag{3.15}$$

Proof. Since both sides are entire functions on $\mathbb{C}$, it suffices to check the inequality for $\Re s \gg 1$. For $\Re s \gg 1$, we have

$$\det(1-\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\mathcal{L}_s^n)\right).$$

On the other hand,

$$Z(\Gamma, s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{[\gamma],k} e^{-n(s+k)\ell(\gamma)}\right).$$

We again consider the orthonormal basis $\phi_k^{\mathfrak{a}}(w) = \sqrt{k+1}\left(\frac{w-w_{\mathfrak{a}}}{R_{\mathfrak{a}}}\right)^k$, where $w_{\mathfrak{a}}$ is the center of $D_{\mathfrak{a}}$ and $R_{\mathfrak{a}}$ is the radius of $D_{\mathfrak{a}}$. We write

$$\mathcal{L}_s = \sum_{\mathfrak{a} \neq \mathfrak{b}^{-1} \in \mathcal{W}} \mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} \quad \text{with} \quad \mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} : \mathcal{H}(D_{\mathfrak{a}}) \to \mathcal{H}(D_{\mathfrak{b}})$$

given by

$$\mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} u(w) = \mathfrak{a}'(w)^s u(\mathfrak{a}(w)) \quad \text{for } w \in D_{\mathfrak{b}}.$$

We have

$$\mathcal{L}_s^n = \sum_{\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}} \in \mathcal{W}^{n+1}} \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}} \quad \text{with} \quad \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}} := \mathcal{L}_{s,\mathfrak{a}_{i_n}\mathfrak{a}_{i_{n+1}}} \cdots \mathcal{L}_{s,\mathfrak{a}_{i_2}\mathfrak{a}_{i_3}} \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}}.$$

The Schwartz kernel of $\mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}}$ is given by

$$\mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}}(w,w') = \sum_k (\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})'(w)^s \phi_k^{\mathfrak{a}_{i_1}}(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}(w)) \overline{\phi_k^{\mathfrak{a}_{i_1}}(w')}, \quad w \in D_{\mathfrak{a}_{i_{n+1}}},\ w' \in D_{\mathfrak{a}_{i_1}}.$$

Its restriction to the diagonal is nonzero if and only if $\mathfrak{a}_{i_{n+1}} = \mathfrak{a}_{i_1}$, since $w = w'$ only when $i_{n+1} = i_1$. Set $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ and $\mathcal{L}_{s,\gamma} := \mathcal{L}_{s,\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n+1}}}$. So, $\mathcal{L}_{s,\gamma} u(w) = \gamma'(w)^s u(\gamma(w))$. A change of variables

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$w = gw'$ gives the operator $(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* : \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_1}}) \to \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_{n+1}}})$ given by

$$(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* u(w') = \gamma'(gw')^s u(g^{-1}\gamma g w').$$

Let $\gamma_1 = g^{-1}\gamma g$, and $\mathcal{L}_{s,\gamma_1} : \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_1}}) \to \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_{n+1}}})$ be given by $\mathcal{L}_{s,\gamma_1} u(w') = \gamma_1'(w)^s u(\gamma_1 w')$. Then

$$(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* u(w') = \mathcal{B}^{-1}\mathcal{L}_{s,\gamma_1}\mathcal{B} \quad \text{with} \quad \mathcal{B}u(w') = (g'(w'))^s u(w').$$

Therefore, by equation (3.13), $\mathrm{tr}\mathcal{L}_{s,\gamma} = \mathrm{tr}\mathcal{L}_{s,\gamma_1}$. So, we may choose the coordinates to make $\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ a diagonal element so that $\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}(w) = e^{-\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}w$. Then

$$(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})'(w)^s = e^{-s\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}.$$

We may assume the center of $D_{\mathfrak{a}_{i_1}}$ is 0. Applying the definition of trace (3.10) to the basis $\phi_k^{\mathfrak{a}_{i_1}}(w)$, we obtain

$$\mathrm{tr}\mathcal{L}_{s,\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n+1}}} = \frac{e^{-s\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}}{\pi R_{\mathfrak{a}_{i_1}}^2}\sum_k (k+1)\int_{|w|\le R_{\mathfrak{a}_{i_1}}}\left(\frac{e^{-\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}w}{R_{\mathfrak{a}_{i_1}}}\right)^k\left(\frac{\bar{w}}{R_{\mathfrak{a}_{i_1}}}\right)^k dm(w) = \sum_k e^{-(s+k)\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}.$$

Suppose the closed loop $\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n} = \mathbf{a}^m$, where $\mathbf{a}$ is primitive. Then

$$\sum_n \frac{1}{n}\mathrm{tr}(\mathcal{L}_s^n) = \sum_m \frac{1}{m}\sum_{[\mathbf{a}]\in\mathcal{P},k} e^{-m(s+k)\ell(\mathbf{a})}.$$

The proof is complete. □

For further applications, we also introduce modified transfer operators that use only discs with a word length $\ge N$. For $\mathbf{a} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n} \in \mathcal{W}^n$, let

$$D_{\mathbf{a}} := \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n-1}}(D_{\mathfrak{a}_{i_n}}).$$

We define the modified transfer operator on $\mathcal{H} = \mathcal{H}_N := \oplus_{\mathbf{b}\in\mathcal{W}^N}\mathcal{H}(D_{\mathbf{b}})$:

$$\mathcal{L}_{s,N}u(w) := \sum_{\mathfrak{a}\neq\mathfrak{a}_{i_1}^{-1}} \mathfrak{a}'(w)^s u(\mathfrak{a}(w)) \quad \text{for} \quad w \in D_{\mathbf{b}} \quad \text{with} \quad \mathbf{b} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}. \tag{3.16}$$

When $N = 1$, it agrees with the previous definition in equation (3.14). When $N \ge 2$, the modified transfer operator acts on a space of functions with a larger domain. We can still use the proof of Lemma 3.10 to show that for any $N \ge 1$,

$$\det(1 - \mathcal{L}_{s,N}) = Z(\Gamma, s).$$

In the computation, we need to use fixed points. By taking larger N, the fixed point of a loxodromic element is still contained in only one disc. The computations are the same.

Now, we can state a much more precise estimate for the transfer operator.

Proposition 3.11. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is a Schottky group with the notations above. Suppose for some $A, B > 0$, $N \in \mathbb{Z}_{>0}$, and $\varphi \in [0,\pi]$, we have for any $\mathfrak{a} \in \mathcal{W}$ and $\mathbf{b} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$,*

- *$e^{-A} \le |\mathfrak{a}'(w)| \le e^A$ and $|\arg\mathfrak{a}'(w)| \le \varphi \le \pi$ for any $w \in D_{\mathbf{b}}$;*
- *$\mathfrak{a}(D_{\mathbf{b}}) \subset e^{-B}D_{\mathbf{a}}$ with $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$.*

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Then the singular values of $\mathcal{L}_{s,N}$ satisfy

$$\mu_\ell(\mathcal{L}_{s,N}) \le C(g,N,B)e^{(A|\Re s|+\varphi|\Im s|-B\ell)/C(g,N)}\sqrt{\ell+1}. \tag{3.17}$$

Proof. The transfer operator $\mathcal{L}_{s,N}$ is the direct sum of operators

$$\mathcal{L}_{s,\mathbf{ab}} : \mathcal{H}(D_{\mathbf{a}}) \to \mathcal{H}(D_{\mathbf{b}}) \quad \text{given by} \quad \mathcal{L}_{s,\mathbf{ab}}u(w) = \mathfrak{a}'(w)^s u(\mathfrak{a}(w)),$$

where $\mathfrak{a} \in \mathcal{W}$ and $\mathbf{b} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$ and $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$. We have

- $|\mathfrak{a}'(w)^s| \le \max(|\mathfrak{a}'(w)|^{\Re s}, |\mathfrak{a}'(w)|^{-\Re s})|\exp(|\arg\mathfrak{a}'(w)||\Im s|) \le e^{A|\Re s|+\varphi|\Im s|}$;
- By Lemma 3.9, the singular values of the pullback operator $\mathfrak{a}^* : \mathcal{H}(D_{\mathbf{a}}) \to \mathcal{H}(D_{\mathbf{b}})$ satisfy

$$\mu_\ell(\mathfrak{a}^*) \le C(1-e^{-B})^{-2}e^{-B\ell}\sqrt{\ell+1}.$$

Then the proposition follows from equation (3.9) and equation (3.12). □

Recall from Lemma 2.23 that, for a family of Schottky groups satisfying condition (★), we have a uniform Schottky basis.

Proposition 3.12. *Suppose we have a degenerating family of Schottky groups $\Gamma_z < \mathrm{SL}_2(\mathbb{R})$ satisfying (★) with their Schottky discs uniformly bounded away from ∞. For any $N \ge 1$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a} \neq \mathfrak{a}_{i_1}^{-1}$, and sufficiently small $0 < |z| \ll_N 1$, we have for any $w \in D_{\mathbf{b},z}$, and the disc $D_{\mathbf{a},z}$ with $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$,*

$$\begin{aligned} &|\log|\mathfrak{a}'(w)|| \lesssim_N \log(1/|z|),\\ &\mathfrak{a}(D_{\mathbf{b},z}) \subset \frac{1}{10}D_{\mathbf{a},z}, \end{aligned} \tag{3.18}$$

in terms of the absolute norm. Moreover, for any $M > 0$, there exists $N(M) \ge 1$ such that if we require $\mathbf{b} \in \mathcal{W}^{N(M)}$ additionally, then

$$|\arg\mathfrak{a}'(w)| \lesssim_M |z|^M.$$

Consequently, given any $C, M > 0$, the family of zeta functions $Z(\Gamma_z, s/\log(1/|z|))$ with $0 < |z| < 1/e$ is uniformly bounded (depending on C and M) in the region

$$|\Re s| \le C, \quad |\Im s| \le C|z|^{-M}\log(1/|z|).$$

Proof. By assumption, $\mathfrak{a} \in \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$, so we can write $\mathfrak{a} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{M}(\mathbb{D})$. Fix any $0 < |z| \ll 1$. The uniform separation $\mathfrak{a}(D_{\mathbf{b},z}) \subset \frac{1}{10}D_{\mathbf{a},z}$ follows from Corollary 3.23.

For any $w \in D_{\mathfrak{b},z}$, we have $\mathfrak{a}(w) = \frac{a_z w + b_z}{c_z w + d_z}$, where the lower subscript means to evaluate the functions at z. Then $\mathfrak{a}'(w) = 1/(c_z w + d_z)^2$. The estimate $|z|^A \le |\mathfrak{a}'(w)|$ for some $A > 0$ follows from the fact that the Schottky discs are uniformly bounded away from ∞. The other estimate $|\mathfrak{a}'(w)| \le |z|^{-A}$ follows from the definition of uniform Schottky discs (2.4), and the convention (2.5) gives

$$\inf_{w \in D_{\mathbf{b},\mathbf{z}}} |w + d_z/c_z| \ge d(\partial D_{\mathfrak{a}^{-1},z}, \mathfrak{a}^{-1}\infty) \ge \mathrm{diam}(D_{\mathfrak{a}^{-1},z})/2.$$

Finally, it follows from Lemma 3.3 that by taking N sufficiently large, we can ensure $\mathrm{diam}(D_{\mathbf{b},z}) = O(|z|^{cN})$. The estimate $|\arg\mathfrak{a}'(w)| \le |z|^M$ for $w \in D_{\mathbf{b},z}$ follows from the observation that

$$\mathfrak{a}'(w) = (c_z w + d_z)^{-2} = c_z^{-2}((\Re w + d_z/c_z) + i\Im w)^{-2}$$

and $D_{\mathbf{b},z} \cap \mathbb{R} \neq \emptyset$.

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For the boundedness of $Z(\Gamma_z, s/\log(1/|z|))$, it suffices to assume $0 < |z| \ll_N 1$, since otherwise, the statement follows from the continuity of the zeta function. Then the hypothesis of Proposition 3.11 is satisfied with $A = A_0 \log(1/|z|)$, where A_0 is fixed and $\varphi = O(|z|^M)$. We can apply the conclusion of Proposition 3.11 and equation (3.11) to obtain the boundedness of $Z(\Gamma_z, s/\log(1/|z|))$. □

3.2.3. Transfer operator: $\mathrm{SL}_2(\mathbb{C})$ case

Let Γ be a classical Schottky group in $\mathrm{SL}_2(\mathbb{C})$ of rank $g \geq 2$. The discussion is similar but has several complications, which we detail below.

Write $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{2g} \rangle$ with $\mathfrak{a}_i \mathfrak{a}_{i+g} = id$ for $i = 1, 2, \cdots, g$. By conjugating Γ if necessary, there exist open discs $D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \cdots, D_{\mathfrak{a}_{2g}} \subset \mathbb{C}$ such that their closures are disjoint, and for each $\mathfrak{a} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$, we have $\mathfrak{a}(\hat{\mathbb{C}} - \overline{D_{\mathfrak{a}^{-1}}}) = D_{\mathfrak{a}}$.

To study the Selberg zeta function, we identify $\mathbb{C}$ with $\mathbb{R}^2$, and given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, complexify the linear fractional transformation $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\gamma(x, y) = \left(\Re \frac{a(x+iy)+b}{c(x+iy)+d}, \Im \frac{a(x+iy)+b}{c(x+iy)+d} \right), \quad (x, y) \in \mathbb{R}^2 \tag{3.19}$$

to

$$\widetilde{\gamma}(w_1, w_2) = \frac{1}{2} \left(\frac{a(w_1+iw_2)+b}{c(w_1+iw_2)+d} + \frac{\bar{a}(w_1-iw_2)+\bar{b}}{\bar{c}(w_1-iw_2)+\bar{d}}, -i\frac{a(w_1+iw_2)+b}{c(w_1+iw_2)+d} + i\frac{\bar{a}(w_1-iw_2)+\bar{b}}{\bar{c}(w_1-iw_2)+\bar{d}} \right) \tag{3.20}$$

with $(w_1, w_2) \in \mathbb{C}^2$.

For each generator, suppose $D_{\mathfrak{a}} = \{|w - w_{\mathfrak{a}}| \leq R_{\mathfrak{a}}\}$. Let $\widetilde{D}_{\mathfrak{a}}$ be the polydisc defined by

$$\widetilde{D}_{\mathfrak{a}} = \{(w_1, w_2) \in \mathbb{C}^2 : \max(|w_1 - \Re w_{\mathfrak{a}}|, |w_2 - \Im w_{\mathfrak{a}}|) \leq R_{\mathfrak{a}}/3\},$$

and we also view $D_{\mathfrak{a}} \cap \widetilde{D}_{\mathfrak{a}}$ as a subset of $\widetilde{D}_{\mathfrak{a}}$.

We introduce the Hilbert space $\widetilde{\mathcal{H}} = \oplus_{i=1}^{2g} \mathcal{H}(\widetilde{D}_{\mathfrak{a}_i})$, where $\mathcal{H}(\widetilde{D}_{\mathfrak{a}_i}) = \{u \in L^2(\widetilde{D}_{\mathfrak{a}_i}) : u \text{ is holomorphic}\}$ with the norm $\|u\|^2_{\mathcal{H}(\widetilde{D}_{\mathfrak{a}_i})} := |\widetilde{D}_{\mathfrak{a}_i}|^{-1} \|u\|^2_{L^2(\widetilde{D}_{\mathfrak{a}_i})}$. Suppose $\mathfrak{a}_i$ maps $\widetilde{D}_{\mathfrak{b}}$ into a compact subset of $\widetilde{D}_{\mathfrak{a}_i}$ for any $\mathfrak{b} \neq \mathfrak{a}_i^{-1}$. The transfer operator $\mathcal{L}_s : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is defined as

$$\mathcal{L}_s u(w_1, w_2) := \sum_{\mathfrak{a}_i \neq \mathfrak{b}^{-1}} [\mathfrak{a}_i'(w_1, w_2)]^s u(\widetilde{\mathfrak{a}}_i(w_1, w_2)) \quad \text{for} \quad (w_1, w_2) \in \widetilde{D}_{\mathfrak{b}} \quad \text{with} \quad \mathfrak{b} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}. \tag{3.21}$$

Here, $[\mathfrak{a}_i'(w_1, w_2)]$ is a holomorphic function in $\widetilde{D}_{\mathfrak{b}}$ such that $[\mathfrak{a}_i'(w_1, w_2)] = |\mathfrak{a}_i'(x+iy)|$ for $(w_1, w_2) = (x, y) \in D_{\mathfrak{b}}$. More precisely, if $\mathfrak{a}_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, then

$$[\mathfrak{a}_i'(w_1, w_2)] = \frac{1}{(c(w_1+iw_2)+d)(\bar{c}(w_1-iw_2)+\bar{d})}. \tag{3.22}$$

Equation (3.21) requires choosing a branch of the complex logarithm such that $\log[\mathfrak{a}_i'(w_1, w_2)]$ is a holomorphic function on $\widetilde{D}_{\mathfrak{b}}$ satisfying $\log[\mathfrak{a}_i'(w_1, w_2)] = \log|\mathfrak{a}_i'(x+iy)|$ for $(w_1, w_2) = (x, y) \in D_{\mathfrak{b}} \cap \widetilde{D}_{\mathfrak{b}}$. This is achievable because the inequality $|w_{\mathfrak{b}} + d/c| > R_{\mathfrak{b}}$ gives $|w_1 + iw_2 + d/c| > R_{\mathfrak{b}}/3$ and hence $[\mathfrak{a}_i'(w_1, w_2)] \neq 0$ for $(w_1, w_2) \in \widetilde{D}_{\mathfrak{b}}$. Define $[\mathfrak{a}_i'(w_1, w_2)]^s := \exp(s \log[\mathfrak{a}_i'(w_1, w_2)])$ on $\widetilde{D}_{\mathfrak{b}}$.

We show that the pullback operator $\widetilde{\gamma}^*$ is again a trace-class operator.

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Lemma 3.13. *Suppose $\widetilde{\gamma} : \widetilde{D}_{\mathfrak{b}} \to \rho\widetilde{D}_{\mathfrak{a}}$ is a holomorphic function between polydiscs in $\mathbb{C}^2$ with $0 < \rho < 1$ and $\mathfrak{a}, \mathfrak{b} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_{2g}\}$. Then the pullback operator*

$$\widetilde{\gamma}^* u(w_1, w_2) = u(\widetilde{\gamma}(w_1, w_2)) : \mathcal{H}(\widetilde{D}_{\mathfrak{a}}) \to \mathcal{H}(\widetilde{D}_{\mathfrak{b}})$$

is a trace-class operator satisfying

$$\mu_\ell(\widetilde{\gamma}^*) \le C_\rho \rho^{\sqrt{\ell}}(\ell + 1).$$

In the following, we will use the multiindex notation:

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\ge 0}, \quad |\alpha| = \alpha_1 + \alpha_2, \quad \text{and} \quad \alpha! = \alpha_1!\alpha_2!.$$

Proof. We may assume $\widetilde{D}_{\mathfrak{a}} := \{(w_1, w_2) \in \mathbb{C}^2 : |w_1| < R_1, |w|_2 < R_2\}$. Consider the orthonormal basis $\phi_\alpha(w_1, w_2) := \sqrt{(\alpha_1 + 1)(\alpha_2 + 1)}(w_1/R_1)^{\alpha_1}(w_2/R_2)^{\alpha_2}$ of $\mathcal{H}(\widetilde{D}_{\mathfrak{a}})$. Then

$$\begin{aligned}\|\widetilde{\gamma}^*\phi_\alpha\|^2_{\mathcal{H}(\widetilde{D}_{\mathfrak{b}})} &= |\widetilde{D}_{\mathfrak{b}}|^{-1} \int_{\widetilde{D}_{\mathfrak{b}}} (\alpha_1 + 1)(\alpha_2 + 1) \frac{|\pi_1(\gamma(w_1, w_2))|^{2\alpha_1}}{R_1^{2\alpha_1}} \frac{|\pi_2(\gamma(w_1, w_2))|^{2\alpha_2}}{R_2^{2\alpha_2}} dm(w_1, w_2) \\ &\le (|\alpha| + 1)^2 \rho^{2|\alpha|},\end{aligned}$$

where $\pi_i : \mathbb{C}^2 \to \mathbb{C}$ is to project on the i-th coordinate. By the minimax property (3.8), we conclude

$$\mu_\ell(\widetilde{\gamma}^*) \le \sum_{|\alpha|^2 \ge \ell - 10} \|\widetilde{\gamma}^*\phi_\alpha\|_{\mathcal{H}(\widetilde{D}_{\mathfrak{b}})} \le \sum_{k^2 \ge \ell - 10} (k + 1)^2 \rho^k \le C_\rho \rho^{\sqrt{\ell}}(\ell + 1).$$

Thus,

$$\sum_\ell \mu_\ell(\widetilde{\gamma}^*) \le C_\rho \sum_\ell \rho^{\sqrt{\ell}}(\ell + 1) < \infty,$$

which implies $\widetilde{\gamma}^*$ is a trace-class operator. □

Similar to the $\mathrm{SL}_2(\mathbb{R})$ case, we have

$$\det(1 - \mathcal{L}_s) = Z(\Gamma, s) \quad \text{for any} \quad s \in \mathbb{C}.$$

We refer to [14] for more details.

As in the $\mathrm{SL}_2(\mathbb{R})$ case, we also consider the modified transfer operators to obtain further applications. The precise construction is as follows. For $\mathbf{a} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \mathcal{W}^n$, define the polydisc $\widetilde{D}_{\mathbf{a}}$ in $\mathbb{C}^2$ associated with $D_{\mathbf{a}} = D(w_{\mathbf{a}}, R_{\mathbf{a}}) \subset \mathbb{C}$ by

$$\widetilde{D}_{\mathbf{a}} = \{(w_1, w_2) : \max(|w_1 - \Re w_{\mathbf{a}}|, |w_2 - \Im w_{\mathbf{a}}|) < R_{\mathbf{a}}/3\}.$$

For any $N \in \mathbb{N}$, the modified transfer operator $\mathcal{L}_{s,N}$ on $\mathcal{H} = \mathcal{H}_N := \oplus_{b \in \mathcal{W}^N} \mathcal{H}(\widetilde{D}_{\mathbf{b}})$ is defined by

$$\mathcal{L}_{s,N} u(w_1, w_2) := \sum_{\mathfrak{a} \ne \widetilde{\mathfrak{a}}_{i_1}} [\mathfrak{a}'(w_1, w_2)]^s u(\widetilde{\mathfrak{a}}(w_1, w_2)) \quad \text{for} \quad (w_1, w_2) \in \widetilde{D}_{\mathbf{b}} \quad \text{with} \quad \mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}.$$

Again, we have for any $N \in \mathbb{N}$,

$$\det(1 - \mathcal{L}_{s,N}) = Z(\Gamma, s) \quad \text{for any} \quad s \in \mathbb{C}.$$

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We state a proposition similar to Proposition 3.11 for $\mathrm{SL}_2(\mathbb{C})$.

Proposition 3.14. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{C})$ is a Schottky group with the notations above. Suppose for some $A, B > 0$, $N \in \mathbb{Z}_{>0}$, and $\varphi \in [0, \pi]$, we have for any $\mathfrak{a} \in \mathcal{W}$ and $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$,*

- *$e^{-A} \leq |[\mathfrak{a}'(w_1, w_2)]| \leq e^A$ and $|\arg[\mathfrak{a}'(w_1, w_2)]| \leq \varphi \leq \pi$ for any $(w_1, w_2) \in \widetilde{D}_{\mathbf{b}}$;*
- *$\widetilde{\mathfrak{a}}(\widetilde{D}_{\mathbf{b}}) \subset e^{-B}\widetilde{D}_{\mathbf{a}}$, where $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$.*

Then the singular values of $\mathcal{L}_s$ satisfy

$$\mu_\ell(\mathcal{L}_{s,N}) \leq C(g, N, B) e^{(A|\Re s| + \varphi|\Im s| - B\ell^{1/2})/C(g,N)}(\ell + 1).$$

Consequently, the operator $\mathcal{L}_{s,N}$ is a trace-class operator.

Proof. The proof is similar to that of Proposition 3.11. The operator $\mathcal{L}_{s,N}$ is a direct sum of the following operators: Consider each component of $\mathcal{L}_{s,N}$:

$$\mathcal{L}_{s,\mathbf{ab}} : \mathcal{H}(\widetilde{D}_{\mathbf{a}}) \to \mathcal{H}(\widetilde{D}_{\mathbf{b}}), \quad \mathcal{L}_{s,\mathbf{ab}} u(w_1, w_2) := [\mathfrak{a}'(w_1, w_2)]^s u(\widetilde{\mathfrak{a}}(w_1, w_2)), \quad \mathbf{a} = \mathfrak{a}\hat{\mathbf{b}},$$

where $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$, and $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$.

We have

- $|[\mathfrak{a}'(w_1, w_2)]^s| \leq \max\left(|[\mathfrak{a}'(w_1, w_2)]|^{\Re s}, |[\mathfrak{a}'(w_1, w_2)]|^{-\Re s}\right) \exp(|\arg[\mathfrak{a}'(w_1, w_2)]||\Im s|)$
  $\leq e^{A|\Re s| + \varphi|\Im s|}$;
- By Lemma 3.13, we know

$$\mu_\ell(\widetilde{\mathfrak{a}}^*) \leq C(B) e^{-B\ell^{1/2}}(\ell + 1).$$

Then the proposition follows from equation (3.9) and equation (3.12). □

As in Proposition 3.12, we have the following.

Proposition 3.15. *Suppose we have a degenerating family of Schottky groups $\Gamma_z < \mathrm{SL}_2(\mathbb{C})$ satisfying (★) with their Schottky discs uniformly bounded away from ∞. For $N \geq 1$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a} \neq \mathfrak{a}_{i_1}^{-1}$, and sufficiently small $0 < |z| \ll_N 1$, we have for any $(w_1, w_2) \in \widetilde{D}_{\mathbf{b},z}$, and the disc $\widetilde{D}_{\mathbf{a},z}$ with $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$,*

$$\begin{aligned} &|\log|[\mathfrak{a}'(w_1, w_2)]|| \lesssim \log(1/|z|), \\ &\widetilde{\mathfrak{a}}(\widetilde{D}_{\mathbf{b},z}) \subset \frac{1}{10}\widetilde{D}_{\mathbf{a},z}, \end{aligned} \tag{3.23}$$

in terms of the absolute norm. Moreover, for any $M > 0$, there exists $N(M) \geq 1$ such that if we require $\mathbf{b} \in \mathcal{W}^{N(M)}$ additionally, then

$$|\arg[\mathfrak{a}'(w_1, w_2)]| \lesssim |z|^M.$$

Consequently, given any $C, M > 0$, the family of zeta functions $Z(\Gamma_z, s/\log(1/|z|))$ with $0 < |z| < 1/e$ is uniformly bounded (depending on C and M) in the region

$$|\Re s| \leq C, \quad |\Im s| \leq C|z|^{-M} \log(1/|z|).$$

Proof. By Corollary 3.23, there exists $\nu > 0$ such that for any $N \in \mathbb{N}$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}\mathfrak{a}_{i_1} \neq id$, and sufficiently small $0 < |z| \ll 1$, we have

$$\mathfrak{a}(D_{\mathbf{b},z}) \subset |z|^\nu D_{\mathbf{a},z}, \tag{3.24}$$

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where $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$.

Fix $0 < |z| \ll 1$. We first show equation (3.23). Suppose $D_{\mathbf{a},z} = D(w_{\mathbf{a},z}, R_{\mathbf{a},z})$ and $D_{\mathbf{b},z} = D(w_{\mathbf{b},z}, R_{\mathbf{b},z})$. Then $|\mathfrak{a}(w) - w_{\mathfrak{a},z}| \le |z|^v R_{\mathbf{a},z}$ for $w \in D_{\mathbf{b},z}$. Using the Cauchy integral formula, we estimate the derivatives of $\mathfrak{a}$ on $D_{\mathbf{b},z}$ at $w_{\mathbf{b},z}$: for any $k \in \mathbb{N}$,

$$|\frac{d^k}{dw^k}\mathfrak{a}(w_{\mathbf{b},z})| = k!\left|\int_{\partial D_{\mathbf{b},z}} \frac{\mathfrak{a}(w)}{(w - w_{\mathbf{b},z})^{k+1}}dw\right| = k!\left|\int_{\partial D_{\mathbf{b},z}} \frac{\mathfrak{a}(w) - w_{\mathbf{a},z}}{(w - w_{\mathbf{b},z})^{k+1}}dw\right| \le 2\pi\frac{k!}{R_{\mathbf{b},z}^k}|z|^v R_{\mathbf{a},z}.$$

We can then form the analytic continuation

$$\widetilde{\mathfrak{a}}(w_1, w_2) = \mathfrak{a}(w_{\mathbf{b},z}) + \sum_{\substack{\alpha\in\mathbb{Z}^2_{\ge 0}\\ \text{with } |\alpha|>0}} \frac{1}{\alpha!}(w_1 - \Re w_{\mathbf{b},z})^{\alpha_1}(w_2 - \Im w_{\mathbf{b},z})^{\alpha_2}\left(\partial_x^{\alpha_1}\partial_y^{\alpha_2}\Re\mathfrak{a}(x+iy), \partial_x^{\alpha_1}\partial_y^{\alpha_2}\Im\mathfrak{a}(x+iy)\right)|_{x+iy=w_{\mathbf{b},z}} \in \mathbb{C}^2.$$

Combining the above two equations, we obtain that for any $(w_1, w_2) \in \widetilde{D}_{\mathbf{b},z}$,

$$\begin{aligned}|\widetilde{\mathfrak{a}}(w_1, w_2) - w_{\mathbf{a},z}| &\le |z|^v R_{\mathbf{a},z} + 4\pi\sum_{\alpha}\frac{|\alpha|!}{\alpha! R_{\mathbf{b}}^{|\alpha|}}|z|^v R_{\mathbf{a},z}(R_{\mathbf{b},z}/3)^{|\alpha|}\\ &= |z|^v R_{\mathbf{a},z} + 4\pi\sum_k\left(\frac{2}{3}\right)^k |z|^v R_{\mathbf{a},z} \le (8\pi+1)|z|^v R_{\mathbf{a},z}.\end{aligned}$$

Thus, we have equation (3.23).

Next, we estimate $|\log|[\mathfrak{a}'(w_1, w_2)]||$. We write $\mathfrak{a} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with $a, b, c, d \in \mathbb{M}(\mathbb{D})$. For any $w \in D_{\mathfrak{b},z}$, we have $\mathfrak{a}(w) = \frac{a_z w + b_z}{c_z w + d_z}$ and $\mathfrak{a}'(w) = 1/(c_z w + d_z)^2$, where the lower subscript means to evaluate the functions at z. By equation (3.22), we have for any $(w_1, w_2) \in \widetilde{D}_{\mathfrak{b},z}$,

$$\begin{aligned}[\mathfrak{a}'(w_1, w_2)] &= |c_z|^{-2}(w_1 + iw_2 - w_{\mathfrak{b},z} + w_{\mathfrak{b},z} + d_z/c_z)^{-1}(w_1 - iw_2 - \bar{w}_{\mathfrak{b},z} + \bar{w}_{\mathfrak{b},z} + \bar{d}_z/\bar{c}_z)^{-1}\\ &= |\mathfrak{a}'(w_{\mathbf{b},z})|^2\left(1 + \frac{w_1 + iw_2 - w_{\mathbf{b},z}}{w_{\mathbf{b},z} + d_z/c_z}\right)^{-1}\left(1 + \frac{w_1 - iw_2 - \bar{w}_{\mathbf{b},z}}{\bar{w}_{\mathbf{b},z} + \bar{d}_z/\bar{c}_z}\right)^{-1}. \qquad (3.25)\end{aligned}$$

Using $|w_{\mathfrak{b},z} + d_z/c_z| > R_{\mathfrak{b},z}$, and $\max(|w_1 - \Re w_{\mathfrak{b}_z}|, |w_2 - \Im w_{\mathfrak{b},z}|) \le R_{\mathfrak{b},z}/3$, we have that the absolute values of both the second and third factors are bounded above by $5/3$ and below by $1/3$. Using the argument of the $\mathrm{SL}_2(\mathbb{R})$ case (proof of Proposition 3.12), we have that there exists $A > 0$, such that $|z|^A \le |\mathfrak{a}'(w_{\mathbf{b},z})| \le |z|^{-A}$. Hence, we have established the estimate for $|\log|[\mathfrak{a}'(w_1, w_2)]||$.

We estimate $|\arg[\mathfrak{a}'(w_1, w_2)]|$. It follows from Corollary 3.23 that $D_{\mathbf{b},z} \subset |z|^v D_{\mathfrak{a}_{i_1},z}$ and $R_{\mathbf{b},z} \le |z|^{vN} R_{\mathfrak{a}_{i_1},z}$. Then, we have

$$\frac{|w_1 + iw_2 - w_{\mathbf{b},z}|}{|w_{\mathbf{b},z} + d_z/c_z|} \le \frac{2}{3}|z|^v(1 - |z|^v)^{-1},$$

which yields the estimate $|\arg[\mathfrak{a}'(w_1, w_2)]| \lesssim |z|^M$ using equation (3.25).

Finally, we use Proposition 3.14 to obtain the boundedness of the zeta functions. □

3.3. *Convergence on the whole plane*

Lemma 3.16. *Let Ω be a connected open set in $\mathbb{C}$ and $f(z)$, $f_n(z)$ be a family of holomorphic functions on Ω. Suppose*

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- $|f_n(z)| \le C$ *for any* $z \in \Omega$ *for some constant* $C > 0$*;*
- $f_n(z) \to f(z)$ *pointwise in some open set* $\Omega' \subset \Omega$*, as* $n \to \infty$*.*

Then $f_n(z) \to f(z)$ *uniformly on any compact subset of* Ω *as* $n \to \infty$*.*

Proof. Since $f_n(z)$ is uniformly bounded, by Montel's theorem, any subsequence of $\{f_n\}$ has a subsequence that converges uniformly on any compact subset. Suppose the lemma is false; then there exists a subsequence of f_n that converges locally uniformly but not to $f(z)$. This contradicts the second condition. □

Therefore, the convergence part of Theorem 1.3 follows from Lemma 3.16 by combining Theorem 3.1, Proposition 3.12, and Proposition 3.15.

3.4. Uniform separation

In this subsection, we verify the uniform separation property in equation (3.18) of Proposition 3.12 and equation (3.24) of Proposition 3.15 using the continuity of the hybrid model. The idea is to relate discs in the Archimedean and non-Archimedean cases and then compute the discs under the action of Γ in both cases. The key input is Lemma 2.23.

We need [32, Lemma 3.2.2].[9]

Lemma 3.17. *Let* k *be a complete valued field. If* k *is Archimedean, we suppose* $(k, |\cdot|)$ *is isometrically embedded into* $(\mathbb{C}, |\cdot|_\infty^\epsilon)$ *for some* $\epsilon \in (0, 1]$*, where* $|\cdot|_\infty$ *is the usual absolute value on* $\mathbb{C}$*. Let* $\rho > 0$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$*. If* $\gamma D^+(\alpha, \rho) \subset \mathbb{A}_k^{1,an}$*, then* $|\alpha + (d/c)| > \rho$ *and*

$$\gamma D^+(\alpha,\rho) = \begin{cases} D^+\left(\left(\frac{a}{c} - \frac{ad-bc}{c^2}\frac{\overline{\alpha+d/c}}{|\alpha+d/c|^{2/\epsilon}-\rho^{2/\epsilon}}, \frac{|ad-bc|\rho}{|c|^2(|\alpha+d/c|^{2/\epsilon}-\rho^{2/\epsilon})^\epsilon}\right)\right), & \text{if } k \text{ is Archimedean;} \\ D^+\left(\frac{b+a\alpha}{c(\alpha+d/c)}, \frac{|ad-bc|\rho}{|c|^2|\alpha+d/c|^2}\right), & \text{otherwise.} \end{cases} \tag{3.26}$$

Recall that for the non-Archimedean case, an open disc $D^-(\beta, \lambda)$ is contained in a closed disc $D^+(\alpha, \rho)$ with $\alpha, \beta \in k$ and $\lambda, \rho > 0$ iff

$$\lambda < \rho \quad \text{and} \quad |\alpha - \beta| < \rho.$$

We recall the continuous map $p : \overline{\mathbb{D}}_r \to \mathbb{A}_{A_r}^{3g-3,an}$ in the hybrid model: for $0 < |z| < r$, the Archimedean norm $|\cdot|_z$ evaluates a function $f \in \mathbb{C}((t))$ which is holomorphic on $\mathbb{D}_r^*$ and meromorphic at 0 by the formula

$$|f|_z = |f(z)|_\infty^{1/\log(1/|z|)},$$

and we have as $|z| \to 0$

$$|f|_z \to |f|_{na}.$$

A ball with center α and radius ρ satisfies

$$D_z^\pm(\alpha,\rho) := D_\infty^\pm(\alpha, \rho^{\log(1/|z|)}) = D_\infty^\pm(\alpha, |z|^{\log(1/\rho)}).$$

Here, we use the lower subscript z (resp. na) to emphasize that the ball is measured using the norm $|\cdot|_z$ (resp. $|\cdot|_{na}$).

[9] In [32, Lemma 3.2.2], they forget to take into account the ϵ in the Archimedean case.

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Lemma 3.18. *Let $\gamma \in \mathrm{PGL}_2(\mathbb{C}((t)))$, and $\alpha, \beta \in \mathbb{C}((t))$ which are holomorphic on $\mathbb{D}_r^*$ and meromorphic at 0. Suppose for some $\rho, \lambda > 0$, we have $\gamma D_{na}^-(\alpha, \rho) \subset D_{na}^+(\beta, \lambda)$. Then there exists $v > 0$ such that for any $0 < |z| \ll 1$, we have*

$$\gamma_z D_z^-(\alpha(z), \rho) \subset D_z^+(\beta(z), \lambda e^{-v}),$$

and in terms of the absolute norm, we have

$$\gamma_z D_\infty^-(\alpha(z), |z|^{\log(1/\rho)}) \subset |z|^v D_\infty^+(\beta(z), |z|^{\log(1/\lambda)}).$$

Proof. Assume $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}((t))$. From Lemma 3.17 and $\gamma D_{na}^-(\alpha, r) \subset \mathbb{A}_k^{1,an}$ with $k = \mathbb{C}((t))$, we have

$$|\alpha + d/c|_{na} > \rho \quad \text{and} \quad \frac{|ad - bc|_{na}\rho}{|c|_{na}^2 |\alpha + d/c|_{na}^2} < \lambda. \tag{3.27}$$

For the Archimedean radius, let $\epsilon = 1/\log(1/|z|)$. Then by Lemma 3.17, the radius of $\gamma_z D_z^-(\alpha(z), \rho)$ is equal to

$$\frac{|ad - bc|_z \rho}{|c|_z^2 (|\alpha + d/c|_z^{2/\epsilon} - \rho^{2/\epsilon})^\epsilon} = \frac{|(ad - bc)|_z \rho}{|c|_z^2 |\alpha + d/c|_z^2} \frac{1}{(1 - (\rho/|\alpha + d/c|_z)^{2/\epsilon})^\epsilon}.$$

Due to $\rho/|\alpha + d/c|_z \to \rho/|\alpha + d/c|_{na} < 1$ and $\epsilon \to 0$ as $|z| \to 0$, we obtain that

$$(1 - (\rho/|\alpha + d/c|_z)^{2/\epsilon})^\epsilon \to 1, \tag{3.28}$$

as $|z| \to 0$. Hence, the Archimedean radius converges to the non-Archimedean radius, and by continuity, we obtain that for $|z|$ small, the Archimedean radius also satisfies

$$\frac{|ad - bc|_z \rho}{|c|_z^2 (|\alpha + d/c|_z^{2/\epsilon} - \rho^{2/\epsilon})^\epsilon} < \lambda.$$

In particular, there exists $v_0 > 0$ such that for $|z|$ small, we have the stronger inequality

$$\frac{|ad - bc|_z \rho}{|c|_z^2 (|\alpha + d/c|_z^{2/\epsilon} - \rho^{2/\epsilon})^\epsilon} < \lambda e^{-v_0}.$$

In terms of the absolute norm, for $|z|$ sufficiently small, the Archimedean radius (the left-hand side of the following inequality) satisfies

$$\left(\frac{|ad - bc|_z r}{|c|_z^2 (|\alpha + d/c|_z^{2/\epsilon} - \rho^{2/\epsilon})^\epsilon} \right)^{1/\epsilon} \leq |z|^{v_0} \lambda^{1/\epsilon}.$$

Next, we consider the center of the ball $\gamma_z D_z^+(\alpha, r)$ for $0 \leq |z| \ll 1$. For the non-Archimedean case (i.e., $z = 0$), from the hypothesis, we have $|\frac{b + a\alpha}{c(\alpha + d/c)} - \beta|_{na} < \lambda$. Using the convergence $|\cdot|_z \to |\cdot|_{na}$ as $|z| \to 0$, we obtain a constant $v_1 > 0$ such that for $0 < |z| \ll 1$, we have

$$\left| \frac{b + a\alpha}{c(\alpha + d/c)} - \beta \right|_z < e^{-v_1} \lambda, \quad \text{that is,} \quad \left| \frac{b + a\alpha}{c(\alpha + d/c)} - \beta \right|_\infty < |z|^{v_1} \lambda^{1/\epsilon}.$$

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By Lemma 3.17, the distance between the Archimedean center δ_a and the non-Archimedean center δ_{na} (extending their domain of definition and viewing them as functions of z) is given by

$$\begin{aligned}\delta_{na}-\delta_a &= \frac{b+a\alpha}{c(\alpha+d/c)} - \left(\frac{a}{c} - \frac{ad-bc}{c^2}\frac{\overline{\alpha+d/c}}{|\alpha+(d/c)|_z^{2/\epsilon}-\rho^{2/\epsilon}}\right)\\ &= -\frac{ad-bc}{c^2(\alpha+d/c)}\left(1-\frac{1}{1-(\rho/|\alpha+d/c|_z)^{2/\epsilon}}\right)\\ &= \frac{ad-bc}{c^2(\alpha+d/c)}\frac{(\rho/|\alpha+d/c|_z)^{2/\epsilon}}{1-(\rho/|\alpha+d/c|_z)^{2/\epsilon}}.\end{aligned}$$

Its $|\cdot|_z$ norm equals (notice that for a real number $|x|_z = |x|^\epsilon$)

$$\begin{aligned}|\delta_{na}-\delta_a|_z &= \left|\frac{ad-bc}{c^2(\alpha+d/c)}\right|_z\left(\frac{\rho}{|(\alpha+d/c)|_z}\right)^2\frac{1}{(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon}\\ &= \rho\left|\frac{ad-bc}{c^2(\alpha+d/c)^2}\right|_z\frac{\rho}{|(\alpha+d/c)|_z}\frac{1}{(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon} < \lambda,\end{aligned} \tag{3.29}$$

where the inequality is due to equation (3.27) and equation (3.28), and holds for $|z|$ small. As in the estimate for the radius, we can replace λ by $e^{-\nu_2}\lambda$ for some $\nu_2 > 0$ in the previous inequality, which holds for $|z|$ small. Combining the estimates of the radius and of the distance between the centers, we obtain for $|z|$ sufficiently small and any $x \in \gamma_z D_z^-(\alpha(z),\rho)$,

$$\begin{aligned}|x-\beta|_z = |x-\beta|_\infty^\epsilon \le (|x-\delta_a|_\infty + |\delta_a-\delta_{na}|_\infty + |\delta_{na}-\beta|_\infty)^\epsilon\\ \le (|z|^{\nu_0}\lambda^{1/\epsilon} + |z|^{\nu_2}\lambda^{1/\epsilon} + |z|^{\nu_1}\lambda^{1/\epsilon})^\epsilon \le 3^\epsilon\lambda e^{-\nu} < \lambda e^{-\nu/2},\end{aligned}$$

where $\nu = \min\{\nu_0,\nu_1,\nu_2\}$. This completes the proof. □

Remark 3.19. From equation (3.29), as $z \to 0$, the $|\cdot|_z$ norm of the difference of the two centers does not go to zero.

We are ready to prove the main result of this part:

Proposition 3.20. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★). There exists $\nu > 0$ such that for any $0 < |z| \ll 1$, the corresponding Schottky group and the Schottky basis given by Lemma 2.23 satisfy*

$$\mathfrak{a}_z(D_{\mathfrak{b},z}) \subset |z|^\nu D_{\mathfrak{a},z},$$

in terms of the absolute norm, for any generators $\mathfrak{a},\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} \neq id$.

Proof. Recall that Lemma 2.23 gives a uniform Schottky basis for all $|z|$ small. In particular, for $z = 0$, the Schottky basis for $\Gamma < \mathrm{PGL}_2(k)$ with $k = \mathbb{C}((t))$ equipped with the norm $|\cdot|_{na}$ is given by

$$\mathcal{B} = \{D^+_{\delta_i,\lambda_i}, D^+_{\delta_i^{-1},\lambda_i^{-1}},\ i = 1,\cdots,g\}.$$

Here, the closed discs are given as in equation (2.4). Writing $\delta_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i\end{pmatrix}$, we have

$$\begin{aligned}&D^+_{\delta_i,\lambda_i} := D^+_{na}\left(\delta_i^{-1}\infty, \lambda_i^{1/2}\left|\frac{(a_id_i-b_ic_i)}{c_i^2}\right|_{na}^{1/2}\right),\\ &\delta_i^{-1}D^-_{\delta_j,\lambda_j} \subset D^+_{\delta_i,\lambda_i} \quad \text{for any} \quad i \neq j.\end{aligned} \tag{3.30}$$

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Then we apply Lemma 3.18 to obtain the uniform separation. More precisely, take any $\delta_i = \mathfrak{a}^{-1}$ and $\delta_j = \mathfrak{b}^{-1}$ with $i \neq j$. For the non-Archimedean case $z = 0$, the Schottky discs are given by

$$D_{\mathfrak{a},0} = D^-_{\delta_i,\lambda_i} \quad \text{and} \quad D_{\mathfrak{b},0} = D^-_{\delta_j,\lambda_j}. \tag{3.31}$$

The Schottky discs for the Archimedean case are given by balls (see equation (2.4) and equation (2.5))

$$D_{\mathfrak{a},z} = D^-_z\left((\delta_i^{-1}\infty)(z), \lambda_i^{1/2}\left|\frac{(a_id_i - b_ic_i)}{c_i^2}\right|^{1/2}_z\right),$$
$$D_{\mathfrak{b},z} = D^-_z\left((\delta_j^{-1}\infty)(z), \lambda_j^{1/2}\left|\frac{(a_jd_j - b_jc_j)}{c_j^2}\right|^{1/2}_z\right).$$

Set

$$\lambda_{na} = \lambda_i^{1/2}\left|\frac{(a_id_i - b_ic_i)}{c_i^2}\right|^{1/2}_{na} \quad \text{and} \quad \rho_{na} = \lambda_j^{1/2}\left|\frac{(a_jd_j - b_jc_j)}{c_j^2}\right|^{1/2}_{na}.$$

Similarly, we introduce the notations $\lambda(z)$ and $\rho(z)$ by replacing $|\cdot|_{na}$ with $|\cdot|_z$.

We can apply Lemma 3.18 to the discs $D_{\mathfrak{b},0}$ and $D_{\mathfrak{a},0}$ due to the property (3.30) and the convention (3.31), and use the convergence of the radii $\rho(z) \to \rho_{na}$ and $\lambda(z) \to \lambda_{na}$ as $z \to 0$. This yields a constant $v' > 0$ such that for any $0 < |z| \ll 1$, we have

$$\mathfrak{a}_z D^-_z\Big((\delta_j^{-1}\infty)(z), \rho(z)\Big) = (\delta_i^{-1})_z D^-_z\Big((\delta_j^{-1}\infty)(z), \rho(z)\Big) \subset D^-_z\Big((\delta_i^{-1}\infty)(z), e^{-v'}\lambda(z)\Big), \tag{3.32}$$

and in terms of the absolute norm, we have

$$\mathfrak{a}_z D_{\mathfrak{b},z} \subset |z|^{v'} D_{\mathfrak{a},z}.$$

Repeating this argument for all possible pairs, we finish the proof. □

Remark 3.21. This proposition also implies the radius estimate in Lemma 3.3.

Remark 3.22. If the Schottky group Γ is indeed inside $\mathrm{SL}_2(\mathbb{R}((t)))$, then for $z \in \mathbb{R}$, the discs $D_{\mathfrak{a},z}$ are centered in $\mathbb{R}$ and the group Γ_z preserves $\mathbb{R} \cup \infty$. Therefore, the intersections $I_{\mathfrak{a},z} = D_{\mathfrak{a},z} \cap \mathbb{R}$ give a Schottky figure for Γ_z as a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Corollary 3.23. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★). There exists $v_1 > 0$ such that for any $0 < |z| \ll 1$ and for any $n \in \mathbb{N}$, the corresponding Schottky group and the Schottky basis given by Lemma 2.23 satisfy*

$$\mathfrak{a}_z(D_{\mathbf{b},z}) \subset |z|^{v_1} D_{\mathbf{a},z},$$

in terms of the absolute norm, where $\mathbf{a} = \mathfrak{a}\hat{\mathbf{b}}$ with $\mathfrak{a}$ a generator and $\mathbf{b} = \mathfrak{b}_{i_1}\cdots\mathfrak{b}_{i_n} \in \mathcal{W}^n$ satisfying $\mathfrak{a}\mathfrak{b}_{i_1} \neq id$.

Proof. Notice that $D_{\mathbf{a},z} = \mathfrak{a}\mathfrak{b}_{i_1}\cdots\mathfrak{b}_{i_{n-2}} D_{\mathbf{b}_{i_{n-1}},z}$, $\mathfrak{a}_z(D_{\mathbf{b},z}) = \mathfrak{a}\mathfrak{b}_{i_1}\cdots\mathfrak{b}_{i_{n-2}} D_{\mathbf{b}_{i_{n-1}}\mathbf{b}_{i_n},z}$. The idea is to apply the distortion estimate of Lemma 3.4 to

$$\gamma := \mathfrak{a}\mathfrak{b}_{i_1}\cdots\mathfrak{b}_{i_{n-2}}, \quad D(\alpha(z),\lambda(z)) := D_{\mathfrak{b}_{i_{n-1}},z}, \quad D(\beta(z),\rho(z)) := D_{\mathfrak{b}_{i_{n-1}}\mathfrak{b}_{i_n},z},$$

and use the result that $D(\beta(z),\rho(z)) \subset D(\alpha(z), e^{-v}\lambda(z))$ for some $v > 0$ due to equation (3.32). More precisely, let w_0 be the center of $D(\alpha(z),\lambda(z))$. For any points $w_1 \in \partial D(\alpha(z),\lambda(z))$ and

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$w_2 \in \partial D(\beta(z), \rho(z))$, we have

$$(1-e^{-v})\lambda(z) \le |w_1 - w_2|_z \le \int_0^1 \left|(\gamma^{-1}f(t))'\right|_z dt \le |\gamma w_1 - \gamma w_2|_z \cdot |\gamma' w_0|_z^{-1} \cdot C\exp(R/(1-c)), \tag{3.33}$$

where $f : [0,1] \to \mathbb{C}$ is given by $f(t) = (1-t)\gamma w_2 + t\gamma w_1$, and the constants $C, R > 0$ and $c \in (0,1)$ are given as in Lemma 3.4.

Let $w_3 \in D(\alpha(z), \lambda(z))$ be such that γw_3 is the center of $\gamma D(\alpha(z), \lambda(z))$. By a similar argument, we have

$$|\gamma w_3 - \gamma w_1|_z \le 2\lambda(z) \cdot |\gamma'(w_0)|_z \cdot C\exp(R/(1-c)). \tag{3.34}$$

Combining equation (3.33) and equation (3.34), we obtain

$$|\gamma w_1 - \gamma w_2|_z \ge \frac{1}{2}C^{-2}\exp(-2R/(1-c)) \cdot (1-e^{-v}) \cdot |\gamma w_3 - \gamma w_1|_z,$$

which implies there exists some $v_1 > 0$ such that for $0 < |z| \ll 1$,

$$\mathfrak{a}_z(D_{\mathbf{b},z}) \subset |z|^{v_1} D_{\mathbf{a},z},$$

in terms of the absolute norm. □

4. Effective convergence of zeta function

Let $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ be a family of Schottky groups satisfying (★). In this section, we establish a logarithmic rate for the convergence $Z(\Gamma, s/\log(1/|z|)) \to Z_I(\Gamma, s)$.

Theorem 4.1. *Given any bounded region $K \subset \mathbb{C}$, we have for any $0 < |z| < 1/e$,*

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_I(\Gamma, s)| \lesssim_K \frac{1}{\log(1/|z|)} \quad \textit{for} \quad s \in K.$$

4.1. Expansion of lengths

The key to obtaining an effective convergence result such as Theorem 4.1 lies in establishing the following expansion of $\ell(\gamma_z)$ with the leading term $\ell^{na}(\gamma)\log(1/|z|)$ and exponential bounds for the coefficients $a_j(\gamma)$.

Proposition 4.2. *There exists a constant $C > 0$ that depends only on Γ such that for any $\gamma \in \Gamma\backslash\{id\}$, we have that for any $0 < |z| < e^{-C\ell^{na}(\gamma)}$, the function $\ell(\gamma_z)$ in z has the expansion*

$$\ell(\gamma_z) = \ell^{na}(\gamma)\log(1/|z|) + \Re\left(\sum_{j\ge 0} a_j(\gamma) z^j\right), \tag{4.1}$$

where the $a_j(\gamma)$'s are complex numbers satisfying

$$|a_0(\gamma)| \le C\ell^{na}(\gamma) \quad \textit{and} \quad |a_j(\gamma)| \le Ce^{C\ell^{na}(\gamma)(j+1)} \quad \textit{for} \quad j \in \mathbb{N}. \tag{4.2}$$

The proof of Proposition 4.2 is based on the following Laurent expansion of the trace.

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Proposition 4.3. *There exists a constant $C > 0$ that depends only on Γ such that for any $\gamma \in \Gamma\backslash\{id\}$, the function $\mathrm{tr}(\gamma_z)$ in z has the Laurent expansion*

$$\mathrm{tr}(\gamma_z) = z^{-\ell^{na}(\gamma)/2}\left(\sum_{j\geq 0} A_j(\gamma) z^j\right), \tag{4.3}$$

where the $A_j(\gamma)$'s are complex numbers satisfying

$$|A_0(\gamma)| \geq e^{-C\ell^{na}(\gamma)}/C \quad \text{and} \quad |A_j(\gamma)| \leq Ce^{j+C\ell^{na}(\gamma)} \quad \text{for} \quad j \in \mathbb{Z}_{\geq 0}. \tag{4.4}$$

Proof. Since each entry of γ is an element of $\mathbb{C}((t))$, $\mathrm{tr}(\gamma)$ is also an element of $\mathbb{C}((t))$, and the first term on the right-hand side of equation (4.3) is obtained using the definition of $\ell^{na}(\gamma)$ (Definition 2.16).

For the upper bound of $A_j(\gamma)$, recall that $\{\mathfrak{a}_k,\ 1 \leq k \leq 2g\}$ is a finite symmetric generating set of Γ. Write $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_m}$, and we may assume that the word on the right-hand side is cyclically reduced, since $\mathrm{tr}(\gamma_z)$ is invariant under conjugation. It follows from Lemma 3.7 that $m \lesssim \ell^{na}(\gamma)$. Each entry $\gamma(i,l)$ of γ is a sum of at most 2^m terms, with each term being a product of m entries of generators. By subadditivity and submultiplicativity of the hybrid norm, the entries of γ satisfy[10]

$$\sup_{1\leq i,l\leq 2} \|\gamma(i,l)\|_{\mathrm{hyb}} \leq 2^m \sup_{\substack{1\leq k\leq 2g\\ 1\leq i,l\leq 2}} \{\|\mathfrak{a}_k(i,l)\|_{\mathrm{hyb}}\}^m \leq e^{C\ell^{na}(\gamma)}.$$

Each entry of γ has the form $\sum_{j\geq j_0} c_j z^j$. By the definition of the hybrid norm, we have $|c_j| \leq \|\sum_{j\geq j_0} c_j z^j\|_{\mathrm{hyb}} e^j$. Combining these two estimates, we obtain the upper bound of $A_j(\gamma)$.

Now, we prove the lower bound. By Lemma 3.5, there exists $c > 0$ such that for any conjugacy class $[\gamma]$ with $\ell^{na}(\gamma) > 1/c$ and $0 < |z| < c$, we have

$$\ell(\gamma_z) \geq c \log(1/|z|)\ell^{na}(\gamma).$$

Let $\lambda_1(\gamma_z), \lambda_2(\gamma_z)$ be the two eigenvalues of γ_z with $|\lambda_1(\gamma_z)| > 1$. Notice that $2\log|\lambda_1(\gamma_z)| = \ell(\gamma_z) = -2\log|\lambda_2(\gamma_z)|$. For $0 < |z| < c$,

$$|\mathrm{tr}(\gamma_z)| \geq e^{\ell(\gamma_z)/2} - e^{-\ell(\gamma_z)/2} \geq e^{c\log(1/|z|)\ell^{na}(\gamma)/2} - e^{-c\log(1/|z|)\ell^{na}(\gamma)/2} \geq \frac{1}{2}(1/|z|)^{c\ell^{na}(\gamma)/2} \geq 1. \tag{4.5}$$

Recall that

$$\mathrm{tr}(\gamma_z) = z^{-\ell^{na}(\gamma)/2}(A_0(\gamma) + A_1(\gamma)z + \cdots). \tag{4.6}$$

Let $f(z) = A_0(\gamma) + A_1(\gamma)z + \cdots$, which is an analytic function on $|z| < 1/e$. Using equations (4.5) and (4.6), we obtain for $0 < |z| < c$,

$$\log|f(z)| \geq -\frac{1}{2}\ell^{na}(\gamma)\log(1/|z|).$$

Since γ_z is loxodromic for all $z \in \mathbb{D}^*$, the analytic function $f(z)$ has no zeros in $\mathbb{D}$. Applying the maximum principle to the harmonic function $\log|f(z)|$, we have

$$\log|A_0(\gamma)| = \log|f(0)| \geq \min_{|z|=c/2} \log|f(z)| \geq -\frac{1}{2}\ell^{na}(\gamma)\log(2/c).$$

Therefore, we obtain an exponential lower bound of $|A_0(\gamma)|$ in terms of $\ell^{na}(\gamma)$. □

[10]The finiteness of the hybrid norm follows from the fact that the entries are meromorphic functions on the unit disc with a possible pole at 0.

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Proof of Proposition 4.2. We use the same notations as in the proof of Proposition 4.3. We have $\lambda_1(\gamma_z) = \mathrm{tr}(\gamma_z)\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mathrm{tr}(\gamma_z)^2}}\right)$[11] is a meromorphic function at 0. This is because equation (4.5) gives $\lambda_2(\gamma_z)/\lambda_1(\gamma_z) \to 0$ as $z \to 0$, which implies $\log(\frac{1}{4} - \frac{1}{\mathrm{tr}(\gamma_z)^2}) = \log\left(\frac{1-\lambda_2(\gamma_z)/\lambda_1(\gamma_z)}{2(1+\lambda_2(\gamma_z)/\lambda_1(\gamma_z))}\right)^2 = 2\log\left(\frac{1-\lambda_2(\gamma_z)/\lambda_1(\gamma_z)}{2(1+\lambda_2(\gamma_z)/\lambda_1(\gamma_z))}\right)$ is holomorphic in a neighborhood of 0. We continue to use equation (4.3) to obtain

$$\begin{aligned}\lambda_1(\gamma_z) &= z^{-\ell^{na}(\gamma)/2}A_0(\gamma)\left(1 + \sum_{j\geq 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right)\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{z^{\ell^{na}(\gamma)}}{A_0(\gamma)^2}\left(1 + \sum_{j\geq 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right)^{-2}}\right)\\ &=: z^{-\ell^{na}(\gamma)/2}A_0(\gamma)\cdot f(z)\cdot g(z).\end{aligned}$$

By the previous estimate, we have $|A_j(\gamma)/A_0(\gamma)| \leq e^{C\ell^{na}(\gamma)+j}$ and $|A_0(\gamma)|^{-1} \leq e^{C\ell^{na}(\gamma)}$ for some $C > 0$. Thus, for $|z| < e^{-10C\ell^{na}(\gamma)}/20$, we have the bounds

$$\left|\sum_{j\geq 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right| \leq 2e^{2C\ell^{na}(\gamma)}|z| \leq \frac{1}{10}, \quad \frac{|z|}{|A_0(\gamma)|^2} \leq e^{2C\ell^{na}(\gamma)}|z| \leq \frac{1}{10}.$$

This implies that on the disc $|z| < e^{-10C\ell^{na}(\gamma)}/20$

$$\log(f(z)g(z)) = \log(f(z))\log(g(z)).$$

Moreover, we apply Cauchy's integral formula to obtain upper bounds of the form of equation (4.2) for the coefficients of the power series expansion of $\log(f(z))$ and $\log(g(z))$. We complete the proof by using the equation

$$\begin{aligned}\ell(\gamma_z) &= 2\log|\lambda_1(\gamma_z)| = \ell^{na}(\gamma)\log(1/|z|) + 2\log|A_0(\gamma)| + 2\log|f(z)| + 2\log|g(z)|\\ &= \ell^{na}(\gamma)\log(1/|z|) + 2\log|A_0(\gamma)| + 2\Re\left(\sum_{j\geq 1} b_j z^j\right),\end{aligned} \tag{4.7}$$

where $\sum_{j\geq 1} b_j z^j$ is the power series expansion of $\log(f(z)) + \log(g(z))$ about 0. □

4.2. Convergence to intermediate zeta functions: statement of results

To obtain a better convergence rate, we introduce the following *intermediate zeta functions*: for each $M \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{D}^*$,

$$Z_M(\Gamma, z, s) = \prod_{[\gamma]\in\mathcal{P}}(1 - e^{-s\ell_M(\gamma,z)}), \tag{4.8}$$

where $\ell_M(\gamma, z)$ is the *M-th expansion of length* defined by

$$\ell_M(\gamma, z) = \ell^{na}(\gamma) + \Re\Big(\sum_{j=0}^{M} a_j(\gamma)z^j/\log(1/|z|)\Big) \tag{4.9}$$

with the $a_j(\gamma)$'s the coefficients given in the expansion (4.1).

[11] On $\mathbb{C}\backslash(-\infty, 0]$, we choose the principal branch of $\log z$: $\log z = \log|z| + i\arg z$ with $|\arg z| < \pi$; set $\sqrt{z} = e^{\log z/2}$.

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A priori, an intermediate zeta function is defined for $\Re s$ large. In Section 5, we will prove that it admits an analytic extension to $s \in \mathbb{C}$.

We establish the following polynomial rate for the convergence of the $\log(1/|z|)$-rescaled zeta functions to an intermediate zeta function.

Theorem 4.4. *For any $C > 0$, $M \in \mathbb{Z}_{\geq 0}$, and $\epsilon > 0$, we have for any $0 < |z| < 1/e$,*

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon} \quad \text{for} \quad |\Re s| \leq C,\ |\Im s| \leq C|z|^{-M}.$$

As a corollary, we obtain the convergence of zeros of zeta functions: in particular, the convergence of the first zeros gives the convergence of the Hausdorff dimensions of limit sets.

Corollary 4.5. *Let $R > 0$, $\epsilon > 0$, and $M \in \mathbb{Z}_{\geq 0}$. Suppose that no zero of $Z_I(\Gamma, s)$ lies on the boundary of $D_R := \{s : |s| < R\}$. Let*

- *$\rho_1, \cdots, \rho_{A_z}$ be the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ in D_R;*
- *$\rho_1^M, \cdots, \rho_{B_z}^M$ be the zeros of $Z_M(\Gamma, z, s)$ in D_R;*
- *$\rho_1^{na}, \cdots, \rho_N^{na}$ be the zeros of $Z_I(\Gamma, s)$ in D_R.*[12]

Then there exists $t_0 > 0$ such that for $0 < |z| < t_0$, we have $A_z = B_z = N$, and we can order the zeros such that for each $j \in \mathbb{N} \cap [1, N]$

$$|\rho_j - \rho_j^M| \leq |z|^{\frac{1}{m_j^{na}} - \epsilon},$$

where m_j^{na} is the multiplicity of ρ_j^{na} as a zero of $Z_I(\Gamma, s)$.

Corollary 4.6. *Let $C > 0$, $\epsilon > 0$, and $M \in \mathbb{Z}_{\geq 0}$. Let $D \subset [-C, C] + i[-C|z|^{-M}, C|z|^{-M}]$ be an open connected set with its boundary a simple closed curve and*

- *$\rho_1, \cdots, \rho_{A_z}$ be the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ in D;*
- *$\rho_1^M, \cdots, \rho_{B_z}^M$ be the zeros of $Z_M(\Gamma, z, s)$ in D.*

Suppose $|Z_M(\Gamma, z, s)| \geq |z|^{1-\epsilon}$ on ∂D for $0 < |z| < t_0$. Then there exists $t_1 \in (0, t_0)$ such that for $0 < |z| < t_1$, we have $A_z = B_z$.

The difference between Corollary 4.5 and Corollary 4.6 lies in the regions where we compare $Z(\Gamma_z, s/\log(1/|z|))$ and $Z_M(\Gamma, z, s)$. Corollary 4.6 will be used in Example 6.8 to recover [34] on the structure of the zeros with large imaginary parts of the Selberg zeta function for a three-funnel surface.

4.3. Convergence to intermediate zeta functions: proofs

In this section, we prove the effective convergence results of Theorem 4.1, Theorem 4.4, Corollary 4.5, and Corollary 4.6.

First, we recall the Hadamard three-circle theorem.

Proposition 4.7. *Let $r_1 < r_2 < r_3 \in (0, \infty)$ and set $\alpha = \frac{\log(r_3/r_2)}{\log(r_3/r_1)}$. Let $f : \{z : r_1 \leq |z - a| \leq r_3\} \to \mathbb{C}$ be a holomorphic function with $a \in \mathbb{C}$. Then*

$$\max_{|z-a|=r_2} |f(z)| \leq \left(\max_{|z-a|=r_1} |f(z)|\right)^{\alpha} \left(\max_{|z-a|=r_3} |f(z)|\right)^{1-\alpha}.$$

Proof. This follows from the fact that $\log|f(z)|$ is a subharmonic function. □

The strategy of the proof of Theorem 4.4 is to prove convergence in the region $\Re s \gg 1$ and then use Proposition 4.7 to obtain convergence in a larger region.

[12] Our convention is to list zeros according to their multiplicities.

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We need a lower bound for the function $\ell_M(\gamma, z)$ given in equation (4.9), whose proof will be provided at the end of Section 5.2.

Lemma 4.8. *There exists $C > 1$ depending only on Γ such that for any $M \in \mathbb{Z}_{\geq 0}$ and for any $\gamma \in \Gamma\backslash\{id\}$, we have for any $0 < |z| \leq e^{-C(M+1)}$,*

$$\ell_M(\gamma, z) \geq \ell^{na}(\gamma)/2.$$

Proposition 4.9. *There exist $s_0 > 1$ and $C_\Gamma > 0$ depending on Γ, such that for any $M \in \mathbb{Z}_{\geq 0}$, we have for any $\log(1/|z|) > C_\Gamma(M+1)$ and any $C_0 > 0$,*

$$\left|\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_M(\Gamma, z, s)} - 1\right| \lesssim_{C_0} |z| \quad \text{for} \quad \Re s > (M+1)s_0,\ |\Im s| \leq C_0|z|^{-M}. \tag{4.10}$$

Proof. We establish the convergence of geodesic lengths. We use C to refer to a constant depending only on Γ, which may vary from line to line.

We first deal with the case $\Re s < 10\log(1/|z|)$. Due to Proposition 4.2, for each $\gamma \in \Gamma\backslash\{id\}$ and $0 < |z| < e^{-C\ell^{na}(\gamma)}$,

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + \Re\left(\sum_{j\geq 0} a_j(\gamma) z^j\right)/\log(1/|z|), \tag{4.11}$$

where the $a_j(\gamma)$'s are complex numbers satisfying

$$|a_0(\gamma)| \leq C\ell^{na}(\gamma), \quad |a_j(\gamma)| \leq Ce^{C\ell^{na}(\gamma)(j+1)}. \tag{4.12}$$

By equation (4.11) and equation (4.12), we have for $\log(1/|z|) \gg C$ and $\ell^{na}(\gamma) \ll C^{-1}\log(1/|z|)$,

$$\ell_M(\gamma, z) \geq \ell^{na}(\gamma)/2, \quad \ell(\gamma_z)/\log(1/|z|) \geq \ell^{na}(\gamma)/2, \tag{4.13}$$

and

$$|\ell_M(\gamma, z) - \ell(\gamma_z)/\log(1/|z|)| \leq C(e^{C\ell^{na}(\gamma)}|z|)^{M+1}/\log(1/|z|). \tag{4.14}$$

Using the following basic inequality: for $|z_1|, |z_2| < 1/2$,

$$|\log(1-z_1) - \log(1-z_2)| \leq 2|z_1 - z_2|, \tag{4.15}$$

we have for $1 \ll \Re s < 10\log(1/|z|)$ and $\ell^{na}(\gamma) \leq \epsilon_0 \log(1/|z|)$ with $\epsilon_0 > 0$ sufficiently small,

$$\begin{aligned}
&\left|\log(1 - e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1 - e^{-s\ell_M(\gamma,z)})\right| \lesssim \left|e^{-s\ell(\gamma_z)/\log(1/|z|)} - e^{-s\ell_M(\gamma,z)}\right| \\
&= e^{-\Re s\ell_M(\gamma,z)}\left|1 - e^{-s\ell(\gamma_z)/\log(1/|z|) + s\ell_M(\gamma,z)}\right| \\
&\lesssim e^{-\Re s\ell^{na}(\gamma)/2}\frac{|s(e^{C\ell^{na}(\gamma)}|z|)^{M+1}|}{\log(1/|z|)} \leq e^{(C(M+1)-\Re s)\ell^{na}(\gamma)/2}\frac{|sz^{M+1}|}{\log(1/|z|)},
\end{aligned}$$

where, to obtain the second-to-last inequality, we use the inequality

$$|1 - e^z| \lesssim |z| \quad \text{for} \quad |\Re z| \leq 1,$$

because equation (4.14) and the assumptions $\Re s < 10\log(1/|z|)$ and $\ell^{na}(\gamma) \leq \epsilon_0\log(1/|z|)$ give

$$|\Re s(e^{C\ell^{na}(\gamma)}|z|)^{M+1}|/\log(1/|z|) \leq 10e^{C\ell^{na}(\gamma)}|z| \leq 1.$$

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Meanwhile, for $\log(1/|z|) \geq C(M+1)$ and $\ell^{na}(\gamma) > \epsilon_0 \log(1/|z|)$, by Lemma 3.5 and Lemma 4.8, we have

$$|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)})| \lesssim e^{-c\Re s \ell^{na}(\gamma)}, \quad |\log(1-e^{-s\ell_M(\gamma,z)})| \lesssim e^{-c\Re s \ell^{na}(\gamma)}, \tag{4.16}$$

where we use $|\log(1-e^{-z})| \lesssim e^{-\Re z}$ for $\Re z > 1$. Thus, for $\log(1/z) \geq C(M+1)$, and for $(M+1)s_0 \leq \Re s < 10\log(1/|z|)$ and $|\Im s| \leq C_0|z|^{-M}$, we have

$$\begin{aligned}
&\sum_{[\gamma]\in\mathcal{P}} |\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1-e^{-s\ell_M(\gamma,z)})| \\
&\lesssim \frac{|sz^{M+1}|}{\log(1/|z|)}\left(\sum_{\ell^{na}(\gamma)\leq\epsilon_0\log(1/|z|)} e^{(C(M+1)-\Re s)\ell^{na}(\gamma)/2}\right) + \sum_{\ell^{na}(\gamma)>\epsilon_0\log(1/|z|)} e^{-c\Re s\ell^{na}(\gamma)} \\
&\lesssim \frac{|sz^{M+1}|}{\log(1/|z|)} + |z| \lesssim_{C_0} |z|,
\end{aligned} \tag{4.17}$$

where we used the counting result (3.7) to obtain the last inequality.

Now, we treat the part of the $k \geq 1$ terms in the Selberg zeta function. We only provide the details for the $\mathrm{SL}_2(\mathbb{R})$ case here, and the $\mathrm{SL}_2(\mathbb{C})$ case is similar. For $\Re s \gg 1$, Lemma 3.5 and equation (4.13) allow us to use equation (4.15) with $z_2 = 0$ to obtain

$$\begin{aligned}
\sum_{k\geq1}(k+1)|\log(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)})| &\lesssim \sum_{k\geq1}(k+1)e^{-(\Re s/\log(1/|z|)+k)\ell(\gamma_z)} \\
&\leq 2e^{-(\Re s/\log(1/|z|)+1)\ell(\gamma_z)}/(1-e^{-\ell(\gamma_z)})^2.
\end{aligned} \tag{4.18}$$

Let $\log(1/|z|) \gg C$ and $\Re s \gg C$. We have the following two cases.

- For $\ell^{na}(\gamma) \leq \epsilon_0 \log(1/|z|)$ with $\epsilon_0 > 0$ sufficiently small, by equation (4.11), we have

$$\ell(\gamma_z) \geq \log(1/|z|)\left(\ell^{na}(\gamma) - \frac{|a_0(\gamma)|}{\log(1/|z|)} + O\left(\frac{e^{C\ell^{na}(\gamma)}|z|}{\log(1/|z|)}\right)\right) \geq \left(1-\frac{2C}{\log(1/|z|)}\right)\log(1/|z|)\ell^{na}(\gamma).$$

  This gives[13]

$$(\Re s/\log(1/|z|)+1)\ell(\gamma_z) \geq (\Re s/\log(1/|z|)+1)\left(1-\frac{2C}{\log(1/|z|)}\right)\log(1/|z|)\ell^{na}(\gamma) \geq \log(1/|z|)\ell^{na}(\gamma).$$

- For $\ell^{na}(\gamma) > \epsilon_0 \log(1/|z|)$, we can continue to apply Lemma 3.5 to equation (4.18) to obtain an upper bound in terms of $\ell^{na}(\gamma)$.

In conclusion, for $\log(1/|z|) \gg C$ and $\Re s \gg C$, we obtain

$$\begin{aligned}
\sum_{[\gamma]\in\mathcal{P}}\sum_{k\geq1}(k+1)|\log(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)})| \lesssim &\sum_{\ell^{na}(\gamma)\leq\epsilon_0\log(1/|z|)} e^{-\ell^{na}(\gamma)\log(1/|z|)} \\
&+ \sum_{\ell^{na}(\gamma)>\epsilon_0\log(1/|z|)} e^{-c(\Re s+\log(1/|z|))\ell^{na}(\gamma)} \lesssim |z|,
\end{aligned} \tag{4.19}$$

where we again used the counting result (3.7) to obtain the last inequality.

[13]For $\Re s, \log(1/|z|) \geq 4C$, we have $(\Re s/\log(1/|z|)+1)(1-2C/\log(1/|z|)) \geq 1+2C/\log(1/|z|)-8C^2/\log(1/|z|)^2 \geq 1$.

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For $\log(1/|z|) \geq C(M+1)$ and $(M+1)s_0 < \Re s < 10\log(1/|z|)$, we obtain equation (4.10) by combining equation (4.17) and equation (4.19). For $\Re s \geq 10\log(1/|z|)$, it remains to bound the right-hand side of the following inequality by $|z|$:

$$\left|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1-e^{-s\ell_M(\gamma,z)})\right| \leq \left|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)})\right| + \left|\log(1-e^{-s\ell_M(\gamma,z)})\right|.$$

This can be achieved by an argument similar to the proof of equation (4.19). □

Now, we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. Fix any $C > 1$, $M \in \mathbb{Z}_{\geq 0}$, and $\epsilon > 0$. Recall the constants $s_0 > 0$ and $C_\Gamma > 0$ in Proposition 4.9.

Set $R = (M+1)s_0 + 1$, and let $C_\epsilon > 1$ be a constant satisfying $C_\epsilon > \max\{10C, R\}$ and

$$\log(R+10C)/\log(R+C_\epsilon) < \epsilon.$$

It suffices to prove the proposition for $\log(1/|z|) > C_\Gamma(M+1)$. This is because the zeta functions depend continuously on the parameters z and s, thus uniformly bounded for $s \in [-C, C] + i[-C|z|^{-M}, C|z|^{-M}]$ and $\log(1/|z|) \leq C_\Gamma(M+1)$ depending on C, M, ϵ.

For any $K_T := [-C, C] + i[T, T+1]$ with $T \in [-C|z|^{-M}, C|z|^{-M}]$, we choose three concentric discs $D_1 \subset D_2 \subset D_3$ with radii $r_1 < r_2 < r_3$ such that $D_1 \subset \{s : \Re s > (M+1)s_0, |\Im s| \leq (C+2)|z|^{-M}\}$, $D_2 \supset K_T$, and $D_3 \subset \{s : \Re s > -C_\epsilon, |\Im s| \leq (C+R+C_\epsilon+1)|z|^{-M}\}$ (see Figure 1). In particular, we can take

$$D_1 = D(a, R-(M+1)s_0), \quad D_2 = D(a, R+10C), \quad D_3 = D(a, R+C_\epsilon),$$

where $a = R + iT$.

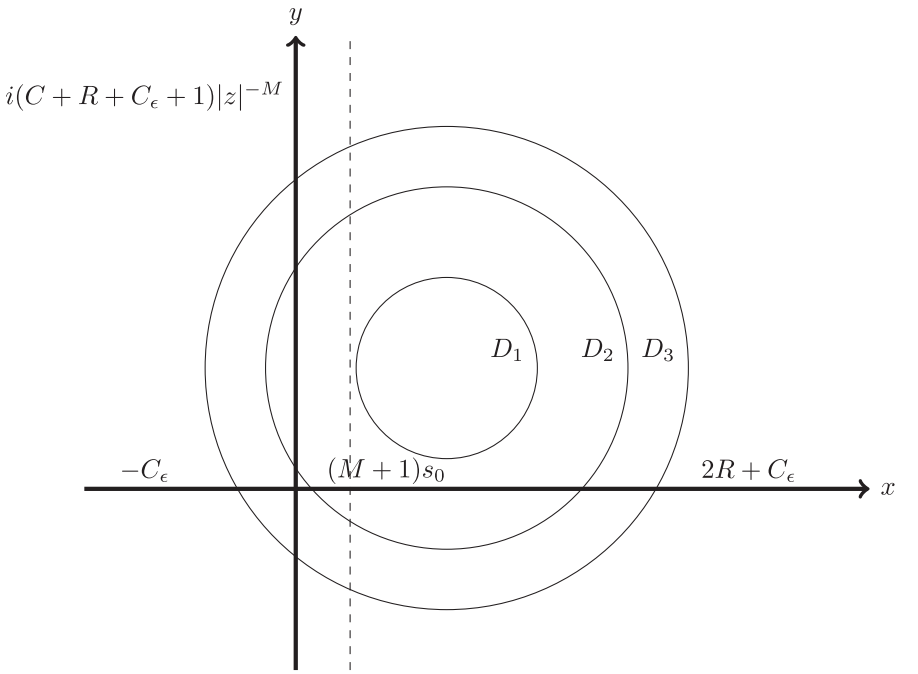


Figure 1. *Choice of discs.*

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By Corollary 5.8, the function

$$F_z(s) = Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)$$

is an entire function. By Proposition 4.9, $F_z(s)$ satisfies the bound $|F_z(s)| \lesssim_C |z|$ for $s \in D_1$ and $s \in D_3 \cap \{s : \Re s > (M+1)s_0\}$. By Proposition 3.12 and Proposition 3.15, $Z(\Gamma_z, s/\log(1/|z|))$ is uniformly bounded on $D_3 \cap \{s : \Re s \le (M+1)s_0\}$. Meanwhile, Corollary 5.8 shows that $Z_M(\Gamma, z, s)$ is uniformly bounded on $D_3 \cap \{s : \Re s \le (M+1)s_0\}$. Therefore, the function $F_z(s)$ is uniformly bounded on D_3, and from Proposition 4.7, we conclude

$$|F_z(s)| \lesssim_{C,M,\epsilon} |z|^{\log(r_3/r_2)/\log(r_3/r_1)}, \quad s \in \partial D_2.$$

The same bound also holds for $s \in D_2$ by the maximum modulus principle since $F_z(s)$ is an entire function. The statement follows from

$$\log(r_3/r_2)/\log(r_3/r_1) = 1 - \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} = 1 - \log(R+10C)/\log(R+C_\epsilon) > 1-\epsilon. \qquad \square$$

Proof of Theorem 4.1. Taking $M = 0$ in Theorem 4.4, we have for $0 < |z| < 1/e$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_0(\Gamma, z, s)| \lesssim_C |z|^{1/2}, \quad \text{for } |\Re s| \le C,\ |\Im s| \le C.$$

It suffices to prove that

$$|Z_0(\Gamma, z, s) - Z_I(\Gamma, s)| \lesssim_K \frac{1}{\log(1/|z|)}, \quad s \in K.$$

By Corollary 5.8,

$$Z_0(\Gamma, z, s) = P_0(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)}),$$

where each $\mu_j(z) = a_j + a_{j,0}/\log(1/|z|)$. By Corollary 5.9,

$$Z_I(\Gamma, s) = P_0(e^{sa:1}, \cdots, e^{sa_J}).$$

Since P_0 is smooth, we conclude that

$$|P_0(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)}) - P_0(e^{sa:1}, \cdots, e^{sa_J})| \lesssim_K \sup_j |\mu_j(z) - a_j| \lesssim_K \frac{1}{\log(1/|z|)}, \quad s \in K.$$

The proof is complete. $\square$

Proof of Corollary 4.5. Given a disc D_R that satisfies the assumption, let ρ^{na} be a zero of $Z_I(\Gamma, s)$ in D_R of multiplicity m. Then there exists a small disc D_0 centered at ρ^{na} such that ρ^{na} is the only zero of $Z_I(\Gamma, s)$ in $\bar{D}_0$. We can write

$$Z_I(\Gamma, s) = (s - \rho^{na})^m f(s),$$

where $f(s)$ is an entire function nonvanishing on D_0.

Since $Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s)$ uniformly for $s \in \partial D_0$, there exists $t_1 > 0$ such that for $0 < |z| < t_1$, we have

$$|Z_I(\Gamma, s)| > |Z_I(\Gamma, s) - Z(\Gamma_z, s/\log(1/|z|))| \quad \text{for} \quad s \in \partial D_0.$$

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By Rouché's theorem, $Z_I(\Gamma, s)$ and $Z(\Gamma_z, s/\log(1/|z|))$ have the same number of zeros in D_0. We can write

$$Z(\Gamma_z, s/\log(1/|z|)) = (s-\rho_1)\cdots(s-\rho_m)f_z(s),$$

where $f_z(s)$ is an entire function nonvanishing on D_0. We claim that $f_z(s) \to f(s)$ uniformly on $\bar{D}_0$. Write $D_0 = D(\rho^{na}, r)$. By the uniform convergence $Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s)$ on $\bar{D}_0$, for any $\epsilon > 0$, there exists $t(\epsilon) > 0$ such that for $0 < |z| < t(\epsilon)$, we have

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_I(\Gamma, s)| < \min\left\{\epsilon, \inf_{\epsilon \le |s-\rho^{na}| \le r} |Z_I(\Gamma, s)|\right\} \quad \text{for} \quad s \in \bar{D}_0.$$

Therefore, $Z(\Gamma_z, s/\log(1/|z|))$ has no zero in $\{s : \epsilon \le |s-\rho^{na}| \le r\}$, which implies $|\rho_j - \rho^{na}| < \epsilon$ for each ρ_j. For $s \in \bar{D}_0$,

$$|f_z(s) - f(s)| \le \sup_{s\in\partial D_0} |Z(\Gamma_z, s/\log(1/|z|))/((s-\rho_1)\cdots(s-\rho_m)) - Z_I(\Gamma, s)/(s-\rho^{na})^m| \lesssim \epsilon.$$

As a consequence, there exist $C > 1$ and $t_2 > 0$ such that for $0 < |z| < t_2$, we have $C^{-1} < |f_z(s)| < C$ for $s \in D_0$.

Similarly, since $Z_M(\Gamma, z, s) \to Z_I(\Gamma, s)$, there exists $t_3 > 0$ such that for $0 < |z| < t_3$, $Z_M(\Gamma, z, s)$ has m zeros in D_0, and we can write

$$Z_M(\Gamma, z, s) = (s-\rho_1^M)\cdots(s-\rho_m^M)f_{M,z}(s),$$

where $f_{M,z}(s)$ is an entire function nonvanishing on D_0. For $0 < |z| < \min\{t_1, t_2, t_3\}$, consider the following functions on D_0:

$$g_z(s) = (s-\rho_1)\cdots(s-\rho_m) \quad \text{and} \quad h_z(s) = g_z(s) - Z_M(\Gamma, z, s)/f_z(s).$$

Thus, we can complete the proof by applying the following Lemma 4.10 to $g_z(s)$ and $h_z(s)$. □

Lemma 4.10. *Let $g(s) = s^d + a_1 s^{d-1} + \cdots + a_d = (s-\rho_1)\cdots(s-\rho_d)$ be a polynomial such that all of its zeros $\rho_1, \ldots, \rho_d$ are in a disc $D(\rho, \epsilon_0)$. Let $h(s)$ be a holomorphic function on $D(\rho, 3\epsilon_0)$ such that $|h(s)| \le \epsilon \le 4^{-d^3-10d}\epsilon_0^d$ for $z \in \partial D(\rho, 2\epsilon_0)$. Then $g(s)+h(s)$ also has d zeros $\rho_1', \cdots, \rho_d'$ in $D(\rho, 2\epsilon_0)$, and we can order the zeros such that for each $j \in [1, d] \cap \mathbb{N}$,*

$$|\rho_j - \rho_j'| < 4^{d^2+3}\epsilon^{1/d}.$$

Proof. Since $|g(s)| \ge \epsilon_0^d$ on $\partial D(\rho, 2\epsilon_0)$, and $|h(s)| \le \epsilon < \epsilon_0^d$, by Rouché's theorem, $g(s)$ and $g(s)+h(s)$ have the same number of zeros in $D(\rho, 2\epsilon_0)$.

We claim there exist finitely many discs D_ℓ of radius $C\epsilon^{1/d}$ for some $C \in [2, 4^{d^2+2}] \cap \mathbb{N}$ such that they cover all the ρ_j's, and $2D_\ell \cap 2D_{\ell'} = \varnothing$[14] for $\ell \ne \ell'$. Let $S = \{|\rho_i - \rho_j| : i, j = 1, 2, \cdots, d\} \subset [0, 2\epsilon_0]$. Then $\#S \le d^2$. By the pigeonhole principle, there exists $C \in [2, 4^{d^2+2}] \cap \mathbb{N}$ such that $[C\epsilon^{1/d}, 4C\epsilon^{1/d}] \cap S = \varnothing$ (note that the assumption $\epsilon \le 4^{-d^3-10d}\epsilon_0^d$ is used here to ensure $4^{d^2+3}\epsilon^{1/d} < \epsilon_0$). Then for any pair ρ_i, ρ_j, we have either $|\rho_i - \rho_j| < C\epsilon^{1/d}$ or $|\rho_i - \rho_j| > 4C\epsilon^{1/d}$. Let $D_1 = D(\rho_1, C\epsilon^{1/d})$. Choose a zero $\rho_{j_2} \notin D_1$, and let $D_2 = D(\rho_{j_2}, C\epsilon^{1/d})$. Using the inequality $|\rho_1 - \rho_{j_2}| > 4C\epsilon^{1/d}$, we obtain $2D_1 \cap 2D_2 = \varnothing$. We repeat this process finitely many times to obtain the desired family of discs.

The fact that $\epsilon \le 4^{-d^3-10d}\epsilon_0^d$ ensures $2D_\ell \subset D(\rho_*, 2\epsilon_0)$. Note that

$$|h(s)| \le \epsilon < (C\epsilon^{1/d})^d \le |g(s)| \quad \text{for} \quad s \in \partial(2D_\ell).$$

[14] Here, if $D_\ell = D(a, r)$, then $2D_\ell = D(a, 2r)$.

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By Rouché's theorem, $g(s) + h(s)$ and $g(s)$ have the same number of zeros in $2D_\ell$, and the proof is complete.

Proof of Corollary 4.6. By Rouché's theorem, it suffices to show

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| < |Z_M(\Gamma, z, s)|, \quad z \in \partial D.$$

Since we assume $|Z_M(\Gamma, z, s)| \geq |z|^{1-\epsilon}$, it follows from Theorem 4.4 that

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon/2} < |z|^{1-\epsilon}$$

for $|z|$ sufficiently small. □

Proof of Theorem 1.3 and Theorem 1.1. Theorem 1.3 is a direct consequence of Theorem 4.1 and Theorem 4.4. The estimate on the Hausdorff dimension of the limit set in Theorem 1.1 is a consequence of Corollary 4.5 and the computation of the intermediate zeta function $Z_0(\Gamma, z, s)$ for symmetric three-funnel hyperbolic surfaces in Example 6.7. □

5. Analytic extension of intermediate zeta functions

Recall the intermediate zeta functions introduced in equation (4.8):

$$Z_M(\Gamma, z, s) = \prod_{[\gamma] \in \mathcal{P}} (1 - e^{-s\ell_M(\gamma, z)}) \quad \text{for} \quad \Re s \gg 1. \tag{5.1}$$

The goal of this section is to show that $Z_M(\Gamma, z, s)$ admits an analytic extension to $\mathbb{C}$.

This can be seen as a generalization of the analytic continuation of the Ihara zeta function. We need to find good combinatorial relations between the $\ell_M(\gamma, t)$'s and then obtain a determinant formula similar to that for the Ihara zeta function.

We will first review the proof of the determinant formula for a weighted version of the Ihara zeta function. Then we use derivative cocycles from the Schottky group to obtain a useful combinatorial relation and apply the determinant formula.

5.1. *Weighted Ihara zeta function*

We briefly review the proof of the analytic continuation of the weighted Ihara zeta function. For more details, see, for example, [16, Section 3].

Let $G = (V, E)$ be a finite graph, where $E = \{e_1, \cdots, e_{2J}\}$ is the set of oriented edges of G with e_j and e_{j+J} in opposite directions. We write $e_{j+J} = e_j^{-1}$ to refer to their relation. For each edge e_j, we associate a complex number h_j. Define $W(s)$ to be a $(2J) \times (2J)$-matrix by

$$W(e_j, e_k)(s) = e^{-s(h_j + h_k)/2}$$

if the endpoint of e_j is the starting point of e_k and $e_j \neq e_k^{-1}$, and $W(e_j, e_k)(s) = 0$ otherwise.

For a nonbacktracking loop $P = (e_{i_1}, \cdots, e_{i_n})$, define $\ell_h(P) = \sum_{j=1}^{n} h_{i_j}$. The weighted Ihara zeta function is defined as a product over primitive loops P:

$$Z_I(G, h, s) = \prod_{[P] \in \mathcal{P}} (1 - e^{-s\ell_h(P)}).$$

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Recall that we use P to denote a nonbacktracking loop with a starting point at a vertex, and $[P]$ to denote a loop forgetting the starting point. For s with a large real part, the product converges absolutely. The following Proposition 5.1 gives the analytic extension of $Z_I(G,h,s)$ to the entire complex plane.

Proposition 5.1. *For $\Re s \gg 1$, we have $Z_I(G,h,s) = \det(I - W(s))$.*

Proof. We expand $\log Z_I(G,h,s)$:

$$\begin{aligned}\log Z_I(G,h,s) &= -\sum_{j=1}^{\infty}\sum_{[P]\in\mathcal{P}}\frac{1}{j}e^{-js\ell_h(P)}\\ &= -\sum_{j=1}^{\infty}\sum_{P\in\mathcal{P}}\frac{1}{j\#(P)}e^{-js\ell_h(P)}\\ &= -\sum_{n=1}^{\infty}\sum_{\substack{P\in\mathcal{P}\\ j:j\#(P)=n}}\frac{1}{n}e^{-js\ell_h(P)},\end{aligned}$$

where the sum $\sum_{P\in\mathcal{P}}$ is over primitive nonbacktracking loops P and $\#(P)$ is the number of edges in the primitive loop P.

We can also expand $\log\det(I - W(s))$:

$$\log\det(I - W(s)) = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}W(s)^n.$$

To conclude the proof, we use the identity $\mathrm{tr}W(s)^n = \sum_{\substack{P\in\mathcal{P}\\ j:j\#(P)=n}} e^{-js\ell_h(P)}$, which holds because the diagonal entries of $W(s)^n$ correspond to nonbacktracking loops of the graph G. □

5.2. M-th expansion of lengths and derivative cocycles

In this section, we prove that the intermediate zeta function $Z_M(\Gamma,z,s)$ has an analytic extension and is computable for concrete examples. We will use the geometry of the Schottky basis. Throughout this section, we fix generators $\mathfrak{a}_1,\cdots,\mathfrak{a}_g$ and a Schottky basis $D_{\mathfrak{a}_j}$ satisfying Proposition 3.20.

For computational convenience, we will moreover assume $D_{\mathfrak{a}_j}$ are contained in $D(0,C)$, that is, uniformly bounded away from ∞. This is always possible by choosing an appropriate coordinate.

The key observation is that the leading terms of the derivative are locally constant, from which we can compute the intermediate zeta function.

Consider pairs of the form (U,f) where U is an open neighborhood of 0, and f is meromorphic on U and holomorphic on $U\backslash\{0\}$. Two such pairs (U,f) and (V,g) are equivalent if there exists an open neighborhood W of 0, contained in $U\cap V$, such that $f|_W = g|_W$. A germ of meromorphic functions at 0 is an equivalence class of such a pair, and it has a representative that is a convergent complex Laurent series about 0, $t^n\sum_{j=0}^{\infty}a_jt^j$ with $n\in\mathbb{Z}, a_0\neq 0$. Let $\mathcal{M}_0$ be the ring of germs of meromorphic functions at 0, which is isomorphic to the ring of all convergent complex Laurent series about 0. The ring $\mathcal{O}_0$ of germs of holomorphic functions at 0 is defined analogously, which is isomorphic to the ring of all convergent complex power series about 0.

Fix any $M\in\mathbb{Z}_{\geq 0}$. We introduce several combinatorial operations.

- For $a = t^n\sum_{j=0}^{\infty}a_jt^j\in\mathcal{M}_0$ with $n\in\mathbb{Z}, a_0\neq 0$, we let $\mathrm{lt}_M(a)$ be the first M leading terms of a:

$$\mathrm{lt}_M(a) := t^n\sum_{j=0}^{M}a_jt^j.$$

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- We define the formal expansion of the logarithm $\mathrm{plog} : \mathcal{M}_0 \to \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z})$ by mapping $a = a_0 t^n(1 + \sum_{j=1}^{\infty} a_j t^j) \in \mathcal{M}_0$ with $n \in \mathbb{Z}$ and $a_0 \neq 0$ to

$$\mathrm{plog}(a) := -n\log(1/t) + \log a_0 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\left(\sum_{j=1}^{\infty} a_j t^j\right)^m. \tag{5.2}$$

  Here, $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(\sum_{j=1}^{\infty} a_j t^j)^m$ is the power series expansion of the holomorphic function $\log(1 + \sum_{j=1}^{\infty} a_j t^j)$ about 0, which is uniquely determined.
- We define $\mathrm{lt}'_M : \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z}) \to \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z})$ by mapping $a = n\log(1/t) + \sum_{j=0}^{\infty} a_j t^j$ to its first M leading terms

$$\mathrm{lt}'_M(a) := n\log(1/t) + \sum_{j=0}^{M} a_j t^j.$$

  And we define $\mathrm{lt}'_{-1} : \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z}) \to \mathbb{Z}\log(1/t)$ by mapping $a = n\log(1/t) + \sum_{j=0}^{\infty} a_j t^j$ to its (-1)-leading term

$$\mathrm{lt}'_{-1}(a) := n\log(1/t).$$

Lemma 5.2. *For any $M \in \mathbb{Z}_{\geq 0}$, and for $x, y \in \mathcal{M}_0$, we have*

$$\begin{aligned} lt'_M \circ \mathrm{plog} \circ lt_M(x) &= lt'_M \circ \mathrm{plog}(x), \\ lt'_M \circ \mathrm{plog} \circ lt_M(xy) &= lt'_M \circ \mathrm{plog}(x) + lt'_M \circ \mathrm{plog}(y). \end{aligned} \tag{5.3}$$

Proof. We recall the definition of plog in equation (5.2). The element $x \in \mathcal{M}_0$ can be written uniquely as $x = x_0(1 + x_1)$ with $x_1 = \sum_{j\geq 1} b_j t^j$, $x_0 = b_0 t^n$ for some $b_0 \neq 0$, $n \in \mathbb{Z}$. Then $\mathrm{plog} x = -n\log(1/t) + \log b_0 + (x_1 - x_1^2/2 + x_1^3/3 - \cdots)$. From this expansion, we see that $\mathrm{lt}'_M \circ \mathrm{plog}(x)$ is determined by x_0 and the first $M-1$ terms of x_1, and hence determined by $\mathrm{lt}_M(x)$. This completes the proof of the first equality.

The second equality uses the observation that for two convergent power series $\sum_{j\geq 1} a_j t^j$, $\sum_{l\geq 1} b_l t^l$, we have $\log((1 + \sum_{j\geq 1} a_j t^j)(1 + \sum_{l\geq 1} b_l t^l)) = \log(1 + \sum_{j\geq 1} a_j t^j) + \log(1 + \sum_{l\geq l} b_l t^l)$ in an open neighborhood of 0. □

In this subsection, it suffices to consider the action of $\mathrm{SL}_2(\mathbb{C}((t)))$ on $\mathbb{P}^1_{\mathbb{C}((t))}$, the set of classical points. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}((t)))$ and $x \neq \gamma^{-1}\infty \in \mathbb{C}((t))$, the derivative of γ at x is given by

$$\gamma' x = 1/(cx + d)^2.$$

Lemma 5.3. *Fix $M \in \mathbb{Z}_{\geq 0}$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}((t)))$ with $\gamma\infty \neq \infty$ and $x, y \in \mathbb{C}((t))\backslash\{\gamma^{-1}\infty\}$ with $|x - y|_{na} < e^{-M}|x - \gamma^{-1}\infty|_{na}$, we have*

$$lt_M(\gamma' x) = lt_M(\gamma' y).$$

Proof. It is sufficient to prove that $|\gamma'(x) - \gamma'(y)|_{na} < e^{-M}|\gamma'(x)|_{na}$, which is equivalent to $\mathrm{lt}_M(\gamma' x) = \mathrm{lt}_M(\gamma' y)$.

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We have

$$\gamma' x - \gamma' y = \frac{1}{(cx+d)^2} - \frac{1}{(cy+d)^2} = \frac{c(y-x)(c(x+y)+2d)}{(cx+d)^2(cy+d)^2}.$$

Due to $\gamma\infty \neq \infty$, we know $c \neq 0$. Therefore,

$$\frac{|\gamma' x - \gamma' y|_{na}}{|\gamma' x|_{na}} = \left|\frac{c(x-y)(c(x+y)+2d)}{(cy+d)^2}\right|_{na} = \left|\frac{(x-y)(c(x+y)+2d)}{(y+d/c)(cy+d)}\right|_{na}.$$

Due to $|x-y|_{na} < e^{-M}|x-\gamma^{-1}\infty|_{na}$, we have that

$$|y+d/c|_{na} = |y-\gamma^{-1}\infty|_{na} = |x-\gamma^{-1}\infty|_{na} > e^{M}|x-y|_{na}$$

and $|cx+d+cy+d|_{na} \leq \max\{|cx+d|_{na}, |cy+d|_{na}\} = |cy+d|_{na}$. The proof is complete. □

Let Γ be a Schottky group in $\mathrm{SL}_2(\mathbb{C}((t)))$. For $\gamma \in \Gamma\backslash\{id\}$, let $\lambda_1(\gamma)$, $\lambda_2(\gamma) \in \mathbb{C}((t))$ be its eigenvalues with $|\lambda_1(\gamma)|_{na} > |\lambda_2(\gamma)|_{na}$. Then there exists $\eta > 0$ depending on the group Γ, such that for any $0 < |z| < \eta$, the evaluation of $\lambda_1(\gamma)$ at z satisfies $\lambda_1(\gamma)(z) = \lambda_1(\gamma_z)$. Hence, $\lambda_1(\gamma)$ is a meromorphic function at 0 (see the proof of Proposition 4.2). Recall that $2\log|\lambda_1(\gamma_z)| = \ell(\gamma_z)$. Using equation (4.7) and the definition of plog, we obtain the following lemma.

Lemma 5.4. *There exists a constant $C > 0$ that depends only on Γ such that for any $\gamma \in \Gamma\backslash\{id\}$, we have for any $0 < |z| < e^{-C\ell^{na}(\gamma)}$,*

$$2\,\Re(\mathrm{plog}(\lambda_1(\gamma))(z)) = \ell^{na}(\gamma)\log(1/|z|) + \Re\left(\sum_{j\geq 0} a_j(\gamma) z^j\right),$$

where $\mathrm{plog}(\lambda_1(\gamma))(z)$ is to evaluate the series $\mathrm{plog}(\lambda_1(\gamma))$ at z, and the $a_j(\gamma)$'s are the complex coefficients given in equation (4.1).

For each $\gamma \in \Gamma\backslash\{id\}$, we define

$$L_M(\gamma) := \mathrm{lt}'_M(2\,\mathrm{plog}(\lambda_1(\gamma))).$$

Then we have that for any $z \in \mathbb{D}^*$, the evaluation of $L_M(\gamma)$ at z satisfies

$$\Re(L_M(\gamma)(z)) = \ell_M(\gamma, z)\log(1/|z|), \tag{5.4}$$

where $\ell_M(\gamma, z)$ is defined in equation (4.8), and hence we obtain $\mathrm{lt}'_{-1}(L_M(\gamma)) = \ell^{na}(\gamma)\log(1/t)$.

Proposition 5.5. *Fix any $M \in \mathbb{Z}_{\geq 0}$. There exists $C > 0$ depending only on Γ such that for $N > C(M+1)$, we have for any reduced word $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}$,*

$$lt_M(\mathfrak{a}'_{i_1}(x)) = lt_M(\mathfrak{a}'_{i_1}(y)) \quad \text{for any} \quad x, y \in \mathbb{C}((t)) \cap D_{\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}}.$$

Denote the common value by $lt_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}))$. Moreover, for any cyclically reduced word $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ with $n > N$, we have

$$-L_M(\gamma) = l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N})) + l_M(\mathfrak{a}_{i_2}, (\mathfrak{a}_{i_3}\cdots\mathfrak{a}_{i_{N+1}})) + \cdots + l_M(\mathfrak{a}_{i_n}, (\mathfrak{a}_{i_{n+1}}\cdots\mathfrak{a}_{i_{n+N-1}})), \tag{5.5}$$

where $i_{k+j} := i_{k+j-n}$ when $k+j > n$, and l_M is defined as

$$l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}}\cdots\mathfrak{a}_{i_{k+N-1}})) := lt'_M \circ \mathrm{plog} \circ lt_M(\mathfrak{a}'_{i_k}(\mathfrak{a}_{i_{k+1}}\cdots\mathfrak{a}_{i_{k+N-1}})).$$

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Remark 5.6. An easy consequence of equation (5.5) is that

$$\mathrm{lt}'_{-1}(-L_M(\gamma)) = \mathrm{lt}'_{-1}\left(\sum_{k=1}^{n} l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}}))\right) = -\ell^{na}(\gamma)\log(1/t),$$

which is useful for the holomorphic extension of the Ihara zeta function.

Remark 5.7. Proposition 5.5 gives another proof of the lower bound of the leading coefficients. In general, it is hard to get a lower bound estimate of the leading coefficients. Similar questions in the Archimedean case are important and hard. In the case of Schottky groups, the geometry and the hyperbolicity enable us to obtain such a lower bound.

Proof of Proposition 5.5. By assumption, ∞ is not contained in any Schottky disc. Due to the properties of the Schottky group, there exists c_0 such that we have a lower bound

$$d_{na}(\mathfrak{a}_i^{-1}\infty, (D_{\bar{\mathfrak{a}}_i})^c) \geq c_0$$

for all generators $\mathfrak{a}_i$. Moreover, the size of Schottky discs $D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}}$ tends uniformly to zero as the word length N goes to infinity (Lemma 3.3 for the non-Archimedean norm). Therefore, we can find N such that the radius of any disc $D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}}$ is less than $e^{-M}c_0$. By Lemma 3.3, we can take $N > C(M+1)$ for a uniform constant C depending only on Γ.

For any cyclically reduced word $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ with $n > N$, recall that $\lambda_1(\gamma), \lambda_2(\gamma) \in \mathbb{C}((t))$ are its eigenvalues with $|\lambda_1(\gamma)|_{na} > |\lambda_2(\gamma)|_{na}$, and let γ_+ be its attracting fixed point, which belongs to $\mathbb{C}((t)) \subset \mathbb{P}^1_{\mathbb{C}((t))}$. Then we have that $\gamma'(\gamma_+) = \lambda_2(\gamma)^2$. It follows from Lemma 5.4 that

$$\begin{aligned} -L_M(\gamma) = \mathrm{lt}'_M \circ \mathrm{plog}(\gamma'(\gamma_+)) &= \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(\gamma'(\gamma_+)) \\ &= \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\gamma_+)\cdots\mathfrak{a}'_{i_n}(\gamma_+)) \\ &= \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M\Big(\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\gamma_+))\cdots\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\gamma_+))\Big), \end{aligned}$$

where for the last equality we use the fact that for any $x, y \in \mathbb{C}((t))$,

$$\mathrm{lt}_M(xy) = \mathrm{lt}_M(\mathrm{lt}_M(x)\mathrm{lt}_M(y)).$$

Since γ is cyclically reduced, the attracting fixed point γ_+ is in the disc $D_\gamma \subset D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}}$, and $\gamma^{-1}\infty \in D_{\mathfrak{a}_{i_n}^{-1}}$. Due to the choice of N and Lemma 5.3, we obtain for any $y \in D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}} \cap \mathbb{C}((t))$,

$$\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\gamma_+)) = \mathrm{lt}_M(\mathfrak{a}'_{i_n}(y)).$$

For any $\mathfrak{a}_{i_j}$, we have $\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma_+ \in \mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}D_\gamma \subset D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma} \subset D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_{j+N-1}}}$, and $a_{i_j}^{-1}\infty \in D_{a_{i_j}}$. Therefore, for any $y \in D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_{j+N-1}}} \cap \mathbb{C}((t))$,

$$\mathrm{lt}_M(\mathfrak{a}'_{i_j}(\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma_+)) = \mathrm{lt}_M(\mathfrak{a}'_{i_j}(y)).$$

In conclusion, we obtain that

$$-L_M(\gamma) = \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M\Big(\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N})\cdots\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\mathfrak{a}_{i_n}\cdots\mathfrak{a}_{n+N-1}))\Big).$$

The proof is completed by applying equation (5.3) to the right-hand side of the above equation. □

As a corollary, we obtain the following.

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Corollary 5.8. *For any $M \in \mathbb{Z}_{\geq 0}$, the intermediate zeta function $Z_M(\Gamma, z, s)$ has a holomorphic extension to the entire complex plane $s \in \mathbb{C}$. More precisely, there exists a polynomial $P_M(x_1, \cdots, x_J)$ such that for any $s \in \mathbb{C}$ and $0 < |z| < 1/e$,*

$$Z_M(\Gamma, z, s) = P_M(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)}), \tag{5.6}$$

where for $j = 1, 2, \cdots, J$, $\mu_j(z)$ is of the form

$$\mu_j(z) = a_j + \frac{1}{\log(1/|z|)} \Re\left(\sum_{k=0}^{M} a_{j,k} z^k\right) \quad \text{with } a_j \in \mathbb{R},\ a_{j,k} \in \mathbb{C} \text{ for } k = 0, \ldots, M. \tag{5.7}$$

Proof. For $\gamma \in \Gamma \backslash \{id\}$, recall from equation (5.4) that for $z \in \mathbb{D}^*$, $\Re(L_M(\gamma)(z)) = \ell_M(\gamma, z)\log(1/|z|)$.

We introduce an edge matrix $W(s)$ with coordinates given by reduced words of length N, that is, words of the form $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, where $N > 0$ is the constant given in Proposition 5.5. The entries are given by

$$W(\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}, \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_{N+1}})(s) = \exp\left(\frac{s}{2\log(1/|z|)} \Re\left(l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))(z) + l_M(\mathfrak{a}_{i_2}, (\mathfrak{a}_{i_3} \cdots \mathfrak{a}_{i_{N+1}}))(z)\right)\right),$$

and the other entries are zero. Using the definition of $Z_M(\Gamma, z, s)$, Proposition 5.1, and Proposition 5.5, we can show that $\det(I - W(s))$ is the holomorphic extension of $Z_M(\Gamma, z, s)$ to the entire complex plane.

It follows from Proposition 5.5 that for any reduced word $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$,

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(\mathfrak{a}'_{i_1}(x)) \quad \text{for any } x \in D_{\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}} \cap \mathbb{C}((t)).$$

Note that $\mathfrak{a}'_{i_1}(x)$ is of the form $t^m \sum_{j \geq 0} b_j t^j$ with $m \in \mathbb{Z}$, $b_0 \neq 0$. Using the definitions of lt'_M, plog, lt_M, we can show that $\det(I - W(s))$ is a polynomial as described in the statement of the corollary. □

Remark. We define a graph $G_{N,g}$ with vertices given by

$$\{\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}} : \mathfrak{a}_{i_j} \mathfrak{a}_{i_{j+1}} \neq 1\}$$

and there is a (directed) edge between two vertices of the form $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}$ and $\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}$. We consider the weighted Ihara zeta function of this graph equipped with the edge between $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}$ and $\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}$ with weight given by $-\frac{1}{\log(1/|z|)} \Re l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$. By a similar proof as Proposition 5.1, we have $Z_M(\Gamma, z, s) = \det(I - V(s))$, where

$$V(\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}, \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})(s) = \exp\left(\frac{s}{\log(1/|z|)} \Re l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))\right),$$

which is smaller and easier to compute.

Continuing with Corollary 5.8, we prove that any two intermediate zeta functions are related in the following way.

Corollary 5.9. *Fix any $M \in \mathbb{Z}_{\geq 0}$. Let $\mu_j(z)$ be given as in equation (5.7) for $j = 1, \ldots, J$. Then we have that for any $M' \in \mathbb{Z}_{\geq 0}$ with $M' \leq M$, the holomorphic extension of $Z_{M'}(\Gamma, z, s)$ to $\mathbb{C}$ is given by*

$$Z_{M'}(\Gamma, z, s) = P_M(e^{s\mu_1^{M'}(z)}, \cdots, e^{s\mu_J^{M'}(z)}), \tag{5.8}$$

where each $\mu_j^{M'}(z)$ is the first M' leading terms of $\mu_j(z)$:

$$\mu_j^{M'}(z) = a_j + \frac{1}{\log(1/|z|)} \Re\left(\sum_{k=0}^{M'} a_{j,k} z^k\right)$$

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with a_j, $a_{j,k} \in \mathbb{C}$ defined in equation (5.7). Similarly, the holomorphic extension of the Ihara zeta function to $\mathbb{C}$ is given by

$$Z_I(\Gamma, s) = P_M(e^{sa:1}, \cdots, e^{sa:J}). \tag{5.9}$$

Proof. Recall that in the proof of Proposition 5.5, we showed that for any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, $l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$ is of the following form:

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M} b_j t^j,$$

with $m \in \mathbb{Z}$ and $b_0 \neq 0$. We have the following general formula for combinatorial operations: for $x \in \mathbb{C}((t))$,

$$\mathrm{lt}'_{M'} \circ \mathrm{plog} \circ \mathrm{lt}_{M'}(x) = \mathrm{lt}'_{M'} \circ \mathrm{plog} \circ \mathrm{lt}_M(x) = \mathrm{lt}'_{M'} \circ \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(x),$$

where the first equality is due to equation (5.3). By the definition of l_M, this implies $l_{M'}(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = \mathrm{lt}'_{M'} l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$, and hence

$$l_{M'}(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M'} b_j t^j.$$

To obtain equation (5.8), when applying the proof of Corollary 5.8 to $Z_{M'}(\Gamma, z, s)$, we consider an edge matrix $W(s)$ similar to the one for $Z_M(\Gamma, z, s)$ with l_M replaced by $l_{M'}$.

For the holomorphic extension of the Ihara zeta function, recall that for any $\gamma \in \Gamma \backslash \{id\}$, $\ell^{na}(\gamma) \log(1/t) = \mathrm{lt}'_{-1}(L_M(\gamma))$. Remark 5.6 allows us to apply the proof of Corollary 5.8 to $Z_I(\Gamma, s)$ with $l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$ replaced by $\mathrm{lt}'_{-1}\big(l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))\big)$, which yields equation (5.9). □

Proposition 5.10. *There exists $C > 0$ depending on Γ such that for any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, we have*

$$\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}(\infty)) = t^m \sum_{j=0}^{\infty} a_j t^j, \tag{5.10}$$

where the a_j's are complex numbers satisfying

$$|a_0| \geq e^{-C} \quad \text{and} \quad |a_j| \leq e^{C(N+j)} \quad \text{for} \quad j \in \mathbb{Z}_{\geq 0}.$$

Moreover, we have for any $M \in \mathbb{Z}_{\geq 0}$ and $N \geq C_1(M+1)$ (from Proposition 5.5),

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M} b_j t^j, \tag{5.11}$$

where the b_j's are complex numbers satisfying

$$|b_j| \leq e^{C(N+1)j} \quad \text{for} \quad j = 0, \ldots, M.$$

For the proof, please see Section A.2.

Proof of Lemma 4.8. By equation (5.4) and Proposition 5.5, there exists $C_1 > 0$ depending only on Γ such that for $N = \lceil C_1(M+2) \rceil$ and for any cyclically reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n}$ with $n > N$, we have

$$\log(1/|z|) \ell_M(\gamma, z) = -\Re\left(\sum_{k=1}^{n} l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}}))(z) \right).$$

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By Proposition 5.10, there exists $C_2 > 0$ depending only on Γ, such that for $k = 1, \dots, n$

$$l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}})) - \mathrm{lt}'_{-1}\big(l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}}))\big) = \sum_{j=0}^{M} b_j t^j,$$

where each b_j satisfies $|b_j| \le e^{C_2(N+1)j}$. This implies there exists $C_3 > 0$ depending only on Γ, such that for any $|z| < e^{-2C_2(N+1)} \le e^{-2C_2(C_1(M+2)+1)}$, we have

$$\left| \sum_{j=0}^{M} b_j z^j \right| \le C_3.$$

Recall Remark 5.6. Combining these together, we obtain $C > 0$ depending only on Γ such that for any $0 < |z| < e^{-C(M+1)}$,

$$\log(1/|z|)\ell_M(\gamma, z) \ge \log(1/|z|)\ell^{na}(\gamma) - nC_3 \ge \frac{1}{2}\log(1/|z|)\ell^{na}(\gamma),$$

where the second inequality is due to Lemma 3.7.

For any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n}$ with $n \le N = \lceil C_1(M+1) \rceil$, we can prove the statement for γ by using Lemma 3.7 and Proposition 4.2. □

6. Examples

Recall that $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★).

6.1. Two-generator case

In this section, we discuss the case when Γ is generated by two generators to illustrate the 0-th intermediate zeta function. In this case, we can obtain explicit expressions for the leading coefficient $A_0(\gamma)$ using the Fricke relation of $\mathrm{SL}_2(\mathbb{C}((t)))$:

$$\mathrm{tr}(gh) + \mathrm{tr}(gh^{-1}) = \mathrm{tr}(g)\mathrm{tr}(h), \quad g, h \in \mathrm{SL}_2(\mathbb{C}((t))). \tag{6.1}$$

For convenience, we denote the leading term of the trace of $\gamma \in \Gamma \backslash \{id\}$ by

$$\mathrm{lt}(\gamma) := \mathrm{lt}_0(\mathrm{tr}(\gamma)) = A_0(\gamma)t^{-\ell^{na}(\gamma)/2}$$

(see equation (4.3) for the Laurent expansion of $\mathrm{tr}(\gamma)$). Recall the 0-th expansion of length, $\ell_0(\gamma, t)$, defined in equation (4.9).

Lemma 6.1. *For any $\gamma \in \Gamma \backslash \{id\}$, we have*

$$\ell_0(\gamma, z) = 2\log|lt(\gamma)|/\log(1/|z|),$$

and hence

$$Z_0(\Gamma, z, s) = \prod_{[\gamma] \in \mathcal{P}} (1 - e^{-2s\log|lt(\gamma)|/\log(1/|z|)}).$$

Proof. Given any $\gamma \in \Gamma \backslash \{id\}$, recall that Proposition 4.3 gives

$$\mathrm{tr}(\gamma) = t^{-\ell^{na}(\gamma)/2}(A_0(\gamma) + A_1(\gamma)t + \cdots),$$

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and Proposition 4.2 and equation (4.7) give

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + 2\log|A_0(\gamma)|/\log(1/|z|) + \Re\left(\sum_{j\geq 1} a_j(\gamma)z^j/\log(1/|z|)\right).$$

The proof is complete. □

When the Schottky group Γ is generated by two generators, that is, $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \rangle$ where $\mathfrak{a}_1\mathfrak{a}_3 = \mathfrak{a}_2\mathfrak{a}_4 = id$, we have three different possibilities (Figures 2, 3, and 4) for the corresponding finite graph Σ_X that we will discuss below. (In Figures 2, 3, and 4, we use $\mathfrak{a}$ and $\mathfrak{b}$ to denote the two generators and $\mathfrak{a}_\pm$ for its attracting and repelling fixed points in $\mathbb{P}^1_{\mathbb{C}((t))}$. The tree on the left is the subtree connecting these points in $\mathbb{P}^{1,an}_{\mathbb{C}((t))}$. The graph on the right is the corresponding Mumford curve of the Schottky group.)

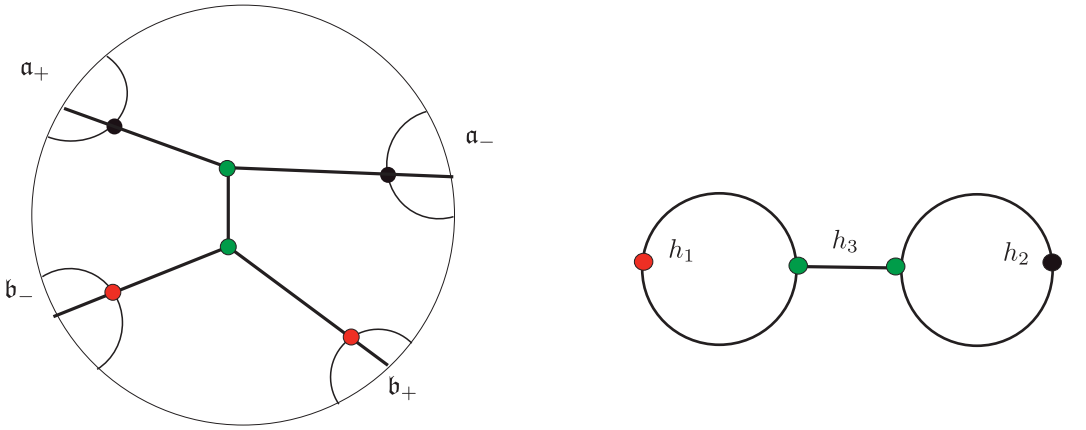


Figure 2. *Dumbbell graph.*

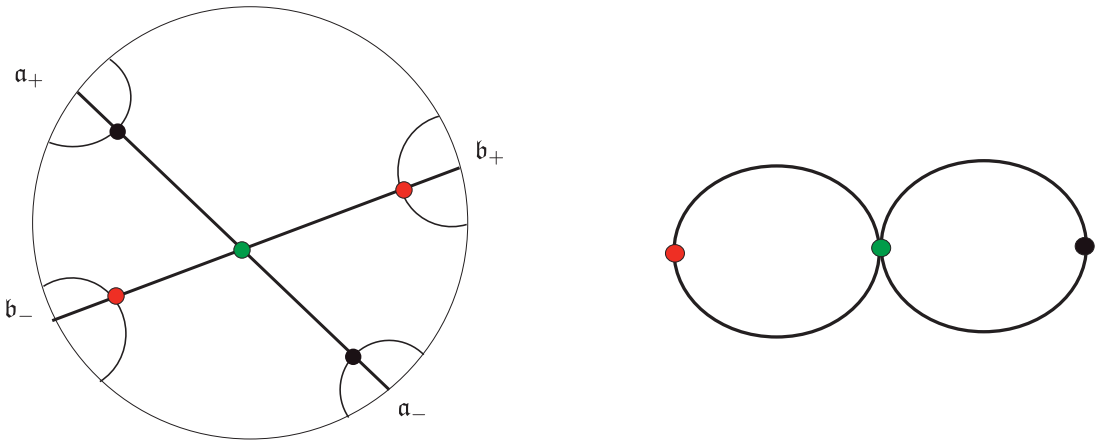


Figure 3. *Figure eight graph.*

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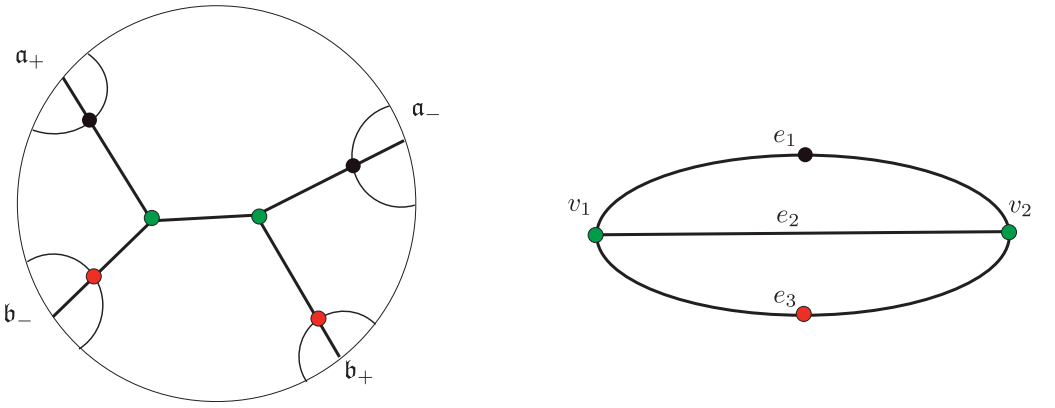


Figure 4. *Figure* Θ *graph.*

We will need the following lemma.

Lemma 6.2. *Assume* $\Gamma = \langle \mathfrak{a}_1, \dots, \mathfrak{a}_4 \rangle < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ *is a family of Schottky groups satisfying* ($\bigstar$).[15] *The following identity holds for any cyclically reduced word:*

$$lt(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n})^2 = \prod_{j=1}^{n} lt(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}) \tag{6.2}$$

with the convention that $i_{n+1} = i_1$.

Proof. Suppose we have a cyclically reduced word $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$. By equation (6.1), we have

$$\mathrm{tr}(\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2) = \mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\mathfrak{a}_1\gamma_2) - \mathrm{tr}(\gamma_2^{-1}\gamma_1).$$

Claim: We have

$$|\mathrm{tr}(\gamma_2^{-1}\gamma_1)|_{na} < |\mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\gamma_2^{-1}\mathfrak{a}_1^{-1})|_{na} = |\mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\mathfrak{a}_1\gamma_2)|_{na}. \tag{6.3}$$

Proof of the claim: The claim only works for the two-generator case, and we provide an ad hoc proof.

For the figure eight graph, we have $\ell^{na}(\mathfrak{a}_1\gamma_1) > \ell^{na}(\gamma_1)$. Since $2\log|\mathrm{tr}(\gamma)|_{na} = \ell^{na}(\gamma)$, we have

$$|\mathrm{tr}(\mathfrak{a}_1\gamma_1)|_{na} > |\mathrm{tr}(\gamma_1)|_{na}.$$

Therefore,

$$\begin{aligned} |\mathrm{tr}(\gamma_2^{-1}\gamma_1)|_{na} &= \exp(\ell^{na}(\gamma_2^{-1}\gamma_1)/2) \le \exp((\ell^{na}(\gamma_2^{-1}) + \ell^{na}(\gamma_1))/2) \\ &= |\mathrm{tr}(\gamma_1)|_{na}|\mathrm{tr}(\gamma_2^{-1})|_{na} < |\mathrm{tr}(\mathfrak{a}_1\gamma_1)|_{na}|\mathrm{tr}(\gamma_2^{-1}\mathfrak{a}_1^{-1})|_{na}. \end{aligned}$$

For the figure Θ graph (Figure 4): let h_1, h_2, h_3 be the lengths of the three edges e_1, e_2, e_3 connecting the two vertices v_1, v_2. For each generator, we use a loop with the starting point v_1 in the graph. For example, the loops of $\mathfrak{a}_1, \mathfrak{a}_2$ are represented by $e_1e_5 = e_1e_2^{-1}$ and $e_2e_6 = e_2e_3^{-1}$, respectively. For a cyclically reduced word γ, we connect the loops corresponding to each letter through v_1 to get a loop of γ. The only possible backtracking part in the loop is the middle edge in the graph Θ, which corresponds to the pairs $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_4\mathfrak{a}_3$ in γ. We have that

$$\ell^{na}(\gamma) = n_1(\gamma)(h_1 + h_2) + n_2(\gamma)(h_2 + h_3) - n_3(\gamma)(2h_2), \tag{6.4}$$

[15] This lemma also works for a general Schottky group in $\mathrm{SL}_2(\mathbb{C}((t)))$.

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with $n_1(\gamma)$ the number of $\mathfrak{a}_1, \mathfrak{a}_3$ in γ, $n_2(\gamma)$ the number of $\mathfrak{a}_2, \mathfrak{a}_4$, and $n_3(\gamma)$ the number of pairs $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_4\mathfrak{a}_3$ in γ.[16]

Lemma 6.3. *For cyclically reduced $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, we have*[17]

$$2\ell^{na}(\gamma) = \sum_{1 \le j \le N} \ell^{na}(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}). \tag{6.5}$$

Proof. It suffices to check the equality for n_1, n_2, n_3 in equation (6.4). The corresponding equality for n_1, n_2 is trivial. For n_3, let n_4 be the number of pairs of $\mathfrak{a}_2\mathfrak{a}_1$ and $\mathfrak{a}_3\mathfrak{a}_4$ in γ. Let $I = \{\mathfrak{a}_1, \mathfrak{a}_4\}$ and $II = \{\mathfrak{a}_2, \mathfrak{a}_3\}$. A cyclically reduced word corresponds to an element in the fundamental group. It can hence be represented as a loop on the graph with vertices $\{\mathfrak{a}_i, 1 \le i \le 4\}$, and all the edges are allowed except the edges $\mathfrak{a}_i\mathfrak{a}_{i+2}$. Then n_3 is the number of edges in the loop from vertices in I to II, and n_4 is the number of edges in the loop from vertices in II to I. Since γ represents a loop, we have $n_3(\gamma) = n_4(\gamma)$. Notice that $n_3(\mathfrak{a}_2\mathfrak{a}_1) = n_3(\mathfrak{a}_1\mathfrak{a}_2) = 1$. We have

$$2n_3(\gamma) = n_3(\gamma) + n_4(\gamma) = \sum_{1 \le j \le N} n_3(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}),$$

where the second equality holds because both sides represent the number of $\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{a}_4\mathfrak{a}_3$, $\mathfrak{a}_2\mathfrak{a}_1$, and $\mathfrak{a}_3\mathfrak{a}_4$ in γ. The proof is complete. □

For a noncyclically reduced word γ, we also have

$$2\ell^{na}(\gamma) \le \sum_{1 \le j \le N} \ell^{na}(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}), \tag{6.6}$$

since, to obtain a cyclically reduced representation of γ, we are doing induction, and we only need to treat cases such as $\cdots \mathfrak{b}_1\mathfrak{a}_1\mathfrak{a}_1^{-1}\mathfrak{b}_2 \cdots$.

We go back to the claim in equation (6.3). For any $\mathfrak{b}_1, \mathfrak{b}_2$ in the set of generators, we have

$$\ell^{na}(\mathfrak{b}_1\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_2) \ge \ell^{na}(\mathfrak{b}_1\mathfrak{b}_2), \tag{6.7}$$

which is true by considering all 16 possibilities of $\mathfrak{b}_1, \mathfrak{b}_2$, and the equality holds if and only if

$$\mathfrak{b}_1 = \mathfrak{a}_3 \text{ or } \mathfrak{b}_2 = \mathfrak{a}_1. \tag{6.8}$$

Therefore, suppose $\mathfrak{a}_1\gamma_1 = \mathfrak{a}_1\mathfrak{b}_1 \cdots \mathfrak{b}_2$ and $\gamma_2^{-1}\mathfrak{a}_1^{-1} = \mathfrak{b}_3 \cdots \mathfrak{b}_4\mathfrak{a}_1^{-1}$ with $\mathfrak{b}_i$ from the set of generators. Then due to equation (6.5) and equation (6.6),

$$\begin{aligned}
&2\ell^{na}(\mathfrak{a}_1\gamma_1) + 2\ell^{na}(\gamma_2^{-1}\mathfrak{a}_1^{-1}) - 2\ell^{na}(\gamma_2^{-1}\gamma_1) \\
&\quad \ge \ell^{na}(\mathfrak{a}_1\mathfrak{b}_1) + \ell^{na}(\mathfrak{b}_2\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_3) + \ell^{na}(\mathfrak{b}_4\mathfrak{a}_1^{-1}) - \ell^{na}(\mathfrak{b}_4\mathfrak{b}_1) - \ell^{na}(\mathfrak{b}_2\mathfrak{b}_3) \\
&\quad = \ell^{na}(\mathfrak{b}_1\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_4) - \ell^{na}(\mathfrak{b}_1\mathfrak{b}_4) + \ell^{na}(\mathfrak{b}_2\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_3) - \ell^{na}(\mathfrak{b}_2\mathfrak{b}_3) \ge 0,
\end{aligned}$$

where the last inequality is due to equation (6.7). Since $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$ is cyclically reduced, we obtain $\mathfrak{b}_1, \mathfrak{b}_2 \ne \mathfrak{a}_3$ and $\mathfrak{b}_3, \mathfrak{b}_4 \ne \mathfrak{a}_1$. Therefore, neither $(\mathfrak{b}_1, \mathfrak{b}_4)$ nor $(\mathfrak{b}_2, \mathfrak{b}_3)$ satisfies equation (6.8). Hence, the equality of the last equation cannot hold, and the proof is complete.

For the dumbbell graph (Figure 2), we can prove by the same method as for figure Θ: first, we show that equation (6.5) holds; then, equation (6.7) holds; finally, we can conclude that equation (6.3) holds.

[16]For example, for $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, the pair $\mathfrak{a}_{i_N}\mathfrak{a}_{i_1}$ should also be considered.
[17]It is possible that the word is simply $\mathfrak{a}_{i_1}$; then we use that $2\ell^{na}(\mathfrak{a}_{i_1}) = \ell^{na}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_1})$.

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Back to the proof: Hence,[18]

$$\mathrm{lt}(\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2) = \mathrm{lt}(\mathfrak{a}_1\gamma_1)\mathrm{lt}(\mathfrak{a}_1\gamma_2). \tag{6.9}$$

Moreover, if $\mathfrak{a}_1\gamma_1$ and $\mathfrak{a}_1\gamma_2$ satisfy equation (6.2), then by equation (6.9), the longer word $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$ also satisfies equation (6.2).

For any cyclically reduced word of length greater than 5, we can find some generator that appears at least twice. Therefore, we can use equation (6.9) to reduce the length of the word and perform induction.

The only cyclically reduced words without any generator appearing twice are the commutator $\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4$, $\mathfrak{a}_i\mathfrak{a}_j$, and $\mathfrak{a}_i$. For the commutator, we have the well-known formula from the Fricke relation

$$\begin{aligned}\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}) &= \mathrm{tr}(\mathfrak{a}_1)^2 + \mathrm{tr}(\mathfrak{a}_2)^2 + \mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)^2 - \mathrm{tr}(\mathfrak{a}_1)\mathrm{tr}(\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2) - 2\\ &= \mathrm{tr}(\mathfrak{a}_1)^2 + \mathrm{tr}(\mathfrak{a}_2)^2 - \mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2^{-1}) - 2.\end{aligned}$$

By checking the three graphs of the two-generator case, we have

$$|\mathrm{tr}(\mathfrak{a}_1)^2|_{na} = |\mathrm{tr}(\mathfrak{a}_1^2)|_{na} < |\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_2^{-1}\mathfrak{a}_1)|_{na}.$$

Hence,

$$\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}) = -\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2^{-1}).$$

The proof is complete. □

6.1.1. Figure Θ graph

We first consider the case where the graph Σ_X is similar to Θ in Figure 4. Without loss of generality, we suppose each edge of the graph Σ_X has length 2 (for example, the symmetric three-funnel case). Suppose $\mathrm{lt}(\mathfrak{a}_1) = Az^{-2}$, $\mathrm{lt}(\mathfrak{a}_2) = Bz^{-2}$, and $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2) = Cz^{-2}$ with $A, B, C \in \mathbb{C}^*$. For the weighted graph, we can define three weights by

$$h_j = 2 + \frac{2\alpha_j}{\log(1/|z|)},$$

with the α_j's determined by

$$\alpha_1 + \alpha_2 = \log|A|, \quad \alpha_2 + \alpha_3 = \log|B|, \quad \alpha_1 + \alpha_3 = \log|C|.$$

For a reduced word γ of word length two, from the choice of α_j, we can verify that the weighted length ℓ_h satisfies

$$\ell_h(\gamma) = 2\log|\mathrm{lt}(\gamma)|/\log(1/|z|) = \ell_0(\gamma, z). \tag{6.10}$$

For example, for the word $\mathfrak{a}_1\mathfrak{a}_2^{-1}$, from the Fricke relation, we have $\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2^{-1}) = \mathrm{tr}(\mathfrak{a}_1)\mathrm{tr}(\mathfrak{a}_2) - \mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)$; then $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2^{-1}) = \mathrm{lt}(\mathfrak{a}_1)\mathrm{lt}(\mathfrak{a}_2)$, which implies $\ell_0(\mathfrak{a}_1\mathfrak{a}_2^{-1}, z) = h_1 + 2h_2 + h_3 = \ell_h(\mathfrak{a}_1\mathfrak{a}_2^{-1})$. By equation (6.2), we know this relation also holds for any cyclically reduced γ. Therefore, the weighted Ihara zeta function of the weight h gives the intermediate zeta function Z_0.

For this example, by Proposition 5.1, the zeta function $Z_0(\Gamma, z, s)$ can be computed and is given by $(-2abc+a^2+b^2+c^2-1)(2abc+a^2+b^2+c^2-1)$ with $a = e^{-s(h_2+h_3)/2}$, $b = e^{-s(h_1+h_3)/2}$, $c = e^{-s(h_1+h_2)/2}$.

[18]This is not a general equation: for three generators with graph (||) consisting of two vertices and four edges between them, the sum of the lengths of $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_2^{-1}\mathfrak{a}_3^{-1}$ equals the length of $\mathfrak{a}_1\mathfrak{a}_3^{-1}$.

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6.1.2. Dumbbell graph

The case when the graph Σ_X is like a dumbbell, as in Figure 2, is similar to Section 6.1.1. Suppose each edge of the graph has a length of two. For example, $\mathrm{lt}(\mathfrak{a}_1) = Az^{-1}$, $\mathrm{lt}(\mathfrak{a}_2) = Bz^{-1}$, and $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2) = Cz^{-4}$ with $A, B, C \in \mathbb{C}^*$. We define the three weights by

$$h_j = 2 + \frac{2\beta_j}{\log(1/|z|)},$$

with the β_j's determined by

$$\beta_1 = \log|A|, \quad \beta_2 = \log|B|, \quad \beta_1 + 2\beta_3 + \beta_2 = \log|C|.$$

As in the previous case, the weighted Ihara zeta function equals the intermediate zeta function Z_0.

For this example, by Proposition 5.1, the zeta function $Z_0(\Gamma, z, s)$ can be computed and is given by $-(c-1)(c+1)(a-1)(a+1)(4a^2b^4c^2 - a^2c^2 + a^2 + c^2 - 1)$ with $a = e^{-sh_1/2}$, $b = e^{-sh_3/2}$, $c = e^{-sh_2/2}$. For $t = 0$, $s = \frac{1}{2}\log 2$ is the solution with the maximal real part.

Example 6.4. If $A = B = C = 1$, then the intermediate zeta function coincides with the Ihara zeta function.

6.1.3. Figure eight graph

The case when the graph Σ_X is like a figure eight (as shown in Figure 3) is not directly applicable to the weighted Ihara zeta function, since equation (6.10) does not hold. But using the identity (6.2), we can compute

$$\begin{aligned} -\log Z_0(\Gamma, z, s) &= \sum_{j=1}^{\infty} \sum_{[\gamma]\in\mathcal{P}} \frac{1}{j} e^{-2js\log|\mathrm{lt}(\gamma)|/\log(1/|z|)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j\#(\gamma)=n} e^{-js\log|\mathrm{lt}(\gamma)^2|/\log(1/|z|)}. \end{aligned}$$

Consider the matrix $W \in M_{12\times 12}$ given by $W(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}, \mathfrak{a}_{i_2}\mathfrak{a}_{i_3}) = \exp\left(-s\frac{\log|\mathrm{lt}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2})|+\log|\mathrm{lt}(\mathfrak{a}_{i_2}\mathfrak{a}_{i_3})|}{2\log(1/|z|)}\right)$. Then the above expression is equal to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} W^n = -\log\det(I - W).$$

Remark. This is indeed the path zeta function in Section 4 of [16]. Consider the matrix $V \in M_{4\times 4}$ given by $V(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}) = \exp\left(-s\frac{\log|\mathrm{lt}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2})|}{\log(1/|z|)}\right)$ for $\mathfrak{a}_{i_1}\mathfrak{a}_{i_2} \neq id$. Then the above expression is also equal to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} V^n = -\log\det(I - V).$$

Example 6.5. Recall the example of a funneled torus in [5, Section 16]. Here, the matrices are given by

$$S_1 = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}, \; S_2 = \begin{pmatrix} \cosh(\ell/2) - \cos\phi \;\sinh(\ell/2) & \sin^2\phi \;\sinh(\ell/2) \\ \sinh(\ell/2) & \cosh(\ell/2) + \cos\phi \;\sinh(\ell/2) \end{pmatrix}$$

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with ϕ the angle between the two shortest geodesics. We change variables by setting $z = e^{-\ell/2}$ and obtain a Schottky family.[19] Moreover,

$$\mathrm{lt}(S_1) = z^{-1},\ \mathrm{lt}(S_2) = z^{-1},\ \mathrm{lt}(S_1S_2) = (1+\cos\phi)z^{-2}/2.$$

The graph is a figure eight with edge length 2.

From the first paragraph of Section 6.1.3, with $\phi = \pi/2$, the intermediate zeta function can be computed and is given by

$$Z_0(\Gamma, z, s) = -(-a+2b+1)(a+2b-1)(a-1)^2,$$

with $a = e^{-2s},\ b = e^{-s(2-\log 2/\log(1/|z|))}$. Hence, the Hausdorff dimension δ_z of the limit set is given by the first zero of

$$e^{-2s} + 2e^{-s(2-\log 2/\log(1/|z|)} - 1 = 0$$

divided by $\log(1/|z|)$. For the case $\ell = 10$ or $z = e^{-5}$, the first zero of $Z_0(\Gamma, e^{-5}, s)$ divided by $\log(1/|z|)$ is close to 0.115, which is close to the numerics in [5, Fig 16.8]. The main term $\frac{\log 3}{10}$ of [9] gives 0.1099.

Remark 6.6. In [37, Theorem 1.1], the author considered the case of three funnels with widths given by $n_1\ell, n_2\ell, n_3\ell,\ n_1, n_2, n_3 \in \mathbb{N}$, and $\ell \to \infty$. Under an extra triangle condition $n_i + n_j > n_k$ with $\{i,j,k\} = \{1,2,3\}$, the convergence to the Ihara zeta function is proved in [37]. In fact, the triangle condition means the graph is a figure Θ. If $n_1 + n_2 = n_3$, then the graph is a figure eight. If $n_1 + n_2 < n_3$, then the graph is a dumbbell. For all these three cases, by the change of variable with $z = e^{-\ell/2}$, our computation of the Ihara zeta functions/intermediate zeta function Z_0 and the convergence all work.

We also remark that the type of graph (figure eight, dumbbell, and figure Θ) depends on the choice of the lengths of the generators. In contrast, the topological type of the surface depends on the configuration of the Schottky discs. In the case of two generators $\mathfrak{a}_1$ and $\mathfrak{a}_2$, if $D_{\mathfrak{a}_1}$ and $D_{\mathfrak{a}_1^{-1}}$ are adjacent (Figures 2 and 4), then the hyperbolic surface is a three-funnel surface; if $D_{\mathfrak{a}_1}$ and $D_{\mathfrak{a}_1^{-1}}$ are separated by $D_{\mathfrak{a}_2}$ and $D_{\mathfrak{a}_2^{-1}}$ (Figure 3), then the hyperbolic surface is a funneled torus.

6.2. Symmetric three-funnel surface

In this section, we discuss how our main theorem applies to the symmetric three-funnel surface. In particular, we recover the result of [34].

Example 6.7. Recall $X(\ell)$ from the introduction. In [5, Section 16], an explicit form is given by

$$S_1 = \begin{pmatrix} \cosh(\ell/2) & \sinh(\ell/2) \\ \sinh(\ell/2) & \cosh(\ell/2) \end{pmatrix},\ S_2 = \begin{pmatrix} \cosh(\ell/2) & a^{-1}\sinh(\ell/2) \\ a\sinh(\ell/2) & \cosh(\ell/2) \end{pmatrix}$$

and $a \in \mathbb{R}$ such that $\mathrm{tr}(S_1S_2) = -2\cosh(\ell/2)$. We make the change of variable with $z = e^{-\ell/4}$; then the group becomes a Schottky family with

$$\mathrm{lt}(S_1) = \mathrm{lt}(S_2) = -\mathrm{lt}(S_1S_2) = z^{-2}.$$

That this family satisfies condition ($\bigstar$) can be verified in the following way:

$$\mathrm{tr}(S_1S_2) = 2((z^{-2}+z^2)/2)^2 + (a+a^{-1})((z^{-2}-z^2)/2)^2 = -2(z^{-2}+z^2)/2.$$

So, we obtain a solution

$$a^{-1} = -1 + 2z + O(|z^2|).$$

[19] The verification of condition ($\bigstar$) can be done similarly as in Example 6.7.

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The fixed points of S_1 are at 1 (attracting) and -1 (repelling). The fixed points of S_2 are at a^{-1} and $-a^{-1}$. The cross-ratios of these four points are

$$\left(\frac{1 \pm a^{-1}}{1 - \pm a^{-1}}\right)^2,$$

which have order greater than -2. The contracting ratios of S_1 and S_2 are z^4. From these, we can verify that this family satisfies condition (★).

The graph has two vertices and three edges connecting them, each of length 2. From the discussion of the two-generator case, the intermediate zeta function Z_0 with $M = 0$ is exactly the graph zeta function for the symmetric three-funnel surface.

Since Theorem 4.4 also works for the $\mathrm{SL}_2(\mathbb{R})$ case, combined with the computation of Z_0, we obtain the convergence of zeta functions in Theorem 1.1. The statement on the Hausdorff dimension follows directly.

Example 6.8. We want to compute the zeta function for other M and explain that we recover the result of [34].

Continue from the previous example. The three-funnel surfaces can be described by four discs, each of radius z^2. Therefore, the separation between $d_{na}(\mathfrak{a}_i\infty, (D_{\mathfrak{a}_i})^c) = e^{-2}$ for the generators.

After doing computations using the algorithm in Proposition 5.5 and Corollary 5.8, we obtain: Let $x_1 = e^{-2s}$, $x_2 = e^{sz^2/2\log(1/|z|)}$, $x_3 = e^{sz^4/2\log(1/|z|)}$. Then

$$Z_2(\Gamma, z, s) = ((x_2^4 + x_1^4(-1 + x_2^2)^2 + x_1^2(x_2^2 - 2x_2^4))^2(x_2^4 + x_1^4(-1 + x_2^2)^2 - 2x_1^2(x_2^2 + x_2^4)))/x_2^{12}$$

and

$$\begin{aligned} Z_4(\Gamma, z, s) = \frac{1}{x_3^{18}x_2^{12}}&(x_1^8 - 4x_3^2x_1^8 + x_3^8x_1^4(-1 + x_1^2)^2 + x_3^4(-2x_1^6 + 6x_1^8) + x_3^3x_1^4(-2 + x_1^2)x_2^2 - 2x_3^5x_1^4(-1 + x_1^2)x_2^2 \\ &+ x_3^7x_1^2(-1 + x_1^2)^2x_2^2 - x_3^6(-1 + x_1^2)(4x_1^6 + x_2^4 - x_1^2x_2^4))^2 \\ &\times (x_1^8 - 4x_3^2x_1^8 + x_3^8x_1^4(-1 + x_1^2)^2 + x_3^4(-2x_1^6 + 6x_1^8) + 4x_3^5x_1^4(-1 + x_1^2)x_2^2 \\ &- 2x_3^7x_1^2(-1 + x_1^2)^2x_2^2 - 2x_3^3x_1^4(1 + x_1^2)x_2^2 - x_3^6(-1 + x_1^2)(4x_1^6 + x_2^4 - x_1^2x_2^4)). \end{aligned}$$

In particular, factorizing Z_2, we can recover the result of [34] by Corollary 4.6. In Z_2, if we let $x_2 = 1$, then we obtain the Ihara zeta function of the symmetric three-funnel surface, which is predicted by Corollary 5.9.

Remark 6.9. From previous computations, we conjecture that the term x_1 in $Z_{2M}(\Gamma, z, s)$ has degree $6 \cdot 2^M$ with coefficient $(1 - x_{M+1}^2)^{3\cdot 2^M}$, where $x_{M+1} = e^{sz^{2M}/2\log(1/|z|)}$. This conjecture provides some support for the fractal Weyl law, which states that the number of resonances in the region $\{s \in \mathbb{C} : \Re s > -C, |\Im s| \le T\}$ for any fixed $C > 0$ grows asymptotically like $T^{1+\delta_\Gamma}$, with δ_Γ being the critical exponent of the Schottky group Γ.

Below is a heuristic argument: In the region $\Re s \in [-C, \delta]$, $\Im s \in [2T\pi, 2(T+1)\pi]$ with $T \approx |z|^{-2M}$, the leading coefficient $(1-x_{M+1}^2)^{3\cdot 2^M}$ is of constant size. Since z is small, the terms $x_{j+1} = e^{sz^{2j}/2\log(1/|z|)}$ with $j \ge 1$ are almost constant. The only variable in Z_{2M} is x_1. Hence, as a polynomial in x_1, Z_{2M} is of degree $6 \cdot 2^M$, and the ratio between the coefficients and the leading coefficient is bounded. Therefore, this polynomial has $6 \cdot 2^M$ zeros of bounded size. Then $s = -\frac{1}{2}\log x_1$ is in the box $\{\Re s \in [-C, \delta],\ \Im s \in [2T\pi, 2(T+1)\pi]\}$. The number of zeros predicted by the fractal Weyl law in this region is

$$(|z|^{-2M})^\delta \approx (|z|^{-2M})^{\log 2/2\log(1/|z|)} = 2^M.$$

For the region $\Re s \in [-C, \delta], \Im s \in [2T\pi, 2(T+1)\pi]$ with $T \ll |z|^{-2M}$, the leading coefficient $(1 - x_{M+1}^2)^{3\cdot 2^M}$ is so small that Z_{2M} may have large zeros of x_1, and the corresponding s is outside

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the region. Since $Z_{2M'}(s)$ for some $M' < M$ with $M' \approx \log T/2\log(1/|z|)$ is a good approximation to $Z_{2M}(s)$ in the region, the number of zeros of $Z_{2M}(s)$ can be computed from $Z_{2M'}(s)$, which is approximately $2^{M'} \approx T^{\delta}$.

In conclusion, with the intermediate zeta function Z_{2M}, for $T \leq |z|^{-2M}$, we observe that the number of zeros in the region $\{\Re s \in [-C, \delta], \Im s \in [2T\pi, 2(T+1)\pi]\}$ is approximately T^{δ}, which fits the prediction of the fractal Weyl law.

A. Appendix

A.1. Critical exponent of the non-Archimedean Poincaré series

In this appendix, we explain that the critical exponent of the Poincaré series is equal to the growth rate of the number of periodic geodesics for non-Archimedean k.

Let Γ be a Schottky group in $\mathrm{PGL}_2(k)$ of g generators. Recall from [32, Theorem II.3.18] the nonwandering domain O of Γ on $\mathbb{P}^{1,an}_k$, the Mumford curve $X = \Gamma\backslash O$, and their skeleta Σ_O and Σ_X. Then Σ_X is a graph of rank g. Let p be the quotient map from O to X. Then $p^{-1}\Sigma_X = \Sigma_O$ and $\Sigma_X = p(\Sigma_O) = \Gamma\backslash\Sigma_O$. Recall that we have a distance d_a on Σ_O; the Γ action on Σ_O preserves the distance, and we have a quotient distance on Σ_X.

Recall the definition of the length of a periodic primitive geodesic in Lemma 2.17. Let $\mathcal{G}_\Gamma(L)$ be the number of periodic primitive geodesics of length less than L in Σ_X, which is also the least translation length of the corresponding conjugacy class on Σ_O. Let $\mathcal{N}_\Gamma(L, o)$ be the number of $\gamma \in \Gamma$ such that $d_a(o, \gamma o) \leq L$ for $o \in \Sigma_O$.

Since Σ_O is a locally finite tree with a metric, it is CAT(-1). We can apply [35] to obtain the critical exponent of the Poincaré series

$$\delta(\Gamma) = \lim_{L\to\infty} \frac{1}{L} \log \mathcal{N}_\Gamma(L, o).$$

Moreover, by [35, Corollaire 2], we have the following.

Proposition A.1. *We have*

$$\lim_{L\to\infty} \frac{1}{L} \log \mathcal{G}_\Gamma(L) = \delta(\Gamma).$$

By [16, Theorem 2.10], we have the following.

Corollary A.2. *The first zero of the Ihara zeta function of the graph is equal to the critical exponent.*

A.2. Proof of Proposition 5.10

Before the proof, we state three elementary lemmas about estimates of the coefficients of the Laurent series after multiplication, division, and logarithm.

Lemma A.3. *Suppose $f(t), g(t) \in \mathbb{C}((t))$ are two formal Laurent series:*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j \quad \textit{with} \quad |f_j| \leq e^{C(j+1)} \textit{ for } j \in \mathbb{Z}_{\geq 0};$$

$$g(t) = t^{m'} \sum_{j=0}^{\infty} g_j t^j \quad \textit{with} \quad |g_j| \leq e^{C(j+1)} \textit{ for } j \in \mathbb{Z}_{\geq 0}.$$

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Then $f(t)g(t) \in \mathbb{C}((t))$ has the Laurent series expansion

$$f(t)g(t) = t^{m+m'} \sum_{j=0}^{\infty} h_j t^j \quad \text{with} \quad |h_j| \le (j+1)e^{C(j+2)} \text{ for } j \in \mathbb{Z}_{\ge 0}.$$

Proof. We have

$$|h_n| = \left| \sum_{j+k=n} f_j g_k \right| \le (n+1)e^{C(n+2)}. \qquad \square$$

Lemma A.4. *Suppose $f(t), g(t) \in \mathbb{C}((t))$ are two formal Laurent series with*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j, \quad f_0 \ne 0, \ |f_j| \le e^{C(j+1)};$$
$$g(t) = t^{m'} \sum_{j=0}^{\infty} g_j t^j, \quad |g_0| \ge e^{-A}, \ |g_j| \le e^{C(j+1)}.$$

Then $f(t)/g(t) \in \mathbb{C}((t))$ has the Laurent series expansion

$$f(t)/g(t) = t^{m-m'} \sum_{j=0}^{\infty} h_j t^j, \quad |h_j| \le e^{(3C+C_1+A)(j+1)}$$

for some universal constant C_1.

Proof. We may first take $\tilde{g}_0 = 1$ and $|\tilde{g}_j| = |g_j/g_0| \le e^{C(j+1)+A}$. Then

$$\frac{g_0 t^{m'}}{g(t)} = \sum_{n=0}^{\infty} \left(- \sum_{j=1}^{\infty} \tilde{g}_j t^j \right)^n = \sum_{n=0}^{\infty} g'_n t^n,$$

where

$$|g'_n| \le \sum_{j_1+\cdots+j_k=n} |\tilde{g}_{j_1} \cdots \tilde{g}_{j_k}| \le e^{C_0 n} e^{2Cn+An} \le e^{(2C+C_0+A)n}$$

for some universal constant C_0. Finally, we multiply by $f(t)$:

$$\frac{f(t)}{g(t)} = \frac{f(t)}{g_0 t^{m'}} \frac{g_0 t^{m'}}{g(t)} = g_0^{-1} t^{m-m'} \left(\sum_{j=0}^{\infty} f_j t^j \right) \left(\sum_{n=0} g'_n t^n \right) = t^{m-m'} \sum_{j=0}^{\infty} h_j t^j,$$

where

$$|h_n| \le |g_0|^{-1} \sum_{j+k=n} |f_j g'_k| \le (n+1)e^A e^{C(n+1)} e^{(2C+C_0+A)n} \le e^{(3C+C_1+A)(n+1)}$$

for some universal constant C_1. $\square$

Lemma A.5. *Suppose $f(t) \in \mathbb{C}((t))$ has the expansion*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j, \quad f_0 \ne 0, \quad |f_j| \le e^{C(j+1)}.$$

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Then as a formal series, $\log(f(t))$ *has the expansion*

$$\log(f(t)) = m\log t + \log(f_0) + \sum_{j=0}^{\infty} h_j f_0^{-j} t^j, \quad |h_j| \le e^{(3C+C_0)j},$$

where C_0 *is a universal constant.*

Proof. We use the formula for log:

$$\begin{aligned}\log(f(t)) &= m\log t + \log(f_0) + \log\left(1 + \sum_{j=1}^{\infty} \frac{f_j}{f_0} t^j\right) \\ &= m\log t + \log(f_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\sum_{j=1}^{\infty} \frac{f_j}{f_0} t^j\right)^n.\end{aligned}$$

Therefore,

$$|h_n| \le \sum_{j_1+\cdots+j_k=n} \frac{1}{k} |f_{j_1} \cdots f_{j_k}| \cdot |f_0|^{n-k} \le e^{C_0 n + 3Cn}. \qquad \square$$

Proof of Proposition 5.10. First, we check the estimate for $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))$. Each entry of $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}$ is a sum of at most 2^{N-1} terms, with each term a product of $N-1$ entries of generators. By subadditivity and submultiplicativity of the hybrid norm, for each entry $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(i,l)$, we have

$$\sup_{1\le i,l\le 2} \|\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(i,l)\|_{\mathrm{hyb}} \le 2^{N-1} \sup_{\substack{\mathfrak{a}\in\mathcal{A}\\ 1\le i,l\le 2}} \|\mathfrak{a}(i,l)\|_{\mathrm{hyb}}^{N-1} \lesssim e^{CN}.$$

Suppose $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For some $0 < r < 1/e$ independent of N and $|t| < r$, the Schottky discs $D_{\mathfrak{a}}$ are disjoint for $\mathfrak{a} \in \mathcal{A}$. If $a = \sum_{n\ge m} a_n t^n \in \mathbb{C}((t))$ with $a_m \ne 0$, then $\|a\|_{\mathrm{hyb}} \le e^{CN}$ implies that $e^{-m} \le \|a\|_{\mathrm{hyb}} \le e^{CN}$ and

$$|a_z| \le \sum_{n\le 0} |a_n|(r/2)^n + \sum_{n>0} |a_n| e^{-n} \le \max\{(re/2)^m, 1\}\|a\|_{\mathrm{hyb}} \le e^{C'N}, \quad \frac{r}{2} \le |z| \le 1/e.$$

In the last step, we use $(re/2)^m = e^{m(\log(r/2)+1)}$ and $\log(r/2) + 1 < 0$.

Let $f(z) = \sum_{j=0}^{\infty} c_j z^j$, where c_j is the coefficient in the Laurent series $c = t^m \sum_{j=0}^{\infty} c_j t^j \in \mathbb{C}((t))$ with $c_0 \ne 0$. Since ∞ does not lie in $D_{\bar{\mathfrak{a}}_{i_N},z}$ and $a/c = \mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(\infty)$, a_z/c_z is in the disc $D_{\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N},z} \subset D_{\mathfrak{a}_{i_2},z}$, which we assume is bounded uniformly from ∞. We conclude that $|a_z/c_z| \le C$ and $c_z \ne 0$ for $|z| < r$. Thus, $\log|f(z)|$ is harmonic in the disc $D(0,r)$. Since $ad - bc = 1$, we have

$$1 \le |a_z d_z| + |b_z c_z| \le C(|a_z| + |b_z|)|c_z| \le e^{CN}|c_z|, \quad \frac{r}{2} \le |z| < r.$$

From the Laurent series expansion $c = t^m \sum_{j=0}^{\infty} c_j t^j \in \mathbb{C}((t))$, we have

$$e^{-m} \le \|c\|_{\mathrm{hyb}} \le e^{CN},$$

which implies $m \ge -CN$.

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By the mean value theorem on $D(0, r/2)$, we have

$$\log|c_0| = \log|f(0)| \geq \min_{|z|=r/2} \log|f(z)| = \min_{|z|=r/2} \log|c_z/z^m| \geq -CN - m\log(r/2) \geq -C'N.$$

In the last step, we use $\log(r/2) < 0$ and $m \geq -CN$. Therefore, $|c_0| \geq e^{-CN}$. Recalling that $\|a\|_{\mathrm{hyb}} \leq e^{CN}$ and $\|c\|_{\mathrm{hyb}} \leq e^{CN}$, we have a Laurent series expansion

$$\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty))) = \frac{a}{c} = t^{m'} \sum_{j=0}^{\infty} a'_j t^j \text{ with } |a'_j| \leq e^{C(N+1)j}, \tag{A.1}$$

where in the last step we use Lemma A.4.

Suppose $\mathfrak{a}_{i_1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then $\mathfrak{a}'_{i_1}(w) = (c'w + d')^{-2}$. We would like to show

$$\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}(\cdots \mathfrak{a}_{i_{N-1}}(\mathfrak{a}_{i_N}(\infty)))) = t^m \sum_{j=0}^{\infty} a_j t^j, \quad |a_j| \leq e^{C(N+1)j}.$$

By Lemma A.3, it suffices to show a similar expansion for $(\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}(\infty) + d'/c')^{-1}$. By Lemma A.4 and equation (A.1), it suffices to show that the leading coefficient π_0 in the expansion

$$\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}(\infty) + d'/c' = t^m \sum_{j=0}^{\infty} \pi_j t^j, \quad \pi_0 \neq 0$$

satisfies $|\pi_0| \geq e^{-CN}$. By Proposition 3.20, $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty))) \in \mathfrak{a}_{i_2} D_{\mathfrak{a}_{i_3},z} \subset |z|^{\nu'} D_{\mathfrak{a}_{i_2},z}$ for some $\nu' > 0$ and $0 < |z| < r$. Hence, the point $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))$ is $|z|^{\nu}$ separated from $D_{\overline{\mathfrak{a}}_{i_1},z}$ (which contains $-d'/c'$) for some $\nu > 0$ and $0 < |z| < r$. By considering the separation with the non-Archimedean norm, we know $|m| \leq C$ is uniformly bounded. Consider the holomorphic function $f(z) = \sum_{j=0}^{\infty} \pi_j z^j$. Then $\log|f(z)|$ is a harmonic function in a small disc $|z| < r$. By the mean value theorem,

$$\log|\pi_0| \geq \min_{|z|=r/2} \log|f(z)| \geq \min_{|z|=r/2} \log|z^m f(z)| - m\log(r/2) \geq \nu\log(r/2) - m\log(r/2).$$

Since m and ν are uniform, we actually prove $|\pi_0| \geq e^{-C}$ holds for some uniform constant $C > 0$. Therefore, Lemma A.4 gives equation (5.10). Similarly, since the discs are bounded uniformly away from ∞, we have

$$\log|\pi_0| \leq \max_{|z|=r/2} \log|f(z)| \leq \max_{|z|=r/2} \log|z^m f(z)| - m\log(r/2) \leq \log C - m\log(r/2).$$

Thus, $|\pi_0| \leq e^C$ for some uniform constant $C > 0$. Since we know the leading coefficient a_0 in equation (5.10) is given by π_0^{-2} times the leading coefficient of c'^{-2}, we conclude

$$|a_0| \geq e^{-2C'}|\pi_0|^{-2} \geq e^{-2C-2C'}.$$

Equation (5.11) then follows from equation (5.10) and the lower bound $|a_0| \geq e^{-C}$. □

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