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Mixing on a class of rank-one transformations

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Abstract. We prove that a rank-one transformation satisfying a condition called restricted growth is a mixing transformation if and only if the spacer sequence for the transformation is uniformly ergodic. Uniform ergodicity is a generalization of the notion of ergodicity for sequences, in the sense that the mean ergodic theorem holds for a family of what we call dynamical sequences. The application of our theorem shows that the class of polynomial rank-one transformations, rank-one transformations where the spacers are chosen to be the values of a polynomial with some mild conditions on the polynomials, that have restricted growth are mixing transformations, implying, in particular, Adams' result on staircase transformations. Another application yields a new proof that Ornstein's class of rank-one transformations constructed using 'random spacers' are almost surely mixing transformations.

1. *Introduction*

1.1. *Background.* Rank-one transformations are transformations defined inductively using at each step a single Rohlin tower or column. They gained importance with Ornstein's construction of a mixing transformation with no square root [**Or72**] and have been used as a source of examples and counterexamples in ergodic theory. Rank-one mixing transformations have been shown to enjoy many interesting properties. It is now known that rank-one mixing transformations are mixing of all orders [**Ka84, Ry93**] and have the property of minimal self-joinings [**Ki88**] and, in particular, they are prime and have a trivial centralizer [**R79**]. We refer to [**Fr70**] for rank-one constructions and [**Fe97**] for a more recent survey of results.

1.2. *History.* Ornstein's construction of a rank-one mixing transformation uses the notion of 'random spacers' and yields a class of transformations so that almost surely a transformation in this class is mixing. These methods were used by Rudolph [**R79**] to construct the first examples of transformations with minimal self-joinings. Bourgain [**Bo93**] showed that transformations in Ornstein's class almost surely have a singular spectrum, and more recently, El Abdalaoui [**Ab99**] showed that transformations in

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Ornstein's class almost surely are mutually singular. However, Ornstein's method does not exhibit a specific rank-one mixing transformation. It was conjectured by Smorodinsky that a specific rank-one map, now called the classical staircase transformation, is mixing. Adams and Friedman [**AF92**] constructed explicit mixing staircase transformations but the conjecture remained open until Adams [**Ad98**] showed that transformations in a class of staircase transformations, which includes the classical staircase, are mixing. The classical staircase has been shown to have a singular spectrum [**Kl96**], and the constructions in [**Ad98**] have been generalized to mixing staircase $\mathbb{Z}^d$-actions in [**AS99**]. More recently, Fayad [**Fa**] has constructed rank-one mixing smooth flows on the 2-torus.

1.3. *Result.* Our main result (Theorem 6) is that a rank-one transformation, satisfying a condition we call restricted growth, is mixing if and only if its spacer sequence is uniformly ergodic. A rank-one transformation is specified by a sequence of positive integers called the cut sequence and a doubly-indexed sequence of integers, called the spacer sequence: this is a specific instance of what we call a dynamical sequence. We generalize the notion of ergodicity of a sequence (meaning that the mean ergodic theorem holds along the sequence, as in the Blum–Hanson theorem [**BH60**]) to dynamical sequences, yielding notions of ergodicity and uniform ergodicity for dynamical sequences. We introduce a condition called rank-one uniform mixing and show that it implies mixing. The proof of our main theorem is then accomplished by showing that uniform ergodicity of the spacer sequence implies rank-one uniform mixing.

1.4. *Applications.* We apply our result to the class of polynomial rank-one transformations and to Ornstein's class of rank-one transformations constructed using 'random spacers'. Polynomial rank-one transformations were first considered by Adams and Friedman [**AF92**] and are rank-one transformations where the spacer sequences are given by the values of polynomials; staircase transformations are rank-one transformations where the spacer sequences are given by linear polynomials and so are a subclass of polynomial rank-one transformations. We show that polynomial rank-one transformations having restricted growth are mixing transformations, providing a large class of concretely constructed rank-one mixing transformations and obtaining Adams' result on mixing staircase transformations [**Ad98**] as a corollary. Utilizing Ornstein's Probabilistic Lemma for 'random spacers' (see [**Na98**]) and the notion of double ergodicity [**Fu81, BFMS01**], we apply our main theorem to provide an alternative proof that almost surely transformations in the class are mixing transformations. After this paper was accepted for publication, Ryzhikov wrote to us that he had independently obtained a result similar to our Theorem 11, and that in Corollary 13 (of Adams) he has removed the condition that T be of restricted growth; he also wrote that his proofs are not written up.

2. *Preliminaries*

2.1. *Dynamical systems.* Let (X, μ) be a probability measure space isomorphic to the unit interval in $\mathbb{R}$ under Lebesgue measure. We concern ourselves with invertible, measurable and measure-preserving transformations $T : X \to X$ on $(X, \mathcal{B}, \mu)$ and use the

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term *transformation* to refer exclusively to such. The space $(X, \mathcal{B}, \mu)$ and a transformation T on it form the *dynamical system* $(X, \mathcal{B}, \mu, T)$.

2.2. *Ergodicity.* A transformation $T : X \to X$ is *ergodic* when every T-invariant set is either null or conull—any measurable set A such that $T(A) = A$ must have measure zero or full measure. The *Weak Ergodic Theorem* states that given an ergodic transformation T, for any measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(T^j(A) \cap B) - \mu(A)\mu(B) = 0,$$

while the *Von Neumann (Mean) Ergodic Theorem* guarantees convergence in the mean—for any measurable set B,

$$\lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-j} - \mu(B) \right|^2 d\mu = 0.$$

Here χ_B denotes the characteristic function of a given measurable set B defined by $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ when $x \in X \setminus B$. Given an ergodic transformation T, a sequence of integers $\{a_n\}$ is *ergodic with respect to* T when for any measurable set B,

$$\lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-a_j} - \mu(B) \right|^2 d\mu = 0$$

and is *weak ergodic with respect to* T when, for any measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(T^{a_j}(A) \cap B) - \mu(A)\mu(B) = 0.$$

Note that a sequence that is ergodic with respect to T is weak ergodic with respect to T: the proof is left to the reader. That the converse does not hold in general is due to Friedman [**Fr83**].

2.3. *(Strong) mixing.* A transformation T is *mixing* when for any measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^n(A) \cap B) - \mu(A)\mu(B) = 0.$$

Given a transformation T, a strictly increasing sequence of positive integers $\{t_n\}$ is *mixing with respect to* T when, for any measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^{t_n}(A) \cap B) - \mu(A)\mu(B) = 0.$$

Mixing and ergodicity on sequences are related by the Blum–Hanson theorem [**BH60**].

THEOREM. (Blum–Hanson) *Let T be an ergodic transformation. Then T is a mixing transformation if and only if every strictly increasing sequence of integers is ergodic with respect to T.*

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2.4. *Weak mixing.* A transformation T is *weakly mixing* when there exists a mixing sequence with respect to T. The following conditions are equivalent to weak mixing:

(i) there exists a density one mixing sequence with respect to T;

(ii) for any sequence of measurable sets $\{A_n\}$ and any measurable set B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T^j(A_n) \cap B) - \mu(A_n)\mu(B)| = 0;$$

(iii) the transformation $T \times T$ is ergodic; and

(iv) *double ergodicity*: for any measurable sets A and B of positive measure there exists a positive integer n such that $\mu(T^n(A) \cap A)\mu(T^n(A) \cap B) > 0$.

The first three were known to von Neumann and Kakutani, the last was first proved in [**Fu81**] and later used in [**BFMS01**] in the context of infinite measure-preserving transformations.

2.5. *Notation.* To show that expressions converge to the same limit, we use the $\pm$ notation in the following manner. The reader may verify that the $\pm$ 'quantity' obeys the 'rules' of arithmetic.

Notation. For any quantities $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, the notation

$$\boldsymbol{A} = \boldsymbol{B} \pm (\boldsymbol{C}) \quad \text{shall mean} \quad |\boldsymbol{A} - \boldsymbol{B}| \leq |\boldsymbol{C}|.$$

3. *Dynamical sequences*

3.1. *Basic notions.* We introduce the notion of dynamical sequences of integers, which arises when considering the spacer levels added to each column in the construction of rank-one transformations. We use dynamical sequences to generalize the Cesàro averaging over a static sequence in the definitions of the previous section to a moving Cesàro average.

Definition 1. Given a non-decreasing sequence of positive integers $\{r_n\}$, a *dynamical sequence* of integers $\{s_{n,j}\}_{\{r_n\}}$ is a doubly-indexed collection of integers $s_{n,j}$ for $n \geq 0$ and $0 \leq j < r_n$. The positive integer $s_{n,j}$ is the jth *element* of the dynamical sequence at the nth *stage* and the sequence of positive integers $\{r_n\}$ is the *index sequence* of the dynamical sequence.

Definition 2. Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence of integers. The *sequence of ranges* for $\{s_{n,j}\}_{\{r_n\}}$, denoted $\{\widetilde{s}_n\}$, is given, for all positive integers n, by

$$\widetilde{s}_n = \max_{0\leq j<r_n} s_{n,j} - \min_{0\leq j<r_n} s_{n,j}$$

and the *sequence of averages* for $\{s_{n,j}\}_{\{r_n\}}$, denoted $\{\overline{s}_n\}$, is given by, for all positive integers n,

$$\overline{s}_n = \left\lfloor \frac{1}{r_n} \sum_{j=0}^{r_n-1} s_{n,j} \right\rfloor.$$

The *normalized dynamical sequence* of $\{s_{n,j}\}_{\{r_n\}}$, denoted $\{\widehat{s}_{n,j}\}_{\{r_n\}}$, is given by $\widehat{s}_{n,j} = s_{n,j} - \overline{s}_n$ for all n and j.

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Note that the sequence of ranges and the index sequence for the normalized dynamical sequence of a given dynamical sequence are identical to those of the given dynamical sequence.

Definition 3. Given a dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$, for any positive integers n and k, the kth *partial sums* of $\{s_{n,j}\}_{\{r_n\}}$ at the nth stage are the integers $\{s^{(k)}_{n,j}\}$ indexed by $j \in \mathbb{Z}_{r_n-k}$ and defined as

$$s^{(k)}_{n,j} = \sum_{z=0}^{k-1} s_{n,j+z}.$$

For a fixed positive integer k, the kth *partial sum dynamical sequence* is the dynamical sequence of positive integers $\{s^{(k)}_{n,j}\}_{\{r_n-k\}}$ (n 'begins' at the smallest value such that $r_n \geq k$) whose elements are the kth partial sums of $\{s_{n,j}\}_{\{r_n\}}$.

Definition 4. Given a dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ and a sequence of positive integers $\{k_n\}$ such that $k_n < r_n$ for all positive integers n, the $\{k_n\}$th *partial sum dynamical sequence* of $\{s_{n,j}\}_{\{r_n\}}$ is the dynamical sequence of positive integers $\{s^{(k_n)}_{n,j}\}_{\{r_n-k_n\}}$ where $s^{(k_n)}_{n,j}$ are the k_nth partial sums of $\{s_{n,j}\}_{\{r_n\}}$ at the nth stage for each positive integer n.

Note that the notation $s^{(k)}_{n,j}$ is used implicitly to represent the kth partial sums at the nth stage of a dynamical sequence of positive integers $\{s_{n,j}\}_{\{r_n\}}$.

Definition 5. A dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *pathological* when the index sequence $\{r_n\}$ has a finite limit point ($\liminf_{n\to\infty} r_n < \infty$).

Unless explicitly stated otherwise, for the remainder of the paper, the term dynamical sequence is assumed to mean a non-pathological dynamical sequence (see Proposition 5.4).

3.2. *Monotonic dynamical sequences.* We generalize the property of monotonicity used in the Blum–Hanson theorem to dynamical sequences yielding three distinct, related concepts. (The symbol # denotes cardinality.)

Definition 6. A dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *strictly increasing* when for all n and all $0 \leq j < r_n - 1$, the term $s_{n,j} < s_{n,j+1}$. The dynamical sequence is *non-decreasing* when, for all n and all $0 \leq j < r_n - 1$, the term $s_{n,j} \leq s_{n,j+1}$.

Definition 7. A dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *square monotone* when, for all fixed positive integers M,

$$\lim_{n\to\infty} \frac{1}{r_n^2} \#\{(j,\ell) \in \mathbb{Z}_{r_n} \times \mathbb{Z}_{r_n} : |s_{n,j} - s_{n,\ell}| < M\} = 0.$$

Definition 8. A dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *weak monotone* when for all fixed positive integers M,

$$\lim_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : |s_{n,j}| < M\} = 0.$$

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PROPOSITION 3.1. *If a given dynamical sequence is strictly increasing, then it is square monotone; if it is square monotone, then it is weak monotone.*

Proof. Given a strictly increasing dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$, for any n and any $0 \leq \ell \leq j < r_n$, we have that $s_{n,j} - s_{n,\ell} \geq 1 + s_{n,j-1} - s_{n,\ell} > \cdots \geq (j-\ell) + s_{n,j-(j-\ell)} - s_{n,\ell} = j - \ell$; thus, $\{s_{n,j}\}_{\{r_n\}}$ is square monotone. That a square monotone dynamical sequence is weak monotone is left to the reader. □

4. *Dynamical Cesàro averaging*

4.1. *Dynamical sequence ergodicity.* The application of dynamical sequences to expressions like those found in the ergodic theorems and the definition of an ergodic sequence is crucial to our main result. We explore notions of ergodicity and weak mixing for dynamical sequences and present a generalized version of the Blum–Hanson theorem.

Definition 9. Given an ergodic transformation T, a dynamical sequence of non-negative integers $\{s_{n,j}\}_{\{r_n\}}$ is *ergodic with respect to* T when, for any measurable set B,

$$\lim_{n\to\infty} \int \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \right|^2 d\mu = 0.$$

We refer to this sequence of functions as *dynamical Cesàro averages* for the L^2 function $\chi_B - \mu(B)$ over the dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$.

When studying mixing sequences on rank-one transformations, we will apply this definition to specific partial sum dynamical sequences, written $\{s^{(k_n)}_{n,j}\}_{\{r_n-k_n\}}$, of some given dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$.

Definition 10. Given an ergodic transformation T, a dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *weak ergodic with respect to* T when, for any measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{s_{n,j}}(A) \cap B) - \mu(A)\mu(B) = 0.$$

Note that a dynamical sequence that is ergodic with respect to a given transformation must be weak ergodic with respect to the transformation but that the converse does not hold in general. The following definitions will be needed when studying mixing on rank-one transformations.

Definition 11. Given an ergodic transformation T, a dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *uniformly ergodic with respect to* T when for any sequence of positive integers $\{k_n\}$ such that $\lim_{n\to\infty} r_n - k_n = \infty$, the $\{k_n\}$th partial sum dynamical sequence $\{s^{(k_n)}_{n,j}\}_{\{r_n-k_n\}}$ is ergodic with respect to T, that is, for any measurable set B,

$$\lim_{n\to\infty} \int \left| \frac{1}{r_n - k_n} \sum_{j=0}^{r_n-k_n-1} \chi_B \circ T^{-s^{(k_n)}_{n,j}} - \mu(B) \right|^2 d\mu = 0.$$

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Definition 12. Given an ergodic transformation T, a dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ is *totally ergodic with respect to* T when, for any measurable set B and each fixed positive integer k,

$$\lim_{n\to\infty} \int \left| \frac{1}{r_n - k} \sum_{j=0}^{r_n-k-1} \chi_B \circ T^{-s_{n,j}^{(k)}}(B) - \mu(B) \right| = 0.$$

Note that uniform ergodicity implies total ergodicity which implies ergodicity for a given dynamical sequence with respect to a given transformation.

4.2. *Generalized Blum–Hanson theorem.* The following theorem is a generalization of the Blum–Hanson theorem characterizing the ergodicity of dynamical sequences with respect to mixing transformations. Note that a dynamical sequence of positive integers that is not square monontone cannot be mixing with respect to any ergodic transformation: the proof is left to the reader. For generalizations of the Blum–Hanson theorem to sequences of transformations the reader is referred to [**BB84, BB86**].

THEOREM 1. *Let T be an ergodic transformation. Then T is mixing if and only if every square monotone dynamical sequence of integers is ergodic with respect to T.*

Proof. Let T be a mixing transformation and $\{s_{n,j}\}_{\{r_n\}}$ a square monotone dynamical sequence of integers. For any measurable set B and any $\epsilon > 0$, there exists $M > 0$ such that for all $m \geq M$ or $m \leq -M$, $|\mu(T^m(B) \cap B) - \mu(B)\mu(B)| < \epsilon$. Since $\{s_{n,j}\}_{\{r_n\}}$ is square monotone, there exists $N > 0$ such that, for all $n \geq N$, $\#\{(j,\ell) \in \mathbb{Z}_{r_n} \times \mathbb{Z}_{r_n} : |s_{n,j} - s_{n,\ell}| < M\} < \epsilon r_n^2$. Then,

$$\begin{aligned}
&\int \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \right|^2 d\mu \\
&\quad = \frac{1}{r_n^2} \sum_{j,\ell=0}^{r_n-1} \int (\chi_B \circ T^{-s_{n,j}} - \mu(B))(\chi_B \circ T^{-s_{n,\ell}} - \mu(B))\, d\mu \\
&\quad = \frac{1}{r_n^2} \sum_{j,\ell=0}^{r_n-1} \mu(T^{s_{n,j}}(B) \cap T^{s_{n,\ell}}(B)) - \mu(B)\mu(B) \\
&\quad = \frac{1}{r_n^2} \sum_{j,\ell=0}^{r_n-1} \mu(T^{s_{n,j}-s_{n,\ell}}(B) \cap B) - \mu(B)\mu(B) \\
&\quad \leq \frac{1}{r_n^2} \sum_{|s_{n,j}-s_{n,\ell}|\geq M} |\mu(T^{s_{n,j}-s_{n,\ell}}(B) \cap B) - \mu(B)\mu(B)| + \frac{1}{r_n^2} \epsilon r_n^2 \mu(B) \\
&\quad < \frac{1}{r_n^2} \sum_{|s_{n,j}-s_{n,\ell}|\geq M} \epsilon + \epsilon\mu(B) \leq \epsilon(1 + \mu(B)).
\end{aligned}$$

Hence, $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T.

For the converse, since mixing is equivalent to Rényi mixing: for all measurable sets B, $\mu(T^m(B) \cap B) \to \mu(B)\mu(B)$, if T is not mixing, then there exist a measurable set B,

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a $\delta > 0$ and a strictly increasing sequence of positive integers $\{t_m\}$ such that for all m, $\mu(T^{t_m}(B) \cap B) - \mu(B)\mu(B) \geq \delta$ (or $\leq -\delta$ in which case take the complement of the set B to obtain $\geq \delta$). Define the dynamical sequence of integers $\{s_{n,j}\}_{\{r_n\}}$ by $r_n = n$ and $s_{n,j} = t_j$ for all n and j. Then, first using the Hölder inequality and then the triangle inequality twice,

$$\begin{aligned}\left(\int \left| \frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B)\right|^2 d\mu\right)^{\frac{1}{2}} &\geq \int \left| \frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B)\right| d\mu \\ &\geq \int_B \left| \frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-t_j} - \mu(B)\right| d\mu \\ &\geq \frac{1}{n}\left| \sum_{j=0}^{n-1} \int_B \chi_B \circ T^{-t_j} - \mu(B)\, d\mu \right| \\ &\geq \left| \frac{1}{n}\sum_{j=0}^{n-1} \mu(T^{t_j}(B) \cap B) - \mu(B)\mu(B)\right| \\ &\geq \delta.\end{aligned}$$

Thus, $\{s_{n,j}\}_{\{r_n\}}$ is not ergodic with respect to T. It remains only to show that $\{s_{n,j}\}_{\{r_n\}}$ is square monotone. But, since $\{t_m\}$ is strictly increasing, $\{s_{n,j}\}_{\{r_n\}}$ is strictly increasing and, thus, square monotone by Proposition 3.1. □

The following theorem generalizes the weak Blum–Hanson theorem to dynamical sequences: the proof is left to the reader.

THEOREM 2. *Let T be an ergodic transformation. Then T is a mixing transformation if and only if every weak monotone dynamical sequence of positive integers is weak ergodic with respect to T.*

5. *Rank-one transformations*

5.1. *Construction of rank-one transformations.* Rank-one transformations as defined here are a class of ergodic transformations on the unit interval in $\mathbb{R}$ under standard Lebesgue measure constructed as follows. An ordered collection of intervals all the same length is termed a *column* and each interval a *level* where the *height* of the column is the number of levels in the column. The associated *column map* is defined by mapping each interval to the interval *above* (next in the order on the collection of intervals) it; hence, the column map is defined from all but the *top* (last in the order) level onto all but the *bottom* (first in the order) level.

We describe the procedure for *cutting* and *stacking* a column, C_n, to obtain a new column, C_{n+1}. Fix a column C_n with height h_n, levels $I_{n,i}$ (indexed from bottom to top by $i = 0, \ldots, h_n - 1$) and column map T_n: $C_n = \{I_{n,i}\}_{i=0}^{h_n-1}$ where $T_n(I_{n,i}) = I_{n,i+1}$ for $i \neq h_n - 1$. For some given $r_n \geq 2$, *cut* C_n into r_n subcolumns by cutting each level $I_{n,i}$ into r_n subintervals or *sublevels*, $\{I_{n,i}^{[j]}\}_{j=0}^{r_n-1}$, of equal length, $(1/r_n)\mu(I_{n,i})$, where $I_{n,i}^{[0]}$ is the leftmost sublevel and $I_{n,i}^{[r_n-1]}$ is the rightmost. Then, the *subcolumns* of C_n are $C_n^{[j]} = \{I_{n,i}^{[j]}\}_{i=0}^{h_n-1}$. By preserving the order on the levels, each subcolumn, $C_n^{[j]}$,

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is a column in its own right with the associated map $T_n^{[j]}$ which is the restriction of T_n to $C_n^{[j]}$.

Given an indexed collection of non-negative integers $\{s_{n,j}\}_{j=0}^{r_n-1}$, the spacer values for C_n, place *spacer levels* ('new' intervals the size of each sublevel) above each subcolumn by adding $s_{n,j}$ levels above $C_n^{[j]}$ and *stack* the resulting subcolumns with spacers right on top of left yielding a new column C_{n+1} with height $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$. Denote the *union of spacer levels* added to C_n by S_n, the collection of levels in C_{n+1} that are not sublevels of levels in C_n; hence, $C_{n+1} = C_n \cup S_n$; and denote by $S_n^{[j]}$ the collection of spacers added over $C_n^{[j]}$, so that $S_n = \bigcap_{j=0}^{r_n-1} S_n^{[j]}$. We note that some authors index the spacers added to C_n by S_{n+1} rather than S_n used here. The associated column map T_{n+1} restricts to T_n on the levels in C_n, as before, and extends it to the spacer levels added as well as all but the leftmost sublevel of the bottom level and the topmost spacer level over the rightmost subcolumn of C_n. We will use implicitly the following facts in the sequel.

LEMMA 5.1. *For any sublevel $I_{n,i}^{[j]}$ in C_n, $I_{n,i}^{[j]} = I_{n+1,i+jh_n+\sum_{z=0}^{j-1} s_{n,z}}$ is a level in C_{n+1}.*

LEMMA 5.2. *For any sublevel $I_{n,i}^{[j]}$ in C_n where $j \neq r_n - 1$,*

$$T_{n+1}^{h_n+s_{n,j}}(I_{n,i}^{[j]}) = I_{n,i}^{[j+1]}.$$

Thus, given a non-decreasing sequence of positive integers $\{r_n\}$, the *sequence of cuts*, and a dynamical sequence of non-negative integers $\{s_{n,j}\}_{\{r_n\}}$, the *sequence of spacers*, we may construct an infinite sequence of columns $\{C_n\}$ where $C_0 = \{I_{0,0}\}$ is defined to be a single level and each C_{n+1} is constructed by cutting and stacking each C_n as described earlier using r_n cuts and $\{s_{n,j}\}_{r_n}$ spacers. The *sequence of heights* is then defined recursively by $h_0 = 1$ and $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$.

Since the initial level $I_{0,0}$ has some finite length, the length of each level in C_n approaches zero as n becomes large. Hence, the associated sequence of column maps, $\{T_n\}$, approaches a map T defined on all but a measure zero subset of the union of the initial levels and all the spacer levels added at each column. Note that T_n is the restriction of T to the levels in C_n (except the top) for each $n \geq 0$. When T is defined on a finite measure space (i.e. if the measure of all the spacers added is finite), choose $I_{0,0}$ to have length such that the total measure of the space T is defined on is 1.

A transformation T is formally a *rank-one transformation* when T can be realized as the limit of a sequence of maps defined by the cut and stack construction applied repeatedly to a single column as described earlier. The reader may verify that rank-one transformations are invertible, measurable, measure-preserving and ergodic.

Given a rank-one transformation T with height sequence $\{h_n\}$ and spacer sequence $\{s_{n,j}\}_{\{r_n\}}$, we define the *normalized height sequence* for T to be the sequence of positive integers $\{\widehat{h}_n\}$ given by $\widehat{h}_n = h_n + \overline{s}_n$. Note that $\lim_{n\to\infty} \widehat{h}_n/h_n = 1$ when T is finite measure-preserving. We define the *normalized spacer sequence* for T to be the normalized dynamical sequence of $\{s_{n,j}\}_{\{r_n\}}$, denoted $\{\widehat{s}_{n,j}\}_{\{r_n\}}$, given by $\widehat{s}_{n,j} = s_{n,j} - \overline{s}_n$.

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LEMMA 5.3. *For any sublevel $I_{n,i}^{[j]}$ in C_n where $j \neq r_n - 1$, we have that $T^{h_n+s_{n,j}}(I_{n,i}^{[j]}) = I_{n,i}^{[j+1]}$ and, equivalently, $T^{\widehat{h}_n+\widehat{s}_{n,j}}(I_{n,i}^{[j]}) = I_{n,i}^{[j+1]}$.*

Definition 13. A rank-one transformation T has an *ergodic spacer sequence* when the spacer sequence for T is ergodic with respect to T. Similarly, we define *weakly ergodic spacer sequence*, *totally ergodic spacer sequence* and *uniformly ergodic spacer sequence*.

Note that the spacer sequence is ergodic (respectively, totally or uniformly ergodic) with respect to T if and only if the normalized spacer sequence is ergodic (respectively, totally or uniformly ergodic) with respect to T. We use implicitly the following observation: the proof is standard.

PROPOSITION 5.4. *Let T be a rank-one transformation with a sequence of cuts $\{r_n\}$ having a finite limit point. Then T is partially rigid, i.e. there exists a sequence $\{a_n\}$ and $1 \geq \alpha > 0$ such that for all measurable sets A,*

$$\liminf_{n\to\infty} \mu(T^{a_n}(A) \cap A) \geq \alpha\mu(A),$$

and, therefore, T cannot be mixing. Hence, if the spacer sequence for T is a pathological dynamical sequence, then T is not mixing.

5.2. *Restricted growth rank-one transformations.* We introduce the class of restricted growth rank-one transformations characterized by adding spacer levels whose maximum variation in height approaches zero relative to the height of the column being cut and stacked. Note that adding spacer levels whose total height approaches zero relative to the height of the column *resulting from the cut and stack procedure* is a necessary condition for the space the transformation is defined on to be finite.

Definition 14. Given a rank-one transformation T with normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ and normalized height sequence $\{\widehat{h}_n\}$, the transformation T has *restricted growth* when for any sequences of positive integers $\{j_n\}$ and $\{k_n\}$ such that $0 \leq j_n < r_n - k_n$ and $\lim_{n\to\infty} r_n - k_n = \infty$, we have

$$\lim_{n\to\infty} \frac{1}{\widehat{h}_n} \widehat{s}_{n,j_n}^{(k_n)} = 0;$$

equivalently, we say $(1/\widehat{h}_n)\widehat{s}_{n,j}^{(k)} \to 0$ as $n \to \infty$ uniformly over positive integers j and k such that $0 \leq j < r_n - k$ and $r_n - k \to \infty$ as $n \to \infty$.

The following property that implies restricted growth is used in the sequel and is in general easier to demonstrate when it applies.

PROPOSITION 5.5. *Given a rank one transformation T with normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ and normalized height sequence $\{\widehat{h}_n\}$, if*

$$\lim_{n\to\infty} \frac{1}{\widehat{h}_n} \sum_{j=0}^{r_n-1} \left|\widehat{s}_{n,j}\right| = 0,$$

then T has restricted growth.

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Proof. For any positive integers n, j and k such that $j + k < r_n$, the partial sum

$$|\widehat{s}^{(k)}_{n,j}| = \left| \sum_{z=0}^{k-1} \widehat{s}_{n,j+z} \right| \leq \sum_{j=0}^{r_n-1} |\widehat{s}_{n,j}|. \qquad \square$$

For completeness, we provide an example of a rank-one transformation on a finite space that does not have restricted growth. Construct T using the sequence of cuts $\{r_n\}$ given by $r_n = 2(2^n - 1)$ and the spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,0} = h_n$ and $s_{n,j} = 0$ for $0 < j < r_n$, where $\{h_n\}$ is the sequence of heights. Observe that

$$\frac{1}{\widehat{h}_n} |\widehat{s}^{(1)}_{n,0}| = \frac{1}{\widehat{h}_n} \left| h_n - \frac{h_n}{r_n} \right| = \left(1 - \frac{1}{r_n}\right) \frac{h_n}{\widehat{h}_n}$$

and so T does not have restricted growth since $h_n/\widehat{h}_n \to 1$ as $n \to \infty$ because T is finite measure-preserving. That T is finite measure-preserving is left to the reader. Examples of mixing restricted growth rank-one transformations appear in the final two sections. We note that from the fact that T is finite measure-preserving, we automatically have that $\lim_{n\to\infty}(1/h_n)s^{(k)}_{n-1,j} = 0$ uniformly over positive integers j and k such that $j + k < r_{n-1}$.

5.3. *Rank-one uniform mixing transformations.* We introduce the notion of rank-one uniform mixing by considering the sums of the mixing values over increasingly fine levels. Our main theorem implies that a rank-one transformation with restricted growth is mixing if and only if the transformation is rank-one uniform mixing but note that the analogous result does not necessarily hold over sequences.

Definition 15. Given an ergodic transformation T, a sequence of positive integers $\{a_n\}$ is *uniformly mixing with respect to* T when

$$\lim_{n\to\infty} \sum_{i=0}^{h_{p_n}-1} |\mu(T^{a_n}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)| = 0$$

where $\{h_m\}$ is the height sequence for T, p_n is the unique positive integer such that $h_{p_n} \leq a_n < h_{p_n+1}$ and $\{I_{p_n,i}\}$ are the levels in the p_nth column for T. The transformation T is *rank-one uniform mixing* when the sequence $\{a_n\}$ given by $a_n = n$ for all $n \in \mathbb{N}$ is uniformly mixing with respect to T.

PROPOSITION 5.6. *Let T be a rank-one transformation and let $\{t_n\}$ be a strictly increasing sequence of positive integers. If $\{t_n\}$ is uniformly mixing with respect to T, then $\{t_n\}$ is mixing with respect to T. Consequently, if T is a rank-one uniform mixing transformation, then T is a mixing transformation.*

Proof. Let T be a rank-one transformation with cut sequence $\{r_n\}$, height sequence $\{h_n\}$ and levels $\{I_{n,i}\}$ and let $\{t_n\}$ be a strictly increasing sequence of positive integers that is uniformly mixing with respect to T. For any positive integer n, let p_n denote the unique positive integer satisfying $h_{p_n} \leq t_n < h_{p_n+1}$. Since the finite unions of levels generate the measurable sets, it suffices to show that $\{t_n\}$ is mixing with respect to T on finite unions

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of levels. Let A and B be unions of levels in some column C_N for some fixed positive integer N. Write $A = \bigcup_{i=0}^{\beta-1} I_{N,\alpha_i}$. Then, for any integer n such that $t_n > h_{N+1}$,

$$\begin{aligned}
&|\mu(T^{t_n}(A) \cap B) - \mu(A)\mu(B)| \\
&\quad = \left| \sum_{i=0}^{\beta-1} \mu(T^{t_n}(I_{N,\alpha_i}) \cap B) - \mu(I_{N,\alpha_i})\mu(B) \right| \\
&\quad \le \sum_{i=0}^{\beta-1} |\mu(T^{t_n}(I_{N,\alpha_i}) \cap B) - \mu(I_{N,\alpha_i})\mu(B)| \\
&\quad \le \sum_{i=0}^{h_N-1} |\mu(T^{t_n}(I_{N,i}) \cap B) - \mu(I_{N,i})\mu(B)| \\
&\quad = \sum_{i=0}^{h_N-1} \left| \sum_{z=0}^{r_N \ldots r_{p_n-1}-1} \mu(T^{t_n}(I_{p_n,i+\gamma(p_n,z)}) \cap B) - \mu(I_{p_n,i+\gamma(p_n,z)})\mu(B) \right| \\
&\quad \le \sum_{i=0}^{h_{p_n}-1} |\mu(T^{t_n}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)|,
\end{aligned}$$

which approaches zero as $n \to \infty$ since $\{t_n\}$ is uniformly mixing with respect to T. In this expression, $\gamma(p_n, z)$ is the height in column C_{p_n} of the zth sublevel of $I_{N,0}$. □

The concept of summing the mixing values over the levels can be applied to the ergodic averages as well. The sums over the levels of the ergodic averages may be regarded as a specific Riemann sum for the ergodic integral: the following proposition makes this explicit.

PROPOSITION 5.7. *Let T be a rank-one transformation with height sequence $\{h_n\}$ and levels $\{I_{n,i}\}$ and let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence of non-negative integers. Then $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T if and only if for any measurable set B and any unbounded non-decreasing sequence of positive integers $\{p_n\}$ such that $(1/h_{p_n} r_n) \sum_{j=0}^{r_n-1} s_{n,j} \to 0$ as $n \to \infty$,*

$$\lim_{n\to\infty} \sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B) \right| = 0.$$

Proof. Let T, $\{h_n\}$, $\{I_{n,i}\}$, $\{s_{n,j}\}_{\{r_n\}}$, B and $\{p_n\}$ be as before. Clearly, for each positive integer n, by the triangle inequality,

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B) \right| \\
&\qquad \le \int \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \right| d\mu
\end{aligned}$$

and so if $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T then this condition holds.

Conversely, assume this condition holds. We may assume that B is a union of levels. Then, for sufficiently large n, we may write $B = \bigcup_{i=0}^{\beta_n-1} I_{p_n,b_{n,i}}$ for appropriate positive

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integers β_n and $b_{n,i}$. Following the techniques of Blum and Hanson,

$$\begin{aligned}
&\int \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \right|^2 d\mu \\
&\quad = \frac{1}{r_n^2} \sum_{j,\ell=0}^{r_n-1} \mu(T^{s_{n,\ell}}(B) \cap T^{s_{n,j}}(B)) - \mu(B)\mu(B) \\
&\quad = \frac{1}{r_n^2} \sum_{j,\ell=0}^{r_n-1} \sum_{i=0}^{\beta_n-1} \mu(T^{s_{n,\ell}}(I_{p_n,b_{n,i}}) \cap T^{s_{n,j}}(B)) - \mu(I_{p_n,b_{n,i}})\mu(B) \\
&\quad \le \frac{1}{r_n} \sum_{\ell=0}^{r_n-1} \sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{s_{n,\ell}}(I_{p_n,i}) \cap T^{s_{n,j}}(B)) - \mu(I_{p_n,i})\mu(B) \right| \\
&\quad \le \frac{1}{r_n} \sum_{\ell=0}^{r_n-1} \left[\sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(I_{p_n,i} \cap T^{s_{n,j}}(B)) - \mu(I_{p_n,i})\mu(B) \right| + 2s_{n,\ell}\mu(I_{p_n,0}) \right] \\
&\quad \le \sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B) \right| + \frac{2}{h_{p_n}} \frac{1}{r_n} \sum_{\ell=0}^{r_n-1} s_{n,\ell}.
\end{aligned}$$

Thus, since $(1/h_{p_n} r_n) \sum_{j=0}^{r_n-1} s_{n,j} \to 0$ as $n \to \infty$, we have that $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T. □

6. *Mixing on rank-one transformations with restricted growth*

6.1. *Mixing height sequences.* Useful preliminary results to our main theorem are the following two theorems equating ergodicity of the spacer sequence and mixing of the height sequence for any rank-one transformation.

THEOREM 3. *Let T be a rank-one transformation with spacer sequence given by $\{s_{n,j}\}_{\{r_n\}}$. Then the height sequence $\{h_n\}$ is mixing with respect to T if and only if $\{s_{n,j}\}_{\{r_n\}}$ is weak ergodic with respect to T. Equivalently, substitute the normalized height sequence $\{\widehat{h}_n\}$ for the height sequence and the normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ for the spacer sequence.*

Proof. Let T, $\{s_{n,j}\}_{\{r_n\}}$ and $\{h_n\}$ be as before. Denote the columns defining T by C_n and the levels by $\{I_{n,i}\}$. Let A and B be finite unions of levels in C_N for some fixed $N > 0$ and consider A as a union of levels in C_n for $n > N$: $A = \bigcup_{i=0}^{\alpha_n-1} I_{n,a_{n,i}}$ for some positive integers α_n and $a_{n,i}$ less than h_n (when using the normalized heights and spacers we ignore the part of A that is in the top $\widehat{h}_n - h_n$ levels of C_n which have measure approaching zero as $n \to \infty$). Then,

$$\begin{aligned}
&\mu(T^{h_n}(A) \cap B) - \mu(A)\mu(B) \\
&\quad = \sum_{i=0}^{\alpha_n-1} \sum_{j=0}^{r_n-1} \mu(T^{h_n}(I_{n,a_{n,i}}^{[j]}) \cap B) - \mu(I_{n,a_{n,i}}^{[j]})\mu(B) \\
&\quad = \left[\sum_{i=0}^{\alpha_n-1} \sum_{j=0}^{r_n-2} \mu(T^{-s_{n,j}}(I_{n,a_{n,i}}^{[j+1]}) \cap B) - \mu(I_{n,a_{n,i}}^{[j+1]})\mu(B) \right] \pm \left(\frac{1}{r_n} \right)
\end{aligned}$$

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$$= \sum_{j=0}^{r_n-2} \Bigg[\sum_{\substack{i=0 \\ a_{n,i} \geq s_{n,j}}}^{\alpha_n-1} \mu(I_{n,a_{n,i}-s_{n,j}}^{[j+1]} \cap B) - \mu(I_{n,a_{n,i}-s_{n,j}}^{[j+1]})\mu(B)$$

$$+ \sum_{\substack{i=0 \\ a_{n,i} < s_{n,j}}}^{\alpha_n-1} \mu(T^{-s_{n,j}}(I_{n,a_{n,i}}^{[j+1]}) \cap B) - \mu(I_{n,a_{n,i}}^{[j+1]})\mu(B) \Bigg] \pm \left(\frac{1}{r_n}\right)$$

$$= \sum_{j=0}^{r_n-2} \sum_{\substack{i=0 \\ a_{n,i} \geq s_{n,j}}}^{\alpha_n-1} \frac{1}{r_n} [\mu(I_{n,a_{n,i}-s_{n,j}} \cap B) - \mu(I_{n,a_{n,i}-s_{n,j}})\mu(B)]$$

$$\pm \left(\sum_{j=0}^{r_n-2} |\mu(S_n^{[j]} \cap B) - \mu(S_n^{[j]})\mu(B)| + \frac{1}{r_n} \right)$$

$$= \frac{1}{r_n} \sum_{j=0}^{r_n-2} \sum_{\substack{i=0 \\ a_{n,i} \geq s_{n,j}}}^{\alpha_n-1} \mu(T^{-s_{n,j}}(I_{n,a_{n,i}}) \cap B) - \mu(I_{n,a_{n,i}})\mu(B) \pm \left(\mu(S_n) + \frac{1}{r_n}\right)$$

$$= \frac{1}{r_n} \sum_{j=0}^{r_n-1} \sum_{i=0}^{\alpha_n-1} \mu(T^{-s_{n,j}}(I_{n,a_{n,i}}) \cap B) - \mu(I_{n,a_{n,i}})\mu(B) \pm \left(2\mu(S_n) + \frac{2}{r_n}\right)$$

$$= \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(A) \cap B) - \mu(A)\mu(B) \pm \left(2\mu(S_n) + \frac{2}{r_n}\right),$$

which approaches zero if and only if $\{s_{n,j}\}_{\{r_n\}}$ is weak ergodic with respect to T, since T is finite measure-preserving and so $\mu(S_n) \to 0$ as $n \to \infty$. We may replace $\{h_n\}$ by $\{\widehat{h}_n\}$ and $\{s_{n,j}\}_{\{r_n\}}$ by $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ since $\widehat{h}_n/h_n \to 1$ as $n \to \infty$ and T is measure-preserving. □

THEOREM 4. *Let T be a rank-one transformation with spacer sequence given by $\{s_{n,j}\}_{\{r_n\}}$. Then the height sequence $\{h_n\}$ is uniformly mixing with respect to T if and only if $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T. Equivalently, substitute the normalized height sequence $\{\widehat{h}_n\}$ for the height sequence and the normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ for the spacer sequence.*

Proof. Let T, $\{s_{n,j}\}_{\{r_n\}}$ and $\{h_n\}$ be as before. Denote the columns defining T by C_n and the levels by $\{I_{n,i}\}$. Let B be a union of levels in some column C_N for a fixed $N > 0$ (as above, when using the normalized heights and spacers we ignore the top $\widehat{h}_n - h_n$ levels in C_n which have measure tending to zero so that the sums over i below range only up to h_n even when considering the mixing behavior of $\{\widehat{h}_n\}$). Then, using the same arguments as in the previous theorem,

$$\sum_{i=0}^{h_n-1} |\mu(T^{h_n}(I_{n,i}) \cap B) - \mu(I_{n,i})\mu(B)| = \sum_{i=0}^{h_n-1} \left| \sum_{j=0}^{r_n-1} \mu(T^{h_n}(I_{n,i}^{[j]}) \cap B) - \mu(I_{n,i}^{[j]})\mu(B) \right|$$

$$= \sum_{i=0}^{h_n-1} \left| \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(I_{n,i}) \cap B) - \mu(I_{n,i})\mu(B) \right| \pm \left(2\mu(S_n) + \frac{2}{r_n}\right).$$

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Using that T is finite measure-preserving we have that $(1/r_n h_n)\sum_{j=0}^{r_n-1} s_{n,j} \to 0$ as $n \to \infty$, so by Proposition 5.7 this quantity approaches zero if and only if $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T. The equivalent formulations follow as before. □

We provide an example of a rank-one transformation that has a uniformly mixing height sequence but that is not a mixing transformation. The spacer sequence will be chosen to be a permutation of the spacer sequence for a staircase transformation as follows. Let T be the rank-one transformation with $r_n = 2n$ and spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given by

$$s_{n,2j} = j \quad \text{for integers } j \text{ such that } 0 \le j < n$$
$$s_{n,2j-1} = r_n - j \quad \text{for integers } j \text{ such that } 0 < j \le n.$$

Note that T has restricted growth. For any measurable set B, clearly

$$\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} = \frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-j},$$

and so $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T and, hence, $\{h_n\}$, the height sequence for T, is uniformly mixing with respect to T by Theorem 4.

We see that, for any level $I_{n,i}$ in C_n,

$$\mu(T^{2h_n+r_n}(I_{n,i}) \cap I_{n,i}) = \frac{r_n - 1}{2r_n}\mu(I_{n,i}),$$

and, thus, $\{2h_n + r_n - 1\}$ is a partially rigid sequence for T and, therefore, T is not mixing. One can also see that the sequence $\{2h_n\}$ is not uniformly mixing.

6.2. *Mixing sequences.* Under the assumption that the rank-one transformation in question has restricted growth, the transformation's mixing behavior on sequences is related to the ergodicity of the partial sums of the spacer sequence as follows; our main result (Theorem 6) follows as a consequence.

THEOREM 5. *Let T be a rank-one transformation with normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ that is ergodic with respect to T and normalized height sequence $\{\widehat{h}_n\}$, and let $\{t_n\}$ be a strictly increasing sequence of positive integers. For each positive integer n, let p_n be the unique positive integer such that $\widehat{h}_{p_n} \le t_n < \widehat{h}_{p_n+1}$ and let k_n be the largest positive integer such that $k_n\widehat{h}_{p_n} \le t_n$. If the following hold:*

(i) *the $\{k_n\}$th and $\{k_n + 1\}$th partial sum dynamical sequences $\{\widehat{s}^{(k_n)}_{p_n,j}\}_{\{r_{p_n}-k_n\}}$ and $\{\widehat{s}^{(k_n+1)}_{p_n,j}\}_{\{r_{p_n}-k_n-1\}}$ are both ergodic with respect to T; and*

(ii) *$(1/\widehat{h}_{p_n})\widehat{s}^{(k_n+1)}_{p_n,j} \to 0$ as $n \to \infty$ uniformly over $j \in \mathbb{Z}_{r_{p_n}-k_n-1}$, specifically, for all $\epsilon > 0$ there exists a positive integer N such that for all integers $n \ge N$ and all $j \in \mathbb{Z}_{r_{p_n}-k_n-1}$, $(1/\widehat{h}_{p_n})\widehat{s}^{(k_n)}_{p_n,j} < \epsilon$;*

then $\{t_n\}$ is uniformly mixing with respect to T.

Proof. Let $\{C_n\}$, $\{h_n\}$, $\{I_{n,i}\}$ and $\{S_n\}$ be the sequences of columns, heights, levels and unions of spacer levels of T, respectively. Let B be a union of levels in some column C_N for some positive integer N. Let n be any positive integer such that $t_n \ge h_N$. Define p_n

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and k_n as before and let q_n be such that $t_n = k_n\widehat{h}_{p_n} + q_n$. Then $q_n < \widehat{h}_{p_n}$. First, assume that $k_n < r_{p_n} - 1$. Following the techniques in our theorems on mixing height sequences,

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-q_n-1} |\mu(T^{t_n}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)| \\
&\quad = \sum_{i=0}^{h_{p_n}-q_n-1} \left| \sum_{j=0}^{r_{p_n}-1} \mu(T^{k_n\widehat{h}_{p_n}+q_n}(I_{p_n,i}^{[j]}) \cap B) - \mu(I_{p_n,i}^{[j]})\mu(B) \right| \\
&\quad = \sum_{i=0}^{h_{p_n}-q_n-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{k_n\widehat{h}_{p_n}}(I_{p_n,i+q_n}^{[j]}) \cap B) - \mu(I_{p_n,i+q_n}^{[j]})\mu(B) \right| \\
&\qquad \pm \left(\sum_{i=0}^{h_{p_n}-q_n-1} \sum_{j=r_{p_n}-k_n}^{r_{p_n}-1} |\mu(T^{k_n\widehat{h}_{p_n}}(I_{p_n,i+q_n}^{[j]}) \cap B) - \mu(I_{p_n,i+q_n}^{[j]})\mu(B)| \right).
\end{aligned}$$

For all positive integers m and k such that $k < r_m$, set

$$\zeta_m^{(k)} = \max_{0 \le j < r_m - k} \widehat{s}_{m,j}^{(k)}.$$

Note that $(1/\widehat{h}_{p_n})\zeta_{p_n}^{(k_n)} \to 0$ as $n \to \infty$ by Hypothesis (ii). Then, for the first summand,

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-q_n-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{k_n\widehat{h}_{p_n}}(I_{p_n,i+q_n}^{[j]}) \cap B) - \mu(I_{p_n,i+q_n}^{[j]})\mu(B) \right| \\
&\quad = \sum_{i=q_n}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-\widehat{s}_{p_n,j}^{(k_n)}}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
&\quad = \sum_{\substack{i=q_n \\ i \ge \zeta_{p_n}^{(k_n)}}}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-\widehat{s}_{p_n,j}^{(k_n)}}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
&\qquad + \sum_{\substack{i=q_n \\ i < \zeta_{p_n}^{(k_n)}}}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-\widehat{s}_{p_n,j}^{(k_n)}}(I_{p_n,i}^{[j+k_n]}) \cap B) - \mu(I_{p_n,i}^{[j+k_n]})\mu(B) \right| \\
&\quad = \sum_{\substack{i=q_n \\ i \ge \zeta_{p_n}^{(k_n)}}}^{h_{p_n}-1} \left| \sum_{j=0}^{r_{p_n}-k_n-1} \frac{1}{r_{p_n}} (\mu(T^{-\widehat{s}_{p_n,j}^{(k_n)}}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B)) \right| \\
&\qquad \pm (\zeta_{p_n}^{(k_n)} |\mu(I_{p_n,0} \cap B) - \mu(I_{p_n,0})\mu(B)|) \\
&\quad \le \sum_{i=0}^{h_{p_n}-1} \left| \frac{1}{r_{p_n}-k_n} \sum_{j=0}^{r_{p_n}-k_n-1} \mu(T^{-s_{p_n,j}^{(k_n)}}(I_{p_n,i}) \cap B) - \mu(I_{p_n,i})\mu(B) \right| + \frac{\zeta_{p_n}^{(k_n)}}{\widehat{h}_{p_n}}
\end{aligned}$$

which approaches zero as $n \to \infty$ by Proposition 5.7 and Theorem 4 since $\{\widehat{s}_{p_n,j}^{(k_n)}\}_{\{r_{p_n}-k_n\}}$ is ergodic with respect to T.

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Using the triangle inequality and that sublevels in C_{p_n} are levels in C_{p_n+1},

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-q_n-1}\sum_{j=r_{p_n}-k_n}^{r_{p_n}-1}|\mu(T^{k_n\widehat{h}_{p_n}}(I_{p_n,i}^{[j]})\cap B)-\mu(I_{p_n,i}^{[j]})\mu(B)|\\
&\quad\le\sum_{i=0}^{h_{p_n}-1}\sum_{j=0}^{k_n-1}|\mu(T^{k_n\widehat{h}_{p_n}+(r_{p_n}-k_n)\widehat{h}_{p_n}+\widehat{s}_{p_n,j}^{(r_{p_n}-k_n)}}(I_{p_n,i}^{[j]})\cap B)-\mu(I_{p_n,i}^{[j]})\mu(B)|\\
&\quad=\sum_{i=0}^{h_{p_n}-1}\sum_{j=0}^{k_n-1}|\mu(T^{h_{p_n+1}}(T^{-\widehat{s}_{p_n,0}^{(j)}-\widehat{s}_{p_n,r_{p_n}-k_n+j}^{(k_n-j)}}(I_{p_n,i}^{[j]}))\cap B)-\mu(I_{p_n,i}^{[j]})\mu(B)|\\
&\quad\le\sum_{i=0}^{h_{p_n+1}-1}|\mu(T^{h_{p_n+1}}(I_{p_n+1,i})\cap B)-\mu(I_{p_n+1,i})\mu(B)|+\mu(S_{p_n}),
\end{aligned}$$

which approaches zero as $n\to\infty$ since $\{h_n\}$ is uniformly mixing with respect to T by Theorem 4 and since T is finite measure-preserving.

Similarly, we have that

$$\begin{aligned}
&\sum_{i=h_{p_n}-q_n}^{h_{p_n}-1}|\mu(T^{t_n}(I_{p_n,i})\cap B)-\mu(I_{p_n,i})\mu(B)|\\
&\quad\le\sum_{i=0}^{q_n-1}\left|\frac{1}{r_{p_n}-k_n-1}\sum_{j=0}^{r_{p_n}-k_n-2}\mu(T^{-\widehat{s}_{p_n,j}^{(k_n+1)}}(I_{p_n,i})\cap B)-\mu(I_{p_n,i})\mu(B)\right|\\
&\qquad+\left(\frac{2}{\widehat{h}_{p_n}}\zeta_{p_n}^{(k_n+1)}+\sum_{i=0}^{h_{p_n+1}-1}|\mu(T^{h_{p_n+1}}(I_{p_n+1,i})\cap B)-\mu(I_{p_n+1,i})\mu(B)|\right.\\
&\qquad\left.+(\widehat{h}_{p_n}-h_{p_n})\mu(I_{p_n,0})\right)
\end{aligned}$$

which approaches zero as $n\to\infty$ as before, substituting k_n+1 for k_n. Thus, the subsequence of $\{t_n\}$ consisting of all t_n such that $k_n<r_{p_n}-1$ is uniformly mixing with respect to T.

When $k_n\ge r_{p_n}-1$, we see that $t_n\ge(r_{p_n}-1)\widehat{h}_{p_n}$. Then,

$$\frac{h_{p_n+1}-t_n}{h_{p_n+1}}\le\frac{h_{p_n+1}-(r_{p_n}-1)\widehat{h}_{p_n}}{h_{p_n+1}}\le\frac{\widehat{h}_{p_n+1}-(r_{p_n}-1)\widehat{h}_{p_n}}{h_{p_n+1}}=\frac{1}{h_{p_n+1}}\sum_{j=0}^{r_{p_n}-1}\widehat{s}_{p_n,j}+\frac{\widehat{h}_{p_n}}{h_{p_n+1}},$$

which approaches zero as $n\to\infty$ since T is finite measure-preserving. Following the previous techniques,

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-1}|\mu(T^{t_n}(I_{p_n,i})\cap B)-\mu(I_{p_n,i})\mu(B)|\\
&\quad\le\sum_{i=0}^{h_{p_n+1}-1}|\mu(T^{t_n}(I_{p_n+1,i})\cap B)-\mu(I_{p_n+1,i})\mu(B)|\\
&\quad\le\sum_{i=\widehat{h}_{p_n+1}-t_n}^{h_{p_n+1}-1}|\mu(T^{t_n}(I_{p_n+1,i})\cap B)-\mu(I_{p_n+1,i})\mu(B)|+(\widehat{h}_{p_n+1}-t_n)\mu(I_{p_n+1,0})
\end{aligned}$$

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$$= \sum_{i=0}^{t_n-1} |\mu(T^{\widehat{h}_{p_n+1}}(I_{p_n+1,i}) \cap B) - \mu(I_{p_n+1,i})\mu(B)| + \frac{\widehat{h}_{p_n+1} - t_n}{h_{p_n+1}} \mu(C_{p_n+1})$$

$$\leq \sum_{i=0}^{h_{p_n+1}-1} |\mu(T^{\widehat{h}_{p_n+1}}(I_{p_n+1,i}) \cap B) - \mu(I_{p_n+1,i})\mu(B)| + \frac{\widehat{h}_{p_n+1} - t_n}{h_{p_n+1}},$$

which approaches zero as $n \to \infty$ using Proposition 5.7 and Theorem 4 as before and using our observation that $(\widehat{h}_{p_n+1} - t_n/h_{p_n+1}) \to 0$ as $n \to \infty$. Hence, the subsequence of $\{t_n\}$ consisting of all t_n such that $k_n \geq r_{p_n} - 1$ is also uniformly mixing with respect to T. □

6.3. *Mixing transformations with restricted growth.* Our main result follows from the previous theorem relating mixing on sequences to ergodic averages of the partial sums of the spacer sequence.

THEOREM 6. *Let T be a restricted growth rank-one transformation. Then the following are equivalent:*

(i) *T is a mixing transformation;*
(ii) *T is a rank-one uniform mixing transformation;*
(iii) *the spacer sequence for T is uniformly ergodic with respect to T; and*
(iv) *the normalized spacer sequence for T is uniformly ergodic with respect to T.*

Proof. Let $\{s_{n,j}\}_{\{r_n\}}$ be the spacer sequence and $\{h_n\}$ the height sequence for T and let $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ and $\{\widehat{h}_n\}$ be the normalized versions of them. We first show that (iii) implies (ii). Assume that $\{s_{n,j}\}_{\{r_n\}}$ is uniformly ergodic with respect to T. Let $\{t_n\}$ be any strictly increasing sequence of positive integers and let $\{p_n\}$, $\{k_n\}$ and $\{q_n\}$ be chosen as in Theorem 5. Note that $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T and that T has restricted growth so Hypothesis (ii) of Theorem 5 is satisfied using Proposition 5.5. If $r_{p_n} - k_n \to \infty$ as $n \to \infty$, then the uniform ergodicity of $\{s_{n,j}\}_{\{r_n\}}$ implies the ergodicity of the dynamical sequences $\{s^{(k_n)}_{p_n,j}\}_{\{r_{p_n}-k_n\}}$ and $\{s^{(k_n+1)}_{p_n,j}\}_{\{r_{p_n}-k_n-1\}}$ so Theorem 5 gives us that $\{t_n\}$ is mixing with respect to T. If $r_{p_n} - k_n$ is bounded over some subsequence of $n \to \infty$ by some positive integer M, then for each of the subsequence of positive integers n,

$$\begin{aligned} h_{p_n+1} - t_n &= r_{p_n} h_{p_n} + \sum_{j=0}^{r_{p_n}-1} s_{p_n,j} - k_n \widehat{h}_{p_n} - q_n \\ &= r_{p_n} \widehat{h}_{p_n} + \sum_{j=0}^{r_{p_n}-1} \widehat{s}_{p_n,j} - k_n \widehat{h}_{p_n} - q_n \\ &= (r_{p_n} - k_n)\widehat{h}_{p_n} + \sum_{j=0}^{r_{p_n}-1} \widehat{s}_{p_n,j} - q_n \\ &\leq M\widehat{h}_{p_n} + \sum_{j=0}^{r_{p_n}} \widehat{s}_{p_n,j} - q_n \\ &\leq M\widehat{h}_{p_n} + \sum_{j=0}^{r_{p_n}-1} \widehat{s}_{p_n,j}. \end{aligned}$$

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Then,

$$\frac{h_{p_n+1}-t_n}{h_{p_n+1}} \le M\frac{\widehat{h}_{p_n}}{h_{p_n+1}} + \frac{1}{h_{p_n+1}}\sum_{j=0}^{r_{p_n}-1}\widehat{s}_{p_n,j} \to 0 \quad \text{as } n\to\infty,$$

since $(\widehat{h}_{p_n}/h_{p_n+1}) \approx (1/r_{p_n}) \to 0$ as $n\to\infty$, M is fixed and T is finite measure-preserving so $(1/h_{p_n+1})\widehat{s}_{p_n,j}^{(r_{p_n})} \approx \mu(S_{p_n}) \to 0$ as $n\to\infty$ (here, as before, S_n is the union of spacer levels added to the nth column when constructing T).

Given a union of levels B in some column C_N for some fixed positive integer N, for any positive integer n such that $p_n \ge N$,

$$\begin{aligned}
&\sum_{i=0}^{h_{p_n}-1} |\mu(T^{t_n}(I_{p_n,i})\cap B) - \mu(I_{p_n,i})\mu(B)| \\
&\quad\le \sum_{i=0}^{h_{p_n+1}-1} |\mu(T^{t_n}(I_{p_n+1,i})\cap B) - \mu(I_{p_n+1,i})\mu(B)| \\
&\quad\le \sum_{i=h_{p_n+1}-t_n}^{h_{p_n+1}-1} |\mu(T^{t_n}(I_{p_n+1,i})\cap B) - \mu(I_{p_n+1,i})\mu(B)| + (h_{p_n+1}-t_n)\mu(I_{p_n+1,0}) \\
&\quad= \sum_{i=0}^{t_n-1} |\mu(T^{h_{p_n+1}}(I_{p_n+1,i})\cap B) - \mu(I_{p_n+1,i})\mu(B)| + \frac{h_{p_n+1}-t_n}{h_{p_n+1}}\mu(C_{p_n+1}) \\
&\quad\le \sum_{i=0}^{h_{p_n+1}-1} |\mu(T^{h_{p_n+1}}(I_{p_n+1,i})\cap B) - \mu(I_{p_n+1,i})\mu(B)| + \frac{h_{p_n+1}-t_n}{h_{p_n+1}},
\end{aligned}$$

which approaches zero as $n\to\infty$ since $(h_{p_n+1}-t_n/h_{p_n+1}) \to 0$ as $n\to\infty$ and $\{h_n\}$ is a uniformly mixing sequence with respect to T because $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T. Thus, the sequence $\{t_n\}$ is uniformly mixing with respect to T regardless of whether $r_{p_n}-k_n \to \infty$ as $n\to\infty$. Hence, T is a rank-one uniform mixing transformation since all strictly increasing sequences of positive integers are then uniformly mixing with respect to T.

The statement (ii) implies (i) is a consequence of Proposition 5.6 so all that remains to be shown is that (i) implies (iii). Suppose that $\{s_{n,j}\}_{\{r_n\}}$ is not uniformly ergodic with respect to T, so along some sequence of positive integers $\{k_n\}$ such that $r_n-k_n\to\infty$ as $n\to\infty$, the dynamical sequence $\{s_{n,j}^{(k_n)}\}_{\{r_n-k_n\}}$ is not ergodic with respect to T. If $\{s_{n,j}\}_{\{r_n\}}$ is not ergodic with respect to T then it is not uniformly ergodic with respect to T so T then cannot be mixing because $\{h_n\}$ is then not uniformly mixing with respect to T; hence, we may assume that $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T. There exists a $\delta>0$ and a union of levels B in some column C_N for some positive integer N such that for infinitely many positive integers n,

$$\int \left| \frac{1}{r_n-k_n}\sum_{j=0}^{r_n-k_n-1} \chi_B\circ T^{-s_{n,j}^{(k_n)}} - \mu(B)\right|^2 d\mu \ge \delta.$$

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For any positive integer $n > N$, write B as a union of levels in C_n; $B = \bigcup_{i=0}^{\beta_n-1} I_{n,b_{n,i}}$ for appropriate β_n and $\{b_{n,i}\}$. Using the techniques in our earlier theorems,

$$\begin{aligned}
&\int \left| \frac{1}{r_n - k_n} \sum_{j=0}^{r_n-k_n-1} \chi_B \circ T^{-s_{n,j}^{(k_n)}} - \mu(B) \right|^2 d\mu \\
&\quad = \frac{1}{(r_n-k_n)^2} \sum_{j,\ell=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} - s_{n,\ell}^{(k_n)}}(B) \cap B) - \mu(B)\mu(B) \\
&\quad = \sum_{i=0}^{\beta_n-1} \frac{1}{(r_n-k_n)^2} \sum_{j,\ell=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} - s_{n,\ell}^{(k_n)}}(I_{n,b_{n,i}}) \cap B) - \mu(I_{n,b_{n,i}})\mu(B) \\
&\quad = \frac{r_n}{r_n-k_n} \sum_{i=0}^{\beta_n-1} \frac{1}{r_n-k_n} \sum_{j,\ell=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} - s_{n,\ell}^{(k_n)}}(I_{n,b_{n,i}}^{[\ell+k_n]}) \cap B) - \mu(I_{n,b_{n,i}}^{[\ell+k_n]})\mu(B) \\
&\quad = \frac{r_n}{r_n-k_n} \sum_{i=0}^{\beta_n-1} \frac{1}{r_n-k_n} \sum_{j,\ell=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} + k_n h_n}(I_{n,b_{n,i}}^{[\ell]}) \cap B) - \mu(I_{n,b_{n,i}}^{[\ell]})\mu(B) \\
&\qquad \pm \left(\frac{1}{h_n} \left(\max_j \widehat{s}_{n,j}^{(k_n)} \right) \right) \\
&\quad = \frac{r_n}{r_n-k_n} \sum_{i=0}^{\beta_n-1} \frac{1}{r_n-k_n} \sum_{j=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} + k_n h_n}(I_{n,b_{n,i}}) \cap B) - \mu(I_{n,b_{n,i}})\mu(B) \\
&\qquad \pm \left(\frac{1}{h_n} \left(\max_j \widehat{s}_{n,j}^{(k_n)} \right) + \sum_{i=0}^{h_{n+1}-1} |\mu(T^{h_{n+1}}(I_{n+1,i}) \cap B) - \mu(I_{n+1,i})\mu(B)| \right) \\
&\quad = \frac{r_n}{r_n-k_n} \frac{1}{r_n-k_n} \sum_{j=0}^{r_n-k_n-1} \mu(T^{s_{n,j}^{(k_n)} + k_n h_n}(B) \cap B) - \mu(B)\mu(B) \\
&\qquad \pm \left(\frac{1}{h_n} \left(\max_j \widehat{s}_{n,j}^{(k_n)} \right) + \sum_{i=0}^{h_{n+1}-1} |\mu(T^{h_{n+1}}(I_{n+1,i}) \cap B) - \mu(I_{n+1,i})\mu(B)| \right)
\end{aligned}$$

and so since $\{h_n\}$ is a uniformly mixing sequence and T has restricted growth, the dynamical sequence $\{s_{n,j}^{(k_n)} + k_n h_n\}_{\{r_n-k_n\}}$ is not weak ergodic with respect to T. Since $\{s_{n,j}^{(k_n)} + k_n h_n\}_{\{r_n-k_n\}}$ is weak monotone (and non-pathological), by the generalized weak Blum–Hanson theorem (Theorem 2), this means that T is not mixing.

The equivalence of (iii) and (iv) follows directly from the measure-preserving property of T since $\widehat{s}_{n,j} = s_{n,j} - \overline{s}_{n0}$. □

7. *Preliminary mixing properties*

We use the criteria in this section for rank-one transformations to be totally ergodic and power uniform ergodic when applying our main theorem to the class of polynomial rank-one transformations in the next section. These preliminary mixing properties allow us to demonstrate the uniform ergodicity of the spacer sequences of polynomial rank-one transformations.

7.1. *Total ergodicity on rank-one transformations.*

We will need the following criterion for total ergodicity on a rank-one transformation in the following. The authors' original

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proof used a characterization of the eigenvalues of a rank-one transformation in [**Na98**, §15.56]; we are indebted to the referee for suggesting we write a more elementary proof.

THEOREM 7. *Let T be a rank-one transformation with (possibly normalized) non-pathological spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ such that for all fixed positive integers $L > 1$,*

$$\limsup_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n-1} : L \text{ divides } s_{n,j+1} - s_{n,j}\} < 1.$$

Then T is a totally ergodic transformation.

Proof. Let T and $\{s_{n,j}\}_{\{r_n\}}$ be as before. Suppose that T is not totally ergodic. Then there exist a positive integer $L > 1$ and a measurable set B, $0 < \mu(B) < 1$, such that $T^L(B) = B$. For all $\ell \in \mathbb{Z}$, let $B_\ell = T^\ell(B)$. Then $B_\ell = B_{\ell \bmod L}$. We may assume that L is the smallest such positive integer and observe that there exists $\delta > 0$ such that for all $\ell \in \mathbb{Z}_L$, $\ell \neq 0$, $\mu(B_\ell \cap B) \leq (1-\delta)\mu(B)$.

Consider the modulo L distributions of $\{s_{n,j}\}_{\{r_n\}}$, the maps $\rho_n : \mathbb{Z}_L \to [0,1]$, given by $\rho_n(\ell) = (1/r_n)\#\{j \in \mathbb{Z}_{r_n} : s_{n,j} = \ell \bmod L\}$. Note that for any $j \in \mathbb{Z}_{r_n-1}$, L divides $s_{n,j+1} - s_{n,j}$ if and only if $s_{n,j+1} = s_{n,j} \bmod L$. By hypothesis, there exists $\delta_1 > 0$ such that $\#\{j \in \mathbb{Z}_{r_n-1} : s_{n,j+1} - s_{n,j} \text{ is divisible by } L\} \leq (1-\delta_1)r_n$ for sufficiently large n. Hence, for at least $\delta_1 r_n$ values of j, we have that $s_{n,j+1} \neq s_{n,j} \bmod L$. Thus, for all $\ell \in \mathbb{Z}_L$, we have $\rho_n(\ell) \leq 1 - \delta_1$.

Since $B_0, \ldots, B_{L-1}$ is a finite set, for any $\epsilon > 0$, there exists sufficiently large integers n and unions of levels $B_{n,\ell}$ in C_n, the nth column for T, for $0 \leq \ell < L$ such that $\mu(B_{n,\ell} \triangle B_\ell) < \epsilon$. Write $B_{n,\ell} = \bigcup_{i=0}^{\beta_n - 1} I_{n,b_{n,\ell,i}}$ where $I_{n,i}$ is the ith level in C_n for appropriate β_n and $b_{n,\ell,i}$. Let $\{h_n\}$ be the height sequence for T and let w_n and v_n be the unique integers such that $h_n = w_n L + v_n$ with $0 \leq v_n < L$. Following the methods of the previous section,

$$\begin{aligned}
&\mu(T^{w_n L}(B) \cap B) \\
&\quad = \mu(T^{h_n}(B) \cap T^{v_n}(B)) \\
&\quad = \mu(T^{h_n}(B_{n,0}) \cap B_{v_n}) \pm (\epsilon) \\
&\quad = \sum_{i=0}^{\beta_n - 1} \sum_{j=0}^{r_n - 1} \mu(T^{h_n}(I^{[j]}_{n,b_{n,0,i}}) \cap B_{n,v_n}) \pm (2\epsilon) \\
&\quad = \sum_{i=0}^{\beta_n - 1} \sum_{j=0}^{r_n - 2} \mu(T^{-s_{n,j}}(I^{[j+1]}_{n,b_{n,0,i}}) \cap B_{n,v_n}) \pm \left(2\epsilon + \frac{1}{r_n}\right) \\
&\quad = \sum_{i=0}^{\beta_n - 1} \sum_{j=0}^{r_n - 2} \frac{1}{r_n} \mu(T^{-s_{n,j}}(I_{n,b_{n,0,i}}) \cap B_{n,v_n}) \pm \left(2\epsilon + \frac{1}{r_n} + 2\mu(S_n)\right) \\
&\quad = \frac{1}{r_n} \sum_{j=0}^{r_n - 1} \mu(T^{-s_{n,j}}(B_{n,0}) \cap B_{n,v_n}) \pm \left(2\epsilon + \frac{2}{r_n} + 2\mu(S_n)\right) \\
&\quad = \frac{1}{r_n} \sum_{j=0}^{r_n - 1} \mu(T^{-s_{n,j}}(B_0) \cap B_{v_n}) \pm \left(4\epsilon + \frac{2}{r_n} + 2\mu(S_n)\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{\ell=0}^{L-1} \rho_n(\ell)\mu(T^{-\ell}(B) \cap T^{v_n}(B)) \pm \left(4\epsilon + \frac{2}{r_n} + 2\mu(S_n)\right) \\
&= \sum_{\ell=0}^{L-1} \rho_n(\ell)\mu(T^{-\ell-v_n}(B) \cap B) \pm \left(4\epsilon + \frac{2}{r_n} + 2\mu(S_n)\right) \\
&\leq (1-\delta_1)\mu(B) + \delta_1(1-\delta)\mu(B) + 4\epsilon + \frac{2}{r_n} + 2\mu(S_n) \\
&= (1-\delta\delta_1)\mu(B) + 4\epsilon + \frac{2}{r_n} + 2\mu(S_n),
\end{aligned}
$$

where the inequality follows from the facts that $\rho_n(\ell) \leq 1-\delta_1$ for all $\ell \in \mathbb{Z}_L$ and $\mu(B_\ell \cap B) \leq (1-\delta)\mu(B)$ for all $\ell \in \mathbb{Z}_L$, $\ell \neq 0$. Since $r_n \to \infty$ and $\mu(S_n) \to 0$ as $n \to \infty$, letting $\epsilon \to 0$, we have that $\mu(T^{w_n L}(B) \cap B) \leq (1-\delta_2)\mu(B)$ for some $\delta_2 > 0$. But this contradicts that $T^L(B) = B$ since then $\mu(T^{w_n L}(B) \cap B) = \mu(B)$. □

7.2. *Power uniform ergodicity on rank-one transformations.* We show that restricted growth rank-one transformations having a totally ergodic spacer sequence are power uniform ergodic: this will serve as a base case for the induction theorem in the following section. A similar property called uniform Cesàro was introduced in [**AF92**]. The argument follows the methods used by Adams [**Ad98**] to show that staircase transformations having restricted growth are mixing.

Definition 16. An ergodic transformation T is *power uniform ergodic* when for any measurable set B and any sequence of positive integers $\{\ell_n\}$ such that $\limsup_{n\to\infty}(\ell_n/n) < \infty$,

$$\lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-j\ell_n} - \mu(B) \right|^2 d\mu = 0.$$

The following pair of lemmas are from [**Ad98**]: the first follows from the measure-preserving property and the proof is left to the reader and we include a proof of the second for completeness.

LEMMA 7.1. (Block Lemma) *Let T be a measure-preserving transformation and B a measurable set. Then, for any positive integers R, L and p,*

$$\int \left| \frac{1}{R} \sum_{i=0}^{R-1} \chi_B \circ T^{-i} - \mu(B) \right| d\mu \leq \int \left| \frac{1}{L} \sum_{i=0}^{L-1} \chi_B \circ T^{-ip} - \mu(B) \right| d\mu + \frac{pL}{R}.$$

LEMMA 7.2. *Let T be an ergodic transformation and $\{v_n\}$ a sequence of positive integers such that for any fixed (non-zero) integer j, the sequence of integers $\{jv_n\}$ is mixing with respect to T. Then, for any $\epsilon > 0$, there exist arbitrarily large positive integers L such that for sufficiently large n,*

$$\int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-jv_n} - \mu(B) \right|^2 d\mu < \epsilon.$$

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Proof. Fix $\epsilon > 0$. For each fixed non-zero integer j, choose a positive integer N_j such that for all $n \geq N_j$, $|\mu(T^{jv_n}(B) \cap B) - \mu(B)\mu(B)| < \epsilon$. Choose a positive integer L such that $1/L < \epsilon$ and set $N = \sup_{-L \leq j \leq L} N_j$. Note that for any $0 < j, \ell \leq L$, $-L \leq j - \ell \leq L$. Then, for all $n \geq N$,

$$\int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-jv_n} - \mu(B) \right|^2 d\mu$$
$$= \frac{1}{L^2} \sum_{j,\ell=0}^{L-1} \mu(T^{(j-\ell)v_n}(B) \cap B) - \mu(B)\mu(B) < \frac{1}{L^2} \sum_{j,\ell=1}^{L-1} \epsilon + \frac{1}{L} < 2\epsilon. \qquad \square$$

PROPOSITION 7.3. *Let T be a restricted growth rank-one transformation with a totally ergodic spacer sequence. Then T is power uniform ergodic.*

Proof. Let T be a restricted growth rank-one transformation with a totally ergodic normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$. Let $\{\ell_n\}$ be any sequence of positive integers such that $\limsup \ell_n/n < \infty$ and let B be any measurable set. Let $\{h_n\}$ denote the sequence of heights for T. Choose q_n to be the (unique) positive integer such that $h_{q_n-1} < \ell_n \leq h_{q_n}$. Choose ρ_n to be the (smallest) positive integer such that $h_{q_n} \leq \ell_n \rho_n < 2h_{q_n}$.

For each positive integer j, we have $jh_{q_n} \leq j\ell_n\rho_n < 2jh_{q_n}$. Note that, for each positive integer j,

$$\frac{1}{h_n} \widehat{s}_{n,z}^{(j)} \leq \cdots \leq \frac{1}{h_n} \widehat{s}_{n,z}^{(2j)} \to 0 \quad \text{as } n \to \infty$$

uniformly over, respectively, $z \in \mathbb{Z}_{r_n-j}, \ldots, z \in \mathbb{Z}_{r_n-2j}$, because T has restricted growth.

For each fixed positive integer j, we see that the total ergodicity of the spacer sequence implies that each of the kth partial sum dynamical sequences of $\{s_{n,j}\}_{\{r_n\}}$ for $k = j, \ldots, 2j$, denoted by $\{s_{n,z}^{(j)}\}_{\{r_n-j\}}, \ldots, \{s_{n,z}^{(2j)}\}_{\{r_n-2j\}}$, are all ergodic with respect to T. We then apply Theorem 5 to the sequence of positive integers $\{j\ell_n\rho_n\}$ and see that for each fixed j, the sequence $\{j\ell_n\rho_n\}$ is mixing with respect to T.

Fix $\epsilon > 0$. An application of Lemma 7.2 to the sequences $\{j\ell_n\rho_n\}$ for each fixed j yields that there exists a positive integer N and an arbitrarily large positive integer L such that for all integers $n \geq N$,

$$\int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-j\ell_n\rho_n} - \mu(B) \right|^2 d\mu < \epsilon.$$

Note that since T is measure-preserving,

$$\frac{r_n h_n}{h_{n+1}} \to 1, \text{ so } \frac{h_n^2}{h_{n+1}} = \frac{r_n h_n}{h_{n+1}} \frac{h_n}{r_n} \to \infty \quad \text{as } n \to \infty.$$

Since $\limsup_{n\to\infty} \ell_n/n < \infty$, we have that

$$\frac{\rho_n}{n} = \frac{\rho_n \ell_n^2}{n\ell_n^2} < \frac{2h_{q_n}}{\ell_n^2} \frac{\ell_n}{n} \leq \frac{2h_{q_n}}{h_{q_n-1}^2} \frac{\ell_n}{n} \to 0 \quad \text{as } n \to \infty.$$

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Hence, there exists a positive integer $\widehat{N}$ such that for all integers $n \geq \widehat{N}$, we have that $L(\rho_n/n) < \epsilon$. Applying the Block lemma (Lemma 7.1) and then the Hölder Inequality yields

$$\begin{aligned}
&\int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-j\ell_n} - \mu(B) \right| d\mu \\
&\quad \leq \int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-j\rho_n\ell_n} - \mu(B) \right| d\mu + \frac{\rho_n L}{n} \\
&\quad \leq \left(\int \left| \frac{1}{L} \sum_{j=0}^{L-1} \chi_B \circ T^{-j\rho_n\ell_n} - \mu(B) \right|^2 d\mu \right)^{\frac{1}{2}} + \frac{\rho_n L}{n} < 2\epsilon.
\end{aligned}$$

Since the functions $(1/n)\sum_{j=0}^{n-1} \chi_B \circ T^{-j\ell_n} - \mu(B)$ are uniformly bounded by 1, the L^1 convergence of these functions implies their L^2 convergence. □

8. *Polynomial rank-one transformations*

8.1. *Construction of polynomial rank-one transformations.* Polynomial rank-one transformations were first considered in [**AF92**], where the authors outlined an algorithm to show that there exists a mixing polynomial rank-one transformation. Here we define a large class of polynomial rank-one transformations and show that all transformations in this class are mixing. A *polynomial rank-one transformation* is a rank-one transformation where the spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ (required to be non-pathological) is given by $s_{n,j} = p_n(j)$ where the p_n are polynomials of degree less than or equal to some fixed positive integer D in j with rational coefficients and mapping integers to integers. We use the term *polynomial* to mean exclusively those having rational coefficients and mapping integers to integers: these polynomials are used in our argument later in a similar manner as in [**Be87**]. We shall further require that the polynomials defining polynomial rank-one transformations have the property that $c_{n,a} \leq Q$ for some fixed Q over all n and a where $c_{n,a}$ is the ath coefficient of p_n and the property that the range of $p_n(x+1) - p_n(x)$ is not contained in density by a proper ideal of $\mathbb{Z}$; that is for all positive integers $L > 1$,

$$\limsup_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : L \text{ divides } p_n(j+1) - p_n(j)\} < 1.$$

Note that, in the case of staircase transformations, this is equivalent to assuming that the lead coefficient of the linear polynomial in question is one. A related notion called the uniform polynomial Cesàro was introduced in [**AF92**].

8.2. *Polynomial uniform ergodicity.* We extend the notion of power uniform ergodicity as the ergodic averages over arithmetic progressions converging to zero uniformly to the notion of polynomial uniform ergodicity, which is the ergodic averages over polynomial sequences converging to zero uniformly.

Definition 17. An ergodic transformation T is *polynomial uniform ergodic* when for any measurable set B, any fixed positive integer D, and any sequence of polynomials $\{p_n\}$ of

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positive degree less than or equal to D and with Dth degree coefficients $c_{n,D}$ such that $\limsup_{n\to\infty} c_{n,D}/n < \infty$,

$$\lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-p_n(j)} - \mu(B) \right|^2 d\mu = 0.$$

Note that power uniform ergodicity corresponds to polynomial uniform ergodicity for linear polynomials where the condition that ℓ_n/n be bounded below infinity is the requirement that the leading coefficient over n be bounded below infinity.

8.3. *Induction theorem for polynomial uniform ergodicity.* Using the concepts developed by Bergelson in [**Be87**], we show that polynomial uniform ergodicity is equivalent to power uniform ergodicity. The proof is by induction on the maximum degree D of the polynomials. Power uniform ergodicity corresponds to the case when $D = 1$ since the measure-preserving property allows us to ignore any constant terms in the polynomials. The proof relies on (the measure-theoretic formulation of) Van Der Corput's Fundamental Inequality (see [**KN74**] for proof and details).

LEMMA 8.1. (Van Der Corput's Fundamental Inequality) *Let T be a transformation and B a measurable set. Then, for any positive integers N and L such that $L \le N$ and any map $F : \mathbb{Z} \to \mathbb{Z}$,*

$$\int \left| \frac{1}{N} \sum_{j=0}^{N-1} \chi_B \circ T^{F(j)} - \mu(B) \right|^2 d\mu \le \frac{N+L-1}{LN} \mu(B) + \frac{2(N+L-1)}{LN} \sum_{\ell=1}^{L-1} \frac{L-\ell}{LN} \sum_{j=0}^{N-\ell-1} \mu(T^{F(j)-F(j+\ell)}(B) \cap B) - \mu(B)\mu(B).$$

THEOREM 8. *Let T be a rank-one transformation that is power uniform ergodic. Then T is polynomial uniform ergodic.*

Proof. We proceed by induction of the degree of the polynomials. Assume that T is uniformly ergodic for polynomials having positive degree less than some fixed positive integer $D > 1$ and such that the lead coefficients of the polynomials over n is bounded below infinity. Let $\{p_n\}$ be a sequence of polynomials having positive degree less than or equal to D and such that $\limsup_{n\to\infty} c_{n,D}/n < \infty$ where $c_{n,D}$ is the Dth degree coefficient of p_n. Let B be any measurable set. Fix $\epsilon > 0$. Fix a positive integer L such that $1/L < \epsilon$. For each positive integer n and each positive integer $\ell < L$, define the polynomial $P_{n,\ell}$ by $P_{n,\ell}(j) = p_n(j+\ell) - p_n(j)$ (that the $P_{n,\ell}$ have rational coefficients and map integers to integers is left to the reader). Since p_n has positive degree less than or equal to D, write $p_n(j) = \sum_{a=0}^{D} c_{n,a} j^a$ where the $c_{n,a}$ are the coefficients of p_n and observe that for any positive integer n and any positive integer $\ell < L$,

$$P_{n,\ell}(j) = p_n(j+\ell) - p_n(j) = \sum_{a=0}^{D} c_{n,a}[(j+\ell)^a - j^a]$$
$$= \sum_{a=0}^{D} c_{n,a} \sum_{b=0}^{a-1} \binom{a}{b} j^b \ell^{a-b} = \sum_{b=0}^{D-1} \left(\sum_{a=b+1}^{D} c_{n,a} \binom{a}{b} \ell^{a-b} \right) j^b$$

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is a polynomial of degree less than or equal to $D-1$ with $(D-1)$st coefficient $C_{n,\ell,D-1} = c_{n,D}D\ell$. Then, for each positive integer $\ell < L$,

$$\limsup_{n\to\infty} \frac{C_{n,\ell,D-1}}{n} = \limsup_{n\to\infty} \frac{c_{n,D}D\ell}{n} \le \limsup_{n\to\infty} \frac{c_{n,D}}{n} DL < \infty,$$

since $\limsup_{n\to\infty} c_{n,D}/n < \infty$. Applying the inductive hypothesis for each positive integer $\ell < L$, there exists a positive integer N_ℓ such that for all integers $n \ge N_\ell$,

$$\int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{P_{n,\ell}(j)} - \mu(B) \right|^2 d\mu < \epsilon.$$

Set $N = \max_{0<\ell<L} N_\ell + L$. Then, for all integers $n \ge N$, first using Van Der Corput's Fundamental Inequality (Lemma 8.1) and then the Hölder Inequality,

$$\begin{aligned}
&\int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{p_n(j)} - \mu(B) \right|^2 d\mu \\
&\quad \le \frac{n+L}{nL} \left(\mu(B) + 2 \sum_{\ell=1}^{L-1} \frac{L-\ell}{Ln} \sum_{j=0}^{n-\ell-1} \mu(T^{p_n(j+\ell)-p_n(j)}(B) \cap B) - \mu(B)\mu(B) \right) \\
&\quad \le \frac{n+L}{nL} \left(\mu(B) + 2 \sum_{\ell=1}^{L-1} \frac{L-\ell}{L} \left| \frac{1}{n-\ell} \sum_{j=0}^{n-\ell-1} \mu(T^{P_{n,\ell}(j)}(B) \cap B) - \mu(B)\mu(B) \right| \right) \\
&\quad \le \frac{n+L}{nL} \left(\mu(B) + 2 \sum_{\ell=1}^{L-1} \left| \int \frac{1}{n-\ell} \sum_{j=0}^{n-\ell-1} \chi_B \circ T^{P_{n,\ell}(j)} \chi_B - \mu(B)\mu(B)\, d\mu \right| \right) \\
&\quad \le \frac{n+L}{nL} \left[\mu(B) + 2 \sum_{\ell=1}^{L-1} \left(\int \left| \frac{1}{n-\ell} \sum_{j=0}^{n-\ell-1} \chi_B \circ T^{P_{n,\ell}(j)} - \mu(B) \right|^2 d\mu \right)^{\frac{1}{2}} \mu(B)^{\frac{1}{2}} \right] \\
&\quad < \left(1 + \frac{L}{n} \right) \left(\frac{1}{L}\mu(B) + 2\frac{1}{L} \sum_{\ell=1}^{L-1} \epsilon^{\frac{1}{2}} \mu(B)^{\frac{1}{2}} \right) \\
&\quad < 2(\epsilon\mu(B) + 2\epsilon^{\frac{1}{2}}\mu(B)^{\frac{1}{2}}).
\end{aligned}$$

□

8.4. *Polynomial uniform ergodicity on polynomial rank-one transformations.* We now show that polynomial rank-one transformations are power uniform ergodic and, hence, are polynomial uniform ergodic. We start with the following theorem, which is essentially Furstenberg's Polynomial Ergodic theorem [**Fu81**, p. 70]: the only difference is that instead of having a fixed polynomial we consider a sequence of polynomials. Our original proof used induction and Van Der Corput's Fundamental Inequality. We are indebted to the referee for mentioning that the theorem follows as a consequence of Weyl's theorem on equidistribution of polynomials and the spectral theorem.

THEOREM 9. (Furstenberg) *Let $\{p_n\}$ be a sequence of polynomials (with rational coefficients and mapping integers to integers) of positive degree less than or equal to some fixed positive integer D such that the coefficients of the p_n are all bounded above by some fixed Q and let T be a totally ergodic transformation. Then, for any sequence of*

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positive integers $\{r_n\}$ having no finite limit points, the dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = p_n(j)$ for all n and j is ergodic with respect to T.

Proof. The ergodicity of the previous dynamical sequences $\{s_{n,j}\}_{\{r_n\}}$ is determined by the averages $\int |1/r_n \sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B)|^2 \, d\mu$ which are isomorphic to the averages $\int |1/r_n \sum_{j=0}^{r_n-1} z^{p_n(j)}|^2 \, d\omega_B(z)$ where ω_B is the spectral measure of B under T. Weyl's theorem on the equidistribution of polynomial sequences states that $1/r_n \sum_{j=0}^{r_n-1} z^{p_n(j)} \to 0$ as $n \to \infty$ for all $z \in S^1 \setminus \mathbb{Q}$. The result follows. □

COROLLARY 10. *Let T be a totally ergodic transformation and let $\{s_{n,j}\}_{\{r_n\}}$ be the spacer sequence for a polynomial rank-one transformation. Then for any fixed positive integer t, the tth partial sum sequence for $\{s_{n,j}\}_{\{r_n\}}$ is ergodic with respect to T; i.e. the spacer sequence is totally ergodic with respect to T.*

Proof. Let T be a totally ergodic transformation and let $\{s_{n,j}\}_{\{r_n\}}$ be the spacer sequence for a polynomial rank-one transformation given by $s_{n,j} = p_n(j)$ where the p_n are polynomials of positive degree less than or equal to some fixed positive integer D. Let $\{s^{(k)}_{n,j}\}_{\{r_n-k\}}$ denote, as usual, the partial sum sequences for the spacer sequence and let t be any fixed positive integer. Write $p_n(j) = \sum_{a=0}^{D} c_{n,a} j^a$ where the $c_{n,a}$ are the coefficients of p_n. Then, the tth partial sum sequence is given by

$$\begin{aligned} s^{(t)}_{n,j} &= \sum_{z=0}^{t-1} s_{n,j+z} = \sum_{z=0}^{t-1} p_n(j+z) = \sum_{z=0}^{t-1} \sum_{a=0}^{D} c_{n,a}(j+z)^a \\ &= \sum_{a=0}^{D} c_{n,a} \sum_{z=0}^{t-1} \sum_{b=0}^{a} \binom{a}{b} j^b z^{a-b} \\ &= \sum_{b=0}^{D} \left(\sum_{a=b}^{D} c_{n,a} \binom{a}{b} \sum_{z=0}^{t-1} z^{a-b} \right) j^b = \sum_{b=0}^{D} C_{n,t,b} j^b, \end{aligned}$$

where the $C_{n,t,b}$ are bounded above since the $c_{n,a}$ are bounded above and t is fixed. Thus, the tth partial sum dynamical sequence for the spacer sequence is given by $s^{(t)}_{n,j} = P_{n,t}(j)$ where $P_{n,t}(j) = \sum_{b=0}^{D} C_{n,t,b} j^b$ is a sequence of polynomials of degree D (the reader may verify that $P_{n,t}$ maps integers to integers) with leading coefficients bounded above. Theorem 9 then guarantees that $\{s^{(t)}_{n,j}\}_{\{r^{(t)}_n\}}$ is ergodic with respect to T. □

PROPOSITION 8.2. *Let T be a restricted growth polynomial rank-one transformation. Then T is polynomial uniform ergodic.*

Proof. Let T be a restricted growth polynomial rank-one transformation. Since we have required that the polynomials p_n defining T have the property that the range of $p_n(x+1) - p_n(x)$ is not contained in density by a proper ideal of $\mathbb{Z}$, Theorem 7 yields that T is a totally ergodic transformation. Corollary 10 then implies that T has a totally ergodic spacer sequence. Then Proposition 7.3 implies that T is power uniform ergodic. Hence, Theorem 8 yields the result. □

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8.5. *Mixing on restricted growth polynomial rank-one transformations.* We now apply our main theorem and the previous results to show that polynomial rank-one transformations having restricted growth are mixing transformations. Since staircase transformations are defined as those rank-one transformations having spacer sequence given by a linear polynomial and since the restricted growth property is equivalent to Adams' condition for staircase transformations to be mixing, we achieve Adams' result [**Ad98**] as a corollary.

THEOREM 11. *Let T be a restricted growth polynomial rank-one transformation. Then T is a mixing transformation.*

Proof. Let T be a polynomial rank-one transformation having restricted growth. Let $\{s_{n,j}\}_{\{r_n\}}$ be the spacer sequence for T and $\{s^{(k)}_{n,j}\}_{\{r_n-k\}}$ be the partial sum sequences of it. Let p_n be the polynomials of positive degree less than or equal to D defining the spacer sequence: $s_{n,j} = p_n(j) = \sum_{a=0}^{D} c_{n,a} j^a$ where the $c_{n,a}$ are the coefficients of p_n. For any sequence of positive integers $\{k_n\}$ such that k_n/r_n is bounded above zero and below one, the $\{k_n\}$th partial sum sequence is given by

$$\begin{aligned} s^{(k_n)}_{n,j} &= \sum_{z=0}^{k_n-1} s_{n,j+z} = \sum_{z=0}^{k_n-1} p_n(j+z) = \sum_{z=0}^{k_n-1}\sum_{a=0}^{D} c_{n,a}(j+z)^a \\ &= \sum_{a=0}^{D}\sum_{z=0}^{k_n-1}\sum_{b=0}^{a} c_{n,a}\binom{a}{b} j^b z^{a-b} \\ &= \sum_{b=0}^{D}\left(\sum_{a=b}^{D} c_{n,a}\binom{a}{b}\sum_{z=0}^{k_n-1} z^{a-b}\right) j^b = P_{n,k_n}(j), \end{aligned}$$

where P_{n,k_n} is a polynomial of degree D in j (verification that the coefficients are rational and that integers are mapped to integers is left to the reader) with Dth degree coefficient $C_{n,k_n,D} = c_{n,D}k_n$ so $\limsup_{n\to\infty} C_{n,k_n,D}/(r_n - k_n) = \limsup_{n\to\infty} c_{n,D}k_n/(r_n - k_n) < \infty$ since $\limsup_{n\to\infty} k_n/r_n < 1$ and $c_{n,D}$ is bounded above. Proposition 8.2 then implies that T is polynomial uniform ergodic, so for any measurable set B,

$$\lim_{n\to\infty}\int \left| \frac{1}{r_n - k_n}\sum_{j=0}^{r_n-k_n-1} \chi_B \circ T^{-P_{n,k_n}(j)} - \mu(B) \right|^2 d\mu = 0.$$

Thus, since $s^{(k_n)}_{n,j} = P_{n,k_n}(j)$ for all n and j, we have that $\{s_{n,j}\}_{\{r_n\}}$ is uniformly ergodic with respect to T so our Main Theorem (Theorem 6) implies that T is a mixing transformation. □

COROLLARY 12. *Let D be a positive integer and let T be the rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given, for all n and j, by*

$$r_n = n \quad \text{and} \quad s_{n,j} = j^D.$$

Then T is a mixing transformation.

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Proof. Since the height sequence $\{h_n\}$ for T increases exponentially (a fact true of all rank-one transformations with non-pathological spacer sequences),

$$\frac{1}{h_n}\sum_{j=0}^{r_n-1} s_{n,j} = \frac{1}{h_n}\sum_{j=0}^{n-1} j^D \le \frac{1}{h_n} n^{D+1} \to 0 \quad \text{as } n \to \infty.$$

Thus, T has restricted growth so Theorem 11 implies that T is mixing. □

COROLLARY 13. (Adams) *Let T be a staircase transformation having restricted growth. Then T is a mixing transformation.*

9. *Ornstein's 'random spacers' method*

9.1. *Construction with 'random spacers'.* We conclude with a discussion of the mixing rank-one transformations due to Ornstein [**Or72**] using a 'random spacers' method for cutting and stacking. The reader is referred to [**Na98**] for a detailed account of this method. The transformations are defined by choosing a set of values $\{x_{n,j}\}_{j=0}^{r_n-1}$ using the uniform distribution on the set of integers between $-\frac{1}{2}\widetilde{s}_n$ and $\frac{1}{2}\widetilde{s}_n$ where $\{\widetilde{s}_n\}$ is a given sequence of positive integers with no finite limit points (Ornstein's original constructions used $\widetilde{s}_n = h_{n-1}$) and letting the spacer sequence for the transformation T be given by $s_{n,j} = \widetilde{s}_n + x_{n,j+1} - x_{n,j}$ where $x_{n,r_n} = x_{n,0}$. The sequence of cuts $\{r_n\}$ is a sequence of positive integers specified later to show mixing properties.

Note that

$$\overline{s}_n = \left\lfloor \frac{1}{r_n}\sum_{j=0}^{r_n-1} s_{n,j} \right\rfloor = \left\lfloor \frac{1}{r_n}\sum_{j=0}^{r_n-1} (\widetilde{s}_n + x_{n,j+1} - x_{n,j}) \right\rfloor$$

$$= \left\lfloor \widetilde{s}_n + \frac{1}{r_n}(x_{n,r_n} - x_{n,0}) \right\rfloor = \widetilde{s}_n.$$

The normalized height sequence $\{\widehat{h}_n\}$ is then given, letting $\{h_n\}$ denote the height sequence, by $\widehat{h}_n = h_n + \widetilde{s}_n$ and the normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ is given by, for all n and $j < r_n$,

$$\widehat{s}_{n,j} = s_{n,j} - \overline{s}_n = \widetilde{s}_n + x_{n,j+1} - x_{n,j} - \overline{s}_n = x_{n,j+1} - x_{n,j}.$$

For any positive integers n, j and k such that $j + k < r_n$, the partial sum of the normalized spacer sequence is given by

$$\widehat{s}_{n,j}^{(k)} = \sum_{z=0}^{k-1} \widehat{s}_{n,j+z} = \sum_{z=0}^{k-1} x_{n,j+z+1} - x_{n,j} = x_{n,j+k} - x_{n,j}.$$

Thus, $|\widehat{s}_{n,j}^{(k)}| \le \widetilde{s}_n$ for all j and k so T has restricted growth since $(\widetilde{s}_n / h_n) \to 0$ as $n \to \infty$ is a necessary condition in the choice of $\{\widetilde{s}_n\}$ for T to be finite measure-preserving. We thank El Abdalaoui for his questions clarifying the application of our theorems to Ornstein's transformations.

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9.2. *Probabilistic lemma.* To show mixing, we will need the following lemma used by Ornstein [**Or72**]; the proof may be found in [**Na98**].

LEMMA 9.1. *Let H be a positive integer and $X = \{j \in \mathbb{Z} : |j| \leq \frac{1}{2}H\}$. For any positive integer m, let $\Omega_m = X^m$ and let P_m be the uniform distribution on Ω_m. Let $\omega = (\omega_1, \ldots, \omega_m) \in \Omega_m$ and let x_j, $0 < j \leq m$, denote the coordinates of the random variable on Ω_m. For each integer ℓ, set $C_{k,\ell} = \#\{0 < j \leq m : x_{j+k}(\omega) - x_j(\omega) = \ell\}$. Then given $\alpha > 1$, $\epsilon > 0$ and a positive integer N, there exists an integer $m \geq N$ such that*

$$P_m\left(\bigcap_{k=1}^{(1-\epsilon)m} \bigcap_{\ell\in\mathbb{Z}} \left\{\omega : C_{k,\ell} \leq \frac{\alpha}{H}(m-k)\right\}\right) > 1-\epsilon.$$

9.3. *Weak mixing using 'random spacers'.* We first show that almost surely such a transformation is weak mixing when the spacer sequence has positive upper density. Note that El Abdalaoui has shown that almost surely such transformations are totally ergodic without requiring our density condition relating the growth rates of the sequences of cuts and ranges [**Na98**, §16.13]: this fact also follows from our criteria for total ergodicity on rank-one transformations (Theorem 7). Our proof is accomplished using the techniques in [**BFMS01**]; the reader is referred to that work for details on double ergodicity.

THEOREM 14. *Let T be a rank-one transformation constructed using 'random spacers' as before with sequence of cuts $\{r_n\}$ and sequence of ranges for the spacer sequence $\{\widetilde{s}_n\}$ such that $\limsup_{n\to\infty} r_n/\widetilde{s}_n > 0$—the spacer sequence for T has positive (upper) density. Then, almost surely T is a weakly mixing transformation.*

Proof. Let T, $\{r_n\}$, and $\{\widetilde{s}_n\}$ be as before and let $\epsilon > 0$. Let A and B be measurable sets with $\mu(A) > 0$ and $\mu(B) > 0$. Then, there exist a positive integer N and levels I and J in the column C_N such that I and J are $(1-\epsilon)$-full of A and B, respectively. (Recall that a set E is $(1-\epsilon)$-full of a set F if $\mu(E \cap F) > (1-\epsilon)\mu(E)$.) Let ℓ be the distance between I and J in C_N (ℓ positive when I is above J). Then $\mu(I \cap A) + \mu(J \cap B) > 2(1-\epsilon)\mu(I)$. Choose $\gamma > 0$ such that $\mu(I \cap A) + \mu(J \cap B) - 2(1-\epsilon)\mu(I) \geq 4\epsilon\gamma\mu(I) > 0$. For any integer n such that $n \geq N$ and $2/r_n \leq \gamma$, write I and J as unions of levels in C_{n+1}; and denote the sublevels of I in C_{n+1} by I_t for $0 \leq t < R_n$ where $R_n = \prod_{z=N}^{n} r_z$. Order the I_t so that each block of sublevels I_{mr_n+k} for $0 \leq k < r_n$ corresponding to each fixed m forms a whole level in C_N such that I_{mr_n+k} is a level in the subcolumn $C_n^{[k]}$. Similarly, we have J_t for $0 \leq t < R_n$ ordered such that I_t is ℓ above J_t. For $t < 0$ and $t \geq R_n$, let I_t and J_t denote $I_{t \bmod R_n}$ and $J_{t \bmod R_n}$, respectively. Then

$$\begin{aligned}\sum_{k=0}^{R_n-1} &\mu(I_t \cap A) + \mu(I_{t+1} \cap A) + 2\mu(J_{t+2} \cap B) - 4(1-\epsilon)\mu(I_t)\\ &= 2(\mu(I \cap A) + \mu(J \cap B) - 2(1-\epsilon)\mu(I)) \geq 8\epsilon\gamma\mu(I).\end{aligned}$$

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Let x_n denote the number of values of t for $0 \le t < R_n$ with $t \bmod r_n \ne r_n - 1$ and $t \bmod r_n \ne r_n - 2$ such that $\mu(I_t \cap A) + \mu(I_{t+1} \cap A) + 2\mu(J_{t+2} \cap B) - 4(1-\epsilon)\mu(I_t) > 0$. Then,

$$\begin{aligned}\sum_{t=0}^{R_n-1} &\mu(I_t \cap A) + \mu(I_{t+1} \cap A) + 2\mu(J_{t+2} \cap B) - 4(1-\epsilon)\mu(I_t) \\ &\le (R_n - x_n)0 + \left(\frac{2R_n}{r_n} + x_n\right)(4\mu(I_t) - 4(1-\epsilon)\mu(I_t)) \\ &= \left(\frac{2R_n}{r_n} + x_n\right) 4\epsilon \frac{1}{R_n}\mu(I) = \frac{8\epsilon\mu(I)}{r_n} + \frac{4\epsilon x_n \mu(I)}{R_n}.\end{aligned}$$

Thus,

$$8\epsilon\gamma\mu(I) \le \frac{8\epsilon\mu(I)}{r_n} + \frac{4\epsilon x_n\mu(I)}{R_n}$$

so

$$\frac{4\epsilon x_n\mu(I)}{R_n} \ge 8\epsilon\mu(I)\left(\gamma - \frac{2}{r_n}\right) \ge 8\epsilon\mu(I)\left(\gamma - \frac{\gamma}{2}\right);$$

hence, $x_n \ge \gamma R_n$. Consider the possible values of $t \bmod r_n$ for the at least γR_n values of t such that $\mu(I_t \cap A) + \mu(I_{t+1} \cap A) + 2\mu(J_{t+2} \cap B) - 4(1-\epsilon)\mu(I_t) > 0$. At most R_{n-1} choices for t have the same value (mod r_n), so there must exist at least γr_n distinct values of k with $0 \le k < r_n$ such that $\mu(I_t \cap A) + \mu(I_{t+1} \cap A) + 2\mu(J_{t+2} \cap B) - 4(1-\epsilon)\mu(I_t) > 0$ where $t = mr_n + k$ for some m. Observe that since P_n is the uniform distribution on the integers between $-\frac{1}{2}\widetilde{s}_n$ and $\frac{1}{2}\widetilde{s}_n$, using the notation from Lemma 9.1,

$$P_n(\{\omega : s_{n,k} - s_{n,k+1} \ne \ell \quad \text{for all } \lfloor \gamma r_n \rfloor \text{ values } k\}) = \left(1 - \frac{1}{\widetilde{s}_n}\right)^{\lfloor \gamma r_n \rfloor}.$$

Thus,

$$\begin{aligned}P\Bigg(\bigcup_{n=N}^{\infty} &\{\omega : s_{n,k} - s_{n,k+1} = \ell \quad \text{for some } k \text{ of the } \lfloor \gamma r_n \rfloor \text{ values}\}\Bigg) \\ &= 1 - P\Bigg(\bigcap_{n=N}^{\infty} \{\omega : s_{n,k} - s_{n,k+1} \ne \ell \quad \text{for all } \lfloor \gamma r_n \rfloor \text{ values } k\}\Bigg) \\ &= 1 - \prod_{n=N}^{\infty} P_n(\{\omega : s_{n,k} - s_{n,k+1} \ne \ell \quad \text{for all } \lfloor \gamma r_n \rfloor \text{ values } k\}) \\ &= 1 - \prod_{n=N}^{\infty} \left(1 - \frac{1}{\widetilde{s}_n}\right)^{\lfloor \gamma r_n \rfloor}.\end{aligned}$$

Using the approximation $\log(1-x) \approx -x$ for small x, we have that

$$\begin{aligned}\log \prod_{n=N}^{\infty} \left(1 - \frac{1}{\widetilde{s}_n}\right)^{\lfloor \gamma r_n \rfloor} &= \sum_{n=N}^{\infty} \lfloor \gamma r_n \rfloor \log\left(1 - \frac{1}{\widetilde{s}_n}\right) \\ &\approx \sum_{n=N}^{\infty} \gamma r_n \frac{-1}{\widetilde{s}_n} = -\gamma \sum_{n=N}^{\infty} \frac{r_n}{\widetilde{s}_n}.\end{aligned}$$

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By our requirement that $r_n/\widetilde{s}_n$ be bounded away from zero along some subsequence, this implies that

$$\prod_{n=N}^{\infty}\left(1-\frac{1}{\widetilde{s}_n}\right)^{\lfloor \gamma r_n\rfloor}=0.$$

Hence, we have that

$$P\Bigg(\bigcup_{n=N}^{\infty}\{\omega : s_{n,k}-s_{n,k+1}=\ell \quad \text{for some } k \text{ of the } \lfloor \gamma r_n\rfloor \text{ values}\}\Bigg)=1.$$

Therefore, almost surely there exists some n, some $k<r_n-2$ and some m where I_t, I_{t+1} and J_{t+2}, letting $t=mr_n+k$, are levels such that $\mu(I_t\cap A)+\mu(I_{t+1}\cap A)+2\mu(J_{t+2}\cap B)>4(1-\epsilon)\mu(I_t)$ and $s_{n,k}-s_{n,k+1}=\ell$. In this case, we have that

$$\begin{aligned}T^{h_n+s_{n,k}}(I_t)&=I_{t+1};\\ T^{h_n+s_{n,k}}(I_{t+1})&=T^{s_{n,k}-s_{n,k+1}}(I_{t+2})=T^{\ell}(I_{t+2})=J_{t+2}.\end{aligned}$$

Since $\mu(I_t\cap A)>(1-4\epsilon)\mu(I_t)$, $\mu(I_{t+1}\cap A)>(1-4\epsilon)\mu(I_{t+1})$ and $\mu(J_{t+2}\cap B)>(1-4\epsilon)\mu(J_{t+2})$, we then have that

$$\begin{aligned}\mu(T^{h_n+s_{n,k}}(A)\cap A)&\geq(1-4\epsilon)\mu(T^{h_n+s_{n,k}}(I_t)\cap A)\\ &=(1-4\epsilon)\mu(I_{t+1}\cap A)\geq(1-4\epsilon)^2\mu(I_{t+1})>0\end{aligned}$$

and, similarly,

$$\begin{aligned}\mu(T^{h_n+s_{n,k}}(A)\cap B)&\geq(1-4\epsilon)\mu(T^{h_n+s_{n,k}}(I_{t+1})\cap B)\\ &=(1-4\epsilon)\mu(J_{t+2}\cap B)\geq(1-4\epsilon)^2\mu(J_{t+2})>0.\end{aligned}$$

Thus, T is doubly ergodic which is equivalent to weak mixing. □

9.4. *Mixing using 'random spacers'.* Assume that the transformation T has been partially constructed up to the column C_{n-1} using the 'random spacers' method. Apply Lemma 9.1 with $H=\widetilde{s}_n$, $N=r_{n-1}$, a fixed $\alpha>1$ and an $\epsilon_n>0$ such that $\epsilon_n\to 0$ as $n\to\infty$ to obtain $r_n=m$. Set $C_{k,\ell}^n=\#\{0<j\leq r_n-k : x_{j+k}(\omega)-x_j(\omega)=\ell\}$ and

$$L_n=\bigcap_{k=1}^{(1-\epsilon_n)r_n}\bigcap_{\ell=-\frac{1}{2}\widetilde{s}_n}^{\frac{1}{2}\widetilde{s}_n}\left\{\omega : C_{k,\ell}^n\leq\frac{\alpha}{\widetilde{s}_n}(r_n-k)\right\}$$

so that $P_{r_n}(L_n)>1-\epsilon_n$. Using $\omega\in L_n$, the normalized spacer sequence for T is given by $\widehat{s}_{n,j}=x_{n,j+1}(\omega)-x_{n,j}(\omega)$ and the partial sums of the normalized spacer sequence are given by $\widehat{s}_{n,j}^{(k)}=x_{n,j+k}(\omega)-x_{n,j}(\omega)$. Then, for any $0<k<r_n-1$ and any fixed measurable set B and any sequence of measurable sets $\{A_n\}$,

$$\begin{aligned}&\frac{1}{r_n-k}\sum_{j=0}^{r_n-k-1}|\mu(T^{\widehat{s}_{n,j}^{(k)}}(A_n)\cap B)-\mu(A_n)\mu(B)|\\ &\quad=\frac{1}{r_n-k}\sum_{\ell=-\frac{1}{2}\widetilde{s}_n}^{\frac{1}{2}\widetilde{s}_n}C_{k,\ell}^n|\mu(T^{\ell}(A_n)\cap B)-\mu(A_n)\mu(B)|\end{aligned}$$

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$$\leq \frac{1}{r_n - k} \sum_{\ell=-\frac{1}{2}\widetilde{s}_n}^{\frac{1}{2}\widetilde{s}_n} \frac{\alpha}{2\widetilde{s}_n}(r_n - k)|\mu(T^\ell(A_n) \cap B) - \mu(A_n)\mu(B)|$$

$$= \frac{\alpha}{\widetilde{s}_n} \sum_{\ell=-\frac{1}{2}\widetilde{s}_n}^{\frac{1}{2}\widetilde{s}_n} |\mu(T^\ell(A_n) \cap B) - \mu(A_n)\mu(B)|.$$

Note that $\widetilde{s}_n \to \infty$ as $n \to \infty$ and T is weak mixing and let $\{k_n\}$ be any sequence of positive integers such that $\limsup_{n\to\infty} k_n/r_n < 1$. Consider the dynamical sequence $\{\widehat{s}_{n,j}^{(k_n)}\}_{\{r_n - k_n\}}$ and observe that

$$\lim_{n\to\infty} \frac{1}{r_n - k_n} \sum_{j=0}^{r_n - k_n - 1} |\mu(T^{\widehat{s}_{n,j}^{(k_n)}}(A_n) \cap B) - \mu(A_n)\mu(B)|$$

$$\leq \alpha \lim_{n\to\infty} \frac{1}{\widetilde{s}_n} \sum_{\ell=-\frac{1}{2}\widetilde{s}_n}^{\frac{1}{2}\widetilde{s}_n} |\mu(T^\ell(A_n) \cap B) - \mu(A_n)\mu(B)|$$

$$= \alpha \lim_{n\to\infty} \frac{1}{2n} \sum_{\ell=-n}^{n} |\mu(T^\ell(A_{f_n}) \cap B) - \mu(A_{f_n})\mu(B)| = 0,$$

where the sequence $\{f_n\}$ represents appropriate choices for the index change in the limits. Hence, $\{\widehat{s}_{n,j}^{(k_n)}\}_{\{r_n - k_n\}}$ is ergodic with respect to T (the proof is obtained by considering the sets

$$A_n = \left\{ x : \left| \frac{1}{r_n - k_n} \sum_{j=0}^{r_n - k_n - 1} \chi_B \circ T^{-s_{n,j}^{(k_n)}} - \mu(B) \right| \geq \epsilon \right\}$$

for large n and $\epsilon \to 0$; details are left to the reader). The normalized spacer sequence $\{\widehat{s}_{n,j}\}_{\{r_n\}}$ is then uniformly ergodic with respect to T since this holds for all sequences $\{k_n\}$ with k_n/r_n bounded below one. Theorem 6 then yields the following theorem originally in [**Or72**].

THEOREM 15. (Ornstein) *Let T be a rank-one transformation constructed using 'random spacers' with sequence of cuts increasing sufficiently fast as before. Then almost surely T is a mixing transformation.*

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