THE POTENTIAL OF A CIRCULAR CURRENT

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1. *Summary.* The 3-dimensional potential due to a charged circle can be derived in various ways which lead to the same expression in terms of the toroidal coordinate σ through Legendre functions of s ($= \cosh\sigma$). Applying the same methods to the magnetic potential of a circular current we find various results:

(*a*) Legendre's equation yields no solution (§ 4·1).

(*b*) The appropriate transformation from cylindrical to toroidal coordinates gives a Fourier series (equation (48)) the coefficients $R_{n-\frac{1}{2}}(s)$ of which, though not Legendre functions, are related to them by the identity (§ 4·2)

$$R_m(s) + R_{-m-1}(s) = P_m(s).$$

By this method we also derive expressions in elliptic integrals (equations (56), (58)).

(*c*) Direct integration of the solid angle gives an alternative expression in elliptic integrals (equation (59)).

2. *Solutions in cylindrical coordinates.* Following the treatment given by Bateman in *Partial Differential Equations* ((1), p. 409) we observe that potential functions axially symmetrical about the z-axis which on the positive part of the axis are of the form

$$f(z) = \int_0^\infty e^{-lz} F(l)\, dl \tag{1}$$

can be represented by the integral (z positive)

$$V = \frac{1}{\pi}\int_0^\infty \int_0^\pi e^{-l(z+i\rho\cos\alpha)} F(l)\, d\alpha\, dl \tag{2}$$

$$= \int_0^\infty e^{-lz} J_0(l\rho)\, F(l)\, dl \tag{3}$$

at the point whose coordinates are (z, ρ). In particular, since

$$\int_0^\infty e^{-lz} J_0(la)\, dl = (z^2 + a^2)^{-\frac{1}{2}}, \tag{4}$$

the potential

$$V_R \equiv \int_0^\infty e^{-lz} J_0(l\rho)\, J_0(la)\, da \tag{5}$$

is that of a circle of radius a carrying unit electrical charge; and again, since

$$\int_0^\infty e^{-lz} J_1(la)\, dl = \frac{1}{a}\left(1 - \frac{z}{(z^2+a^2)^{\frac{1}{2}}}\right) \tag{6}$$

the potential

$$V_S \equiv 2\pi a \int_0^\infty e^{-lz} J_0(l\rho)\, J_1(la)\, dl \tag{7}$$

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is the magnetic potential of a circle of radius a carrying unit electric current in the right-handed sense. These potentials are closely related by the equation

$$V_S = -\frac{\partial}{\partial z}\int_0^a 2\pi a_1 V_R(a_1)\,da_1, \tag{8}$$

but the connexion becomes less obvious when the potentials are expressed in toroidal coordinates.

3. *The charged ring.* The known result for this problem will now be derived (equation (25)).

3·1. *Transformation from cylindrical to toroidal coordinates.* Equation (5) may be expressed by means of the integral representation of Bessel functions as

$$V_R = \frac{1}{\pi^2}\int_0^\infty \int_0^\pi \int_0^\pi e^{-l(z+i\rho\cos\alpha+ia\cos\beta)}\,d\beta\,d\alpha\,dl. \tag{9}$$

Performing the integrations in inverse order we get

$$V_R = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{d\beta\,d\alpha}{z+i\rho\cos\alpha+ia\cos\beta}, \tag{10}$$

unless the denominator of the integrand vanishes. In the α-plane the integrand is holomorphic except at the poles given by

$$\cos\alpha_1 = -\frac{z+ia\cos\beta}{i\rho}, \tag{11}$$

and accordingly, by integration round the contour shown in Fig. 1 we may obtain the result

$$V_R = \frac{1}{\pi}\int_0^\pi \frac{d\beta}{\{(z+ia\cos\beta)^2+\rho^2\}^{\frac{1}{2}}} \tag{12}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{dc}{(1-c^2)^{\frac{1}{2}}\{(z+iac)^2+\rho^2\}^{\frac{1}{2}}}, \tag{13}$$

where c is written for $\cos\beta$.

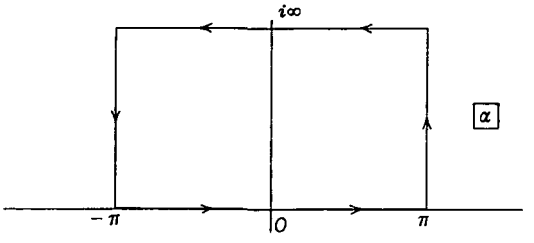


Fig. 1

In the c-plane the integrand is holomorphic except at the branch-points

$$\pm 1, \quad c_2 = (\rho+iz)/a, \quad c_1 = (-\rho+iz)/a,$$

provided that the plane is cut between the members of each pair of branch-points, e.g. as in Fig. 2.

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Since on a large circle centred on the origin the integrand is $O(c^{-2})$, we can conclude that

$$\frac{2}{ia\pi}\int_{-1}^{1}\frac{dc}{\{(1-c^2)(c-c_1)(c-c_2)\}^{\frac{1}{2}}} = -\frac{1}{ia\pi}\int_{C}\frac{dc}{\{(1-c^2)(c-c_1)(c-c_2)\}^{\frac{1}{2}}}. \tag{14}$$

If now we put

$$c = i\cot\tfrac{1}{2}\zeta_1 = i\cot\tfrac{1}{2}(\psi+i\sigma_1), \tag{15}$$

$$c_1 = i\cot\tfrac{1}{2}(\psi-i\sigma), \tag{16}$$

$$c_2 = i\cot\tfrac{1}{2}\zeta = i\cot\tfrac{1}{2}(\psi+i\sigma), \tag{17}$$

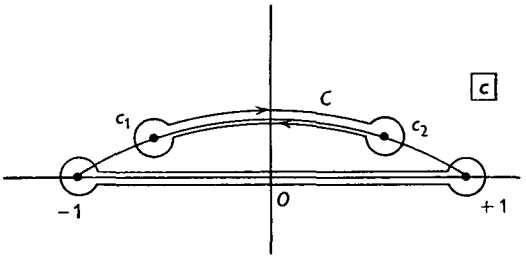


Fig. 2

this is equivalent to taking toroidal coordinates ψ, σ, where

$$z-i\rho = a\cot\tfrac{1}{2}(\psi+i\sigma), \tag{18}$$

so that

$$\rho = \frac{a\sinh\sigma}{s-\tau}, \tag{19}$$

$$z = \frac{a\sin\psi}{s-\tau} \tag{20}$$

and

$$\psi+i\sigma = i\ln\frac{\rho+iz+a}{\rho+iz-a}, \tag{21}$$

where s is written for $\cosh\sigma$ and τ for $\cos\psi$.

The contour C is chosen to follow a cut on which ψ = constant; this is a circular arc in the c-plane which, if produced, would pass through -1 and $+1$. We then find that (13)–(17) give us

$$V_R = \frac{1}{2\pi a}\int_{-\sigma}^{\sigma}\frac{\operatorname{cosec}\tfrac{1}{2}(\psi+i\sigma_1)\,d\sigma_1}{\{\cot\tfrac{1}{2}(\psi+i\sigma_1)-\cot\tfrac{1}{2}(\psi+i\sigma)\}^{\frac{1}{2}}\{\cot\tfrac{1}{2}(\psi+i\sigma_1)-\cot\tfrac{1}{2}(\psi-i\sigma)\}^{\frac{1}{2}}} \tag{22}$$

$$= \frac{(s-\tau)^{\frac{1}{2}}}{2\pi a}\int_{-\sigma}^{\sigma}\frac{d\sigma_1}{(\cosh\sigma-\cosh\sigma_1)^{\frac{1}{2}}}. \tag{23}$$

The integral

$$\frac{\sqrt{2}}{\pi}\int_0^{\sigma}\frac{d\sigma_1}{(\cosh\sigma-\cosh\sigma_1)^{\frac{1}{2}}}$$

is the equivalent for $s > 1$ of Mehler's integral for Legendre functions of degree $-\frac{1}{2}$, i.e. ((2), p. 315)

$$\frac{\sqrt{2}}{\pi}\int_0^{\theta}\frac{d\theta_1}{(\cos\theta_1-\cos\theta)^{\frac{1}{2}}} = P_{-\frac{1}{2}}(\cos\theta) \quad (s=\cos\theta). \tag{24}$$

So

$$V_R = \frac{(s-\tau)^{\frac{1}{2}}}{\sqrt{2}\,a}P_{-\frac{1}{2}}(s). \tag{25}$$

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3·2. *Solution of Laplace's equation.* This known result was obtained by Heine ((3), pp. 283–301) from the direct solution of Laplace's equation in toroidal coordinates, viz.

$$\frac{\partial}{\partial\sigma}\frac{\sinh\sigma}{s-\tau}\frac{\partial V}{\partial\sigma}+\frac{\partial}{\partial\psi}\frac{\sinh\sigma}{s-\tau}\frac{\partial V}{\partial\psi}+\frac{1}{(s-\tau)\sinh\sigma}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{26}$$

This equation is solved by separation of the variables with the aid of the substitution

$$V=(s-\tau)^{\frac{1}{2}}v \tag{27}$$

which gives

$$\frac{\partial}{\partial s}\left\{(s^2-1)\frac{\partial v}{\partial s}\right\}+\frac{\partial^2 v}{\partial\psi^2}+\frac{1}{s^2-1}\frac{\partial^2 v}{\partial\phi^2}+\frac{v}{4}=0. \tag{28}$$

Equation (28) is satisfied by functions of the axially symmetric type $e^{in\psi}P_{n-\frac{1}{2}}(s)$ where $v\ (=P_{n-\frac{1}{2}}(s))$ satisfies the Legendre equation of degree $(n-\frac{1}{2})$

$$\frac{\partial}{\partial s}\left\{(s^2-1)\frac{\partial v}{\partial s}\right\}-(n^2-\tfrac{1}{4})v=0. \tag{29}$$

Mehler's expression for $Q_{n-\frac{1}{2}}(s)$, viz.

$$Q_{n-\frac{1}{2}}(s)=\frac{1}{\sqrt{2}}\int_0^\pi\frac{\cos n\theta}{(s-\cos\theta)^{\frac{1}{2}}}d\theta \tag{30}$$

shows that $2(\sqrt{2/\pi})\,Q_{n-\frac{1}{2}}(s)$ is the nth Fourier coefficient in the expansion of $(s-\tau)^{-\frac{1}{2}}$, and accordingly a potential which is constant on the toroid $s=s_0$ is

$$\sum_{n=-\infty}^{\infty}(s-\tau)^{\frac{1}{2}}e^{in\psi}P_{n-\frac{1}{2}}(s)\frac{Q_{n-\frac{1}{2}}(s_0)}{P_{n-\frac{1}{2}}(s_0)}. \tag{31}$$

As $s_0\to\infty$ we find that the term involving $P_{-\frac{1}{2}}(s)$ becomes the leading term in the Fourier series.

3·3. *Direct approach in toroidal coordinates.* In a third derivation of the ring formula we use the reciprocal distance r^{-1} from the given point at which the potential is to be calculated to the point on the fundamental circle with third coordinate ϕ. This is ((1), p. 465)

$$ar^{-1}=\frac{(s-\tau)^{\frac{1}{2}}}{\sqrt{2}}(s-(s^2-1)^{\frac{1}{2}}\cos\phi)^{-\frac{1}{2}}. \tag{32}$$

The potential of unit charge is therefore given by

$$V_R=\frac{(s-\tau)^{\frac{1}{2}}}{2\sqrt{2}\,\pi a}\int_0^{2\pi}(s-(s^2-1)^{\frac{1}{2}}\cos\phi)^{-\frac{1}{2}}d\phi \tag{33}$$

$$=\frac{\sqrt{2}}{\pi a}(s-\tau)^{\frac{1}{2}}e^{-\sigma/2}K((1-e^{-2\sigma})^{\frac{1}{2}}), \tag{34}$$

where $K(k)$ is the complete elliptic integral of the first kind to modulus k. Making use of Laplace's integral for Legendre functions ((2), p. 313)

$$P_n(s)=\frac{1}{\pi}\int_0^\pi(s\pm(s^2-1)^{\frac{1}{2}}\cos\phi)^n d\phi \tag{35}$$

we find that

$$V_R=\frac{(s-\tau)^{\frac{1}{2}}}{\sqrt{2}\,a}P_{-\frac{1}{2}}(s)$$

as before.

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3·4. *The inversion method.* This is described for example in Bateman's *Partial Differential Equations* ((1), pp. 416–7) and gives, in our symbols,

$$V_R = \frac{\sqrt{2}}{\pi a}(s-\tau)^{\frac{1}{2}} \operatorname{sech}\frac{\sigma}{2} K\left(\tanh\frac{\sigma}{2}\right) \tag{36}$$

which is related to equation (34) by Landen's Transformation ((2), p. 507).

4. *The circular current.* Similar methods may be applied to this case, but the results are more diverse than for the charged ring.

4·1. *Legendre's equation.* The general series

$$\sum_{n=-\infty}^{\infty} (s-\tau)^{\frac{1}{2}} e^{in\psi}\{a_n P_{n-\frac{1}{2}}(s) + b_n Q_{n-\frac{1}{2}}(s)\} \tag{37}$$

with suitable constants a_n, b_n might seem to cover fairly well the range of cases of practical importance, in which a single valued or cyclic potential can be represented as a Fourier series in ψ, at least in the range $-\pi$ to π. However, this would only be true if on substitution of the Fourier series for v, say

$$v = \sum_{n=-\infty}^{\infty} e^{in\psi} f(n, s), \tag{38}$$

the expression

$$(s-\tau)^{\frac{1}{2}} \sum_{-n}^{n} e^{in\psi} L_n f(n, s) \tag{39}$$

did in fact converge as $n \to \infty$. We write here L_n for the linear operator

$$\frac{\partial}{\partial s}(s^2-1)\frac{\partial}{\partial s} - (n^2 - \tfrac{1}{4}).$$

It will be seen that for the circular current the summation diverges as $n \to \infty$, so relaxing the condition that $f(n, s)$ must be a Legendre function of degree $(n-\frac{1}{2})$. This inapplicability of Legendre functions was noted by Hicks (4).

Such non-Legendrian solutions can easily be constructed.

Let $S(\psi)$ be a step function such that

$$S(\psi) = -1 \quad (0 > \psi > -\pi),$$
$$S(\psi) = +1 \quad (0 < \psi < \pi).$$

This is obviously a potential function except on the plane $z = 0$. Then

$$(s-\tau)^{\frac{1}{2}} \frac{S(\psi)}{(s-\tau)^{\frac{1}{2}}}$$

is a potential function for which

$$v = \frac{S(\psi)}{(s-\tau)^{\frac{1}{2}}}.$$

Thus the nth Fourier coefficient is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{S(\theta)\sin n\theta}{(s-\cos\theta)^{\frac{1}{2}}}\, d\theta = \frac{2}{\pi}\int_0^{\pi} \frac{\sin n\theta}{(s-\cos\theta)^{\frac{1}{2}}}\, d\theta,$$

but unlike

$$\frac{1}{\sqrt{2}}\int_0^{\pi} \frac{\cos n\theta}{(s-\cos\theta)^{\frac{1}{2}}}\, d\theta$$

$(= Q_{n-\frac{1}{2}}(s)$; see (1), p. 464) this function is not a Legendre function.

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4·2. *Transformation from cylindrical to toroidal coordinates.* For this case we transform V_S (equation (7)) by writing

$$V_S = \frac{2\pi a}{\pi^2}\int_0^\infty\int_0^\pi\int_0^\pi i\cos\beta\, e^{-l(z+i\rho\cos\alpha+ia\cos\beta)}\,d\beta\,d\alpha\,dl \tag{40}$$

$$= \frac{2a}{\pi}\int_0^\pi\int_0^\pi \frac{i\cos\beta\,d\beta\,d\alpha}{z+i\rho\cos\alpha+ia\cos\beta} \tag{41}$$

$$= 2\int_0^\pi \frac{ia\cos\beta\,d\beta}{\{(z+ia\cos\beta)^2+\rho^2\}^{\frac{1}{2}}} \tag{42}$$

$$= 2\int_{-1}^1 \frac{c\,dc}{\{(1-c^2)(c-c_1)(c-c_2)\}^{\frac{1}{2}}}. \tag{43}$$

Since the integrand is now $O(c^{-1})$ at infinity, we can conclude that

$$2\int_{-1}^1 \frac{c\,dc}{\{(1-c^2)(c-c_1)(c-c_2)\}^{\frac{1}{2}}} = -\int_c \frac{c\,dc}{\{(1-c^2)(c-c_1)(c-c_2)\}^{\frac{1}{2}}} + 2\pi \tag{44}$$

$$= -(s-\tau)^{\frac{1}{2}}\int_{-\sigma}^{\sigma}\frac{\cot\frac{1}{2}(\psi+i\sigma_1)\,d\sigma_1}{(\cosh\sigma-\cosh\sigma_1)^{\frac{1}{2}}} + 2\pi. \tag{45}$$

This expression may be developed as an infinite series of definite integrals by writing, for any ζ_1 not purely real

$$\left.\begin{aligned} i\cot\zeta_1 &= -\frac{e^{i\zeta_1}+e^{-i\zeta_1}}{e^{i\zeta_1}-e^{-i\zeta_1}} \\ &= 1+2e^{2i\zeta_1}+2e^{4i\zeta_1}+\ldots \qquad (0<\arg\zeta_1<\pi), \\ i\cot\zeta_1 &= -(1+2e^{-2i\zeta_1}+2e^{-4i\zeta_1}+\ldots) \quad (-\pi<\arg\zeta_1<0). \end{aligned}\right\} \tag{46}$$

We find on making this substitution for $\cot\frac{1}{2}(\psi+i\sigma_1)$ in (45) that

$$V_S = -i(s-\tau)^{\frac{1}{2}}\sum_{n=1}^\infty\left\{-2e^{in\psi}\int_0^\sigma\frac{e^{-n\sigma_1}\,d\sigma_1}{(\cosh\sigma-\cosh\sigma_1)^{\frac{1}{2}}}+2e^{-in\psi}\int_0^\sigma\frac{e^{-n\sigma_1}\,d\sigma_1}{(\cosh\sigma-\cosh\sigma_1)^{\frac{1}{2}}}\right\}+2\pi \tag{47}$$

$$= -4\sqrt{2}\,\pi(s-\tau)^{\frac{1}{2}}\sum_{n=1}^\infty \sin n\psi R_{n-\frac{1}{2}}(s)+2\pi, \tag{48}$$

where

$$R_{n-\frac{1}{2}}(s) = \frac{1}{\sqrt{2}\,\pi}\int_0^\sigma\frac{e^{-n\sigma_1}}{(s-\cosh\sigma_1)^{\frac{1}{2}}}\,d\sigma_1. \tag{49}$$

The terms of this series are not in the ordinary sense toroidal potentials since they do not individually satisfy equations (26)–(28); $R_{n-\frac{1}{2}}(s)$ is not a Legendre function of s. The expression (48), however, satisfies Laplace's equation for values of ψ between 0 and π, and therefore, also, for values between 0 and $-\pi$, notwithstanding that the parent equation (7) is valid for only positive values of z.

It is readily seen by comparison with Mehler's integral,

$$P_{n-\frac{1}{2}}(s) = \frac{1}{\sqrt{2}\,\pi}\int_{-\sigma}^{\sigma}\frac{e^{-n\sigma_1}\,d\sigma_1}{(s-\cosh\sigma_1)^{\frac{1}{2}}} \tag{50}$$

$$= \frac{\sqrt{2}}{\pi}\int_0^\sigma\frac{\cosh n\sigma_1\,d\sigma_1}{(s-\cosh\sigma_1)^{\frac{1}{2}}}, \tag{51}$$

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that
$$P_{n-\frac{1}{2}}(s) = R_{n-\frac{1}{2}}(s) + R_{-n-\frac{1}{2}}(s), \tag{52}$$

or
$$P_m(s) = R_m(s) + R_{-m-1}(s) \tag{53}$$

for all values real of m.

There is some connexion between the R-functions and those discussed at the end of the previous section. Integrating $e^{in\theta}/(s-\cos\theta)^{\frac{1}{2}}$ in the θ-plane along the contour shown in Fig. 3, we can show that if $s > 1$,

$$\int_0^\sigma \frac{e^{-n\beta}}{(s-\cosh\beta)^{\frac{1}{2}}} - \int_0^\pi \frac{\sin n\alpha}{(s-\cos\alpha)^{\frac{1}{2}}}\, d\alpha = e^{in\pi} \int_0^\infty \frac{e^{-n\beta}}{(s+\cosh\beta)^{\frac{1}{2}}}\, d\beta. \tag{54}$$

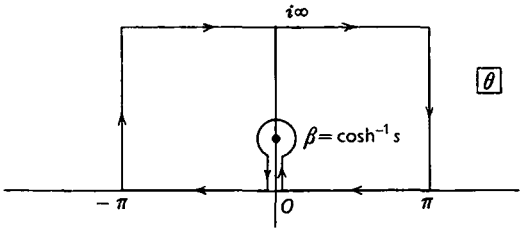


Fig. 3

Each term of this equation corresponds to a Fourier coefficient in the development of potentials V_S, $2\pi S$, V_T respectively where V_T is physically equivalent to V_S but has the discontinuity of 4π for $\psi = \pi$ instead of $\psi = 0$. All three terms under the operation L_n give ultimately increasing functions of n; the resulting infinite series consequently diverge.

4·3. *Alternative applications without series development.* It is also noted that a more compact expression for V_S is obtained without the use of infinite series, by reducing (45) to the form

$$V_S = 2\pi - 2(s-\tau)^{\frac{1}{2}} \sin\psi \int_0^\sigma \frac{d\sigma_1}{(\cosh\sigma_1 - \tau)\,(s-\cosh\sigma_1)^{\frac{1}{2}}} \tag{55}$$

which follows at once from the substitution (equations (19), (20))

$$\cot\tfrac{1}{2}(\psi + i\sigma_1) = \frac{\sin\psi - i\sinh\sigma_1}{\cosh\sigma_1 - \tau}$$

if we note that the imaginary term is an odd function of σ_1. Equation (55) can be further transformed by the substitutions

$$\cosh\sigma_1 = 2u^2 + 1, \quad t^2 = 1 - \frac{2}{s-1}\, u^2$$

into
$$V_S = 2\pi - \frac{4\sin\psi}{(s+1)^{\frac{1}{2}}\,(s-\tau)^{\frac{1}{2}}}\, \Pi\left(\frac{1-s}{s-\tau}, k_1\right), \tag{56}$$

where $\Pi(n, k)$ stands for Legendre's complete elliptic integral of the third kind

$$\int_0^1 \frac{dt}{(1+nt^2)\,(1-t^2)^{\frac{1}{2}}\,(1-k^2t^2)^{\frac{1}{2}}}$$

and where
$$k_1 = \left(\frac{s-1}{s+1}\right)^{\frac{1}{2}}.$$

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Alternatively V_S may be further transformed by an appropriate contour integration in the plane of σ_1 to give

$$V_S = 2\pi\left(1-\frac{\psi}{|\psi|}\right)+2(s-\tau)^{\frac{1}{2}}\sin\psi\int_0^\infty \frac{d\sigma_1}{(\cosh\sigma_1+\tau)(s+\cosh\sigma_1)^{\frac{1}{2}}} \tag{57}$$

and this in turn by the substitutions

$$\cosh\sigma_1 = 2u^2-1, \quad t^2=\frac{u^2-1}{u^2}$$

to give (for positive ψ)

$$V_S = 2(s-\tau)^{\frac{1}{2}}\sin\psi\left\{\frac{4}{(s+1)^{\frac{1}{2}}(1-\tau^2)}\Pi(n,k_1)-\frac{2}{(s+1)^{\frac{1}{2}}(1-\tau)}K(k_1)\right\} \tag{58}$$

where $n=(1-\tau)/(1+\tau)$ and $K(k_1)$ is the complete elliptic integral of the first kind to modulus k_1 ($=(s-1)^{\frac{1}{2}}/(s+1)^{\frac{1}{2}}$ as before).

4·4. *Direct solution in elliptic integrals.* Direct integration of the solid angle subtended by the disk element of area $a\,da\,d\phi$ can be effected in cylindrical coordinates; substitution of toroidals then gives Hicks's formula ((4), p. 628)

$$V_S = 2\pi-\frac{k\sin\psi}{\sqrt{(\sinh\sigma)}}\left\{\frac{(s+\tau)^{\frac{1}{2}}+(s-\tau)^{\frac{1}{2}}}{(s^2-\tau^2)^{\frac{1}{2}}-\sinh\sigma}\Pi(n_1,k)+\frac{(s+\tau)^{\frac{1}{2}}-(s-\tau)^{\frac{1}{2}}}{(s^2-\tau^2)^{\frac{1}{2}}+\sinh\sigma}\Pi(n_2,k)\right\}, \tag{59}$$

where $$k^2=1-e^{-2\sigma}, \quad n_1=\frac{2\sinh\sigma}{(s^2-\tau^2)^{\frac{1}{2}}-\sinh\sigma}, \quad n_2=\frac{-2\sinh\sigma}{(s^2-\tau^2)^{\frac{1}{2}}+\sinh\sigma}$$

and these expressions give us an identity between the complete elliptic integrals, in which k and k_1 are related by the equations

$$k_1=\frac{1-k'}{1+k'}, \quad k^2+k'^2=1,$$

for on expressing τ in terms of n and s in terms of k' we find

$$\beta(1+k_1)\left\{\frac{(n+1)^2}{n}\Pi(n,k_1)-\frac{n+1}{n}K(k_1)\right\}$$
$$=\frac{\pi(n+1)}{\sqrt{n}}-\left\{(\alpha+\beta)\frac{k'n_1}{k^2}\Pi(n_1,k)-(\alpha-\beta)\frac{k'n_2}{k^2}\Pi(n_2,k)\right\}, \tag{60}$$

where $$n_1=\frac{2k^2}{\alpha\beta-k^2}, \quad n_2=\frac{-2k^2}{\alpha\beta+k^2},$$

$$\alpha=\left(1+k'^2+2k'\frac{1-n}{1+n}\right)^{\frac{1}{2}}, \quad \beta=\left(1+k'^2-2k'\frac{1-n}{1+n}\right)^{\frac{1}{2}}.$$

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