Forum of Mathematics, Pi (2026), **Vol. 14**:e4 1–22
doi:10.1017/fmp.2026.10024

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Bounded geometry for PCF-special subvarieties

Laura DeMarco[1], Niki Myrto Mavraki[2] and Hexi Ye[3]

[1]Harvard University, USA; E-mail: demarco@math.harvard.edu (Corresponding author).
[2]University of Toronto, Canada; E-mail: myrto.mavraki@utoronto.ca.
[3]Zhejiang University, China; E-mail: yehexi@zju.edu.cn.

Received: 6 November 2024; **Revised:** 11 November 2025; **Accepted:** 11 November 2025

2020 Mathematics Subject Classification: *Primary* – 37F46, 37P30

Abstract

For each integer $d \geq 2$, let M_d denote the moduli space of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d. We study the geometric configurations of subsets of postcritically finite (or PCF) maps in M_d. A complex-algebraic subvariety $Y \subset \mathrm{M}_d$ is said to be PCF-special if it contains a Zariski-dense set of PCF maps. Here we prove that there are only finitely many positive-dimensional irreducible PCF-special subvarieties in M_d with degree $\leq D$. In addition, there exist constants $N = N(D, d)$ and $B = B(D, d)$ so that for any complex algebraic subvariety $X \subset \mathrm{M}_d$ of degree $\leq D$, the Zariski closure $\overline{X \cap \mathrm{PCF}_d}$ has at most N irreducible components, each with degree $\leq B$. We also prove generalizations of these results for points with small critical height in $\mathrm{M}_d(\overline{\mathbb{Q}})$.

1. Introduction

For each integer $d \geq 2$, let M_d denote the moduli space of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, defined over the field $\mathbb{C}$ of complex numbers. The space M_d consists of Möbius conjugacy classes of maps f; it is constructed as a complex orbifold in [Mi1] and as a scheme over $\mathrm{Spec}\,\mathbb{Z}$ in [Si1]. Let $\mathrm{PCF}_d \subset \mathrm{M}_d$ denote the subset of postcritically finite maps; that is, the conjugacy classes of maps f for which each critical point has a finite forward orbit. We say that an irreducible complex algebraic subvariety $X \subset \mathrm{M}_d$ is **PCF-special** if $\mathrm{PCF}_d \cap X$ is Zariski dense in X. Note that M_d is itself PCF-special, as is the polynomial subvariety $\mathrm{MPoly}_d \subset \mathrm{M}_d$, as are all subvarieties defined by orbit relations among the critical points; see, for example, [BD, Proposition 2.6] and [De2, Theorem 6.2]. Conjecturally, all PCF-special subvarieties are determined by critical orbit relations (in a general sense, accounting for symmetries among the maps) [BD, Conjecture 1.10].

In this article, we study the geometry of the set PCF_d by examining how it intersects with families of varieties in M_d, and we prove uniform bounds on the degrees of PCF-special subvarieties in these families. Here, and throughout this article, we measure degree with respect to a fixed choice of very ample line bundle on M_d, for example providing the projective compactification of M_d defined in [Si1]. The degree bounds will depend on the choice, but the following result holds for any such choice:

Theorem 1.1. *Fix integers $d \geq 2$ and $D \geq 1$. There exist constants $B = B(D, d)$ and $N = N(D, d)$ so that for any complex algebraic subvariety $X \subset \mathrm{M}_d$ of degree $\leq D$, the Zariski closure*

$$\overline{X \cap \mathrm{PCF}_d}$$

has at most N irreducible components, each with degree $\leq B$.

© The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

Theorem 1.2. *Fix integers $d \geq 2$ and $D \geq 1$. There are only finitely many positive-dimensional irreducible PCF-special subvarieties in M_d with degree $\leq D$.*

Example 1.3. Consider the sequence of PCF-special curves $\mathrm{Per}_n(0)$ in M_2, defined by the condition that one critical point of a quadratic rational map has exact period n. These were introduced by Milnor in [Mi1] and conjectured to be irreducible for every n. While irreducibility remains open, Theorem 1.2 implies that the degrees of the irreducible components of $\mathrm{Per}_n(0)$ must be growing with n. Theorem 1.2 implies degree growth also of the irreducible components of the analogous hypersurfaces $\mathrm{Per}_n(0)$ in M_d, for every $d \geq 2$.

Example 1.4. Consider the family of curves $\mathrm{Per}_1(\lambda)$ in M_2, parameterized by $\lambda \in \mathbb{C}$, consisting of all quadratic maps having a fixed point of multiplier λ. This family was first introduced by Milnor in [Mi1], and they are lines in the natural coordinates that Milnor introduced on $\mathrm{M}_2 \simeq \mathbb{C}^2$. It is known that $\mathrm{Per}_1(\lambda)$ is PCF-special if and only if $\lambda = 0$ [DWY]. And although PCF_d is Zariski-dense in M_2, Theorem 1.1 combined with the main result of [DWY] implies: *There is a uniform bound N so that*

$$\#\,\mathrm{Per}_1(\lambda) \cap \mathrm{PCF}_d \;\leq\; N$$

for all $\lambda \neq 0$ in $\mathbb{C}$. Theorem 1.1 also implies the corresponding result for the family of curves $\mathrm{Per}_n(\lambda)$ in the moduli space MPoly_3 of cubic polynomials, for any fixed $n \geq 1$, in view of [BD, Theorem 1.1] [FG2, Theorem B].

In fact we prove stronger versions of Theorem 1.1 and Theorem 1.2 in the spirit of the Bogomolov Conjecture [Ul, Zh1] or, more recently, of the "height gap principle" [GGK, Gao] in the setting of abelian varieties, for subvarieties of M_d defined over $\overline{\mathbb{Q}}$. The **critical height** of a conjugacy class $[f] \in \mathrm{M}_d(\overline{\mathbb{Q}})$ is defined by

$$\hat{h}_{\mathrm{crit}}(f) = \sum_{\{c\,:\,f'(c)=0\}} \hat{h}_f(c) \;\geq\; 0,$$

for any representative f defined over $\overline{\mathbb{Q}}$, where the critical points are counted with multiplicity and $\hat{h}_f$ is the Call-Silverman canonical height on $\mathbb{P}^1(\overline{\mathbb{Q}})$ [CS] [Si2]. Ingram proved that $\hat{h}_{\mathrm{crit}}$ is comparable to a Weil height on the open subset $\mathrm{M}_d \setminus \mathcal{L}_d$ where $\mathcal{L}_d$ is the locus of flexible Lattès maps [In]. A more explicit version of Ingram's result was later established in [GOV, Theorem C]. A different approach is given by the Yuan–Zhang theory of adelic line bundles; see [YZ, Remark 6.3.3]. The PCF points in $\mathrm{M}_d(\overline{\mathbb{Q}})$ are precisely those with $\hat{h}_{\mathrm{crit}}(f) = 0$.

To state our results, we fix a projective embedding of M_d defined over $\mathbb{Q}$. We also fix height functions h_{Ch} on the associated Chow varieties $\mathrm{Ch}(\mathrm{M}_d, r, D)$ of cycles of dimension r and degree $\leq D$ in M_d, for example, a Philippon height [Ph]; see §2.5 for definitions. The constants B, N, and ϵ of the following theorems will depend on these choices.

Theorem 1.5. *Fix integers $d \geq 2$ and $D \geq 1$. There exist constants $B = B(D,d)$, $N = N(D,d)$, and $\epsilon = \epsilon(D,d) > 0$ so that for any irreducible algebraic subvariety $X \subset \mathrm{M}_d$ of degree $\leq D$ defined over $\overline{\mathbb{Q}}$, and every $0 \leq \epsilon' < \epsilon$, the Zariski closure*

$$\overline{X \cap \{z \in \mathrm{M}_d(\overline{\mathbb{Q}}) \;:\; \hat{h}_{\mathrm{crit}}(z) \leq \epsilon' \max\{1, h_{\mathrm{Ch}}(X)\}\}}$$

has at most N irreducible components, each with degree $\leq B$.

The **essential minimum** of the critical height on a subvariety $Y \subset \mathrm{M}_d$ defined over $\overline{\mathbb{Q}}$ is

$$\mathrm{ess.min.}(\hat{h}_{\mathrm{crit}}|_Y) \;:=\; \sup_{Y' \subset Y} \inf_{y \in (Y \setminus Y')(\overline{\mathbb{Q}})} \hat{h}_{\mathrm{crit}}(y) \;\geq\; 0,$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

where the supremum is taken over all proper subvarieties Y' in Y defined over $\overline{\mathbb{Q}}$. Note that all PCF-special subvarieties $Y \subset \mathrm{M}_d$ are defined over $\overline{\mathbb{Q}}$ and have essential minimum 0, because they contain a Zariski-dense set of PCF points that can be defined over $\overline{\mathbb{Q}}$; see §2.2. We also prove the following result, strengthening Theorem 1.2.

Theorem 1.6. *Fix integers $d \geq 2$ and $D \geq 1$. There exists $\epsilon'' = \epsilon''(d, D) > 0$ such that there are only finitely many positive-dimensional irreducible subvarieties $Y \subset \mathrm{M}_d$ defined over $\overline{\mathbb{Q}}$ with degree $\leq D$ and* $\mathrm{ess.min.}(\hat{h}_{\mathrm{crit}}|_Y) \leq \epsilon'' \max\{1, h_{\mathrm{Ch}}(Y)\}$.

1.1. History and context

The study of the PCF-special varieties in M_d grew out of parallels between the theory of complex dynamical systems and the study of abelian varieties. A conjectural classification of PCF-special subvarieties in M_d first appears in [Si2, Chapter 6] where it was given the name of "Dynamical André-Oort" (or "DAO" as it is known in [JX]), because the geometry of the PCF locus in M_d shares features with the locus of CM points in a moduli space of abelian varieties; see [Za] for background on several Unlikely Intersection problems. The first cases of DAO were proved by Baker-DeMarco [BD] and Ghioca-Hsia-Tucker [GHT], followed by a series of results mostly focused on polynomial dynamics where the relations and symmetries are understood, such as [GY, FG2], culminating in a complete classification of PCF-special curves in moduli spaces of polynomials by Favre-Gauthier [FG3] and a characterization of PCF-special curves in M_d by Ji-Xie [JX]. We note, however, that there are currently no known classification results for PCF-special subvarieties of dimensions > 1, even when restricting to spaces of polynomials. Nevertheless, Theorem 1.1 can be viewed as a "Uniform DAO" that holds for PCF-special subvarieties of any dimension.

The questions addressed here are inspired by Pink's conjectures [Pi] in the setting of families of abelian varieties, and the uniformity results about subgroup schemes in families of abelian varieties, for example, as obtained recently in [Kü, DGH, GGK, GH, Yu]. The analogous study of preperiodic subvarieties for regular self-maps of projective spaces (or more generally, for polarized dynamical systems, which then includes the geometry of subgroups in abelian varieties) was initiated by Zhang (see, for example, [Zh2]). Uniform bounds on the geometry of preperiodic points appeared in [DKY1, DKY2, DM2, MaS].

Remark 1.7. We emphasize that the aforementioned uniformity results each relied crucially on the classification of the "special" varieties in their respective settings. In contrast, the classification of PCF-special subvarieties is only conjectural [BD]; we obtain uniformity without proving that PCF-special subvarieties are necessarily defined by critical orbit relations. In particular, we do not recover (or use) the main result of [JX] for curves in M_d, and each of our theorems is new also for curves in M_d. We do not know of any other result in the literature where uniformity is known without the classification.

A far-reaching conjecture in [DM3] implies the classification of PCF-special subvarieties (as formulated in [BD, Conjecture 1.10]) and the uniform bounds on their geometry (our Theorems 1.1 and 1.2) and many results about geometry of torsion points in abelian varieties as special cases. It also contains the (partial) uniformity results obtained in [DM1] and [GTV].

Finally, we remark that we do not give explicit bounds on B and N in Theorem 1.1, nor on the number of PCF-special subvarieties in Theorem 1.2. It would be very interesting to know how large these values must be and how they depend on d and D (and the dimension of the PCF-special subvarieties). In different contexts, some effective bounds have been given in [DKY2, DP, DM2].

1.2. Proof strategy

We build upon the theory of arithmetic equidistribution for heights on quasiprojective varieties [Kü, Ga3, YZ], and we make use of complex-dynamical tools for analyzing bifurcation measures as in [BE, Du].

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

We consider families $\mathcal{X} \to V$ of r-dimensional varieties $X_\lambda \subset \mathrm{M}_d$, parametrized by λ in an irreducible quasi-projective variety V of dimension $\ell \geq 1$. We say that a family $\mathcal{X} \to V$ is **maximally varying** if the natural map from V to the Chow variety $\mathrm{Ch}(\mathrm{M}_d, r, D)$ has finite fibers, where $\mathrm{Ch}(\mathrm{M}_d, r, D)$ is a space of cycles of M_d with dimension r and degree $\leq D$, with respect to a fixed projective embedding of M_d defined over $\mathbb{Q}$; see §2.5.

For any positive integer m, we let $\mathcal{X}_V^m$ denote the m-th fiber power of $\mathcal{X}$ over V. We denote elements of $\mathcal{X}_V^m$ by $(\lambda, f_1, \dots, f_m)$, with $f_i \in X_\lambda$ for each i. Our main theorems are derived from the following result, which is inspired from the Relative Bogomolov Conjecture [DGH2] and by results in [Kü] and [GGK].

Theorem 1.8. *Let $\mathcal{X} \to V$ be a maximally varying family of dimension $r \geq 1$ varieties in M_d, parametrized by an irreducible quasi-projective variety V of dimension $\ell \geq 1$, all defined over $\overline{\mathbb{Q}}$. Assume that the generic fiber of $\mathcal{X} \to V$ is geometrically irreducible. There exists $\epsilon = \epsilon(\mathcal{X}) > 0$ such that the set*

$$Z_\varepsilon(\mathcal{X}) := \left\{ (\lambda, f_1, \dots, f_{\ell+1}) \in \mathcal{X}_V^{\ell+1}(\overline{\mathbb{Q}}) \ : \ \sum_{i=1}^{\ell+1} \hat{h}_{\mathrm{crit}}(f_i) < \epsilon \max\{1, h_{\mathrm{Ch}}(\lambda)\} \right\}$$

is not Zariski dense in $\mathcal{X}_V^{\ell+1}$.

To prove Theorem 1.8, we first pass to a maximally varying family of varieties $\mathcal{X}' \to V$ in a finite branched cover $\mathcal{M}_d^{cm} \to \mathrm{M}_d$ that parameterizes maps on $\mathbb{P}^1$ with marked critical points $c_1, \dots, c_{2d-2} : \mathcal{M}_d^{cm} \to \mathbb{P}^1$. We construct two nonzero bifurcation measures on the fiber power $(\mathcal{X}')_V^{\ell+1}$, from two distinct r-tuples of critical points, and, associated to these measures, we define two adelically metrized line bundles on the quasiprojective $(\mathcal{X}')_V^{\ell+1}$, in the sense of [YZ]. The construction of the measures is carried out in Section 3, and their nonvanishing ultimately comes from properties of the bifurcation current and its powers in M_d from [Mc1, De1, Du, GOV]. The nonvanishing of the measures implies that the two metrized line bundles are non-degenerate (as defined in [YZ]). We are thus able to deduce from the height inequality of [YZ, Theorem 1.3.2] that, if $\epsilon > 0$ is sufficiently small, then "most" points of $Z_\epsilon(\mathcal{X})$ lie in subvarieties X_λ of bounded h_{Ch}-height.

We now proceed by contradiction: if no such ϵ exists, then there will be a generic sequence of points in $(\mathcal{X}')_V^{\ell+1}(\overline{\mathbb{Q}})$ with $\sum_{i=1}^{\ell+1} \hat{h}_{\mathrm{crit}}(f_i)$ tending to 0. We apply [YZ, Theorem 5.4.3] to deduce that the Galois orbits of these points will be equidistributed with respect to the two bifurcation measures, implying the two measures coincide.

The contradiction will come from an analysis of the two measures. More precisely, using a slicing argument as in [MaS] (which was revisited in [DM1]), it suffices to analyze the family of bifurcation measures they induce on the fibers X_λ. Here we appeal to the maximal variation of the family and the dynamical theory of J-stability. We find a parameter $\lambda_0 \in V$ and conjugacy class $[f_0] \in X_{\lambda_0}$ so that one collection of r critical points is pre-repelling for f_0, while one of the critical points in the other r-tuple is in the basin of a parabolic point. We then zoom into the parameter space at f_0, where the first r-tuple of preperiodic critical points yields an asymptotic similarity between the bifurcation measure and the dynamical space, exactly as in the proof of [BE, Main Theorem]. But zooming in at this rate yields 0 for the bifurcation measure defined with the second r-tuple of critical points, because one of the critical points has infinite orbit and is attracted to a parabolic cycle. Details are provided in Section 4. The proofs of Theorems 1.1, 1.2, 1.5, and 1.6 are completed in Section 5.

1.3. Questions

Finally, we remark that our initial (failed) attempts to prove Theorems 1.8 were more in line with previously known cases of DAO, combined with the strategy from [MaS] to obtain uniform bounds, but we encountered several obstacles. We were led to the following questions, and we view each as a measure-theoretic strengthening of DAO.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

Question 1.9. Suppose that Λ is a complex manifold of dimension $r \geq 1$, parameterizing a holomorphic family of maps

$$f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$$

defined by $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$, and suppose that $c_1, \ldots, c_{r+1} : \Lambda \to \mathbb{P}^1$ are marked critical points of f_λ. Let T_i denote the bifurcation current on Λ associated to the pair (f, c_i), defined in §2.3. Suppose that

$$T_1 \wedge \cdots \wedge T_{r-1} \wedge T_r \;=\; \alpha\, T_1 \wedge \cdots \wedge T_{r-1} \wedge T_{r+1} \neq 0$$

on Λ for some $\alpha > 0$. Can we conclude that the pair (c_r, c_{r+1}) of critical points is dynamically related along Λ? In other words, does there exist a proper, complex-analytic subvariety $X \subset \Lambda \times (\mathbb{P}^1 \times \mathbb{P}^1)$ which projects surjectively to Λ, is invariant for the action of (f, f), and contains the graph of (c_r, c_{r+1}) over Λ?

The question has only been answered in a few special cases with $r = 1$: (1) for the 1-parameter families of polynomials treated by Baker-DeMarco in [BD] parameterized by $\Lambda = \mathbb{C}$, and (2) for the curves $\Lambda = \mathrm{Per}_1(\eta)$ in M_2, for each $\eta \in \mathbb{C}$, of Example 1.4 [DWY].

Question 1.10. Suppose that Λ is a smooth and irreducible, quasiprojective, complex algebraic variety of dimension $r \geq 2$ and that $f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ is a morphism that defines an algebraic family of maps on $\mathbb{P}^1$, with marked critical points. With the notation of Question 1.9, suppose that

$$T_1 \wedge \cdots \wedge T_r \neq 0$$

on Λ but

$$T_1 \wedge \cdots \wedge T_{r-1} \wedge T_{r+1} = 0$$

on Λ. Can we conclude that there exists $i \in \{1, \ldots, r-1\}$ so that the pair (c_i, c_{r+1}) is dynamically related along Λ? (We remark here that one can define a dynamical relation for a tuple $(c_1, \ldots, c_{r+1})$ as for a pair; see [De2, §6.2]. But it follows by [MeSc, Proposition 2.21] that dynamical relations always come in pairs.) We know of no cases of this question that have been answered.

Remark 1.11. Versions of Theorems 1.1 and 1.2 are expected to hold in more general settings, for example when marking points on $\mathbb{P}^1$ are not necessarily critical, or upon working with preperiodic points in families of maps on $\mathbb{P}^N$. See [DM3]. Analogues of Questions 1.9 and 1.10 can also be formulated, but we keep our focus in this article on the postcritically-finite maps on $\mathbb{P}^1$ and the bifurcation currents associated to marked critical points.

2. Background on M_d and its subvarieties

In this section, we provide important background on the moduli spaces M_d, and we look into properties of a maximally varying family $\mathcal{X} \to V$ of subvarieties of M_d.

2.1. The moduli space M_d and a critically-marked parameter space

For each degree $d \geq 2$, the moduli space M_d is the space of conformal conjugacy classes of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{C}$. It is an affine algebraic variety of dimension $2d - 2$. Silverman showed that it exists as a geometric quotient over $\mathrm{Spec}\,\mathbb{Z}$, and he introduced a projective compactification $\overline{M}_d$ via the tools of Geometric Invariant Theory which can be defined over $\mathbb{Q}$ [Si1]. We note that $\mathrm{M}_2 \simeq \mathbb{C}^2$ [Mi1] while M_d is singular in all degrees $d > 2$ [We, MSW]. Because of the existence of maps with automorphisms, this M_d is not a fine moduli space, but we can pass to a finite branched cover $\mathcal{M}_d^{cm} \to \mathrm{M}_d$ to build a family

$$f : \mathcal{M}_d^{cm} \times \mathbb{P}^1 \to \mathcal{M}_d^{cm} \times \mathbb{P}^1 \tag{2.1}$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

of maps, and we may choose $\mathcal{M}_d^{cm}$ so that we also have functions

$$c_i : \mathcal{M}_d^{cm} \to \mathbb{P}^1$$

marking each of the critical points, $i = 1, \dots, 2d-2$.

2.2. PCF maps and Thurston rigidity

A map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is **postcritically finite** if each of its critical points has a finite forward orbit. An important class of examples of such maps are the Lattès maps, which are quotients of a self-map of a complex torus; the **flexible Lattès maps** are those that are part of a positive-dimensional family in M_d; see [Mi2] for more information on Lattès maps. We will denote the flexible Lattès locus by $\mathcal{L}_d$ in M_d; it exists only in square degrees. From Thurston's rigidity theorem [DH2], we may deduce that any postcritically finite map which is *not* flexible Lattès is Möbius-conjugate to a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$.

2.3. The bifurcation current and its wedge powers

Following [DF], we can define a **bifurcation current** T_i on $\mathcal{M}_d^{cm}$ associated to the pair (f, c_i) for the map f of (2.1) and each critical point c_i, as follows. We let ω denote a Fubini-Study form on $\mathbb{P}^1$ and let $\hat{\omega}$ be its pullback to $\mathcal{M}_d^{cm} \times \mathbb{P}^1$ by the projection map. Then the sequence of pullbacks $\frac{1}{d^n}(f^n)^*\hat{\omega}$ converge weakly to a current $\hat{T}_f$ on $\mathcal{M}_d^{cm} \times \mathbb{P}^1$. We let

$$T_i = \pi_* \left(\hat{T}_f \wedge [\Gamma_{c_i}] \right), \tag{2.2}$$

where $\pi : \mathcal{M}_d^{cm} \times \mathbb{P}^1 \to \mathcal{M}_d^{cm}$ is the projection and $[\Gamma_{c_i}]$ is the current of integration along the graph of c_i. Then T_i is a positive (1,1)-current with continuous potentials. The sum

$$T_{\mathrm{bif}} = \sum_{i=1}^{2d-2} T_i \tag{2.3}$$

descends to a well-defined positive $(1,1)$-current on the moduli space M_d, and it agrees with the **bifurcation current** introduced in [De1]. Because of its continuous potentials, the wedge powers of T_{bif} are well defined. The **bifurcation measure** $T_{\mathrm{bif}}^{\wedge(2d-2)}$ was first introduced in [BB] and is non-zero on M_d.

If $X \subset \mathcal{M}_d^{cm}$ is an irreducible complex algebraic subvariety of dimension $r > 0$ whose image in M_d is not contained in the flexible Lattès locus $\mathcal{L}_d$, then $T_{\mathrm{bif}} \neq 0$ on X, as a consequence of [Mc1, Lemma 2.2] and [De1, Theorem 1.1]. Moreover, $T_i = 0$ on X if and only if the critical point c_i is persistently preperiodic along X [DF, Theorem 2.5]. It was proved in [GOV, Lemma 6.8] that we also have $T_{\mathrm{bif}}^{\wedge r} \neq 0$ on X; in other words, there always exists an r-tuple of critical points $c_{i_1}, \dots, c_{i_r}$ so that $T_{i_1} \wedge \cdots \wedge T_{i_r} \neq 0$ on X.

2.4. The critical height

For each $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$ defined over $\overline{\mathbb{Q}}$, recall that the canonical height on $\mathbb{P}^1(\overline{\mathbb{Q}})$ is defined by

$$\hat{h}_f(x) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(x)),$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

where h is a Weil height on $\mathbb{P}^1$ [CS]. Counting the critical points with multiplicity, the sum

$$\hat{h}_{\mathrm{crit}}(f) := \sum_{\{c\,:\,f'(c)=0\}} \hat{h}_f(c)$$

defines a function on $\mathcal{M}_d^{cm}$, which Silverman called the **critical height** and conjectured that it is comparable with a Weil height on $\mathrm{M}_d \setminus \mathcal{L}_d$; see, for example, [Si2]. This comparability was proved by Ingram in [In].

The construction of the critical height was revisited by Yuan and Zhang in [YZ, §6.3], where they show that $\hat{h}_{\mathrm{crit}}$ is the height associated to a non-degenerate adelically metrized line bundle on $\mathcal{M}_d^{cm}$ over $\mathbb{Q}$. Indeed, they construct an f-invariant adelic line bundle $\overline{L}_f$ on $\mathcal{M}_d^{cm} \times \mathbb{P}^1$ for the f of (2.1). The critical height is then the height function associated to the "restriction" of $\overline{L}_f$ to the ramification divisor of f; see [YZ, §6.3]. Important for us is that the curvature distribution at the archimedean place is equal to the bifurcation current T_{bif} defined in §2.3. The non-degeneracy of this metrized line bundle follows because the bifurcation measure is non-trivial, as was proved in [BB].

Yuan and Zhang use their general equidistribution theorem [YZ, Theorem 5.4.3] to give an alternative proof of the arithmetic equidistribution of the points in PCF_d to the bifurcation measure $T_{\mathrm{bif}}^{\wedge 2d-2}$ in [YZ, §6.3], a result which was first obtained by Gauthier in [Ga3] (and was proved for the polynomial moduli spaces in [FG1]).

2.5. The Chow variety and maximal variation

For general information about Chow varieties, we refer the reader to [GKZ] [Ko]. Here we fix notation and state some basic properties that we need. We will apply the following discussion to $Y = \mathrm{M}_d$ or $Y = \mathcal{M}_d^{cm}$.

Suppose that Y is an irreducible quasiprojective algebraic variety, equipped with a projective embedding of $Y \hookrightarrow \mathbb{P}^N$ defined over $\mathbb{Q}$. We let $\overline{Y}$ denote the closure of Y in $\mathbb{P}^N$. For each integer $0 \le r < \dim Y$, and with respect to this embedding, we first consider the Chow variety $\mathrm{Ch}(\overline{Y}, r, D)$ of algebraic cycles $X \subset \overline{Y}$ in dimension r with degree $\le D$. It is an algebraic subvariety, possibly reducible, of the Chow variety $\mathrm{Ch}(\mathbb{P}^N, r, D)$, consisting of the cycles with dimension r and degree $\le D$ supported in $\overline{Y}$. As $\overline{Y}$ is defined over $\mathbb{Q}$, the Chow variety $\mathrm{Ch}(\overline{Y}, r, D)$ is also defined over $\mathbb{Q}$. We let $\mathrm{Ch}(Y, r, D)$ denote the Zariski-open subset consisting of cycles with support intersecting Y.

We will say that an irreducible subvariety $X \subset Y$ of dimension r has **degree $\le D$** if its Zariski closure in $\overline{Y}$ is the support of a cycle represented by a point in $\mathrm{Ch}(Y, r, D)$. Note that the degree depends on the choice of the embedding of Y into $\mathbb{P}^N$, but the concept of bounded degree does not.

Suppose that $\pi : \mathcal{X} \to V$ is a family of r-dimensional varieties in Y, parametrized by λ in an irreducible quasi-projective variety V of dimension $\ell \ge 1$. Assume that the generic fiber of $\mathcal{X} \to V$ is geometrically irreducible. We say that $\mathcal{X} \to V$ is **maximally varying** if the natural map from V to the Chow variety $\mathrm{Ch}(Y, r, D)$ has finite fibers, where we send each irreducible fiber $X_\lambda := \pi^{-1}(\lambda)$ to the point of $\mathrm{Ch}(Y, r, D)$ representing the cycle $[\overline{X}_\lambda]$ on $\overline{Y}$.

For any positive integer m, recall that $\mathcal{X}_V^m$ denotes the m-th fiber power of $\mathcal{X}$ over V. There is a natural map

$$\rho_m : \mathcal{X}_V^m \to Y^m$$

defined by $\rho_m(\lambda, f_1, \ldots, f_m) = (f_1, \ldots, f_m)$.

Proposition 2.1. *Suppose that Y is an irreducible quasiprojective complex algebraic variety. Suppose that $\mathcal{X} \to V$ is a family of r-dimensional varieties $X_\lambda \subset Y$, parametrized by λ in a quasi-projective*

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

variety V of dimension $\ell \geq 1$. Assume that V and the generic fiber of $\mathcal{X} \to V$ are irreducible. If $\mathcal{X} \to V$ is maximally varying, then the map

$$\rho_\ell : \mathcal{X}_V^\ell \to Y^\ell$$

is generically finite. In particular $\dim \rho_\ell(\mathcal{X}_V^\ell) = \dim \mathcal{X}_V^\ell$.

Proof. We prove the result by induction on ℓ. For $\ell = 1$, the result is clear, as $\dim \mathcal{X} = r + 1$ and maximal variation implies that the image of $\mathcal{X}$ in Y is not contained in an r-dimensional subvariety, so $\dim \rho_1(\mathcal{X}) = \dim \mathcal{X} = r + 1$.

Now suppose $\ell > 1$ and that we know the result for maximally varying families with base dimension $< \ell$. Note that $\dim \mathcal{X} = \ell + r$ and $\dim \mathcal{X}_V^\ell = \ell(r + 1)$. Choose any smooth point x_0 of $\rho_1(\mathcal{X})$ which is also smooth for a subvariety X_{λ_0} it lies in, and which is regular for the maps $\rho_1 : \mathcal{X} \to \rho_1(\mathcal{X})$ and $\pi_1 : \rho_\ell(\mathcal{X}_V^\ell) \to \rho_1(\mathcal{X})$ where the latter is the restriction of the projection $Y^\ell \to Y$ to the first coordinate. Here, "regular for the maps" means that the fiber over x_0 has the expected dimension and there is a point in the fiber over x_0 mapping surjectively on tangent spaces to that of x_0.

Let $Y_1 \subset \rho_1(\mathcal{X}) \subset Y$ be a subvariety of codimension r in $\rho_1(\mathcal{X})$ containing x_0, intersecting X_{λ_0} (and therefore a general X_λ) in a finite set. Consider the projection $p_1 : \mathcal{X}_V^\ell \to \mathcal{X}$ to the first factor and the projection $p_2 : \mathcal{X}_V^\ell \to \mathcal{X}_V^{\ell-1}$ to the remaining $\ell - 1$ factors. Note that $\rho_1^{-1}(Y_1) \subset \mathcal{X}$ projects dominantly to the base V, and $p_1^{-1}(\rho_1^{-1}(Y_1))$ projects dominantly and generically finitely by p_2 to $\mathcal{X}_V^{\ell-1}$.

Now choose any $Y_1' \subset Y_1$ of codimension 1 containing x_0, so of codimension $r + 1$ in $\rho_1(\mathcal{X})$. Then $\rho_1^{-1}(Y_1')$ has codimension $r + 1$ in $\mathcal{X}$, and its projection to the base V will be a subvariety V_1 of codimension 1. Let $\mathcal{X}|_{V_1} \to V_1$ denote the restricted family over the $(\ell - 1)$-dimensional base. Note that $p_1^{-1}(\rho_1^{-1}(Y_1'))$ is contained in $\mathcal{X}_{V_1}^\ell$ and projects dominantly and generically finitely by p_2 to $\mathcal{X}_{V_1}^{\ell-1}$.

By induction we know that $\dim \rho_{\ell-1}(\mathcal{X}_{V_1}^{\ell-1}) = \dim \mathcal{X}_{V_1}^{\ell-1} = (\ell-1)(r+1) = \ell(r+1)-(r+1)$. Therefore

$$\dim \rho_\ell(p_1^{-1}(\rho_1^{-1}(Y_1'))) = \dim \mathcal{X}_{V_1}^{\ell-1} = \ell(r + 1) - (r + 1).$$

Now consider $\pi_1^{-1}(Y_1')$ in $\rho_\ell(\mathcal{X}_V^\ell)$, which also has codimension $r + 1$ by the choice of x_0. But $\pi_1 \circ \rho_\ell = \rho_1 \circ p_1$, so that

$$\dim \pi_1^{-1}(Y_1') = \dim \rho_\ell(p_1^{-1}(\rho_1^{-1}(Y_1'))),$$

implying that

$$\dim \rho_\ell(\mathcal{X}_V^\ell) = \ell(r + 1)$$

and the proof is complete. □

3. Bifurcation measures on $\mathcal{X}_V^\ell$ and $\mathcal{X}_V^{\ell+1}$

Recall the definition of $\mathcal{M}_d^{cm}$ from §2.1. Throughout this section, we assume that $\mathcal{X} \to V$ is a maximally varying family of r-dimensional varieties $X_\lambda \subset \mathcal{M}_d^{cm}$, parametrized by λ in a quasi-projective variety V of dimension $\ell \geq 1$. Assume that V and the generic fiber of $\mathcal{X} \to V$ are irreducible. For any positive integer m, recall from §1.2 that $\mathcal{X}_V^m$ denotes the m-th fiber power of $\mathcal{X}$ over V, and we denote elements of $\mathcal{X}_V^m$ by $(\lambda, f_1, \ldots, f_m)$, with $f_i \in X_\lambda$ for each i. There is a natural map

$$\rho_m : \mathcal{X}_V^m \to (\mathcal{M}_d^{cm})^m \tag{3.1}$$

defined by $\rho_m(\lambda, f_1, \ldots, f_m) = (f_1, \ldots, f_m)$. From Proposition 2.1, we know that ρ_m is generically finite for all $m \geq \ell$.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

In this section we construct nontrivial measures on $\mathcal{X}_V^\ell$ and $\mathcal{X}_V^{\ell+1}$ that are pullbacks of powers of the bifurcation currents in $\mathcal{M}_d^{cm}$, defined in §2.3. The measure on $\mathcal{X}_V^\ell$ will be defined in terms of $T_{\rm bif}$ of (2.3), while two measures on $\mathcal{X}_V^{\ell+1}$ will additionally involve choices of r-tuples of marked critical points. The proofs build on the strong rigidity result that the power $T_{\rm bif}^{\wedge r}$ of the bifurcation current is nonzero on every r-dimensional subvariety of $\mathrm{M}_d \setminus \mathcal{L}_d$, where $\mathcal{L}_d$ is the flexible Lattès locus; this is proved in [GOV, Lemma 6.8] for all r. (See the discussion in §2.3.)

We also appeal to descriptions of the support of powers of $T_{\rm bif}$ in M_d in terms of critical orbit behavior obtained by Buff-Epstein in [BE] and Dujardin in [Du]. Precisely, suppose that $f_1, \dots, f_m$ is a collection of holomorphic families of maps on $\mathbb{P}^1$ parameterized by λ in a complex manifold Λ, and let $\Phi := (f_1, \dots, f_m) : \Lambda \times (\mathbb{P}^1)^m \to \Lambda \times (\mathbb{P}^1)^m$ be the induced holomorphic map. We say that an m-tuple of marked points $\mathbf{a} = (a_1, \dots, a_m) : \Lambda \to (\mathbb{P}^1)^m$ is **transversely** (respectively, **properly**) **prerepelling for $\boldsymbol{\Phi}$ at $\boldsymbol{\lambda_0}$** over $\boldsymbol{\Lambda}$ if there exist repelling periodic points p_i of f_{λ_0} and integers $n_i \geq 1$ such that $f_{\lambda_0}^{n_i}(a_i(\lambda_0)) = p_i$ for each $i = 1, \dots, m$, and the following holds: if $p_i : U \to \mathbb{P}^1$ denotes the holomorphic continuation of the repelling periodic point $p_i = p_i(\lambda_0)$ over some neighborhood $U \subset \Lambda$ of λ_0 where it remains repelling, then the graphs of the maps

$$\lambda \longmapsto \left(f_{1,\lambda}^{n_1}(a_1(\lambda)), \dots, f_{m,\lambda}^{n_m}(a_m(\lambda))\right) \quad \text{and} \quad \lambda \longmapsto (p_1(\lambda), \dots, p_m(\lambda))$$

intersect transversely (respectively, properly) in $U \times (\mathbb{P}^1)^m$ at λ_0.

For positive integers $m \leq 2d-2$, marked critical points $\mathbf{c} = (c_1, \dots, c_m)$, and $f_1 = \cdots = f_m := f$, Dujardin proved that there is a dense set of parameters in $\operatorname{supp} T_1 \wedge \cdots \wedge T_m$ in $\mathcal{M}_d^{cm}$ at which $\mathbf{c}$ is transversely prerepelling over $\mathcal{M}_d^{cm}$ [Du, Theorem 1.1]; recall that T_i was defined in (2.2). And conversely, if the m-tuple of critical points is properly prerepelling for $(f, \dots, f)$ at some parameter λ_0 over a parameter space of dimension $\geq m$, then $T_1 \wedge \cdots \wedge T_m$ must be nonzero in that parameter space with λ_0 in its support, as first observed in [BE] (where it was proved in the case $m = 2d-2$ and assuming transversely prerepelling) and revisited in [Ga1, Theorem 6.2], [Ga2, Theorem 2.3], and [BBD, Proposition 3.7] in more general contexts and without assuming transversality. See [DM1, Proposition 4.8] for the version we use, which implies that the latter converse result remains true with any m-tuple of maps $(f_1, \dots, f_m)$ of degree d in place of $(f, \dots, f)$.

3.1. Measure on $\mathcal{X}_V^\ell$

Proposition 3.1. *Let $\mathcal{X} \to V$ be a maximally varying family of dimension $r \geq 1$ varieties in $\mathcal{M}_d^{cm}$ with $\dim V = \ell \geq 1$, and let ρ_ℓ be the map of (3.1) for $m = \ell$. Set $\mathcal{Y}_\ell = \rho_\ell(\mathcal{X}_V^\ell)$. Let $T_{\rm bif}$ denote the bifurcation current in $\mathcal{M}_d^{cm}$ and let $\pi_i : (\mathcal{M}_d^{cm})^\ell \to \mathcal{M}_d^{cm}$ be the projection to the i-th factor for $i = 1, \dots, \ell$. Then*

$$\pi_1^*(T_{\rm bif}^{\wedge r+1}) \wedge \cdots \wedge \pi_\ell^*(T_{\rm bif}^{\wedge r+1}) \wedge [\mathcal{Y}_\ell] \neq 0$$

in $(\mathcal{M}_d^{cm})^\ell$, so that

$$\mu_\ell := \rho_\ell^* \left(\pi_1^*(T_{\rm bif}^{\wedge r+1}) \wedge \cdots \wedge \pi_\ell^*(T_{\rm bif}^{\wedge r+1})\right)$$

defines a positive measure on $\mathcal{X}_V^\ell$. Moreover, there is a collection of marked critical points

$$\mathbf{c} = (c_{1,1}, \dots, c_{1,r+1}, \dots, c_{\ell,1}, \dots, c_{\ell,r+1})$$

defined on $\mathcal{X}_V^\ell$ which is transversely prerepelling for the family of product maps

$$\Phi := (f_1, \dots, f_1, \dots, f_\ell, \dots, f_\ell)$$

at a parameter x_0 over $\mathcal{X}_V^\ell$, where each f_i is repeated $r+1$ times.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

If $\mathbf{c}$ satisfies the conclusion of Proposition 3.1 for Φ, we say that it **witnesses** the nonvanishing of μ_ℓ.

Proof. Let $\mathcal{Y} \subset \mathcal{M}_d^{cm}$ be the image of ρ_1, and let k be the dimension of $\mathcal{Y}$, so that

$$2 \leq r+1 \leq k \leq r+\ell = \dim \mathcal{X}.$$

Since the projection $\mathcal{M}_d^{cm} \to \mathrm{M}_d$ has finite fibers, we know from [GOV, Lemma 6.8] that $T_{\mathrm{bif}}^{\wedge k} \neq 0$ on $\mathcal{Y}$.

If $\ell = 1$, then we are done, since $k = r+1$ and $T_{\mathrm{bif}}^{\wedge(r+1)} \wedge [\mathcal{Y}]$ in $\mathcal{M}_d^{cm}$.

Assume $\ell > 1$. Let

$$\Phi = (f_1, \ldots, f_1, \ldots, f_\ell, \ldots, f_\ell)$$

denote the family of product maps on $(\mathbb{P}^1)^{\ell(r+1)}$ parameterized by $\mathcal{Y}_\ell \subset (\mathcal{M}_d^{cm})^\ell$, where each coordinate f_i is repeated $r+1$ times.

Let $U \subset \mathcal{Y}$ be a Zariski-open set so that the fibers of the two maps $\rho_1 : \mathcal{X} \to \mathcal{Y}$ and $\pi_1 : \mathcal{Y}_\ell \to \mathcal{Y}$ to the first factor of $(\mathcal{M}_d^{cm})^\ell$ have the expected dimension over U, namely $r+\ell-k$ for fibers of ρ_1 and $\ell+r\ell-k$ for π_1, and the maps are surjective on tangent spaces at some point of each fiber over elements of U. Note that, by the symmetry of $\mathcal{Y}_\ell$, the fibers will have the same properties for projections π_i for each i. From [Du, Theorem 1.1], there exists a k-tuple of critical points $\{c_1, \ldots, c_k\}$ and a parameter $y_0 \in U$ in the support of $T_{\mathrm{bif}}^{\wedge k}$ at which these critical points are transversely prerepelling for the family of maps f_1 parameterized by U. We may assume that y_0 is a smooth point of $\mathcal{Y}$ and of the subvariety X_{λ_0} for a smooth parameter $\lambda_0 \in V$.

Now choose a collection of $r+1$ critical points $c_{1,1}, \ldots, c_{1,r+1} \in \{c_1, \ldots, c_k\}$ so that $(c_{1,1}, \ldots, c_{1,r})$ is transversely prerepelling at y_0 over X_{λ_0} and the full $(r+1)$-tuple is transversely prerepelling at y_0 over U. Let $Z_1 \subset U$ be an irreducible component of the codimension-$(r+1)$ subvariety containing y_0 where these critical points are persistently preperiodic. Note that y_0 is an isolated point of the intersection of Z_1 with X_{λ_0}. Moreover, we know that $\pi_1^{-1}(Z_1)$ has codimension $r+1$ in $\mathcal{Y}_\ell$.

Note that the preimage $\rho_\ell^{-1}(\pi_1^{-1}(Z_1))$ in $\mathcal{X}_V^\ell$ projects to a subvariety V_1 of codimension 1 in the base V. In fact it maps with finite fibers to the $(\ell-1)$-th fiber power of a subfamily

$$\mathcal{X}|_{V_1} \to V_1$$

where $\dim V_1 = \ell - 1$. Now let $\mathcal{Y}_1$ be the image of this subfamily $\mathcal{X}|_{V_1}$ in $\mathcal{M}_d^{cm}$. We repeat the argument with $\mathcal{Y}_1$ in place of $\mathcal{Y}$. By maximal variation of $\mathcal{X}$, we know that

$$2 \leq r+1 \leq k_1 := \dim \mathcal{Y}_1 \leq r+\ell-1$$

and $T_{\mathrm{bif}}^{\wedge k_1} \neq 0$ on $\mathcal{Y}_1$. We choose another collection of critical points $c_{2,1}, \ldots, c_{2,r+1} \in \{c_1, \ldots, c_k\}$ so that the first r are transversely prerepelling at y_0 over X_{λ_0} and the full $(r+1)$-tuple is transversely prerepelling at y_0 over $\mathcal{Y}_1$. Note that we could take $c_{2,j} = c_{1,j}$ for $j = 1, \ldots, r$. We let $Z_2 \subset \mathcal{Y}_1$ be the codimension-$(r+1)$ subvariety containing y_0 where these critical points are persistently preperiodic. Then $\pi_2^{-1}(Z_2) \cap \pi_1^{-1}(Z_1)$ has codimension $2(r+1)$ in $\mathcal{Y}_\ell$.

In this way, we inductively construct a parameter $y = (y_0, \ldots, y_0) \in \mathcal{Y}_\ell$ and an $\ell(r+1)$-tuple of marked critical points

$$\mathbf{c} = (c_{1,1}, \ldots, c_{1,r+1}, \ldots, c_{\ell,1}, \ldots, c_{\ell,r+1})$$

which is transversely prerepelling for the map Φ at y over $\mathcal{Y}_\ell$. In particular, in the language of [DM1, §4.3], the graph $\Gamma_{\mathbf{c}}$ of $\mathbf{c}$ in $\mathcal{Y}_\ell \times (\mathbb{P}^1)^{\ell(r+1)}$ defines a rigid repeller for the map Φ at this parameter, and we conclude from [DM1, Proposition 4.8] that

$$\hat{T}_\Phi^{\wedge \ell(r+1)} \wedge [\Gamma_{\mathbf{c}}] \neq 0, \tag{3.2}$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

where

$$\hat{T}_\Phi = \sum_{i=1}^{\ell} \sum_{j=1}^{r+1} p_{i,j}^* \hat{T}_{f_i}$$

is the dynamical Green current for Φ on $\Lambda \times (\mathbb{P}^1)^{\ell(r+1)}$, for the projections $p_{i,j} : \Lambda \times (\mathbb{P}^1)^{\ell(r+1)} \to \Lambda \times \mathbb{P}^1$. Unwinding the definitions, we conclude that

$$\pi_1^*(T_{f_1,c_{1,1}} \wedge \cdots \wedge T_{f_1,c_{1,r+1}}) \wedge \cdots \wedge \pi_\ell^*(T_{f_\ell,c_{\ell,1}} \wedge \cdots \wedge T_{f_\ell,c_{\ell,r+1}}) \wedge [\mathcal{Y}_\ell] \neq 0$$

and therefore that

$$\pi_1^*(T_{\mathrm{bif}}^{\wedge r+1}) \wedge \cdots \wedge \pi_\ell^*(T_{\mathrm{bif}}^{\wedge r+1}) \wedge [\mathcal{Y}_\ell] \neq 0$$

so that the measure $\mu_\ell \neq 0$. Here we used the fact that $\dim \mathcal{X}_V^\ell = \dim \mathcal{Y}_\ell$ from the maximal variation.

Finally, since the map from $\mathcal{X}_V^\ell$ to $\mathcal{Y}_\ell$ is generically finite by Proposition 2.1, our initial choice of y_0 guarantees that the marked critical points $\mathbf{c}$ are also transversely prerepelling at a preimage of y, over $\mathcal{X}_V^\ell$. □

3.2. *Measures on $\mathcal{X}_V^{\ell+1}$*

Let μ_ℓ be the measure of Proposition 3.1, witnessed by the marked critical point $\mathbf{c}$. We now use these critical points to construct nonvanishing measures on the next fiber power $\mathcal{X}_V^{\ell+1}$. Let

$$\Phi = (f_1, \ldots, f_1, \ldots, f_\ell, \ldots, f_\ell)$$

be as in Proposition 3.1 parameterized by $\mathcal{X}_V^\ell$ via the map ρ_ℓ of (3.1), and let

$$\Phi_{\ell+1} = (f_1, \ldots, f_1, \ldots, f_\ell, \ldots, f_\ell, f_{\ell+1}, \ldots, f_{\ell+1})$$

be the family of maps on $(\mathbb{P}^1)^{\ell(r+1)+r}$ parameterized by $\mathcal{X}_V^{\ell+1}$, where $f_1, \ldots, f_\ell$ are each repeated $r+1$ times, but the final coordinate $f_{\ell+1}$ is only repeated r times. Specializing at each $\lambda \in V$ we also have a family of maps

$$F^{(\lambda)} := (f, \ldots, f) : X_\lambda \times (\mathbb{P}^1)^r \to X_\lambda \times (\mathbb{P}^1)^r,$$

parametrized by $X_\lambda \subset \mathcal{M}_d^{cm}$, and, focusing on the ℓ-th factor of $\mathcal{X}_V^\ell$ and associated components of $\mathbf{c}$, we will use the marked critical points

$$c_{\ell,1}|_{X_\lambda}, \ldots, c_{\ell,r+1}|_{X_\lambda} : X_\lambda \to \mathbb{P}^1$$

to define two measures μ_ℓ^i and μ_ℓ^j on $\mathcal{X}_V^{\ell+1}$. Conditions (1) and (2) of Proposition 3.2 will allow us to conclude that μ_ℓ^i and μ_ℓ^j are nonzero. Conditions (3) and (4) will allow us to prove in Section 4 that μ_ℓ^i and μ_ℓ^j are not proportional measures.

Proposition 3.2. *Suppose the marked critical point*

$$\mathbf{c} = (c_{1,1}, \ldots, c_{1,r+1}, \ldots, c_{\ell,1}, \ldots, c_{\ell,r+1})$$

witnesses $\mu_\ell \neq 0$ on $\mathcal{X}_V^\ell$. Then there are parameters $x_0 = (\lambda_0, t_1, \ldots, t_\ell)$ and $x_1 = (\lambda_1, s_1, \ldots, s_\ell)$ in the support of μ_ℓ in $\mathcal{X}_V^\ell$ and indices $i \neq j$ in $\{1, \ldots, r+1\}$ so that

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

1. *the r-tuple* $(c_{\ell,1}, \dots, \widehat{c_{\ell,i}}, \dots c_{\ell,r+1})$ *is transversely prerepelling for* $F^{(\lambda_0)}$ *at* t_ℓ *over* X_{λ_0}*;*
2. *the r-tuple* $(c_{\ell,1}, \dots, \widehat{c_{\ell,j}}, \dots c_{\ell,r+1})$ *is properly prerepelling for* $F^{(\lambda_0)}$ *at* t_ℓ *over* X_{λ_0}*;*
3. *the r-tuple* $(c_{\ell,1}, \dots, \widehat{c_{\ell,i}}, \dots c_{\ell,r+1})$ *is transversely prerepelling for* $F^{(\lambda_1)}$ *at* s_ℓ *over* X_{λ_1}*; and*
4. *the critical point* $c_{\ell,i}$ *is attracted to a parabolic periodic point of* f *at* $s_\ell \in X_{\lambda_1}$*,*

where $\widehat{c_{\ell,i}}$ *means that the i-th coordinate is omitted. Let* $\Pi_1 : \mathcal{X}_V^\ell \times_V \mathcal{X} \to \mathcal{X}_V^\ell$ *and* $\Pi_2 : \mathcal{X}_V^\ell \times_V \mathcal{X} \to \mathcal{X}$*. Then*

$$\mu_\ell^i := \Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{c_{\ell,1}} \wedge \cdots \widehat{T_{c_{\ell,i}}} \cdots \wedge T_{c_{\ell,r+1}}) \neq 0$$

and

$$\mu_\ell^j := \Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{c_{\ell,1}} \wedge \cdots \widehat{T_{c_{\ell,j}}} \cdots \wedge T_{c_{\ell,r+1}}) \neq 0$$

on $\mathcal{X}_V^{\ell+1}$*.*

Proof. Via the map $\rho_\ell : \mathcal{X}_V^\ell \to (\mathcal{M}_d^{cm})^\ell$, we view $\mathcal{X}_V^\ell$ as the parameter space for a family of product maps $\Phi = (f_1, \dots, f_1, \dots, f_\ell, \dots, f_\ell)$ on $(\mathbb{P}^1)^{\ell(r+1)}$. From Proposition 3.1, we know that $\mathbf{c}$ is transversely prerepelling for Φ at a parameter $x_0 = (\lambda_0, t_1, \dots, t_\ell) \in \operatorname{supp} \mu_\ell$.

Let $S_{i,j}$ be the irreducible component of the hypersurface in $\mathcal{X}_V^\ell$ containing x_0 along which $c_{i,j}$ is persistently preperiodic for f_i, $i = 1, \dots, \ell$, $j = 1, \dots, r+1$. Consider the component of the intersection

$$S_{1,1} \cap \cdots \cap S_{\ell-1,r+1}$$

containing x_0 in $\mathcal{X}_V^\ell$. Similar to the proof of Proposition 2.1, for dimension reasons, it maps (generically finitely) to a family $\mathcal{X}_1 \to V_1$ of r-dimensional varieties over a 1-dimensional base, where these marked critical points are persistently preperiodic for $f_1, \dots, f_{\ell-1}$. We will continue to write $x_0 = (\lambda_0, t_\ell)$ for the point we started with, but now viewed in $\mathcal{X}_1$.

By the transversality, we know that r of the remaining critical points, say $c_{\ell,1}, \dots, c_{\ell,r}$, are transversely prerepelling at t_ℓ for $F^{(\lambda_0)} = (f, f, \dots, f)$ over X_{λ_0}.

We now aim to show that by modifying the parameter x_0 if necessary, we can arrange so that another r-tuple, say $c_{\ell,2}, \dots, c_{\ell,r+1}$, is also properly prerepelling at t_ℓ for $F^{(\lambda_0)}$ over X_{λ_0}. Note that transversality of the intersection $S_{1,1} \cap \cdots \cap S_{\ell-1,r+1}$ at the original x_0 persists in a neighborhood, so the first $(\ell-1)(r+1)$ critical points remain transversely prerepelling in a neighborhood of x_0 for $(f_1, \dots, f_{\ell-1})$.

Consider the intersection

$$S_{\ell,1} \cap \cdots \cap S_{\ell,r}$$

in $\mathcal{X}_1$. These hypersurfaces intersect transversely at x_0 and so define a smooth holomorphic curve Γ containing x_0 which is locally a section of $\mathcal{X}_1 \to V_1$ near λ_0. The intersection of Γ with X_{λ_0} is $x_0 = (\lambda_0, t_\ell)$. Note further that $c_{\ell,r+1}$ is not persistently preperiodic along Γ (because of the original setup of transverse intersections), and so $c_{\ell,r+1}$ must be active along Γ with x_0 in its bifurcation locus (because it lands at a repelling cycle there). The bifurcation locus for $c_{\ell,r+1}$ in Γ has no isolated points, so there must be an infinite collection of parameters $x \in \Gamma$ where $c_{\ell,r+1}$ is preperiodic to a repelling cycle for f_ℓ, dense in the bifurcation locus of $c_{\ell,r+1}$ in a neighborhood of x_0.

Now omit $S_{\ell,1}$ from the intersection, so that $S_{\ell,2} \cap \cdots \cap S_{\ell,r}$ (in $\mathcal{X}_1$) defines a 1-parameter family of algebraic curves $\mathcal{C}$, for all λ in a small neighborhood of $\lambda_0 \in V_1$, by

$$C_\lambda = (S_{\ell,2} \cap \cdots \cap S_{\ell,r}) \cap X_\lambda.$$

We know that $S_{\ell,r+1}$ intersects the total space of this family $\mathcal{C}$ transversely at the given point x_0. Suppose that it fails to intersect properly when restricted to X_{λ_0}; this would mean that $c_{\ell,r+1}$ is persistently preperiodic along the curve C_{λ_0} in X_{λ_0}. Suppose this happens at all parameters $x \in \Gamma$ where $c_{\ell,r+1}$ is preperiodic to a repelling cycle. This implies that the associated bifurcation current $T_{c_{\ell,r+1}}$ vanishes

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

identically on each of these curves C_x passing through each such x in the family $\mathcal{C}$. By continuity of potentials of the bifurcation currents, we deduce that $T_{c_{\ell,r+1}}$ vanishes along C_x for all x in $\operatorname{supp} T_{c_{\ell,r+1}}|_\Gamma$ near x_0. But this means that $c_{\ell,r+1}$ is persistently preperiodic along each of these curves [DF, Theorem 2.5]. As they form an uncountable family, we see that $c_{\ell,r+1}$ must be preperiodic along the entire family $\mathcal{C}$, which is a contradiction.

We conclude that we can adjust our parameter λ_0 slightly, if needed, so that points $c_{\ell,1},\dots,c_{\ell,r}$ are transversely prerepelling at a new choice of $x_0 \in \Gamma$ along X_{λ_0} while $c_{\ell,2},\dots,c_{\ell,r+1}$ are properly prerepelling at x_0 along X_{λ_0}. This completes the proofs of (1) and (2) of the proposition.

As $c_{\ell,r+1}$ is bifurcating along the curve we called Γ, the characterizations of J-stability (as in [MSS]) imply that there are parameters, everywhere dense in its bifurcation locus, for which the maps have a parabolic cycle; the theory of renormalization (see [DH1]) implies that $c_{\ell,r+1}$ will lie in the basin of the parabolic cycle. We let λ_1 be a parameter close to λ_0 in the support of $\pi_*\mu_\ell$, where π is the projection to V, so that $c_{\ell,r+1}$ is attracted to a parabolic cycle at $x_1 := \Gamma \cap X_{\lambda_1}$, while $c_{\ell,1},\dots,c_{\ell,r}$ remain transversely prerepelling at x_1 along X_{λ_1}. This completes the proof of properties (1)–(4).

To prove the non-vanishing of μ_ℓ^i and μ_ℓ^j, consider the map $\Phi_{\ell+1} = f_1^{\times(r+1)} \times \cdots \times f_\ell^{\times(r+1)} \times f_{\ell+1}^{\times r}$ on $(\mathbb{P}^1)^{\ell(r+1)+r}$ over $\mathcal{X}_V^{\ell+1}$. The argument above, leading to properties (1) and (2), shows that we can label critical points of $f_1,\dots,f_{\ell+1}$ so that the graphs of the marked points

$$\mathbf{c}_i = (c_{1,1},\dots,c_{1,r+1},\dots,c_{\ell,1},\dots,c_{\ell,r+1},c_{\ell+1,1},\dots\widehat{c_{\ell+1,i}}\dots,c_{\ell+1,r+1})$$

and

$$\mathbf{c}_j = (c_{1,1},\dots,c_{1,r+1},\dots,c_{\ell,1},\dots,c_{\ell,r+1},c_{\ell+1,1},\dots\widehat{c_{\ell+1,j}}\dots,c_{\ell+1,r+1})$$

each define a rigid repeller for the map $\Phi_{\ell+1}$ over a neighborhood of a parameter $(\lambda_0,t_1,\dots,t_\ell,t_\ell)$ in $\mathcal{X}_V^{\ell+1}$, in the sense of [DM1, §4.3], and so the measures are nonzero by [DM1, Proposition 4.8]. □

4. Proof of Theorem 1.8

Suppose that $\mathcal{X} \to V$ is a maximally varying family of r-dimensional varieties $X_\lambda \subset M_d$ parametrized by V of dimension $\ell \geq 1$, defined over $\overline{\mathbb{Q}}$. As in §2.1, we may pass to a finite branched cover $\mathcal{M}_d^{cm} \to \mathrm{M}_d$ over which we can define a family of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ with $2d-2$ marked critical points. Pulling back each X_λ and possibly replacing V with a finite branched cover, we will continue to use the notation $\mathcal{X} \to V$ for the family of varieties in $\mathcal{M}_d^{cm}$. Therefore, we may assume that we can define the family of product maps

$$\Phi = (f_1,\dots,f_1,\dots,f_\ell,\dots,f_\ell),$$

on $(\mathbb{P}^1)^{(r+1)\ell}$, parametrized by $\mathcal{X}_V^\ell$ and defined over $\overline{\mathbb{Q}}$, where each coordinate f_i is repeated $r+1$ times and a family

$$\Phi_{(\ell+1)} = (f_{\ell+1},\dots,f_{\ell+1})$$

on $(\mathbb{P}^1)^r$, parametrized by $\mathcal{X}$, where each coordinate is repeated r times. We denote by $\overline{L}_\Phi$ and $\overline{L}_{\Phi_{(\ell+1)}}$ the Φ (respectively $\Phi_{(\ell+1)}$)-invariant extension of the classical polarization as defined by [YZ, Theorem 6.1.1]. In particular, using the notation in [YZ], we have that $\overline{L}_{\Phi_{(\ell+1)}} \in \widehat{\mathrm{Pic}}(\mathcal{X} \times (\mathbb{P}^1)^r)_{\mathbb{Q},\mathrm{nef}}$ and $\overline{L}_\Phi \in \widehat{\mathrm{Pic}}(\mathcal{X}_V^\ell \times (\mathbb{P}^1)^{\dim \mathcal{X}_V^\ell})_{\mathbb{Q},\mathrm{nef}}$.

In what follows it is convenient to work with the Deligne pairing (of relative dimension 0), of the invariant metrized line bundles restricted to graphs of critical marked points (see [YZ, §4.2] for metrics of the Deligne pairing and [YZ, §6.2.2] for the "restriction" of the invariant metrized line bundle to a subvariety). By [YZ, Theorem 4.2.3], to each marked tuple of critical points $\mathbf{c} : \mathcal{X}_V^\ell \to (\mathbb{P}^1)^{\dim \mathcal{X}_V^\ell}$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

we can associate a metrized line bundle $\langle \overline{L}_\Phi|_{\Gamma_\mathbf{c}}\rangle \in \widehat{\mathrm{Pic}}(\mathcal{X}_V^\ell)_{\mathbb{Q},\mathrm{nef}}$, where $\Gamma_\mathbf{c}$ denotes the graph of $\mathbf{c}$ in $\mathcal{X}_V^\ell \times (\mathbb{P}^1)^{\dim \mathcal{X}_V^\ell}$. Moreover, we let

$$\overline{L} := \otimes_\mathbf{c} \langle \overline{L}_\Phi|_{\Gamma_\mathbf{c}}\rangle \in \widehat{\mathrm{Pic}}(\mathcal{X}_V^\ell)_{\mathbb{Q},\mathrm{nef}}$$

be the tensor product over all marked tuples of critical points $\mathbf{c}$ for the maps $f_1, \dots, f_\ell$.

Now let $\mathbf{c} = (c_{1,1}, \dots, c_{\ell,r+1}) : \mathcal{X}_V^\ell \to (\mathbb{P}^1)^{(r+1)\ell}$ be a marked critical point for Φ that witnesses $\mu_\ell \neq 0$, as in Proposition 3.1. Reordering the critical points if needed, we can find $x_0, x_1 \in \operatorname{supp} \mu_\ell$ as in Proposition 3.2, with $i = r+1$ and $j = 1$. We write $c_i := c_{\ell,i} : \mathcal{X} \to \mathbb{P}^1$ in what follows. As before, using [YZ, Theorem 4.2.3], we define two adelic metrized line bundles on $\mathcal{X}$ by

$$\begin{aligned} \overline{L}_1 &:= \langle \overline{L}_{\Phi_{(\ell+1)}}|_{\Gamma_{(c_1,\dots,c_r)}}\rangle \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q},\mathrm{nef}} \\ \overline{L}_2 &:= \langle \overline{L}_{\Phi_{(\ell+1)}}|_{\Gamma_{(c_2,\dots,c_{r+1})}}\rangle \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q},\mathrm{nef}}, \end{aligned} \tag{4.1}$$

and note that their curvature forms (at an archimedean place) are the bifurcation currents

$$\begin{aligned} c_1(\overline{L}_1) &= T_{f_{\ell+1},c_1} + \cdots + T_{f_{\ell+1},c_r} \\ c_1(\overline{L}_2) &= T_{f_{\ell+1},c_2} + \cdots + T_{f_{\ell+1},c_{r+1}}. \end{aligned} \tag{4.2}$$

We can now define two metrized line bundles

$$\overline{M}_i = \Pi_1^*(\overline{L}) + \Pi_2^*(\overline{L}_i) \in \widehat{\mathrm{Pic}}(\mathcal{X}_V^{\ell+1})_{\mathbb{Q},\mathrm{nef}}$$

on $\mathcal{X}_V^{\ell+1}$ for $i = 1, 2$. By [YZ, Lemma 6.2.1, Theorem 4.1.3], the heights associated to $\overline{M}_i$ vanish at PCF points; see also [YZ, Lemma 6.3.2]. More precisely, for a point $(\lambda_0, t_1, \dots, t_{\ell+1}) \in \mathcal{X}_V^{\ell+1}(\overline{\mathbb{Q}})$ we have

$$\begin{aligned} h_{\overline{M}_1}(\lambda_0, t_1, \dots, t_{\ell+1}) &= \sum_{i=1}^{\ell} h_{\mathrm{crit}}(f_i) + \sum_{j=1}^{r} \hat{h}_{f_{\lambda_0,t_{\ell+1}}}(c_j(\lambda_0, t_{\ell+1})) \\ h_{\overline{M}_2}(\lambda_0, t_1, \dots, t_{\ell+1}) &= \sum_{i=1}^{\ell} h_{\mathrm{crit}}(f_i) + \sum_{j=2}^{r+1} \hat{h}_{f_{\lambda_0,t_{\ell+1}}}(c_j(\lambda_0, t_{\ell+1})). \end{aligned} \tag{4.3}$$

We will now show that $\overline{M}_i$ is "non-degenerate" for $i = 1, 2$ in the terminology of [YZ, §6.2.2]. Equivalently, by [YZ, Lemma 5.4], it suffices to show that

$$c_1(\overline{M}_i)^{\wedge \ell(r+1)+r} \neq 0.$$

Recall that $T_{f_i,c_j}^{\wedge 2} = 0$ for all i, j and similarly $(\Pi_2^* c_1(\overline{L}_i))^{\wedge r+1} = (\Pi_2)^*(c_1(\overline{L}_i)^{\wedge r+1}) = 0$ for $i = 1, 2$, by the definition of the bifurcation current as limits of pullbacks, because the map $\Phi_{(\ell+1)}$ acts on the r-dimensional $(\mathbb{P}^1)^r$. Moreover, since $\dim \mathcal{X}_V^\ell = \ell(r+1)$ we also have that $\Pi_1^*(c_1(\overline{L}))^{\wedge \ell(r+1)+1} = 0$. Thus, using also [YZ, Theorem 4.2.3] as before, we have

$$c_1(\langle \overline{L}\rangle)^{\wedge (r+1)\ell} = \alpha_1 \, \mu_\ell \tag{4.4}$$

on $\mathcal{X}_V^\ell$, and

$$\begin{aligned} c_1(\overline{M}_1)^{\wedge \ell(r+1)+r} &= \alpha_2 \, \Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{f_{\ell+1},c_1} \wedge \cdots \wedge T_{f_{\ell+1},c_r}) = \alpha_2 \, \mu_\ell^{r+1} \neq 0, \\ c_1(\overline{M}_2)^{\wedge \ell(r+1)+r} &= \alpha_3 \, \Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{f_{\ell+1},c_2} \wedge \cdots \wedge T_{f_{\ell+1},c_{r+1}}) = \alpha_3 \, \mu_\ell^1 \neq 0, \end{aligned} \tag{4.5}$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

for positive constants $\alpha_1, \alpha_2, \alpha_3 > 0$. Their non-vanishing follows by Proposition 3.2 (and our choice of critical points). Therefore, by the height inequality [YZ, Theorem 1.3.2] we infer that there are positive constants $\epsilon_1 = \epsilon_1(\mathcal{X}) > 0,\ \epsilon_2 = \epsilon_2(\mathcal{X}) > 0$ such that

$$\sum_{i=1}^{\ell+1} \hat{h}_{\mathrm{crit}}(f_i) \geq \epsilon_1 h_{\mathrm{Ch}}(\lambda) - \epsilon_2, \tag{4.6}$$

for all $(\lambda, f_1, \dots, f_{\ell+1}) \in U(\overline{\mathbb{Q}})$ for a Zariski open and dense subset $U \subset \mathcal{X}_V^{\ell+1}$.

It remains to show that for some $\epsilon = \epsilon(\mathcal{X})$, the set

$$\left\{ (\lambda, f_1, \dots, f_{\ell+1}) \in \mathcal{X}_V^{\ell+1}(\overline{\mathbb{Q}}) \ : \ \sum_{i=1}^{\ell+1} \hat{h}_{\mathrm{crit}}(f_i) < \epsilon \right\}$$

is not Zariski dense in $\mathcal{X}_V^{\ell+1}$. We prove this result by contradiction. Suppose that there is a generic Galois invariant sequence $\{x_n\}_n$ of critically small points in $\mathcal{X}_V^{\ell+1}$, so that $h_{\overline{M}_i}(x_n) \to 0$ for $i = 1, 2$. Since $\overline{M}_i$ is nef and non-degenerate, Yuan-Zhang's fundamental inequality [YZ, Theorem 5.3.3] implies that $h_{\overline{M}_i}(\mathcal{X}_V^{\ell+1}) = 0$ for $i = 1, 2$. We can use [YZ, Theorem 5.4.3] with the Galois-invariant critically small set to get that

$$\Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{f_{\ell+1},c_1} \wedge \cdots \wedge T_{f_{\ell+1},c_r}) = \alpha\, \Pi_1^*(\mu_\ell) \wedge \Pi_2^*(T_{f_{\ell+1},c_2} \wedge \cdots \wedge T_{f_{\ell+1},c_{r+1}}) \neq 0,$$

on $\mathcal{X}_V^{\ell+1}$ for some constant $\alpha > 0$.

We now slice these measures with respect to the projection $\mathcal{X}_V^{\ell+1} \to \mathcal{X}_V^{\ell}$; see, for example [BB, Proposition 4.3]. This implies that

$$\int_{\mathcal{X}_V^\ell} \left(\int_{X_{\pi(x)}} \phi \cdot \bigwedge_{i=1}^{r} T_{f_{\ell+1},c_i}|_{X_{\pi(x)}} \right) d\mu_\ell(x) = \alpha \int_{\mathcal{X}_V^\ell} \left(\int_{X_{\pi(x)}} \phi \cdot \bigwedge_{i=2}^{r+1} T_{f_{\ell+1},c_i}|_{X_{\pi(x)}} \right) d\mu_\ell(x) \tag{4.7}$$

for every continuous and compactly supported function ϕ on $\mathcal{X}_V^{\ell+1}$. Here, $\pi : \mathcal{X}_V^{\ell} \to V$ denotes the projection to the base. (Strictly speaking, since the measure μ_ℓ is not smooth, we should approximate μ_ℓ by smooth and compactly-supported forms on $\mathcal{X}_V^{\ell}$ to deduce (4.7); compare the proof of [MaS, Theorem 1.8].)

For each $\lambda \in V$, set

$$\mu_\lambda^{r+1} := T_{f_{\ell+1},c_1} \wedge \cdots \wedge T_{f_{\ell+1},c_r}|_{X_\lambda}$$

and

$$\mu_\lambda^{1} = T_{f_{\ell+1},c_2} \wedge \cdots \wedge T_{f_{\ell+1},c_{r+1}}|_{X_\lambda}.$$

We infer from (4.7) that

$$\mu_\lambda^{r+1} = \alpha\, \mu_\lambda^1 \tag{4.8}$$

as measures on X_λ, for some constant $\alpha > 0$ and for all λ in the support of $\pi_*\mu_\ell$ in V. Indeed, suppose there exists $b \in V(\mathbb{C})$ where (4.8) doesn't hold, and so that $\pi_*\mu_\ell(U) > 0$ for every open neighborhood U of b. Then we can find a continuous, non-negative, compactly supported function ψ on $\mathcal{X}$ such that

$$\int_{X_b} \psi\, d\mu_b^{r+1}(x) \neq \alpha \int_{X_b} \psi\, d\mu_b^1(x).$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

By weak continuity of the measures μ_λ^{r+1} and μ_λ^1 as λ varies (since the bifurcation currents have continuous potentials), we find that

$$\int_{X_\lambda} \psi \, d\mu_\lambda^{r+1}(x) \neq \alpha \int_{X_\lambda} \psi \, d\mu_\lambda^1(x)$$

for all λ in a small neighborhood U of b. Therefore, setting

$$\phi(\lambda, t_1, \dots, t_\ell, t_{\ell+1}) = h_b(\lambda)\psi(\lambda, t_{\ell+1})$$

on $\mathcal{X}_V^{\ell+1}$ for a non-negative function h_b supported in a small neighborhood of b, the equality (4.7) will fail.

The remainder of the proof is devoted to deriving a contradiction from the equality (4.8). We first sketch the idea. Recall from Proposition 3.2 that we have parameter $x_1 = (\lambda_1, s_1, \dots, s_\ell) \in \operatorname{supp} \mu_\ell$ so that the tuple of marked critical points $(c_1, \dots, c_r)$ is transversely prerepelling for $F^{(\lambda_1)} = (f, \dots, f)$ at s_ℓ over X_{λ_1}. This implies that $\mu_{\lambda_1}^{r+1}$ is nonzero on X_{λ_1}, with s_ℓ in its support by [BE]. Moreover, as in [BE, Section 3] and the generalization appearing in [Ga2, Proposition 3.2], we know that if we zoom in on the measure $\mu_{\lambda_1}^{r+1}$ on X_{λ_1} at the parameter s_ℓ at an exponential rate in n (determined by the multipliers of the repelling cycles in the forward orbits of the marked critical points), and rescale the measure appropriately we get a nontrivial limit; see (4.11). On the other hand, we know from Proposition 3.2 that the critical point c_{r+1} is attracted to a parabolic cycle at $s_\ell \in X_{\lambda_1}$. Zooming into the measure $\mu_{\lambda_1}^1$ on X_{λ_1} at s_ℓ at the same exponential rate (which is supposed to give a nonzero limit), we actually get 0 from the following lemma. This will produce our desired contradiction.

Lemma 4.1. *Suppose that $f : \mathbb{D}^r \times \mathbb{P}^1 \to \mathbb{D}^r \times \mathbb{P}^1$ is a holomorphic family of maps with degree $d \geq 2$ parametrized by $\mathbb{D}^r \subset \mathbb{C}^r$ with a marked critical point $c : \mathbb{D}^r \to \mathbb{P}^1$. Assume that $c(0)$ is attracted to a parabolic fixed point at $z = 0$ for f_0. Then for any constant $\rho > 1$, there exists $\delta > 0$ so that the sequence of functions*

$$\alpha_n(x) := f_{x/\rho^n}^n(c(x/\rho^n))$$

converges uniformly to 0 on the polydisk $(\mathbb{D}_\delta)^r$, as $n \to \infty$.

Proof. Let $\rho > 1$. Since $|f_0'(0)| = 1$, we can first fix a small $\varepsilon > 0$ and $\rho_\varepsilon > 1$ so that

$$\left|f_x'(z)\right| \leq \rho_\varepsilon < \rho,$$

for all $x \in \mathbb{D}_\epsilon^r$ and $z \in \mathbb{D}_\varepsilon$. We replace c with an iterate so that $c(0)$ and all its forward iterates under f_0 are in the disk $\mathbb{D}_{\varepsilon/2}$. Choose $M > 0$ so that the gradient of f (with respect to parameter x) satisfies

$$|\nabla_x f_x(z)| \leq M$$

for all $x \in \mathbb{D}^r$ and $z \in \mathbb{D}_\varepsilon$. Note that

$$\alpha_n(x) = f_{x/\rho^n}(\alpha_{n-1}(x/\rho)).$$

Taking derivatives, we have

$$\begin{aligned} |\nabla \alpha_n(x)| &= \left| \frac{1}{\rho^n} \cdot (\nabla_w f_w)(\alpha_{n-1}(x/\rho)) + \frac{1}{\rho} f'_{x/\rho^n}(\alpha_{n-1}(x/\rho)) \nabla_{x/\rho} \alpha_{n-1}(x/\rho) \right| \\ &\leq \frac{M}{\rho^n} + \frac{\rho_\varepsilon}{\rho} \left| \nabla_{x/\rho} \alpha_{n-1}(x/\rho) \right| \end{aligned} \tag{4.9}$$

as long as $\alpha_{n-1}(x/\rho)$ lies in $\mathbb{D}_\varepsilon$.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

Now choose n_0 so that

$$\frac{M}{\rho^{n_0}} < \frac{\rho_\varepsilon}{\rho}.$$

We take a very small δ_1 so that

$$\alpha_{n_0-1}(x) \in \mathbb{D}_\varepsilon$$

for all $x \in \mathbb{D}^r_{\delta_1}$, and set

$$M_1 := \max_{x \in \mathbb{D}^r_{\delta_1}} |\nabla \alpha_{n_0-1}(x)|.$$

From the inequality (4.9), we have

$$|\nabla \alpha_{n_0}(x)| \ \le \ \frac{M}{\rho^{n_0}} + \frac{\rho_\varepsilon}{\rho} |\nabla_{x/\rho} \alpha_{n_0-1}(x/\rho)| \ \le \ \frac{\rho_\varepsilon}{\rho} + \frac{\rho_\varepsilon}{\rho} M_1$$

for each $x \in \mathbb{D}^r_{\delta_1}$. Let

$$M_2 := \frac{\rho_\varepsilon}{\rho} + \frac{\rho_\varepsilon}{\rho} M_1,$$

and define M_i recursively by

$$M_{i+1} = \frac{\rho_\varepsilon}{\rho^i} + \frac{\rho_\varepsilon}{\rho} M_i$$

for $i \ge 2$. Note that $M_i \to 0$ as $i \to \infty$, because

$$M_i = \left(\frac{\rho_\varepsilon}{\rho}\right)^{i-1} (M_1 + 1) + \frac{1}{\rho^i} \sum_{j=1}^{i-2} \rho_\varepsilon^j.$$

Finally, we can show inductively that our derivative estimate (4.9) holds when applied to α_n for all $n > n_0$ on a sufficiently small disk near 0. This will imply that

$$|\nabla \alpha_{n_0+i}(x)| < \frac{M}{\rho^{n_0+i}} + \frac{\rho_\varepsilon}{\rho} |\nabla \alpha_{n_0+i-1}(x/\rho)| \le \frac{\rho_\varepsilon}{\rho^{i+1}} + \frac{\rho_\varepsilon}{\rho} M_{i+1} = M_{i+2}$$

for all x in a small disk around 0. We know it holds for $i = 0$ for all $|x| < \delta_1$. We take $M_0 := \max_{i \ge 2} M_i$, and set

$$\delta := \min(\delta_1, \varepsilon/(2M_0), \varepsilon).$$

As $\alpha_{n_0}(0) \in \mathbb{D}_{\varepsilon/2}$, the inequality (4.9) gives $|\nabla \alpha_{n_0}(x)| < M_2 \le M_0$, so that $\alpha_{n_0}(x)$ is in $\mathbb{D}_\varepsilon$ for each $x \in \mathbb{D}^r_\delta$. By induction, we see that if $\alpha_{n_0+i-1}(x)$ is in $\mathbb{D}_\varepsilon$ with $|\nabla \alpha_{n_0+i-1}| \le M_{i+1}$, then $\alpha_{n_0+i}(x)$ is in $\mathbb{D}_\varepsilon$, because $\alpha_{n_0+i}(0)$ is in $\mathbb{D}_{\varepsilon/2}$ and $\delta < \varepsilon/(2M_0)$. Since $M_i \to 0$ as i tends to infinity, we see that α_n converges to 0 uniformly on $\mathbb{D}^r_\delta$. □

We now return to our proof. Abusing the notation slightly, we write $\{f_s\}_{s \in X_{\lambda_1}}$ to denote the restriction of the family of maps to X_{λ_1}, c_{r+1} for the restriction of c_{r+1} to X_{λ_1}, and $T_{c_{r+1}}$ its associated bifurcation current on X_{λ_1}. We know from Lemma 4.1 that, zooming into s_ℓ in X_{λ_1} at any exponential rate, the graphs of iterates of c_{r+1} are converging to the graph of a constant function.

Let $p_0 \in \mathbb{P}^1$ be the parabolic periodic point of f_{s_ℓ}; replacing f_{s_ℓ} with an iterate, we have $f^n_{s_\ell}(c_{r+1}(s_\ell)) \to p_0$ as $n \to \infty$. After a change of coordinates if necessary, we may assume that

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

$c_{r+1}(s_\ell)$ and p_0 are not ∞. Recalling the definition of the bifurcation current associated to c_{r+1}, we can work in homogeneous coordinates on $\mathbb{P}^1$ over a neighborhood of s_ℓ in X_{λ_1}. We define

$$G_s(X,Y) := \lim_{i\to\infty} \frac{1}{d^i} \log \|F_s^i(X,Y)\|,$$

with a choice of homogeneous lift F_s for f_s normalized so that $G_{s_\ell}(p_0,1) = 0$. Note that $G_s(c_{r+1}(s),1)$ is a potential of the bifurcation current $T_{c_{r+1}}$ on a neighborhood of $s_\ell \in X_{\lambda_1}$. As long as $f_s^n(c_{r+1}(s))$ is not ∞, which is always true when s is in $\mathbb{D}(s_\ell, \delta/\rho^n)$ for small $\delta > 0$, any $\rho > 1$ and big n by Lemma 4.1, we define

$$U_n(s) := G_s(f_s^n(c_{r+1}(s)),1)$$

so that

$$dd^c U_n(s) = d^n \cdot dd^c G_s(c_{r+1}(s),1) = d^n\, T_{c_{r+1}}$$

where defined, as $G_s(F_s^n(c_{r+1}(s),1)) = d^n \cdot G_s(c_{r+1}(s),1)$ differs from $U_n(s)$ by a harmonic function.

Let r_0 be any biholomorphic map from $\mathbb{D}^r$ to a neighborhood of $s_\ell \in X_{\lambda_1}$, with $r_0(0) = s_\ell$. From Lemma 4.1 and the choice of our lift, we deduce that, for any $\rho > 1$, the functions $U_n \circ r_0(y/\rho^n)$ converge uniformly to $G_{s_\ell}(p_0,1) = 0$ on a small neighborhood of $(0,\dots,0) \in \mathbb{D}^r$. As a consequence, for any $\rho_i \in \mathbb{C}$ with $|\rho_i| > 1$, we can pick some $\rho > 1$ with $\rho < \min_{1\le i\le r}(|\rho_i|)$ so that the sequence of functions

$$U_n \circ r_0(y_1/\rho_1^n, \dots, y_r/\rho_r^n) = U_n \circ r_0(A_n(y)/\rho^n) \to 0 \tag{4.10}$$

uniformly, as $A_n(y) := (y_1 \cdot \rho^n/\rho_1^n, \dots, y_r \cdot \rho^n/\rho_r^n)$ converges to zero uniformly for $n \to \infty$.

Following the notation of [Ga2, Proposition 3.2], we now let $r_n : \mathbb{D}^r \to X_{\lambda_1}$ be the sequence of maps that define the zooms near the parameter s_ℓ for the measure $\mu_{\lambda_1}^{r+1}$ corresponding to the transversely prerepelling critical point at x_1, as described before Lemma 4.1. From [Ga2, Proposition 3.2] we know that there is a measure $\mu \neq 0$ (which can be described explicitly) such that

$$d^{nr}\, (r_n)^*(\mu_{\lambda_1}^{r+1}) \to \mu, \tag{4.11}$$

uniformly as $n \to \infty$. On the other hand, it follows from (4.10) that the functions $U_n \circ r_n$ also converge uniformly to 0. We conclude that $d^n\, (r_n)^* T_{c_{r+1}} \to 0$ weakly, and therefore

$$d^{nr}\, (r_n)^* \mu_{\lambda_1}^1 \to 0.$$

Therefore, $\mu_{\lambda_1}^1$ cannot be proportional to $\mu_{\lambda_1}^{r+1}$, contradicting (4.8). This completes the proof of Theorem 1.8.

Remark 4.2. The proof would simplify if we knew that the $(r+1)$-th critical point c_{r+1} were preperiodic to a parabolic cycle, instead of only in the basin of that cycle. We could employ ideas that appeared in [GNY, Proposition 5.2] (written for measures in the dynamical plane) or the proof of [FG3, Proposition 5.12] (for 1-parameter families of polynomials and marked points). But the critical point c_{r+1} is sometimes the only "free" critical point and must therefore have an infinite orbit in the presence of a parabolic cycle.

5. Proofs of Theorems 1.1, 1.2, 1.5, and 1.6

An elementary, but important, geometric observation is the following [GGK, Lemma 4.3]: If Σ is a subset of a complex projective algebraic variety X, equipped with an ample line bundle L, and if the product Σ^M lies in a proper subvariety $Y \subset X^M$ for some integer $M \ge 1$, then Σ must be contained in a proper subvariety Z of X, where the number of components of Z and the degree of each are bounded by

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

constants depending only on M, $\dim X$, the degree of X, and the sum of the degrees of the components of Y. Here, we measure degree in X^M with respect to the ample line bundle $p_1^*L \otimes \cdots \otimes p_M^*L$ on X^M, where $p_i : X^M \to X$ is the projection to the i-th factor.

Fix degree $d \geq 2$. For positive integers r and D, we will apply Theorem 1.8 to families $\mathcal{X} \to V$ over a subvariety $V \subset \mathrm{Ch}(\mathrm{M}_d, r, D)$, all defined over $\overline{\mathbb{Q}}$. We work inductively on the dimension r of the fibers in the family.

First suppose that $\mathcal{C} \to V$ is a family of curves in M_d of degree $\leq D$, with irreducible $V \subset \mathrm{Ch}(\mathrm{M}_d, 1, D)$ defined over $\overline{\mathbb{Q}}$ and of dimension $\ell \geq 1$, and assume that the generic fiber of $\mathcal{C} \to V$ is geometrically irreducible. From Theorem 1.8 applied to the fiber power $\mathcal{C}_V^{\ell+1}$, we know that there exists an $\varepsilon = \varepsilon(\mathcal{C})$ so that points $(\lambda, f_1, \ldots, f_{\ell+1}) \in \mathcal{C}_V^{\ell+1}(\overline{\mathbb{Q}})$ with $\sum_i \hat{h}_{\mathrm{crit}}(f_i) < \varepsilon \max\{1, h_{\mathrm{Ch}}(\lambda)\}$ are not Zariski dense. We let Z_ε denote the Zariski closure of this set in $\mathcal{C}_V^{\ell+1}$, over $\overline{\mathbb{Q}}$. It follows that there is a Zariski open $U \subset V$ (defined over $\overline{\mathbb{Q}}$) so that $Z_\varepsilon \cap C_\lambda^{\ell+1}$ is a proper algebraic subvariety of $(C_\lambda)^{\ell+1}$ for all $\lambda \in U(\mathbb{C})$, with its number of irreducible components bounded over $\lambda \in U(\mathbb{C})$. It is worth noting that all points $(\lambda, f_1, \ldots, f_{\ell+1}) \in \mathcal{C}_V^{\ell+1}(\mathbb{C})$ where each f_i is in $\mathrm{M}_d(\overline{\mathbb{Q}})$ with $\sum_i \hat{h}_{\mathrm{crit}}(f_i) < \varepsilon$ must lie in Z_ε, also for $\lambda \in V(\mathbb{C}) \setminus V(\overline{\mathbb{Q}})$; indeed, any such $(\ell+1)$-tuple must lie in a family of curves in V defined over $\overline{\mathbb{Q}}$ that includes this C_λ.

We now apply [GGK, Lemma 4.3] to the set

$$\Sigma_{\lambda,\varepsilon} := \left\{ f \in C_\lambda \cap \mathrm{M}_d(\overline{\mathbb{Q}}) : \hat{h}_{\mathrm{crit}}(f) < \frac{\varepsilon}{\ell+1} \max\{1, h_{\mathrm{Ch}}(\lambda)\} \right\} \subset C_\lambda$$

for $\lambda \in U(\overline{\mathbb{Q}})$, and to

$$\Sigma_{\lambda,\varepsilon} := \left\{ f \in C_\lambda \cap \mathrm{M}_d(\overline{\mathbb{Q}}) : \hat{h}_{\mathrm{crit}}(f) < \frac{\varepsilon}{\ell+1} \right\} \subset C_\lambda$$

for $\lambda \in U(\mathbb{C}) \setminus U(\overline{\mathbb{Q}})$, so that $\Sigma_{\lambda,\varepsilon}^{\ell+1} \subset Z_\varepsilon \cap C_\lambda^{\ell+1}$ for all $\lambda \in U(\mathbb{C})$. We deduce the existence of a constant N so that $|\Sigma_{\lambda,\varepsilon}| \leq N$ for all $\lambda \in U(\mathbb{C})$. Note here that we have used the fact that the number of components of $Z_\varepsilon \cap C_\lambda^{\ell+1}$ and the degree of each is uniformly bounded for $\lambda \in U(\mathbb{C})$ (controlled by the geometry of its generic fiber). Note also that if $0 \leq \varepsilon' < \varepsilon$, then $\Sigma_{\lambda,\varepsilon'} \subset \Sigma_{\lambda,\varepsilon}$ and hence $|\Sigma_{\lambda,\varepsilon'}| \leq N$ for each $\lambda \in U(\mathbb{C})$.

We then repeat the argument with the families of curves over the positive-dimensional components in the complement of $V \setminus U$; we work with irreducible components so that the generic fibers of these families will be (geometrically) irreducible. Note that each family we treat is defined over $\overline{\mathbb{Q}}$. We increase the bound N and decrease ϵ as needed. We are left with a finite set S in $V(\overline{\mathbb{Q}})$ containing all of the irreducible PCF-special curves with degree $\leq D$ as well as all curves in V with infinitely many points of critical height $< \frac{\varepsilon}{\ell+1} \max\{1, h_{\mathrm{Ch}}(\lambda)\}$ (or $< \frac{\varepsilon}{\ell+1}$ over complex parameters). This proves Theorems 1.2 and 1.6 for the curves in V. If for any of the curves C_λ in this leftover finite set $S \subset V(\overline{\mathbb{Q}})$, we have $N < |\Sigma_{\lambda,\varepsilon}| < \infty$, we just increase the bound N as needed so that Theorem 1.5 and 1.2 hold for all curves in V. We repeat the argument for all irreducible components V of $\mathrm{Ch}(\mathrm{M}_d, 1, D)$ with geometrically irreducible generic fiber and adjust the constants ε, N as needed. This proves Theorems 1.1, 1.2, 1.5 and 1.6 for curves.

Now fix $r > 1$ and suppose we know the conclusion of Theorems 1.1, 1.2, 1.5, and 1.6 for subvarieties in M_d of dimension $< r$. Consider any family $\mathcal{X} \to V$ of r-dimensional varieties in M_d over an irreducible and quasiprojective $V \subset \mathrm{Ch}(\mathrm{M}_d, r, D)$ defined over $\overline{\mathbb{Q}}$ and of dimension $\ell \geq 1$ for which the generic fiber of $\mathcal{X} \to V$ is geometrically irreducible.

As in the case of curves, we infer from Theorem 1.8 that there exists $\varepsilon = \varepsilon(\mathcal{X}) > 0$ so that all points $(\lambda, f_1, \ldots, f_{\ell+1}) \in \mathcal{X}_V^{\ell+1}(\overline{\mathbb{Q}})$ where each f_i is in $\mathrm{M}_d(\overline{\mathbb{Q}})$ with $\sum_i \hat{h}_{\mathrm{crit}}(f_i) < \varepsilon \max\{1, h_{\mathrm{Ch}}(\lambda)\}$ lie in a proper closed subvariety Z_ε in $\mathcal{X}_V^{\ell+1}$. Therefore, as before, from [GGK, Lemma 4.3], there is a Zariski-open subset $U \subset V$ (defined over $\overline{\mathbb{Q}}$) and constants N_1 and B_1 so that the points of

$$\Sigma_{\lambda,\varepsilon}^r := \left\{ f \in X_\lambda \cap \mathrm{M}_d(\overline{\mathbb{Q}}) : \hat{h}_{\mathrm{crit}}(f) < \frac{\varepsilon}{\ell+1} \max\{1, h_{\mathrm{Ch}}(f)\} \right\} \subset X_\lambda$$

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

for $\lambda \in U(\overline{\mathbb{Q}})$, and of

$$\Sigma^r_{\lambda,\varepsilon} := \left\{ f \in X_\lambda \cap \mathrm{M}_d(\overline{\mathbb{Q}}) : \hat{h}_{\mathrm{crit}}(f) < \frac{\varepsilon}{\ell+1} \right\} \subset X_\lambda$$

for $\lambda \in U(\mathbb{C}) \setminus U(\overline{\mathbb{Q}})$, lie in a proper algebraic subvariety $Z_{(\lambda)} \subset X_\lambda$ which has $\leq N_1$ irreducible components, each with degree $\leq B_1$, for all $\lambda \in U(\mathbb{C})$. Note that $\Sigma^r_{\lambda,\epsilon}$ is not necessarily Zariski dense in $Z_{(\lambda)}$, so we apply the induction hypotheses (specifically, the constants from each of the statements of Theorems 1.1, 1.2, 1.5, 1.6 for subvarieties of dimension $< r$) to each irreducible component $Y_{(\lambda)}$ of $Z_{(\lambda)}$, for all $\lambda \in U(\mathbb{C})$. Thus, by reducing ε if needed, we conclude that the Zariski-closure of $\Sigma^r_{\lambda,\varepsilon} \cap Y_{(\lambda)}$ has uniformly bounded number of components, each with uniformly bounded degree for all $\lambda \in U(\mathbb{C})$. Further, adjusting these uniform bounds if needed, using the induction hypothesis, we can arrange that they also bound the geometry of the Zariski closure of $\Sigma^r_{\lambda,\varepsilon'} \cap Y_{(\lambda)} \subset \Sigma^r_{\lambda,\varepsilon} \cap Y_{(\lambda)}$ for all $0 \leq \varepsilon' < \varepsilon$ (and in particular of $Y_{(\lambda)} \cap \mathrm{PCF}_d$). Combined with the initial bound of N_1 on the number of components of $Z_{(\lambda)}$, repeating this procedure provides uniform bounds on the Zariski closures of $\Sigma^r_{\lambda,\varepsilon'}$ for all $\varepsilon' < \varepsilon$ and of $\overline{X_\lambda \cap \mathrm{PCF}_d}$ for all $\lambda \in U(\mathbb{C})$.

We now repeat the argument over each of the irreducible components of the complement $V \setminus U$ corresponding to families with geometrically irreducible generic fibers, and we note that each family we treat is defined over $\overline{\mathbb{Q}}$. Finally, we are left with a finite set S in $V(\overline{\mathbb{Q}})$ consisting of r-dimensional varieties that are either PCF-special or contain a Zariski-dense set of points in $\Sigma^r_{\lambda,\varepsilon}$. We repeat the arguments above for curves, reducing ε and increasing B and N as needed, once again appealing to the induction hypotheses. We have thus completed the proofs of Theorems 1.1, 1.2, 1.5, and 1.6.

Acknowledgments. We would like to thank Dori Bejleri, Xavier Buff, Ziyang Gao, Thomas Gauthier, Zhuchao Ji, Lars Kühne, Harry Schmidt, Gabriel Vigny, and Junyi Xie for helpful discussions. We also thank the anonymous referees for their comments and suggestions. In addition, we acknowledge the Radcliffe Institute for Advanced Study at Harvard, the University of Toronto, and the Fields Institute.

Competing interest. The author has no competing interest to declare.

Financial support. Our research was supported in part by the National Science Foundation, by the Natural Sciences and Engineering Research Council of Canada, by National Key R&D Program of China (2021YFA1003202), and by NSFC (12131016, 12331004).

References

[BD] M. Baker and L. DeMarco, 'Special curves and postcritically-finite polynomials', *Forum Math. Pi* **1** (2013), 35 pages.
[BB] G. Bassanelli and F. Berteloot, 'Bifurcation currents in holomorphic dynamics on $\mathbb{P}^k$', *J. Reine Angew. Math.* **608** (2007), 201–235.
[BBD] F. Berteloot, F. Bianchi and C. Dupont, 'Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of $\mathbb{P}^k$', *Ann. Sci. Éc. Norm. Supér. (4)* **51** (2018), 215–262.
[BE] X. Buff and A. Epstein, 'Bifurcation measure and postcritically finite rational maps', In *Complex Dynamics*, pages 491–512. A K Peters, Wellesley, MA, 2009.
[CS] G. S. Call and J. H. Silverman, 'Canonical heights on varieties with morphisms', *Compositio Math.* **89** (1993), 163–205.
[DP] S. David and P. Philippon, 'Minorations des hauteurs normalisées des sous-variétés des puissances des courbes elliptiques', *Int. Math. Res. Pap. IMRP* **3** (2007), 113.
[De1] L. DeMarco, 'Dynamics of rational maps: a current on the bifurcation locus', *Math. Res. Lett.* **8** (2001), 57–66.
[De2] L. DeMarco, 'Bifurcations, intersections, and heights', *Algebra Number Theory.* **10** (2016), 1031–1056.
[DKY1] L. DeMarco, H. Krieger and H. Ye, 'Uniform Manin-Mumford for a family of genus 2 curves', *Ann. of Math. (2)* **191** (2020), 949–1001.
[DKY2] L. DeMarco, H. Krieger and H. Ye, 'Common preperiodic points for quadratic polynomials', *J. Mod. Dyn.* **18** (2022), 363–413.
[DM1] L. DeMarco and N. Myrto Mavraki, 'Dynamics on $\mathbb{P}^1$: preperiodic points and pairwise stability', *Compos. Math.* **160**(2) (2024), 356–387.
[DM2] L. DeMarco and N. Myrto Mavraki, 'Geometry of PCF parameters in spaces of quadratic polynomials', *Algebra Number Theory.* **19**(11) (2025), 2163–2183.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

[DM3] L. DeMarco and N. Myrto Mavraki, 'The geometry of preperiodic points in families of maps on $\mathbb{P}^N$', *Enseign. Math.* (2026). doi: https://doi.org/10.4171/LEM/1102.

[DWY] L. DeMarco, X. Wang and H. Ye, 'Bifurcation measures and quadratic rational maps', *Proc. London Math. Soc.* **111** (2015), 149–180.

[DGH] V. Dimitrov, Z. Gao and P. Habegger, 'Uniformity in Mordell-Lang for curves', *Ann. of Math. (2)* **194** (2021), 237–298.

[DGH2] V. Dimitrov, Z. Gao and P. Habegger, 'A consequence of the relative Bogomolov conjecture', *J. Number Theory.* **230** (2022), 146–160.

[DH1] A. Douady and J. Hamal Hubbard, 'On the dynamics of polynomial-like mappings', *Ann. Sci. École Norm. Sup. (4)* **18** (1985), 287–343.

[DH2] Adrien Douady and John H. Hubbard, 'A proof of Thurston's topological characterization of rational functions', *Acta Math.* **171** (1993), 263–297.

[Du] R. Dujardin, 'The supports of higher bifurcation currents', *Ann. Fac. Sci. Toulouse Math. (6)* **22** (2013), 445–464.

[DF] R. Dujardin and C. Favre, 'Distribution of rational maps with a preperiodic critical point', *Amer. J. Math.* **130** (2008), 979–1032.

[FG1] C. Favre and T. Gauthier, 'Distribution of postcritically finite polynomials', *Israel J. Math.* **209** (2015), 235–292.

[FG2] C. Favre and T. Gauthier, 'Classification of special curves in the space of cubic polynomials', *Int. Math. Res. Not. IMRN* **2** (2018), 362–411.

[FG3] C. Favre and T. Gauthier, *The Arithmetic of Polynomial Dynamical Pairs*, volume 214 of *Annals of Mathematics Studies* (Princeton University Press, Princeton, NJ, 2022).

[Gao] Z. Gao, 'Recent developments of the Uniform Mordell-Lang Conjecture', Preprint, arXiv:2104.03431v5 [math.NT].

[GGK] Z. Gao, T. Ge, and L. Kühne, 'The Uniform Mordell-Lang Conjecture', Preprint, arXiv:2105.15085v2 [math.NT].

[GH] Z. Gao and P. Habegger, The Relative Manin-Mumford Conjecture. Preprint, arXiv:2303.05045v2 [math.NT].

[Ga1] T. Gauthier, Strong bifurcation loci of full Hausdorff dimension. *Ann. Sci. Éc. Norm. Supér. (4)* **45**(2012), 947–984 (2013).

[Ga2] T. Gauthier, Dynamical pairs with an absolutely continuous bifurcation measure. *Ann. Fac. Sci. Toulouse Math. (6)* **32**(2023), no.2, 203–230.

[Ga3] T. Gauthier, 'Good height functions on quasiprojective varieties: equidistribution and applications in dynamics', *Ann. Fac. Sci. Toulouse Math.* **35** (2026), 57–94.

[GOV] T. Gauthier, Y. Okuyama and G. Vigny, Approximation of non-archimedean Lyapunov exponents and applications over global fields. *Trans. Amer. Math. Soc.* **373**(2020), 8963–9011.

[GTV] T. Gauthier, J. Taflin and G. Vigny, 'Sparsity of postcritically finite maps of $\mathbb{P}^k$ and beyond: a complex analytic approach', *Publ. Math. IHÉS* (to appear).

[GKZ] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications (Birkhäuser Boston Inc., Boston, MA, 1994).

[GHT] D. Ghioca, L.-C. Hsia and T. Tucker, 'Preperiodic points for families of polynomials', *Algebra Number Theory* **7** (2013), 701–732.

[GNY] D. Ghioca, K. D. Nguyen and H. Ye, 'The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for split rational maps', *J. Eur. Math. Soc. (JEMS)* **21** (2019), 1571–1594.

[GY] D. Ghioca and H. Ye, 'A dynamical variant of the André-Oort conjecture', *Int. Math. Res. Not. IMRN* **8** (2018), 2447–2480.

[In] P. Ingram, 'The critical height is a moduli height', *Duke Math. J.* **167** (2018), 1311–1346.

[JX] Z. Ji and J. Xie, 'DAO for curves', Preprint, arXiv:2302.02583v2 [math.DS].

[Ko] J. Kollár, *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* (Springer-Verlag, Berlin, 1996).

[Kü] L. Kühne, 'Equidistribution in families of abelian varieties and uniformity', Preprint, arXiv:2101.10272v3 [math.NT].

[MSS] R. Mañé, P. Sad and D. Sullivan, 'On the dynamics of rational maps', *Ann. Sci. Ec. Norm. Sup.* **16** (1983), 193–217.

[MaS] N. Myrto Mavraki and H. Schmidt, 'On the dynamical Bogomolov conjecture for families of split rational maps', *Duke Math. J.* **174** (2025), 803–856.

[Mc1] C. T. McMullen, 'Families of rational maps and iterative root-finding algorithms', *Ann. of Math. (2)* **125** (1987), 467–493.

[MeSc] A. Medvedev and T. Scanlon, 'Invariant varieties for polynomial dynamical systems', *Ann. of Math. (2)* **179** (2014), 81–177.

[MSW] N. Miasnikov, B. Stout and P. Williams, 'Automorphism loci for the moduli space of rational maps', *Acta Arith.* **180** (2017), 267–296.

[Mi1] J. Milnor, 'Geometry and dynamics of quadratic rational maps', *Experiment. Math.* 2 (1993), 37–83. With an appendix by the author and Lei Tan.

[Mi2] J. Milnor, 'On Lattès maps', in *Dynamics on the Riemann Sphere* (European Mathematical Society, Zürich, 2006), 9–43.

[Ph] P. Philippon, 'Sur des hauteurs alternatives. I', *Math. Ann.* **289** (1991), 255–283.

[Pi] R. Pink, 'A combination of the conjectures of Mordell-Lang and André-Oort', in *Geometric Methods in Algebra and Number Theory*, volume 235 of *Progr. Math.* (Birkhäuser Boston, Boston, MA, 2005), 251–282.

[Si1] J. H. Silverman, 'The space of rational maps on P^1', *Duke Math. J.* **94** (1998), 41–77.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press

[Si2] J. H. Silverman, *Moduli Spaces and Arithmetic Dynamics*, volume 30 of *CRM Monograph Series* (American Mathematical Society, Providence, RI, 2012).
[Ul] E. Ullmo, 'Positivité et discrétion des points algébriques des courbes', *Ann. of Math. (2)* **147** (1998), 167–179.
[We] L. West, 'The moduli space of cubic rational maps', Preprint, arXiv:1408.3247 [math.NT].
[Yu] X. Yuan, 'Arithmetic bigness and a uniform Bogomolov-type result', Preprint, arXiv:2108.05625v2 [math.NT].
[YZ] X. Yuan and S. Zhang, 'Adelic line bundles over quasiprojective varieties', Preprint, arxiv:2105.13587v6 [math.NT].
[Za] U. Zannier, *Some Problems of Unlikely Intersections in Arithmetic and Geometry*, volume 181 of *Annals of Mathematics Studies* (Princeton University Press, Princeton, NJ, 2012). With appendixes by David Masser.
[Zh1] S.-W. Zhang, 'Equidistribution of small points on abelian varieties', *Ann. of Math. (2)* **147** (1998), 159–165.
[Zh2] S.-W. Zhang, 'Distributions in algebraic dynamics', in *Surveys in Differential Geometry. Vol. X*, volume 10 of *Surv. Differ. Geom.* (International Press, Somerville, MA, 2006), 381–430.

https://doi.org/10.1017/fmp.2026.10024 Published online by Cambridge University Press