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Homogeneous orbit closures and applications

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In memory of Dan Rudolph

Abstract. We give new classes of examples of orbits of the diagonal group in the space of unit volume lattices in $\mathbb{R}^d$ for $d \geq 3$ with nice (homogeneous) orbit closures, as well as examples of orbits with explicitly computable but irregular orbit closures. We give Diophantine applications to the former; for instance, we show that, for all $\gamma, \delta \in \mathbb{R}$,

$$\liminf_{|n|\to\infty} |n| \langle n\sqrt[3]{2} - \gamma \rangle \langle n\sqrt[3]{4} - \delta \rangle = 0,$$

where $\langle c \rangle$ denotes the distance of a real number c to the integers.

1. *Introduction and results*

Let G be a Lie group and $\Gamma < G$ be a closed subgroup. The space $X = G/\Gamma$ is a *homogeneous space* on which G acts transitively by left multiplication. In homogeneous dynamics one studies the action of a closed subgroup, $H < G$, on X. One of the basic questions one can ask is to analyze orbit closures, $\overline{Hx}$, for various points $x \in X$. We will shortly restrict our discussion to a specific example, having number theoretic applications in mind, but for the meantime, let us make the following definitions.

Definition 1.1.

(1) An H-orbit Hx is *periodic* if Hx supports an H-invariant probability measure.

(2) An H-orbit Hx is *H-regular* if $\overline{Hx} = Lx$ for some closed subgroup $H < L < G$.

(3) An H-orbit is *H-regular of periodic type* if furthermore Lx is a periodic L-orbit.

A point x is said to be H-periodic, H-regular or H-regular of periodic type if the corresponding H-orbit Hx has this property.

A simple situation where every point is H-regular is given by the action of a closed subgroup $H < \mathbb{R}^d$ on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. It is well known that in this situation any point $x \in \mathbb{T}^d$ is H-regular of periodic type. Moreover, the commutativity of $\mathbb{R}^d$ implies that the group L that satisfies $Lx = \overline{Hx}$ does not depend on x. A much deeper theorem ensuring such regularity is the following fundamental result of Ratner (see [**Rat91b**, Theorems A and B]).

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THEOREM 1.2. (Ratner's orbit closure theorem) *Assume $\Gamma < G$ is a lattice and $H < G$ a closed subgroup generated by one-parameter unipotent subgroups of G. Then any point $x \in G/\Gamma$ is H-regular of periodic type.*

Apart from their considerable intrinsic interest, the study of orbit closures for group actions on homogeneous spaces has numerous applications to other areas of mathematics, notably to number theory and the theory of Diophantine approximations. For example, in the mid 1980s, Margulis established a long-standing conjecture of Oppenheim regarding values of indefinite quadratic forms by analyzing orbit closures for the action of the group preserving such an indefinite form on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ (see [**Mar89, Mar97**]).

In Margulis' proof of the Oppenheim conjecture, the acting group is generated by unipotent one-parameter groups. We shall discuss in this paper the opposite situation where the acting subgroup $H < G$ is diagonalizable. In fact we will confine our discussion to the specific setting of

$$X_d = G/\Gamma, \quad G = \mathrm{SL}_d(\mathbb{R}), \quad \Gamma = \mathrm{SL}_d(\mathbb{Z}), \quad d \geq 2.$$

We denote the image of $e \in G$ under the projection $G \to G/\Gamma$ by e_Γ. More generally, if $g \in G$, we write ge_Γ for the image of g under this projection. The space X_d is identified in a natural way with the space of unit-volume lattices in $\mathbb{R}^d$. Under this identification $ge_\Gamma \in X_d$ corresponds to the lattice spanned by the columns of the matrix g (hence e_Γ corresponds to the standard lattice $\mathbb{Z}^d$), and the action of G on G/Γ coincides with the action of G on the space of lattices induced from the action of G on $\mathbb{R}^d$. Unless stated otherwise, we shall view elements of $\mathbb{R}^d$ as column vectors. We let

$$A = \left\{ \mathrm{diag}(e^{t_1}, \ldots, e^{t_d}) : t_i \in \mathbb{R}, \sum_1^d t_i = 0 \right\} \tag{1.1}$$

denote the subgroup of G consisting of diagonal matrices with positive diagonal entries (the group A depends implicitly on d).

1.1. *Regular and irregular A-orbits in X_d.* It is well known that when $d = 2$ there are many irregular points for the A-action (though by ergodicity of the A-action, almost every $x \in X_2$ has a dense orbit under A, hence in particular is A-regular). Indeed, in this case there are points $x \in X_2$ such that the Hausdorff dimension of the orbit closure $\overline{Ax}$ is not an integer, including points with a bounded A-trajectory.

The situation is expected to change dramatically for $d \geq 3$. For example, for $d \geq 3$ we have the following conjecture essentially due to Cassels and Swinnerton-Dyer [**CSD55**], recast in dynamical terms by Margulis [**Mar97**].

CONJECTURE 1.3. *For $d \geq 3$ every bounded A-orbit in X_d is periodic.*

While this conjecture remains open, Einsiedler *et al* [**EKL06**] have shown that for $d \geq 3$, for any $x \in X_d$ with a bounded A-orbit, the orbit closure $\overline{Ax}$ has the same dimension as A. In contrast to the unipotent case, it is easy to see that even for $d \geq 3$ there are points in X_d with an irregular A-orbit. For example, take any point in X_2 whose orbit under the one-parameter diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$ is not A-regular, and let Λ' denote the

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corresponding lattice in $\mathbb{R}^2$. Then the point in X_3 corresponding to the lattice $\Lambda = \Lambda' \oplus \mathbb{Z}$ has an irregular A-orbit. It seems reasonable to expect that there should be some countable union of explicit proper subvarieties $V_i \subset G$ so that every $x \notin \bigcup_i V_i$ has a regular A-orbit (indeed, a dense A-orbit), but nailing down an explicit conjecture in this direction has proved to be somewhat tricky.

The aim of this paper is to exhibit new explicit examples of A-regular points of periodic type as well as explicit examples of irregular points. We then use the results to obtain non-trivial information on Diophantine approximations of algebraic numbers.

The following theorem gives an explicit construction of interesting A-regular points of periodic type (see §2.3 for definitions and terminology, e.g. of geometric embedding).

THEOREM 1.4. *Let K be a number field of degree $d \geq 3$ that is not a CM field†, and let $\varphi : K \to \mathbb{R}^d$ be a geometric embedding of K. Let $\Lambda \subset K$ be a lattice and $x_\Lambda \in X_d$ be the point corresponding to the lattice $\varphi(\Lambda)$ in $\mathbb{R}^d$ after normalizing its volume. Then x_Λ is A-regular of periodic type.*

Theorem 1.4 is a special case of Theorem 2.1, whose statement is deferred to the next section. When K is totally real (i.e. it has only real embeddings) the A-orbit of the point x_Λ is periodic, hence trivially A-regular of periodic type. Weiss and the first named author [**LW01**] have shown that any point $x \in X_d$ for which $\overline{Ax} \ni x_\Lambda$ (with x_Λ arising as above from a totally real field K, and $d \geq 3$) is A-regular of periodic type, and this theorem can also be used to construct non-obvious explicit A-regular points. Theorem 1.4 and the results of [**LW01**] imply that in fact if $\overline{Ax} \ni x_\Lambda$ then x is A-regular of periodic type whenever K satisfies the conditions of Theorem 1.4 (cf. Corollary 3.8).

In the other direction, in [**Sha**] the second named author established that there exist irregular A-orbits in X_3 not of the form outlined above after Conjecture 1.3. This is somewhat surprising, as it contradicts an influential conjecture regarding the orbit closure of a multidimensional diagonalizable group by Margulis [**Mar00**, Conjecture 1.1] (Maucourant [**Mau**] has already given a counterexample to this conjecture when, instead of taking the full diagonal group A, one takes a suitable multidimensional subgroup; we have learned while finalizing this text that Tomanov has also constructed interesting counterexamples somewhat similar to the class considered here for a different group G). The proof given in [**Sha**] was indirect. In §6 we further analyze these examples and give a full description of the orbit closures in these cases. Keeping notational introduction to the minimum, we state here the following theorem. A more accurate version in the form of Theorem 6.4 appears in §6. For a vector $v \in \mathbb{R}^{d-1}$, we let

$$h_v = \begin{pmatrix} 1 & 0 \\ v & I_{d-1} \end{pmatrix}, \quad g_v = \begin{pmatrix} 1 & v^{\mathrm{t}} \\ 0 & I_{d-1} \end{pmatrix}, \tag{1.2}$$

where I_{d-1} denotes the identity matrix of dimension $d-1$ and the 0s denote the corresponding trivial vectors. Let $x_v, z_v \in X_d$ denote the lattices spanned by the columns of h_v and g_v respectively.

† Recall that a number field K is said to be a CM field if it is a totally complex quadratic extension of a totally real field.

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THEOREM 1.5. *Let $v = (\alpha, \beta)^t \in \mathbb{R}^2$ be such that α, β are irrational and $1, \alpha, \beta$ linearly dependent over $\mathbb{Q}$. Then there exist two reductive groups $H^{(i)}$, $i = 1, 2$ (containing A), and two lattices $y_1, y_2 \in X_3$, such that the orbits $H^{(i)} y_i$ are closed and such that:*
(1) $\overline{Ax_v} \subset Ax_v \cup H^{(1)} y_1 \cup H^{(2)} y_2$;
(2) *the orbit Ax_v is disjoint from $H^{(i)} y_i$; and*
(3) $\overline{Ax_v} \cap H^{(i)} y_i \neq \emptyset$.
A corresponding statement for the lattice z_v holds (with different groups $H^{(i)}$). In particular, x_v, z_v are irregular for the A-action.

Remark 1.6. In fact, it is not hard to see that for (Lebesgue) almost any α, for any β as in Theorem 1.5, one actually has the equality

$$\overline{Ax_v} = Ax_v \cup H^{(1)} y_1 \cup H^{(2)} y_2.$$

1.2. *Diophantine approximations of algebraic vectors.* One of the main motivations that led to the results appearing in this paper was to interpret dynamically the work of Cassels and Swinnerton-Dyer, who proved in [**CSD55**] that if $\alpha, \beta \in \mathbb{R}$ are two algebraic numbers belonging to the same cubic number field, then they satisfy the following conjecture of Littlewood.

CONJECTURE 1.7. (Littlewood, *ca.* 1930) *For any pair of real numbers $\alpha, \beta \in \mathbb{R}$,*

$$\liminf_{|n| \to \infty} |n| \langle n\alpha \rangle \langle n\beta \rangle = 0. \tag{1.3}$$

In dynamical terms, the Cassels and Swinnerton-Dyer result amounts to showing that for $v = (\alpha, \beta)^t$, the orbit of x_v under an appropriate open semigroup of A is unbounded. We fully analyze the orbit closures in this and more general cases, and this stronger statement has further Diophantine implications. In order to state our results on Diophantine approximations, we give the following definition.

Definition 1.8. A vector $v \in \mathbb{R}^d$ is said to have property C (after Cassels) of the first type, if the following statement holds:

$$\text{for all } \vec{\gamma} \in \mathbb{R}^d, \quad \liminf_{|n| \to \infty} |n| \prod_1^d \langle n v_i - \gamma_i \rangle = 0. \tag{1.4}$$

It is said to have property C of the second type, if the following statement holds:

$$\text{for all } \gamma \in \mathbb{R}, \quad \liminf_{\vec{n} \in \mathbb{Z}^d, \prod |n_i| \to \infty} \left(\prod_1^d |n_i| \right) \left\langle \sum_1^d n_i v_i - \gamma \right\rangle = 0. \tag{1.5}$$

For $d = 1$, it was shown by Khinchine in the early 1920s that numbers $v \in \mathbb{R}$ with property C (the two notions of this property coincide in this case) do not exist (see [**Dav51**]). The question of whether in higher dimensions vectors with property C exist was open until recently. In [**Sha**], the second named author proved that almost any vector in $\mathbb{R}^d$ ($d \geq 2$) has property C of both types. Moreover, it was shown there that if $1, \alpha, \beta$ form a basis for a totally real cubic number field, then the vector $(\alpha, \beta)^t$ has property C of both types. We give the following more general result covering the case of non-totally real cubic fields and number field of higher degree.

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THEOREM 1.9. *Let* $1, \alpha_2, \ldots, \alpha_d \in \mathbb{R}$ *be a basis for a number field of degree* $d \geq 3$ *over* $\mathbb{Q}$. *Then the vector* $(\alpha_2, \ldots, \alpha_d)^{\mathrm{t}} \in \mathbb{R}^{d-1}$ *has property C of both types.*

Note that for the vector $(\alpha, \beta)^{\mathrm{t}}$ to have property C of the first type is a much stronger property than for it to satisfy Littlewood's conjecture. Note also that when α, β are linearly dependent over $\mathbb{Q}$, then α, β satisfy Littlewood's conjecture almost trivially, while the vector $(\alpha, \beta)^{\mathrm{t}}$ does not have property C of any type; see [**Sha**, Theorem 1.3]. In this respect, Theorem 1.9 is a strengthening of the aforementioned result of Cassels and Swinnerton-Dyer.

We shall use the following definition from [**Sha**].

Definition 1.10. A lattice $x \in X_d$ is said to be GDP†, if for any vector $w \in \mathbb{R}^d$, the set of products $\{\prod_1^d (u_i + w_i) : u \in x\}$ is dense in $\mathbb{R}$.

In [**Sha**] it is shown that if the lattice x_v (respectively z_v) is GDP, then v has property C of type 1 (respectively 2). Hence, Theorem 1.9 follows from the next two theorems.

THEOREM 1.11. *Let* $d \geq 3$ *and* $x \in X_d$ *be given. If* x *is A-regular of periodic type, then either* x *is A-periodic, or* x *is GDP.*

THEOREM 1.12. *Let* $v = (\alpha_2, \ldots, \alpha_d)^{\mathrm{t}} \in \mathbb{R}^{d-1}$ *be as in Theorem 1.9. Then* x_v, z_v *are A-regular of periodic type but not A-periodic.*

Theorem 1.11 is proved at the end of §3 and Theorem 1.12 is proved in §5.

2. *Lattices coming from number fields*

In this section we study in some detail the lattices coming from number fields, which are the subject of Theorems 1.4 and 2.1. We begin by fixing some of the notation that will accompany us through this paper. Throughout this section we fix $d \geq 2$ and $r, s \geq 0$ to be integers such that $d = r + 2s$.

2.1. *Maximal tori in G.* Given square matrices $B_1, \ldots, B_n$ of any dimensions, we denote by

$$\operatorname{diag}(B_1, \ldots, B_n)$$

the block diagonal square matrix formed by the B_is. For a complex number ω we let

$$R_\omega = \begin{pmatrix} \operatorname{Re}\omega & -\operatorname{Im}\omega \\ \operatorname{Im}\omega & \operatorname{Re}\omega \end{pmatrix}.$$

Let

$$T^{(r,s)} = \{\operatorname{diag}(a_1, \ldots, a_r, R_{\omega_1}, \ldots, R_{\omega_s}) \in G : a_i \in \mathbb{R}_+, \omega_i \in \mathbb{C}\}. \tag{2.1}$$

$T^{(r,s)}$ is the connected component of the identity of a maximal torus in G and any connected component of the identity in a maximal torus in G is conjugate to exactly one of the $T^{(r,s)}$. When $s = 0$ we denote $A = T^{(d,0)}$. The *split part* of $T^{(r,s)}$ is defined to be

$$A_{r,s} = A \cap T^{(r,s)}. \tag{2.2}$$

† GDP is an acronym for all Grids have Dense Products.

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Let $G_{\mathbb{C}} = \mathrm{SL}_d(\mathbb{C})$ and $A_{\mathbb{C}}$ be the group of diagonal matrices in $G_{\mathbb{C}}$. Set $B = 1/\sqrt{2}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$ and denote

$$\theta_{r,s} = \mathrm{diag}\Big(\underbrace{1, \dots, 1}_{r}, \underbrace{B, \dots, B}_{s}\Big) \in G_{\mathbb{C}}. \tag{2.3}$$

Denote conjugation by $\theta_{r,s}$ in $G_{\mathbb{C}}$ by $g \mapsto \tilde{g}$.

For a subgroup $H < G_{\mathbb{C}}$ we also denote

$$\tilde{H} = \theta_{r,s} H \theta_{r,s}^{-1}. \tag{2.4}$$

The reader can easily verify that $\tilde{T}^{(r,s)} \subset A_{\mathbb{C}}$ and more precisely that

$$g = \mathrm{diag}(a_1, \dots, a_r, R_{\omega_1}, \dots, R_{\omega_s}) \Rightarrow \tilde{g} = \mathrm{diag}(a_1, \dots, a_r, \omega_1, \bar{\omega}_1, \dots, \omega_s, \bar{\omega}_s). \tag{2.5}$$

Note that $g \mapsto \tilde{g}$ is the identity map on $A_{r,s}$.

For $1 \le i \ne j \le d$ let $\chi_{i,j} : A_{\mathbb{C}} \to \mathbb{C}^*$ be the character defined by

$$\chi_{ij}(\mathrm{diag}(a_1, \dots, a_d)) = \frac{a_i}{a_j}. \tag{2.6}$$

2.2. *Maximal parabolics.* For $1 \le k \le d-1$ let $a_k(t)$ denote the one-parameter subgroup of A given by

$$a_k(t) = \mathrm{diag}\Big(\underbrace{e^{(d-k)t}, \dots, e^{(d-k)t}}_{k}, \underbrace{e^{-kt}, \dots, e^{-kt}}_{d-k}\Big). \tag{2.7}$$

To the one-parameter group $a_k(t)$ we can attach two maximal parabolic subgroups of G, namely the weak-unstable and weak-stable horospherical subgroups of $a_k(1)$. More precisely, let

$$P_k^+ = \left\{ \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \in G \right\}, \quad P_k^- = \left\{ \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \in G \right\}, \tag{2.8}$$

where in the above equations B and D are square matrices of dimensions $k, d-k$ respectively and C and 0 are rectangular matrices of the obvious dimensions, 0 denoting here the matrix all of whose entries equal zero.

2.3. *Geometric embeddings.* Let K be a number field of degree d over $\mathbb{Q}$. We say that K *is of type* (r, s) if it has r distinct real embeddings $\sigma_i : K \to \mathbb{R}$, $i = 1, \dots, r$, and s non-conjugate complex embeddings $\sigma_i : K \to \mathbb{C}, i = r+1, \dots, r+s$. A *geometric embedding* of K in $\mathbb{R}^d$ is a map $\varphi : K \to \mathbb{R}^d$ whose coordinates are the real embeddings and the real and imaginary parts of the non-conjugate complex embeddings; i.e. up to a permutation of the coordinates it is the following map:

$$\varphi = \Big(\sigma_1, \dots, \sigma_r, \underbrace{\dots, \mathrm{Re}\, \sigma_{r+i}, \mathrm{Im}\, \sigma_{r+i}, \dots}_{2s}\Big). \tag{2.9}$$

We shall always work with geometric embeddings as in (2.9) and will not allow any permutation for ease of notation. Let $\alpha_1, \dots, \alpha_d$ be a basis of K over $\mathbb{Q}$. The $\mathbb{Z}$-module $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{\alpha_i\}$ is called a *lattice* in K. It is well known that the geometric embedding of Λ, $\varphi(\Lambda) \subset \mathbb{R}^d$, is a lattice in $\mathbb{R}^d$. Hence, by normalizing the covolume to be one,

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Λ defines a point in X_d, which we denote by x_Λ. We refer to such a lattice x_Λ as a *lattice coming from a number field of type* (r, s). We now have enough terminology to state one of the main results in this paper. Theorem 1.12 is a consequence of the following theorem, which generalizes Theorem 1.4 when $r > 0$. It is proved together with Theorem 1.4 in the next section.

THEOREM 2.1. *Let $x_\Lambda \in X_d$ be a lattice coming from a number field of type (r, s), let k be a number co-prime to d such that $\{a_k(t)\} < T^{(r,s)}$, and let $p \in P_k^+ \cup P_k^-$. Then px_Λ is A-regular of periodic type.*

2.4. *A few lemmas.* We now describe the connection between lattices x_Λ coming from number fields of type (r, s), and the tori $T^{(r,s)}$. We shall shortly prove that the orbit $T^{(r,s)}x_\Lambda$ is compact and homeomorphic to $\mathbb{T}^{d-1}$. Moreover, we shall analyze to some extent the closure of the orbit $A_{r,s}x_\Lambda$ in $T^{(r,s)}x_\Lambda$. We shall use hereafter the following notation: if a group H acts on a set X then, for $x \in X$, H_x denotes the stabilizer of x in H. As $T^{(r,s)}$ is isomorphic as a group to $\mathbb{R}^{r+s-1} \times \mathbb{T}^s$, we have the following basic lemma, which is left without proof.

LEMMA 2.2. *Let $x \in X_d$ be given. The orbit $T^{(r,s)}x$ is compact if and only if the stabilizer $T_x^{(r,s)}$ contains a free abelian group with $r + s - 1$ generators. Moreover, if $T^{(r,s)}x$ is compact then, for any closed connected subgroup $H < T^{(r,s)}$, $\overline{Hx} = Lx$ for some closed connected subgroup $H < L < T^{(r,s)}$.*

Let K be a number field of type (r, s) with geometric embedding φ as above. Let $\psi : K \to \mathrm{M}_d(\mathbb{R})$ be the map defined by (recall the notation of §2.1)

$$\psi(\alpha) = \mathrm{diag}(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), R_{\sigma_{r+1}(\alpha)}, \ldots, R_{\sigma_{r+s}(\alpha)}). \tag{2.10}$$

Observe that if we denote multiplication by α in K by m_α, then the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \mathbb{R}^d \\ {\scriptstyle m_\alpha}\downarrow & & \downarrow{\scriptstyle \psi(\alpha)} \\ K & \xrightarrow[\varphi]{} & \mathbb{R}^d \end{array} \tag{2.11}$$

The *associated order* of a lattice $\Lambda \subset K$ is defined to be

$$\mathcal{O}_\Lambda = \{\alpha \in K : m_\alpha(\Lambda) \subset \Lambda\}.$$

The reader should easily be able to see that $\mathcal{O}_\Lambda$ is a subring of K and that the group of units of this ring is given by $\mathcal{O}_\Lambda^* = \{\alpha \in K : m_\alpha(\Lambda) = \Lambda\}$. We denote

$$\mathcal{O}_{\Lambda,+}^* = \{\alpha \in \mathcal{O}_\Lambda^* : \sigma_i(\alpha) > 0, i = 1, \ldots, r\}. \tag{2.12}$$

It follows from (2.11) and (2.12) that $\mathcal{O}_{\Lambda,+}^*$ is embedded via ψ in the stabilizer of x_Λ in $T^{(r,s)}$ (note that the determinant of $\psi(\alpha)$ is equal to 1 for any $\alpha \in \mathcal{O}_{\Lambda,+}^*$). In fact it is not hard to verify that this embedding is onto; i.e.

$$T_{x_\Lambda}^{(r,s)} = \psi(\mathcal{O}_{\Lambda,+}^*). \tag{2.13}$$

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Here $\mathcal{O}^*_{\Lambda,+}$ is a subgroup of finite index in $\mathcal{O}^*_\Lambda$ and hence by Dirichlet's unit theorem

$$\mathcal{O}^*_{\Lambda,+} \simeq \mu \times \mathbb{Z}^{r+s-1}, \tag{2.14}$$

where μ is a finite group of roots of unity. Equations (2.13) and (2.14) together with Lemma 2.2 imply the following result.

LEMMA 2.3. *Let $x_\Lambda \in X_d$ be a lattice coming from a number field of type (r, s), then the orbit $T^{(r,s)}x_\Lambda$ is compact.*

In order to state the next lemma we introduce some more terminology. A subgroup $H < A_{\mathbb{C}}$ is an *equiblock diagonal group* if there are numbers d_1, d_2 such that $d = d_1 d_2$ and a partition of the indices $\{1, \dots, d\}$ into subsets I_ℓ, $\ell = 1, \dots, d_2$, of equal size d_1 such that, for any ℓ and any $i \neq j \in I_\ell$, $H < \mathrm{Ker}(\chi_{ij})$. In that case d_1 is referred to as the *size of the block*.

LEMMA 2.4. *Let $x_\Lambda \in X_d$ be a lattice coming from a number field K of type (r, s). Let $H < T^{(r,s)}$ be a subgroup such that Hx_Λ is closed. Suppose $\tilde{H} < \mathrm{Ker}(\chi_{ij})$ (see (2.4) for notation) for some $i \neq j$. Then $\tilde{H}_{x_\Lambda}$ is an equiblock diagonal group with block size $d_1 > 1$. Moreover, if $d = d_1 d_2$ then there is a subfield $K' \subset K$ of degree d_2 over $\mathbb{Q}$ such that $H_{x_\Lambda} \subset \psi(\mathcal{O}_{\Lambda,+} \cap K')$.*

Proof. From (2.5) and (2.13) it follows that

$$\tilde{T}^{(r,s)}_{x_\Lambda} = \left\{ \mathrm{diag}\Big(\sigma_1(\alpha), \dots, \sigma_r(\alpha), \underbrace{\dots, \sigma_{r+i}(\alpha), \bar{\sigma}_{r+i}(\alpha), \dots}_{s} \Big) : \alpha \in \mathcal{O}^*_{\Lambda,+} \right\}.$$

Hence, the assumption $\tilde{H} < \mathrm{Ker}(\chi_{ij})$ implies that there are two distinct embeddings of K, τ, η (corresponding to the ith and jth diagonal entries) such that if $\alpha \in \mathcal{O}^*_{\Lambda,+}$ satisfies $\tilde{\psi}(\alpha) \in \tilde{H}$ then $\tau(\alpha) = \eta(\alpha)$. Let $K' = \{\alpha \in K : \tau(\alpha) = \eta(\alpha)\}$. Note that K' is a proper subfield of K. Let $d_1 = \deg(K/K')$ and $d_2 = \deg(K'/\mathbb{Q})$. The different embeddings of K partition into d_2 sets of equal size such that, if τ' and η' belong to the same partition set, then their restrictions to K' coincide. The lemma now follows. □

The following two lemmas will be needed to complete the proofs of Theorems 1.4 and 2.1. Recall that a number field K is a CM field if K is of type $(0, s)$ and contains a totally real subfield of degree $s = d/2$ over $\mathbb{Q}$.

LEMMA 2.5. *Let $x_\Lambda \in X_d$ be a lattice coming from a number field K of type (r, s) with $s > 0$. Let $H < T^{(r,s)}$ be the subgroup satisfying $\overline{A_{r,s}x_\Lambda} = Hx_\Lambda$. Then the following are equivalent:*

(1) *H strictly contains $A_{r,s}$;*
(2) *$C_G(H) = T^{(r,s)}$; and*
(3) *K is not a CM field.*

Proof. We first show that (3) ⇒ (2). Suppose K is not CM. Then $C_G(H)$ is strictly larger than $T^{(r,s)}$ if and only if H does not contain any regular elements or, equivalently, there exist $1 \leq i \neq j \leq d$ such that $\tilde{H} < \mathrm{Ker}(\chi_{ij})$. Lemma 2.4 implies that in this case $\tilde{H}_{x_\Lambda}$ is an equiblock diagonal group with block size $d_1 > 1$. On the other hand $A_{r,s} = \tilde{A}_{r,s} < \tilde{H}$

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by definition. As the orbit Hx_Λ is compact we conclude that $\tilde{H}/\tilde{H}_{x_\Lambda}$ is compact too and in particular any element of $A_{r,s}$ can be brought to a compact set when multiplied by an appropriate element of $\tilde{H}_{x_\Lambda}$. It follows that $r = 0$, $d_1 = 2$, and that $\tilde{H}_{x_\Lambda} \subset A_{0,s}$. This implies that $A_{0,s}x_\Lambda$ is compact and that $H = A_{0,s}$ by definition. Lemma 2.4 also implies that there is a subfield $K' \subset K$ of degree $d/2 = s$ over $\mathbb{Q}$ such that

$$H_{x_\Lambda} \subset \psi(\mathcal{O}^*_{\Lambda,+} \cap K').$$

As $H = A_{0,s} \simeq \mathbb{R}^{s-1}$ and the quotient H/H_{x_Λ} is compact, we conclude that the rank of the group $\mathcal{O}^*_{\Lambda,+} \cap K'$ must be at least $s - 1$. On the other hand, it follows from Dirichlet's unit theorem that the rank of this group is bounded above by $s - 1$ and equality holds if and only if K' is totally real. This implies that K' is indeed totally real and that K is a quadratic totally complex extension of it, i.e. that K is a CM field, which contradicts our assumption.

The implication (2) $\Rightarrow$ (1) is obvious since we assume that $s > 0$ and so the centralizer of $A_{r,s}$ strictly contains $T^{(r,s)}$.

Finally, to see that (1) $\Rightarrow$ (3), assume that K is a CM field. It follows that $\mathrm{Stab}_{A_{0,s}}(x_\Lambda) = \psi(\mathcal{O}_{\Lambda,+})$. Dirichlet's unit theorem implies that $\mathrm{rank}(\mathcal{O}^*_{\Lambda,+}) = s - 1$, hence $A_{0,s}x_\Lambda$ is compact and by definition $H = A_{0,s}$. □

LEMMA 2.6. *Let $x_\Lambda \in X_d$ be a lattice coming from a number field K of type (r, s). Let k be a number co-prime to d such that $\{a_k(t)\} < A_{r,s}$, and let $H' < T^{(r,s)}$ be the subgroup satisfying $\overline{\{a_k(t)x_\Lambda\}}_{t\in\mathbb{R}} = H'x_\Lambda$. Then $C_G(H') = T^{(r,s)}$.*

Proof. First, $C_G(H')$ strictly contains $T^{(r,s)}$ if and only if $\tilde{H}' < \mathrm{Ker}(\chi_{ij})$ for some $1 \leq i \neq j \leq d$. If this is the case, then Lemma 2.4 implies that $\tilde{H}'_{x_\Lambda}$ is an equiblock group. As $\tilde{H}'/\tilde{H}'_{x_\Lambda}$ is compact, it follows that large elements of the one-parameter group $\{a_k(t)\}_{t\in\mathbb{R}} < \tilde{H}'$ can be brought to a compact set if multiplied by the appropriate elements of the equiblock diagonal group $\tilde{H}'_{x_\Lambda}$. This contradicts the assumption that k is co-prime to d. □

3. *Proofs of Theorems 1.4, 1.11, and 2.1.*

This section is organized as follows. We present below a strategy for proving that a point $x \in X_d$ is A-regular of periodic type, which is the subject of Theorems 1.4 and 2.1. The discussion culminates in Lemma 3.6 below, and then the theorems are derived. At the end of the section we deduce Theorem 1.11 from the results appearing in [**Sha**].

In [**LW01**], Weiss and the first named author proved the following theorem.

THEOREM 3.1. *Let $d \geq 3$ and $x \in X_d$ be such that $\overline{Ax}$ contains a compact A-orbit, then x is A-regular of periodic type.*

The following is a consequence of [**PR72**, Theorem 2.8].

THEOREM 3.2. *Let L be a reductive subgroup of G containing A, and let Lx be a periodic orbit of L in X_d. Then Lx contains a compact A-orbit.*

Ratner's measure classification theorem [**Rat91a**, Theorem 1] gives a classification of measures in X_d that are invariant under a one-parameter unipotent subgroup of G. We shall require the following variant, proved in the next section.

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THEOREM 3.3. *Let μ be an A-invariant and ergodic probability measure on X_d that is invariant under a one-parameter unipotent subgroup of G. Then, it is in fact an L-invariant probability measure supported on a single L-orbit in X_d, for some reductive group L containing A.*

The above three theorems suggest a scheme for proving that a point $x \in X_d$ is A-regular of periodic type. Namely, one should prove that $\overline{Ax}$ contains the support of an A-invariant probability measure that is invariant under a one-parameter unipotent subgroup of G. The proofs of Theorems 1.4 and 2.1 follow this scheme. To obtain an A-invariant measure in our arguments, we start with an initial probability measure ν, which is not A-invariant but is supported inside the orbit closure $\overline{Ax}$, we choose a Følner sequence $F_n \subset A$, and we define the averages

$$\mu_n = \frac{1}{|F_n|} \int_{F_n} a_* \nu \, da, \tag{3.1}$$

where $|F_n|$ denotes the Haar measure of F_n in A. Any weak* limit μ of the sequence μ_n will be an A-invariant measure on X_d. We face two problems.

(1) One needs to prove that μ is a probability measure (i.e. there is no escape of mass).
(2) One needs to prove that μ is invariant under a one-parameter unipotent subgroup of G.

The fact that enables us to overcome the above problems is the nature of the initial probability measure ν. We shall see in the course of the arguments that ν is chosen to be an H-invariant probability measure supported on an orbit $Hy \subset \overline{Ax}$, for some suitable choice of a point $y \in \overline{Ax}$ and a subgroup $H < G$ (having some additional properties). The tool that enables us to resolve problem (1) is the following theorem of Eskin *et al* (see [**EMS97**]).

THEOREM 3.4. *Let H be a reductive subgroup of G and let ν be an H-invariant measure supported on an orbit $Hy \subset X_d$. If the orbit $C_G(H)y$ of the centralizer of H in G is compact, then for any sequence $g_n \in G$, any weak* limit of $(g_n)_*\nu$ is a probability measure.*

The following lemma is needed for the resolution of problem (2). It shows us how to choose the Følner sets in (3.1) in order that μ will indeed be invariant under a unipotent one-parameter subgroup of G. The proof is postponed to the next section.

LEMMA 3.5. *Let $H < G$ be a closed connected subgroup not contained in A and let ν be an H-invariant probability measure. There exists an open cone C in A and a unipotent one-parameter subgroup $u(t)$ in G such that if the Følner sets F_n are contained in C, then any weak* limit of μ_n from (3.1) is $u(t)$-invariant.*

We summarize the above discussion in the form of the following lemma.

LEMMA 3.6. *Let $d \geq 3$ and $x \in X_d$ be given. The following imply that x is A-regular of periodic type: there exist a closed connected reductive subgroup H of G and a point $y \in \overline{Ax}$ with the following properties:*

(1) *H is not contained in A;*
(2) *$Hy \subset \overline{Ax}$ and Hy supports an H-invariant probability measure; and*
(3) *the orbit $C_G(H)y$ is compact.*

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Concluding the proof of Theorem 1.4. If $s = 0$, then Ax_Λ is compact by Lemma 2.3 and hence x_Λ is A-regular of periodic type. Assume that $s > 0$. We wish to use Lemma 3.6 with the following choices of y and H. Let $y = x_\Lambda$ and let $H < T^{(r,s)}$ be the closed subgroup defined by the equation $\overline{A_{r,s}x_\Lambda} = Hx_\Lambda$. As we assume that the number field is not CM, it follows from Lemma 2.5 that H is not contained in A. It is clear that H is connected and reductive, and that the H-orbit $Hx_\Lambda \subset \overline{Ax_\Lambda}$ supports an H-invariant probability measure. Lemma 2.5 implies that $C_G(H) = T^{(r,s)}$ and Lemma 2.3 implies that $C_G(H)x_\Lambda$ is compact. We see that the conditions of Lemma 3.6 are satisfied and the theorem follows. □

Concluding the proof of Theorem 2.1. Let $F = \overline{Apx_\Lambda}$. Observe that F contains

$$a_k(t)pa_k(-t)a_k(t)x_\Lambda.$$

Assume for example that $p \in P_k^-$. Note that as $t \to \infty$, the conjugation $a_k(t)pa_k(-t)$ approaches a limit $p' \in G$, while $a_k(t)x_\Lambda$ has as limit points any point in $H'x_\Lambda$, where $H' < T^{(r,s)}$ is defined as in Lemma 2.6. Denote $y = p'x_\Lambda$ and $H = p'H'p'^{-1}$. We see that H is connected and reductive, and that F contains the orbit Hy that supports an H-invariant probability measure. Moreover, Lemmas 2.6 and 2.3 imply that y has a compact orbit under the action of the centralizer

$$C_G(H) = p'C_G(H')p'^{-1} = p'T^{(r,s)}p'^{-1}. \tag{3.2}$$

The argument now splits into two possibilities. Assume that H is not contained in A. Then Lemma 3.6 applies and the theorem is proved. Assume, on the other hand, that $H < A$. It follows from (3.2) that $A = p'T^{(r,s)}p'^{-1}$, hence $s = 0$, $A = T^{(r,s)}$, and $p' \in N_G(A)$. We conclude that F contains the compact orbit $Ay = p'Ax_\Lambda$ (by Lemma 2.3). Theorem 3.1 applies and the theorem follows. □

Note that Theorems 3.1, 3.2 and 3.3 imply together the following characterization of A-regular points of periodic type in X_d ($d \geq 3$).

THEOREM 3.7. *Let $d \geq 3$ and $x \in X_d$. Then x is A-regular of periodic type if and only if $\overline{Ax}$ contains a compact A-orbit.*

COROLLARY 3.8. (Inheritance) *Let $d \geq 3$ and $x \in X_d$ be such that $\overline{Ax}$ contains a point y that is A-regular of periodic type. Then, x is A-regular of periodic type too.*

We end this section by deducing Theorem 1.11 from the results in [**Sha**].

Proof of Theorem 1.11. In [**Sha**, Theorem 4.5] it is stated that if $d \geq 3$ and $x \in X_d$ is such that $\overline{Ax}$ contains a compact A-orbit, then either Ax is compact, or x is GDP. It now follows from Theorem 3.7 that our assumption that x is A-regular of periodic type implies that $\overline{Ax}$ contains a compact A-orbit, and the theorem follows. □

4. *Proofs of Lemma 3.5 and Theorem 3.3*

4.1. *Preliminaries.* In order to present the proofs of Lemma 3.5 and Theorem 3.3, we need to introduce some terminology. We denote the Lie algebras of G and A by $\mathfrak{g}$ and $\mathfrak{a}$

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respectively. Here $\mathfrak{a}$ consists of traceless diagonal matrices. We have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}, \tag{4.1}$$

where the $\mathfrak{g}_{ij}$ are the one-dimensional common eigenspaces of Ad_a, $a \in A$; and $\mathfrak{g}_{ij}$ is referred to as a *root space*. Given a vector $X \in \mathfrak{g}$ we let $X = X_{\mathfrak{a}} + \sum_{i \neq j} X_{ij}$ denote its decomposition with respect to (4.1). We denote by log the inverse of the exponential map $\exp : \mathfrak{a} \to A$. Given a vector $v \in \mathfrak{a}$, the operator $\mathrm{Ad}_{\exp(v)}$ has $\mathfrak{g}_{ij}$ as a one-dimensional eigenspace and it acts on it by multiplication by $e^{\lambda_{ij}(v)}$, where $\lambda_{ij} : \mathfrak{a} \to \mathbb{R}$ is a linear functional called a *root*. Hence, we have the following identity for $v \in \mathfrak{a}$ and $X \in \mathfrak{g}$:

$$\mathrm{Ad}_{\exp(v)}(X) = X_{\mathfrak{a}} + \sum_{i \neq j} e^{\lambda_{ij}(v)} X_{ij}. \tag{4.2}$$

The reader should easily be able to see that if $L < G$ is a closed connected subgroup with Lie algebra $\mathfrak{l}$, then L is normalized by A if and only if

$$\mathfrak{l} = (\mathfrak{a} \cap \mathfrak{l}) \oplus \bigoplus_{(i,j) \in I} \mathfrak{g}_{ij}, \tag{4.3}$$

for some suitable choice of subset $I \subset \{(i, j) : 1 \leq i \neq j \leq d\}$.

We order the roots in the following way: we say that $\lambda_{ij} > \lambda_{k\ell}$ if $j - i > k - \ell$ or if $j - i = k - \ell$ and $i < k$. In this way the ordering is linear. We say that a root λ_{ij} is *positive* if $i < j$. We fix some norm $\| \cdot \|$ on $\mathfrak{g}$ and some metric $\mathrm{d}(\cdot, \cdot)$ on G (inducing the usual topology). In a metric space Y, we let $B_y^Y(\rho)$ denote the ball of radius ρ around y in Y. If $E, F \subset Y$, we let $\mathrm{d}(E, F)$ denote the distance between the sets E and F. Finally, for any diagonal matrix a (not necessarily traceless), we denote by $p_{\mathfrak{a}}(a)$ its projection to $\mathfrak{a}$, i.e.

$$p_{\mathfrak{a}}(a) = a - \left(\frac{\mathrm{Tr}(a)}{d}, \ldots, \frac{\mathrm{Tr}(a)}{d} \right). \tag{4.4}$$

4.2. *Proofs.*

LEMMA 4.1. *Let $H < G$ be a closed connected subgroup not contained in A. Let $\mathfrak{h}$ denote its Lie algebra. Then, there exist an open cone $\hat{C} < \mathfrak{a}$ and a nilpotent matrix $\mathbf{n} \in \mathfrak{g}$ such that, for any $\delta > 0$, there is a radius $R > 0$ such that, if $v \in \hat{C}$ has norm $> R$, then $\mathrm{d}(\mathbf{n}, \mathrm{Ad}_{\exp(v)}(\mathfrak{h})) < \delta$.*

Deducing Lemma 3.5 from Lemma 4.1. Let $C = \exp(\hat{C})$ and F_n be a Følner sequence in C. Let $\mu_n = (1/|F_n|) \int_{F_n} a_* \nu \, da$ be as in (3.1), and let μ be a weak* limit of the μ_n. We shall prove that μ is invariant under the one-parameter unipotent subgroup of G, given by $u(t) = \exp(t\mathbf{n})$. To prove this, let f be a continuous function with compact support on X_d. We need to show that the following equality holds for any $t \in \mathbb{R}$:

$$\int_{X_d} f(x) \, d\mu = \int_{X_d} f(u(t)x) \, d\mu. \tag{4.5}$$

We show this for $t = 1$ for example, and denote $u = u(1)$. For convenience, we further assume that $\|f\|_\infty \leq 1$. Given $\epsilon > 0$, we can find n_0 and R sufficiently large and $\rho > 0$ sufficiently small so that the following four conditions hold:

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(1) for any $n > n_0$ we have

$$\left| \int_{X_d} f(x)\, d\mu - \int_{X_d} f(x)\, d\mu_n \right| < \epsilon$$

and

$$\left| \int_{X_d} f(ux)\, d\mu - \int_{X_d} f(ux)\, d\mu_n \right| < \epsilon;$$

(2) for any $g \in G$ such that $\mathrm{d}(g, e) < \rho$, and for any $x \in X_d$, we have

$$|f(gx) - f(x)| < \epsilon;$$

(3) for any $a \in C$ with $\|\log(a)\| > R$, there exist $h_a \in H$ and $g_a \in B_\rho^G(e)$ such that

$$u = g_a a h_a a^{-1};$$

(4) finally, for any $n > n_0$ we have

$$\frac{|F_n \cap \exp(B_R^{\mathfrak{a}})|}{|F_n|} < \epsilon.$$

Condition (1) follows from the definition of weak* convergence; (2) follows from the fact that f is continuous and has compact support; (3) is a reformulation of the conclusion of Lemma 4.1; and (4) just follows (if n_0 is sufficiently large) from the fact that $|F_n| \to \infty$ while R is fixed. To conclude the proof we have the following series of estimates which imply (4.5) when taking ϵ to zero. We have marked the equalities and estimates below to indicate which of the above properties is used in each passage. We use the symbol $\alpha \sim_\epsilon \beta$ to denote that α and β are at most ϵ distance apart. The only unmarked equality is in the fifth line and the reason it holds is that, for $a \in A$, the measure $a_*\nu$ is aHa^{-1}-invariant. For $n > n_0$ and R as above, we have

$$\begin{aligned}
&\int_{X_d} f(ux)\, d\mu \overset{(1)}{\sim_\epsilon} \int_{X_d} f(ux)\, d\mu_n \\
&\quad \overset{\text{def}}{=} \frac{1}{|F_n|} \int_{F_n \cap \exp(B_R^{\mathfrak{a}})} \int_{X_d} f(ux)\, da_*\nu\, da + \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R^{\mathfrak{a}})} \int_{X_d} f(ux)\, da_*\nu\, da \\
&\quad \overset{(4)}{\sim_\epsilon} \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R^{\mathfrak{a}})} \int_{X_d} f(ux)\, da_*\nu\, da \\
&\quad \overset{(3)}{=} \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R^{\mathfrak{a}})} \int_{X_d} f(g_a a h_a a^{-1} x)\, da_*\nu\, da \\
&\quad = \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R)^{\mathfrak{a}}} \int_{X_d} f(g_a x)\, da_*\nu\, da \overset{(2)+(3)}{\sim_\epsilon} \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R)^{\mathfrak{a}}} \int_{X_d} f(x)\, da_*\nu\, da \\
&\quad \overset{(4)}{\sim_\epsilon} \frac{1}{|F_n|} \int_{F_n \cap \exp(B_R)^{\mathfrak{a}}} \int_{X_d} f(x)\, da_*\nu\, da + \frac{1}{|F_n|} \int_{F_n \setminus \exp(B_R)^{\mathfrak{a}}} \int_{X_d} f(x)\, da_*\nu\, da \\
&\quad \overset{\text{def}}{=} \int_{X_d} f(x)\, d\mu_n \overset{(1)}{\sim_\epsilon} \int_{X_d} f(x)\, d\mu.
\end{aligned}$$

□

Proof of Lemma 4.1. As we assume that H is not contained in A, we conclude that there is a root space (say of a positive root) $\mathfrak{g}_{ij}$ such that

$$\pi_{ij}(\mathfrak{h}) = \mathfrak{g}_{ij}. \tag{4.6}$$

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Assume that $\lambda_{i_0 j_0}$ is the maximal positive root for which (4.6) is satisfied with respect to the ordering of the roots described in the previous subsection. Let $X = X_{\mathfrak{a}} + \sum_{ij} X_{ij} \in \mathfrak{h}$ be such that $X_{i_0 j_0} \neq 0$. Let

$$v_0 = p_{\mathfrak{a}}\left(\operatorname{diag}\left(\underbrace{j_0,\, j_0-1,\, \ldots,\, 3,\, 2}_{j_0-1},\, \underbrace{0,\, \ldots,\, 0}_{d-j_0+1}\right)\right) \in \mathfrak{a}. \tag{4.7}$$

The reader should easily be able to verify that, for any $(i,\, j) \neq (i_0,\, j_0)$ such that $X_{ij} \neq 0$,

$$\lambda_{i_0 j_0}(v_0) \geq \lambda_{ij}(v_0) + 1. \tag{4.8}$$

It follows from continuity that we can choose an open cone, $\hat{C} \subset \mathfrak{a}$, containing the half line $\{t v_0\}_{t>0}$, such that for some $\epsilon > 0$ the following holds: for any $v \in \hat{C}$ of norm 1 and any $(i,\, j) \neq (i_0,\, j_0)$ such that $X_{ij} \neq 0$,

$$\lambda_{i_0 j_0}(v) \geq \lambda_{ij}(v) + \epsilon. \tag{4.9}$$

Now fix $\delta > 0$ and let $R > 0$ be given. Any vector in $\hat{C}$ of norm $> R$ is of the form tv for $v \in \hat{C}$ of norm 1 and $t > R$. We now estimate the distance between $\mathrm{Ad}_{\exp(tv)}(\mathfrak{h})$ and the nilpotent matrix $X_{i_0 j_0} \neq 0$. It follows from (4.2) and (4.9) that

$$\begin{aligned}
\|\mathrm{Ad}_{\exp(tv)}(e^{-\lambda_{i_0 j_0}(v)t} X) - X_{i_0 j_0}\| &= \left\| e^{-\lambda_{i_0 j_0}(v)t}\left(X_{\mathfrak{a}} + \sum_{ij} e^{\lambda_{ij}(v)t} X_{ij}\right) - X_{i_0 j_0} \right\| \\
&\leq e^{-\lambda_{i_0 j_0}(v)t}\|X_{\mathfrak{a}}\| + \left\| \sum_{(i,j)\neq(i_0,j_0)} e^{(\lambda_{ij}(v)-\lambda_{i_0 j_0}(v))t} X_{ij} \right\| \\
&\leq e^{-\lambda_{i_0 j_0}(v)t}\|X_{\mathfrak{a}}\| + \sum_{(i,j)\neq(i_0,j_0)} e^{-\epsilon t}\|X_{ij}\|.
\end{aligned}$$

As the last expression goes to zero when $R \to \infty$ (recall that $t > R$), the lemma follows because $e^{-\lambda_{i_0 j_0}(v)t} X \in \mathfrak{h}$. □

We now prove Theorem 3.3.

Proof of Theorem 3.3. Let L denote the identity component of the closed subgroup

$$\mathrm{Stab}_G(\mu) < G. \tag{4.10}$$

Let $\mathfrak{l}$ denote its Lie algebra. It follows from (4.3) that $\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{(i,j)\in I} \mathfrak{g}_{ij}$, where I is some subset of $\{(i,\, j) : 1 \leq i \neq j \leq d\}$. Our assumption that μ is invariant under a one-parameter unipotent subgroup implies that I is not trivial and in fact μ is invariant under a one-parameter unipotent group of the form $\{u_{i_0 j_0}(t)\} = \exp(\mathfrak{g}_{i_0 j_0})$. Choose an element $a \in A$ such that $\chi_{i_0 j_0}(a) > 1$. The expanding horospherical subgroup of G with respect to a is

$$G_a^+ = \exp\left(\bigoplus_{\{ij:\chi_{ij}(a)>1\}} \mathfrak{u}_{ij}\right).$$

It follows from [**EL10**, Theorem 7.6] that $\mathrm{h}_\mu(a) = \mathrm{h}_\mu(a,\, G_a^+)$, where $\mathrm{h}_\mu(a,\, G_a^+)$ is the so-called 'entropy contribution' of G_a^+. Also, from [**EL10**, Corollary 9.10] it follows that

$$\mathrm{h}_\mu(a,\, G_a^+) = \sum_{\{ij:\chi_{ij}(a)>0\}} \mathrm{h}_\mu(a,\, \{u_{ij}(t)\}). \tag{4.11}$$

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From [**EL10**, Theorem 7.9] we conclude that, as μ is $\{u_{i_0 j_0}(t)\}$-invariant, the summand $\mathrm{h}_\mu(a, \{u_{i_0 j_0}(t)\})$ on the right-hand side of (4.11) equals $\log|\det rmAd_a|_{\mathfrak{u}_{i_0 j_0}}| > 0$. Hence we deduce that $\mathrm{h}_\mu(a) > 0$. We can now apply [**EKL06**, Theorem 1.3] to conclude that μ is the L-invariant probability measure supported on a periodic L-orbit. Note that, although it is not stated explicitly in [**EKL06**, Theorem 1.3] that L is reductive, it is proved there that this is indeed the case (see also [**LW01**] for a full classification of the possible groups L that could arise in this way). □

Appealing to [**EKL06**, Theorem 1.3] is slightly artificial as the main difficulty in its proof is to use positivity of entropy to deduce invariance under a unipotent. Here we start with a measure that is already invariant under a unipotent. Moreover, [**EKL06**, Theorem 1.3] is only applicable for $d \geq 3$ (which is the case we are interested in), whereas Theorem 3.3 also holds for $d = 2$.

We sketch below an alternative argument, based directly on Ratner's measure classification theorem [**Rat91a**, Theorem 1] (a similar argument can be found in [**MT96**]).

Alternative proof of Theorem 3.3. Suppose that μ is an A-invariant and ergodic probability measure on X_d invariant under a one-parameter unipotent subgroup u_t. Since μ is also invariant under $au_t a^{-1}$ for any $a \in A$, and since the group of $g \in G$ preserving μ is closed, by going to the limit we may assume that $\{u_t\}$ is normalized by A.

Let $\mu = \int_\Xi \mu_\xi \, d\rho(\xi)$ be the ergodic decomposition of μ with respect to the action of u_t. Let L_ξ denote the connected component of identity of $\mathrm{stab}_G \mu_\xi$. By Ratner's measure classification theorem, for ρ-almost every ξ the measure μ_ξ is supported on a single periodic L_ξ-orbit. Since A normalizes $\{u_t\}$, the group A acts on the space of $\{u_t\}$-ergodic components Ξ, and moreover by the A-invariance of μ, for any $a \in A$ and $\xi \in \Xi$, it holds that

$$L_{a\xi} = aL_\xi a^{-1}.$$

By Poincaré recurrence for the action of A it is easy to deduce that L_ξ is almost surely normalized by A, and by ergodicity it follows that there is a connected group L so that $L_\xi = L$ ρ-almost everywhere (a.e.).

If L were not reductive, one can find an element $a \in A$ so that $\det \mathrm{Ad}(a)|_{\mathrm{Lie}\, L} < 1$, and since we have already shown that μ-a.e. $x \in X_d$ lies on a periodic L-orbit, it follows that in this case $a^n x \to \infty$ μ-a.e., in contradiction to Poincaré recurrence. Moreover, if Lx is periodic so is $[L, L]x$. Since $u_t \in [L, L]$, the natural measure on Lx will not be u_t ergodic unless $L = [L, L]$—in contradiction to the construction of L using the ergodic decomposition of μ. Therefore L is semisimple. Similarly Lx periodic for a semisimple L implies that Hx is closed for $H = N_G(L)^0$ (this can be deduced, for example, from Lemma 6.1 below).

Finally, $H = AL$ unless L fixes some vector in $\mathbb{R}^d$. But then, since Lx is periodic, for any $\ell \in L$ we have that $\{(\ell - 1)v : v \in x\}$ is a non-trivial proper additive subgroup of the lattice x, which for an appropriate choice of inner product in $\mathbb{R}^d$ is contained in the orthogonal complement to the subspace of $\mathbb{R}^d$ fixed by L. It follows that x intersects non-trivially an L-invariant proper subspace of $\mathbb{R}^d$, and since L is normalized by A one can find an element $a \in A$ contracting the subspace, hence again $a^n x \to \infty$; which cannot happen for a typical x.

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To conclude we have shown that μ is A- and L-invariant and supported on a single closed orbit of $H = AL$; hence this orbit must be periodic and we are done. □

5. *Application to Diophantine approximations*

The proof of Theorem 1.12, which is the subject of this section, is merely a simple application of Theorem 2.1.

Proof of Theorem 1.12. Let $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{1, \alpha_2, \dots, \alpha_d\}$ be the lattice spanned by the $\mathbb{Q}$-basis $\{1, \alpha_2, \dots, \alpha_d\}$ of the number field K. Let φ be a geometric embedding of K in $\mathbb{R}^d$ (as in (2.9)) and assume that the first embedding is chosen to be the identity. Let $x_\Lambda \in X_d$ be the lattice corresponding to Λ. Let

$$P_1^- = \left\{ \begin{pmatrix} b & 0 \\ \vec{c} & D \end{pmatrix} \in G \right\}$$

be as in (2.8) (i.e. $b \in \mathbb{R}$, $\vec{c} \in \mathbb{R}^{d-1}$ and $D \in \mathrm{GL}_{d-1}(\mathbb{R})$), and $a_1(t)$ be as in (2.7). The reader can easily verify that there exist a non-zero constant $c \in \mathbb{R}$ and $p \in P_1^-$ such that

$$\begin{pmatrix} 1 & v^{\mathrm{t}} \\ 0 & I_{d-1} \end{pmatrix} = cp \begin{pmatrix} \vdots & \vdots & & \vdots \\ \varphi(1) & \varphi(\alpha_2) & \cdots & \varphi(\alpha_d) \\ \vdots & \vdots & & \vdots \end{pmatrix}, \tag{5.1}$$

where the matrix on the right in the above equation has $\varphi(1)$ and $\varphi(\alpha_i)$ as its columns (here c is just the inverse of the determinant of the lattice $\varphi(\Lambda)$). It follows that $z_v = px_\Lambda$. Theorem 2.1 now implies that z_v is A-regular of periodic type (note that indeed the conditions of Theorem 2.1 are satisfied; i.e. 1 is co-prime to d and if K is of type (r, s), then $r \geq 1$ and $\{a_1(t)\} < T^{(r,s)}$). We now argue that x_v is A-regular of periodic type too.

Denote by x^* the dual lattice to a lattice $x \in X_d$ and the involution $g \mapsto (g^{-1})^{\mathrm{t}}$ on G by $g \mapsto g^*$. For any $g \in G$ and $x \in X_d$, $(gx)^* = g^*x^*$, and as e_Γ is self-dual we have that $ge_\Gamma = g^*e_\Gamma$. Hence $(z_{(-v)})^* = x_v$, and it follows that, if $L < G$ is such that $\overline{Az_{(-v)}} = Lz_{(-v)}$ is a finite-volume orbit, then

$$\overline{Ax_v} = \overline{Az^*_{(-v)}} = (\overline{(Az_{(-v)})})^* = (Lz_{(-v)})^* = L^*x_v. \tag{5.2}$$

Hence x_v is A-regular of periodic type. Note that in (5.2) we used the fact that $A^* = A$. In fact, it is not hard to see that $L^* = L$ too. □

6. *Examples of A-irregular points in X_3*

In this section we shall prove Theorem 1.5; indeed we prove the somewhat more precise Theorem 6.4. Recall that for a vector $v = (\alpha, \beta)^{\mathrm{t}} \in \mathbb{R}^2$ we denote by x_v and z_v the lattices in X_3 that are spanned by the columns of the matrices

$$h_v = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}, \quad g_v = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.1}$$

respectively. We first note that any statement about the A-orbit closure of the lattice x_v immediately implies a corresponding statement for the lattice z_{-v}. This is because

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(as in (5.2)) $\overline{Az_{-v}} = (\overline{Ax_v})^*$. Hence from now on we confine our discussion to lattices of the form x_v. Before we turn to state Theorem 6.4, we need to state some lemmas and introduce some notation. The following lemma is well known.

LEMMA 6.1. *Let* $\rho : \mathrm{SL}_d(\mathbb{R}) \to \mathrm{GL}(V)$ *be a* $\mathbb{Q}$*-representation,* $v_0 \in V$ *a rational vector, and* $H = \{g \in \mathrm{SL}_d(\mathbb{R}) : \rho(g)v_0 = v_0\}$. *Then the orbit* $H\,\mathrm{SL}_d(\mathbb{Z})$ *is closed in* $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$.

Let

$$H^{(1)} = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G \right\}, \quad H^{(2)} = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \in G \right\}. \tag{6.2}$$

For each i, the orbit of the group $H^{(i)}$ through the identity coset $e_\Gamma \in X_3$ is closed by Lemma 6.1. For example, for $i = 1$, one takes the appropriate exterior product of the adjoint representation of G on its Lie algebra, and v_0 a rational vector corresponding to the one-parameter subgroup $\{\mathrm{diag}(e^t, e^t, e^{-2t}) : t \in \mathbb{R}\}$ (as $H^{(1)}$ is the centralizer in G of this one-parameter subgroup, it is equal to the stabilizer of v_0). It now follows that, for any matrix $p \in \mathrm{SL}_3(\mathbb{Q})$, the orbits $H^{(i)}pe_\Gamma$ are closed in X_3 (this is done by considering the conjugations of $H^{(i)}$ by p^{-1}). For a positive integer q, let us consider the following closed orbits:

$$\begin{aligned} M_q^{(1)} &= H^{(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q^{-1} & 0 & 1 \end{pmatrix} e_\Gamma \\ &= \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 1/(q \det B) & 0 & 1/(\det B) \end{pmatrix} e_\Gamma \in X_3 : B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}; \end{aligned} \tag{6.3}$$

$$\begin{aligned} M_q^{(2)} &= H^{(2)} \begin{pmatrix} 1 & 0 & 1 \\ q^{-1} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} e_\Gamma \\ &= \left\{ \begin{pmatrix} a & 0 & b \\ 1/(q \det B) & 1/(\det B) & 0 \\ c & 0 & d \end{pmatrix} e_\Gamma \in X_3 : B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}. \end{aligned} \tag{6.4}$$

We will prove that the accumulation points of the orbit Ax_v belong to $M_q^{(i)}$, for certain q; hence we wish to have a convenient characterization of the lattices composing $M_q^{(i)}$. This characterization is given by the following simple lemma.

LEMMA 6.2. *A lattice* $x \in X_3$ *belongs to* $M_q^{(1)}$ *if and only if there exists* $a \in A$, $g \in G$, *and integers* ℓ_1, ℓ_2 *that generate* $\mathbb{Z}/q\mathbb{Z}$ *such that* $x = age_\Gamma$, *and* g *is of the following form:*

$$g = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ \ell_1/q & \ell_2/q & 1 \end{pmatrix}. \tag{6.5}$$

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A lattice $x \in X_3$ belongs to $M_q^{(2)}$ if and only if there exists $a \in A$, $g \in G$, and integers ℓ_1, ℓ_2, that generate $\mathbb{Z}/q\mathbb{Z}$ such that $x = age_\Gamma$, and g is of the following form:

$$g = \begin{pmatrix} * & 0 & * \\ \ell_1/q & 1 & \ell_2/q \\ * & 0 & * \end{pmatrix}. \tag{6.6}$$

Remark 6.3. It is a simple exercise to show that ℓ_1, ℓ_2 generate $\mathbb{Z}/q\mathbb{Z}$ if and only if there exists a matrix $(k_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$ such that $(\ell_1, \ell_2) = (1, 0)(k_{ij}) \bmod q$.

Proof. We shall prove the lemma for $M_q^{(1)}$, leaving the corresponding statement for $M_q^{(2)}$ to the reader. Let $x \in M_q^{(1)}$. It follows from (6.3) that, up to the action of A, $x = ge_\Gamma$, with g in the form of (6.5), with $\ell_1 = 1$, $\ell_2 = 0$. For the other direction, let $x = age_\Gamma$, where $a \in A$ and g is in the form of (6.5). By Remark 6.3, as ℓ_1, ℓ_2 generate $\mathbb{Z}/q\mathbb{Z}$, there exists a matrix $(k_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$ such that $(\ell_1, \ell_2)(k_{ij}) = (1, 0) \bmod q$, i.e. $k_{11}\ell_1 + k_{21}\ell_2 = n_1 q + 1$, and $k_{12}\ell_1 + k_{22}\ell_2 = n_2 q$, for some integers n_1, n_2. Let $\gamma \in \Gamma$ be the matrix

$$\gamma = \begin{pmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ -n_1 & -n_2 & 1 \end{pmatrix}.$$

Then a short calculation shows that the matrix $ag\gamma$, which represents the lattice x, is of the form given in (6.3), i.e. the lattice x belongs to $M^{(1)}$ as desired. □

Finally, let

$$a^{(1)}(t) = \mathrm{diag}(e^{-t}, e^t, 1), \quad a^{(2)}(t) = \mathrm{diag}(e^{-t}, 1, e^t).$$

THEOREM 6.4. *Let $v = (\alpha, \beta)^{\mathrm{t}} \in \mathbb{R}^2$ be such that α, β are irrational and $1, \alpha, \beta$ are linearly dependent over $\mathbb{Q}$. Suppose that*

$$\beta = \frac{p_1}{q}\alpha + \frac{p_2}{q}, \quad \alpha = \frac{p_1'}{q'}\beta + \frac{p_2'}{q'},$$

where these equations are written in reduced forms, i.e. q (respectively q') is a positive integer, and p_1, p_2 (respectively p_1', p_2') generate $\mathbb{Z}/q\mathbb{Z}$ (respectively $\mathbb{Z}/q'\mathbb{Z}$). Then the following hold:

(1) *the orbit Ax_v is disjoint from $M_q^{(1)} \cup M_{q'}^{(2)}$;*

(2) *if $a_n \in A$ is a sequence such that the distance from a_n to the two rays, $\bigcup_{i=1,2}\{a^{(i)}(t)\}_{t>0}$, goes to ∞, then the sequence $a_n x_v$ diverges (i.e. it has no converging subsequences in X_3);*

(3) *let $\Omega_i = \{\overline{\{a^{(i)}(t)x_v\}}_{t>0} \backslash \{a^{(i)}(t)x_v\}_{t>0}\}$, and write $A\Omega_i = \bigcup_{a\in A} a\Omega_i$; then*

$$\overline{Ax_v} \backslash Ax_v = A\Omega_1 \cup A\Omega_2 \quad \text{and} \quad A\Omega_i \neq \emptyset \quad \text{for } i = 1, 2;$$

(4) *finally, $A\Omega_1 \subset M_q^{(1)}$, and $A\Omega_2 \subset M_{q'}^{(2)}$.*

Proof. We first argue why part (1) of the theorem follows from the fact that both α and β are irrational. Working with (6.3), we see that it suffices to show that there is no $\gamma \in \Gamma$

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such that

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \gamma = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 1/(q \det B) & 0 & 1/(\det B) \end{pmatrix} \quad \text{where } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}). \tag{6.7}$$

An analogous statement can be verified when working with (6.4). In order to argue why there is no $\gamma \in \Gamma$ solving (6.7), note first that the rightmost column of γ must be of the form $(0, 0, \det B)^{\mathrm{t}}$. This implies that $\det B$ is an invertible integer, i.e. $\det B = \pm 1$; this follows because the determinant of γ equals $\det B$ times the determinant of the two-by-two upper left block of γ, which is also an integer. It now follows that, as β is irrational, the leftmost and middle columns of γ must be of the form $(0, *, *)^{\mathrm{t}}$. Hence the first row of γ equals zero, a contradiction.

We now prove part (2) of the theorem. Let $a_n = \mathrm{diag}(e^{-t_n - s_n}, e^{s_n}, e^{t_n}) \in A$ be a diverging sequence (i.e. $|t_n| + |s_n| \to \infty$) such that $a_n x_v \to x$, for some $x \in X_3$. Our goal is to show that the sequence $\min\{|t_n|, |s_n|\}$ is bounded. We will use the following fact about converging sequences of lattices: a converging sequence in X_3 has a positive lower bound on the lengths of the shortest non-zero vectors of its elements. We first argue that both t_n and s_n are bounded from below. This is because the lattice x_v contains the standard basis vectors e_2, e_3, and if for instance t_n is not bounded from below, then $a_n x_v$ contains the non-zero vector $a_n e_3$, which is arbitrarily short when t_n is negative and arbitrarily large in absolute value. Hence, we can assume that $t_n, s_n \geq 0$ (this is done by replacing the sequence a_n by a constant multiple of it, $a a_n$, if necessary). We now exclude the possibility of $\min\{t_n, s_n\}$ being unbounded from above. We use Dirichlet's theorem, which asserts that for any real number $\theta \in \mathbb{R}$ and any $T > 0$, there exist $k, m \in \mathbb{Z}$, with $0 < |k| \leq T$, such that

$$|k\theta + m| \leq \frac{1}{kT}.$$

Using this theorem, we wish to produce vectors in $a_n x_v$ that will be arbitrarily short, once $\min\{t_n, s_n\}$ is arbitrarily large. For a given n, choose

$$T = \begin{cases} e^{t_n + (s_n/2)} & \text{if } t_n \geq s_n, \\ e^{s_n + (t_n/2)} & \text{if } t_n < s_n, \end{cases} \tag{6.8}$$

and apply Dirichlet's theorem for α and T to conclude the existence of $k, m \in \mathbb{Z}$ with $0 < |k| < T$, and $|k\alpha + m| < 1/kT$. As $\beta = (p_1/q)\alpha + (p_2/q)$, this implies that

$$|qk\beta + (p_1 m - p_2 k)| = |p_1 k\alpha + p_2 k + p_1 m - p_2 k| \leq \frac{p_1}{kT}.$$

We conclude that there exists a vector in the lattice $a_n x_v$, of the form

$$a_n h_v \begin{pmatrix} qk \\ qm \\ p_1 m - p_2 k \end{pmatrix} = \begin{pmatrix} e^{-t_n - s_n} qk \\ e^{s_n}(qk\alpha + qm) \\ e^{t_n}(qk\beta + (p_1 m - p_2 k)) \end{pmatrix}, \tag{6.9}$$

which has

$$\text{length} \leq \max\left\{ e^{-t_n - s_n} qT, \frac{e^{s_n} q}{T}, \frac{e^{t_n} p_1}{T} \right\} \leq \max\{p_1, q\} e^{-\min\{s_n, t_n\}/2},$$

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where the last inequality follows from (6.8). This quantity is of course arbitrarily small once $\min\{s_n, t_n\}$ is not bounded from above, which concludes the proof of part (2) of the theorem.

We now prove part (4) of the theorem, postponing part (3) to the end. As $M_q^{(1)}$ (respectively $M_{q'}^{(2)}$) is A-invariant, it is enough to prove that $\Omega_1 \subset M_q^{(1)}$ (respectively $\Omega_2 \subset M_{q'}^{(2)}$). We shall prove that if $t_i \nearrow \infty$ is such that $x = \lim_i a^{(1)}(t_i)x_v$, then $x \in M_q^{(1)}$, leaving the analogous statement for $a^{(2)}(t)$ to the reader. Let

$$S = \left(\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \in G \right).$$

From Lemma 6.1, it follows that Se_Γ is a closed orbit, and as $\{a^{(1)}(t)\} < S$ we conclude that $x \in S\Gamma$. This means that x has a basis composing the columns of a matrix in S. It now follows from Lemma 6.2 that part (4) of the theorem will follow once we show the following claims.

CLAIM 1. *Any vector $w \in x$ is of the form $(*, *, \ell/q)^{\mathrm{t}}$, for some $\ell \in \mathbb{Z}$.*

CLAIM 2. *There exist two vectors $w_j = (*, *, \ell_j/q)^{\mathrm{t}} \in x$, $j = 1, 2$, where ℓ_1, ℓ_2 generate $\mathbb{Z}/q\mathbb{Z}$.*

To prove Claim 1, let $w_i \in a^{(1)}(t_i)x_v$ be a sequence of vectors converging to $w \in x$. There are sequences of integers k_i, m_i, n_i such that

$$w_i = a^{(1)}(t_i)h_v \begin{pmatrix} k_i \\ m_i \\ n_i \end{pmatrix} = \begin{pmatrix} e^{-t_i}k_i \\ e^{t_i}(k_i\alpha + m_i) \\ k_i\beta + n_i \end{pmatrix}. \tag{6.10}$$

As $t_i \nearrow \infty$, we conclude that $k_i\alpha + m_i \to 0$. In other words, the distance from $k_i\alpha$ to $\mathbb{Z}$, which we denote by $\langle k_i\alpha \rangle$, approaches zero. This implies that the distance from $k_i\beta = (p_1/q)k_i\alpha + (p_2/q)k_i$ to $(1/q)\mathbb{Z}$ approaches zero as well. Hence the third coordinate of w, which is the limit of $k_i\beta + n_i$, belongs to $(1/q)\mathbb{Z}$ as desired. In fact, a closer look shows that

$$k_i\beta = \frac{1}{q}(p_1m_i + p_2k_i) + \frac{\langle k_i\alpha \rangle}{q}.$$

This shows that in order to derive Claim 2 we need to find two families of sequences of integers $k_i^{(j)}$, $m_i^{(j)}$, $n_i^{(j)}$, $j = 1, 2$, such that the vectors $w_i^{(j)}$ as in (6.10) converge (maybe after passing to a subsequence), and such that there exist some pair $0 \le \ell_1, \ell_2 \le q - 1$, generating $\mathbb{Z}/q\mathbb{Z}$, such that, for any i,

$$\ell_j = p_1m_i^{(j)} + p_2k_i^{(j)} \mod q.$$

Note that the role of $n_i^{(j)}$ is not significant and these might be chosen so as to bring $k_i^{(j)}\beta$ to the unit interval.

To motivate the arguments we note the following. There is a natural projection from the periodic orbit Se_Γ (in which our discussion takes place) to the space of two-dimensional unimodular lattices X_2. This projection is defined by the following formula (as we will now mix dimensions two and three, we will denote hereafter $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{Z})$ by G_d

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and Γ_d respectively):

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \theta_1 & \theta_2 & 1 \end{pmatrix} e_{\Gamma_3} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_{\Gamma_2}.$$

We denote this projection by π. It follows that in our notation

$$\pi(x_v) = x_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} e_{\Gamma_2}.$$

If we denote $a(t) = \operatorname{diag}(e^{-t}, e^t) \in G_2$, then

$$\pi(a^{(1)}(t_i)x_v) = a(t_i)x_\alpha,$$

and from the continuity of π we deduce that $a(t_i)x_\alpha$ converges to $\pi(x)$.

Let $(k_i^{(j)}, m_i^{(j)})^{\mathrm{t}} \in \mathbb{Z}^2$, $j = 1, 2$, be chosen so that the vectors $\hat{w}_i^{(1)}, \hat{w}_i^{(2)} \in a(t_i)x_\alpha$, given by

$$\hat{w}_i^{(j)} = a(t_i) \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} k_i^{(j)} \\ m_i^{(j)} \end{pmatrix}, \tag{6.11}$$

are such that $\hat{w}_i^{(1)}$ is the shortest vector and $\hat{w}_i^{(2)}$ is the second shortest vector (not collinear with $w_i^{(1)}$) in the lattice $a(t_i)x_\alpha$. As the first and second shortest vectors in a two-dimensional lattice always form a basis, it follows that the matrix

$$\gamma = \begin{pmatrix} k_i^{(1)} & k_i^{(2)} \\ m_i^{(1)} & m_i^{(2)} \end{pmatrix}$$

has determinant ± 1. Also, as $a(t_i)x_\alpha$ converges, the lengths of the vectors $\hat{w}_i^{(1)}$ are bounded from below, which in turn implies that the lengths of the vectors $\hat{w}^{(2)}$ are bounded from above (this follows from the fact that the covolume of the lattice $a(t_i)x_\alpha$ is equal to 1). It follows that, possibly after passing to a subsequence, we may assume that both sequences $\hat{w}_i^{(j)}$ converge. By passing to another subsequence if necessary, we may assume that the residue classes $p_1 m_i^{(j)} + p_2 k_i^{(j)} \bmod q$, $j = 1, 2$, are fixed and equal ℓ_j, $j = 1, 2$. It follows from Remark 6.3 that the equality $(\ell_1, \ell_2) = (p_2, p_1)\gamma \bmod q$ forces ℓ_1, ℓ_2 to generate $\mathbb{Z}/q\mathbb{Z}$. We now choose $n_i^{(j)}$ in the manner specified above to define $w_i^{(j)}$ as in (6.10) and note that the third coordinate of the $w_i^{(j)}$ automatically converges to ℓ_j/q; hence the sequences $w_i^{(j)}$ converge to some limits $w_j = (*, *, \ell_j/q)^{\mathrm{t}} \in x$. The proof of part (4) follows.

The argument of part (3) of the theorem is divided into two parts. We first establish the equality $\overline{Ax_v} \backslash Ax_v = A\Omega_1 \cup A\Omega_2$, postponing the proof that the limit points form a non-empty set to the end. It is clear from parts (1) and (4), and the definition of Ω_i that $\overline{Ax_v} \backslash Ax_v \supset A\Omega_1 \cup A\Omega_2$. To establish the opposite inclusion, let $x \in \overline{Ax_v} \backslash Ax_v$ be given. Then, there is a divergent sequence $a_n \in A$, such that $a_n x_v \to x$. From the second part of the theorem it follows that, after passing to a subsequence, there is a sequence $t_n \to \infty$, such that $a_n = a_n' a^{(i)}(t_n)$, for $i = 1$ or $i = 2$, and $a_n' \to a \in A$. We conclude that $a^{(i)}(t_n)x_v$ must converge to some point in Ω_i (namely to $a^{-1}x$), and that $x = a \lim a^{(i)}(t_n)x_v \in A\Omega_i$.

Finally, in order to finish the proof of the theorem we need to argue why $\Omega_j \neq \emptyset$. Note that from parts (1) and (4) it follows that

$$\Omega_j = \{x \in X_3 : x = \lim a^{(j)}(t_i)x_v \text{ for some } t_i \nearrow \infty\}.$$

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Hence we only need to argue why the trajectories $\{a^{(j)}(t)x_v\}_{t\geq 0}$ are not divergent. We argue this for $j = 1$ for example. Using the notation of the proof of part (4), it is not hard to see that the projection $\pi : Se_{\Gamma_3} \to X_2$ is proper. Hence, if the trajectory $\{a^{(1)}(t)x_v\}_{t\geq 0}$ is divergent, then so is its projection under π in X_2, i.e. the trajectory $\{a(t)x_\alpha\}_{t\geq 0}$. It is well known that a geodesic ray in the upper half plane projects to a divergent geodesic ray in the modular surface if and only if its end point is rational. Since α is the end point here and it is irrational, a contradiction emerges. A slightly different argument showing that the orbit Ax_v is not closed could be derived from Margulis' classification of divergent A-orbits given in the appendix to [**TW03**]. □

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