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# BIVARIATE MARSHALL–OLKIN EXPONENTIAL SHOCK MODEL

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The well-known Marshall–Olkin model is known for its extension of exponential distribution preserving lack of memory property. Based on shock models, a new generalization of the bivariate Marshall–Olkin exponential distribution is given. The proposed model allows wider range tail dependence which is appealing in modeling risky events. Moreover, a stochastic comparison according to this shock model and also some properties, such as association measures, tail dependence and Kendall distribution, are presented. The new shock model is analytically quite tractable, and it can be used quite effectively, to analyze discrete–continuous data. This has been shown on real data. Finally, we propose the multivariate extension of the Marshall–Olkin model that has some intersection with the well-known multivariate Archimax copulas.

**Keywords:** distortion copula, Marshall–Olkin model, shock model, stochastic comparison

## 1. INTRODUCTION

Following the seminal paper of Marshall and Olkin [18], a vector $(X_1, X_2)$ has the Marshall–Olkin (MO) distribution whenever

$$(X_1, X_2) = (\min\{W_1, W_3\}, \min\{W_2, W_3\}), \tag{1}$$

where $W_1 \sim E(\lambda_1)$, $W_2 \sim E(\lambda_2)$ and $W_3 \sim E(\lambda_3)$ are independent exponentially random variables. The survival function of (1) is given by the following equation:

$$\bar{F}_{\mathrm{MO}}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max\{x_1, x_2\}\}, \quad x_1, x_2, \lambda_1, \lambda_2, \lambda_3 > 0. \tag{2}$$

Starting with the common stochastic representation (1), many bivariate distributions and extensions of the MO model have been constructed and studied in the literature by supposing that the shocks ($W_i$'s) follow a specific distributions or fail to be independent (see, e.g., [6]). On the other hand, model (2) supports positive dependence and only justifies upper tail dependence. Also, this model has an intuitive interpretation in terms of shock models. The random variables $W_1$ and $W_2$ in (1) can be considered as the arrival times of individual shocks for two different components, while $W_3$ represents the arrival time of a shock which is common to both components. The MO model has been the interest of many researchers. For instance, Arnold [3] presented a multivariate exponential MO distribution

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based on hierarchical successive damage, Lindskog and McNeil [17] derived the multivariate MO model from the common Poisson shock model with applications to insurance and credit risk, Elouerkhaoui [9] introduced a methodology for pricing and hedging portfolio credit derivatives in a dynamic credit model using a single-name MO framework, Meintanis [19] gave a test of fit for MO distributions, Sarhan and Balakrishnan [23] stated an MO model type based on a latent variable using the exponential distribution, Al-Khedhairi and El-Gohary [2] proposed a Gompertz–MO model and derived its properties, Kundu and Dey [12] estimated the parameters of a bivariate Weibull–MO distribution via the EM algorithm, Mu and Wang [20] found a multivariate exponential distribution of MO type and estimated its parameters using the EM algorithm, Kundu and Gupta [13] gave the bivariate Sarhan–Balakrishnan distribution which is of MO type with singular component and explained a procedure for estimating its parameters, Li and Pellerey [15] used a generalized form of the MO model and derived its bivariate ageing properties, Li et al. [16] analyzed some bivariate failure time data with the Weibull–MO model and discussed estimation methods, Gupta et al. [11] obtained some reliability characteristics based on the MO model, Nagaraja and He [21] presented the explicit expression for the elements of Fisher's information matrix based on the MO model for censored data. See also Cherubini et al. [6] and references therein for recent contributions. Based on the MO shock model construction, we propose another shock model that extends the bivariate MO model assuming dependence of the random variables. Moreover, the corresponding copula function of this new model has some intersection with Archimax copulas (see [22]). Also, the proposed model supports upper tail dependence as the well-known MO model.

This paper is organized as follows. In Section 2, the proposed model is constructed with an interpretation applicable to reliability of systems. Based on the given construction, a stochastic comparison is made. Different sub-families of this model are given that shows the wide range of flexibility for this model. In Section 3, we investigate some dependence properties of the proposed model. Section 4 gives an algorithm to generate random numbers with several examples specifying the impact of distortion on the MO model dependence structure. Also, a performance analysis based on maximum likelihood estimation is given. A real data example in engineering is presented in Section 5. In the last section, we present the multivariate transformed MO model. Based on the construction of the new shock model, we present the multivariate version of the presented shock model followed by some multivariate properties.

## 2. THE PROPOSED MODEL

The new model is constructed by taking the component-wise minima from the sequence $\{(X_i^{(1)}, X_i^{(2)})|i \geq 1\}$ distributed from the MO model in (2) that is independent of the discrete random variable $N \in \{1, 2, \ldots\}$. This idea, where $N$ has a geometric distribution and $(X_i^{(1)}, X_i^{(2)})$ follows a given bivariate continuous distribution, originally appeared in Marshall and Olkin [18] as a method for adding new parameters to a given distribution. In this paper, we consider $N$ as a general integer-valued random variables and the pairs $(X_i^{(1)}, X_i^{(2)})$, $i = 1, \ldots, N$ from an MO model. The following example gives us a motivation for choosing the MO model and using this construction.

Example 1: *In a multiprocessing system, consider $N$ independent personal computers (PC) with two processing units (PUs) of type 1 and 2, within each computer. Let $X_i^{(1)}$ and $X_i^{(2)}$ be the lifetimes of the PUs of ith PC, $i = 1, \ldots, N$. These two sub-parts are*

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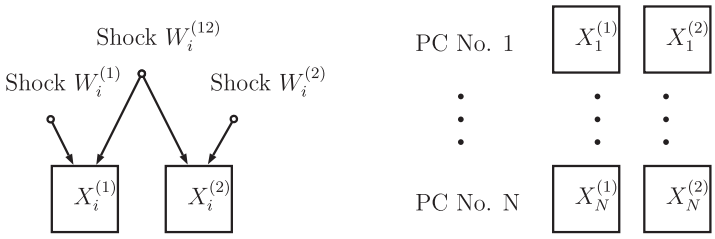



FIGURE **1.** (Left panel) The two PUs of $i$th PC that are in danger from three types of shocks. (Right panel) Random number of $N$ PCs.

*in danger from three kinds of shocks (Figure 1) that are assumed to be distributed as a Poisson process. Therefore, the times of strikes* $W_i^{(1)}$, $W_i^{(2)}$ *and* $W_i^{(12)}$ *for each PC are three independent exponential random variables. Consequently, the pairs* $(X_i^{(1)}, X_i^{(2)}) = (\min\{W_i^{(1)}, W_i^{(12)}\}, \min\{W_i^{(2)}, W_i^{(12)}\})$, $i = 1, \ldots, N$, *is a sample of random size from an MO model. The survival function of the component-wise minima, that is,*

$$\mathbf{J} = (J_1, J_2) = (\min\{X_1^{(1)}, \ldots, X_N^{(1)}\}, \min\{X_1^{(2)}, \ldots, X_N^{(2)}\}) \tag{3}$$

*represents the lifetimes of two types of PUs.*

The following theorem provides the survival function of (3).

PROPOSITION 1: *Let $N$ be a discrete random variable taking values $N = 1, 2, \ldots$ with probability generating function $E[t^N] = g^{-1}(t)$, where $g$ is a continuous increasing and concave function such that $g(0) = 0$ and $g(1) = 1$. Consider the sequence $\{(X_i^{(1)}, X_i^{(2)}),\ i \geq 1\}$ (independent of $N$) of i.i.d. random vectors with survival function (2). If $(J_1, J_2)$ are defined as in (3), then we have*

$$\bar{F}_{J_1,J_2}(t, s) = g^{-1}(\exp\{-\lambda_1 t - \lambda_2 s - \lambda_3 \max(t, s)\}). \tag{4}$$

PROOF: For any discrete random variable $N$ with values $N = 1, 2, \ldots$, we have

$$\begin{aligned}\bar{F}_{\mathbf{J}}(t, s) &= \sum_{n=1}^{\infty} (\bar{F}_{\mathrm{MO}}(t, s))^n P(N = n) \\ &= g^{-1}(\exp\{-\lambda_1 t - \lambda_2 s - \lambda_3 \max(t, s)\}).\end{aligned}$$

Hence, the proof is completed. ■

According to Sklar's theorem (see [22]), for a given joint survival function $\bar{F}$ with the marginals $\bar{F}_1$ and $\bar{F}_2$, there exists a survival copula $\hat{C}$ such that $\bar{F}(x, y) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y))$. The survival copula of the MO model in (2) (denoted by $\mathrm{MO}(\alpha, \beta)$) is given by $\hat{C}_{\alpha,\beta}(u, v) = \min(u^{\alpha} v, u v^{\beta})$, where $\alpha = \lambda_1/(\lambda_1 + \lambda_3)$ and $\beta = \lambda_2/(\lambda_2 + \lambda_3)$. Evidently, the survival copula of (3) is obtained by

$$\hat{C}_{\mathbf{J}}(t, s) = \min\{g^{-1}(g^{\alpha}(t) g(s)), g^{-1}(g(t) g^{\beta}(s))\}, \quad \alpha, \beta \in (0, 1). \tag{5}$$

The survival copula in (5) is an extension of the MO copula and is named the transformation of the MO survival copula denoted by $\mathrm{TMO}(\alpha, \beta; g)$. It should be noted that the TMO model

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in (5) is based on the fact that $(X_i^{(1)}, X_i^{(2)})$ follows from an MO model that is constructed by exponential shocks ($W_i^{(1)}$, $W_i^{(2)}$ and $W_i^{(12)}$, $i = 1, \ldots, N$).

The following remark shows some sub-classes of the extended MO model in (5).

REMARK 1: *Some new sub-classes of* $\mathrm{TMO}(\alpha, \beta; g)$ *are given as follows:*

- *Set* $\theta := \alpha = \beta$. *Let* $P(N = 1) = 1$, *then* $g^{-1}(t) = t$ *and so* $g(t) = t$. *In this case, we get the Cuadras–Augé model [7] as*

$$\hat{C}_{\mathbf{J}}(t, s) = \min\{t^{\theta} s, t s^{\theta}\}, \quad \theta \in (0, 1).$$

- *If* $P(N = 1) = 1$, *we get* $g^{-1}(t) = t$. *In this case, we get the general MO model given in (2).*
- *Let* $\psi$ *be the Laplace transformation corresponding to the distribution function* $\Lambda(\theta)$ *as* $\psi(t) = \int_0^{\infty} e^{-\theta t} \mathrm{d}\Lambda(\theta)$. *The Archimedean copula generator* $\phi$ *is actually the inverse of Laplace transformation* $\psi$. *The generator* $\phi : (0, 1) \to [0, \infty]$ *is a continuous, strictly decreasing and convex function with* $\phi(1) = 0$. *Suppose we have a discrete distribution having the support* $N = 1, 2, \ldots$ *with probability generating function* $g^{-1}(t) = E(t^N) = \psi(-\ln(t)) = \phi^{-1}(-\ln(t))$. *So,* $g(t) = e^{-\phi(t)}$ *and we get the well-known Archimedean family of copulas corresponding to the generator* $\phi$ *as*

$$\hat{C}_{\mathbf{J}}(t, s) = \phi^{[-1]}(\phi(t) + \phi(s)).$$

  *For more information, see Nelsen [22, pp. 109–112].*
- *Consider a discrete valued distribution with support* $N = 1, 2, \ldots$ *with probability generating function* $g^{-1}(t) = E(t^N)$. *If* $\alpha = \beta$, *then we have a new model that is an extension of the Cuadras–Augé model denoted by* $\mathrm{ECA}(\alpha; g)$ *with the form*

$$\hat{C}_{\mathbf{J}}(t, s) = \min\{g^{-1}(g^{\alpha}(t) g(s)), g^{-1}(g(t) g^{\alpha}(s))\}, \quad \alpha \in (0, 1).$$

- *Say we have a discrete valued distribution with support* $N = 1, 2, \ldots$ *and probability generating function* $g^{-1}(t) = E(t^N) = \phi^{-1}(-\ln(t))$. *Thus,* $g(t) = e^{-\phi(t)}$ *and we get a new model from the Archimax family Charpentier et al. [5] of copulas with the equation*

$$\hat{C}_{\mathbf{J}}(t, s) = \min\{\phi^{-1}(\alpha\phi(t) + \phi(s)), \phi^{-1}(\phi(t) + \beta\phi(s))\}.$$

- *Let* $N$ *be distributed from the Zero-truncated Poisson model with parameter* $r > 0$ *with probability mass function*

$$P(N = n) = \frac{r^n e^{-r}}{n!(1 - e^{-r})}, \quad n = 1, 2, \ldots, r > 0.$$

  *Then, its corresponding probability generating function is* $E(t^N) = g^{-1}(t) = (e^{rt} - 1)/(e^r - 1)$. *In this case, we have a new model named Poisson–MO model given as*

$$\hat{C}_{\mathbf{J}}(t, s) = \min\left\{\frac{1}{\gamma}[(\gamma s + 1) e^{(1/r^{\alpha})(\ln(\gamma t + 1))^{\alpha}} - 1], \frac{1}{\gamma}[(\gamma t + 1) e^{(1/r^{\beta})(\ln(\gamma s + 1))^{\beta}} - 1]\right\}, \tag{6}$$

  *where* $\gamma = e^r - 1$.

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- *Let $N$ be distributed from the Logarithmic series model with parameter $r \in (0,1)$ with probability mass function*

$$P(N=n) = \frac{-r^n}{n\ln(1-r)}, \quad n = 1,2,\ldots, r \in (0,1).$$

*The associated probability generating function of $N$ is $E(t^N) = g^{-1}(t) = \ln(1+rt)/\ln(1+r)$. Then, the survival copula of a new model named Logarithmic–MO is*

$$\hat{C}_{\mathbf{J}}(t,s) = \min\left\{\frac{1}{\eta}\ln\left(1+\frac{1}{r^\alpha}(e^{\eta t}-1)^\alpha(e^{\eta s}-1)\right), \frac{1}{\eta}\ln\left[1+\frac{1}{r^\beta}(e^{\eta s}-1)^\beta(e^{\eta t}-1)\right]\right\}, \tag{7}$$

*where $\eta = \ln(1+r)$.*

There is a connection between the mentioned shock model and the well-known distortion function. Consider the p.g.f of an integer-valued random variable $N$, the function $g^{-1}(t) = E(t^N); t \in (0,1)$ satisfies $g^{-1}(1) = 1$, $g^{-1}(0) = 0$, $(d/dt)g^{-1}(t) = E(Nt^{N-1}) > 0$ and $(d^2/dt^2)g^{-1}(t) = E(N(N-1)t^{N-2}) > 0$. Thus, $g^{-1}(\cdot)$ is nondecreasing and convex, or equivalently the function $g(\cdot)$ is nondecreasing and concave. If we consider the function $g : [0,1] \to [0,1]$ as a continuous, nondecreasing and concave function with $g(1) = 1$ and $g(0) = 0$, then $g$ is a distortion function and the proposed model (5) is, in fact, the distorted MO copula. For more details on the distortion of copulas, we refer to Nelsen [22] and the references therein.

Based on the construction in (3), we present a stochastic comparison. As in Example 1, suppose that we have another TMO$(\alpha', \beta'; g)$ random vector as

$$\mathbf{K} = (K_1, K_2) = (\min\{Y_1^{(1)}, \ldots, Y_N^{(1)}\}, \min\{Y_1^{(2)}, \ldots, Y_N^{(2)}\}),$$

that is constructed based on i.i.d. vectors $(Y_i^{(1)}, Y_i^{(2)}) = (\min\{Z_i^{(1)}, Z_i^{(3)}\}, \min\{Z_i^{(2)}, Z_i^{(3)}\})$ such that $\{Z_i^{(1)}, Z_i^{(2)}, Z_i^{(3)}\}$ are distributed as $\{W_i^{(1)}, W_i^{(2)}, W_i^{(3)}\}$, respectively, for $i = 1, \ldots, N$. The next result provides a stochastic comparison regarding to this construction method.

PROPOSITION 2: *Let $Z_i^{(1)} \succ_{st} W_i^{(1)}$, $Z_i^{(2)} \succ_{st} W_i^{(2)}$ and $W_i^{(3)} \succ_{st} Z_i^{(3)}$ for $i = 1, \ldots, N$. Then, $\bar{F}_{J_1,J_2}(t,s) \le \bar{F}_{K_1,K_2}(t,s)$ for all $t, s \in [0,1]$.*

PROOF: Li and Pellerey [15] proved that when $Z_i^{(1)} \succ_{\text{st}} W_i^{(1)}$, $Z_i^{(2)} \succ_{\text{st}} W_i^{(2)}$ and $W_i^{(3)} \succ_{\text{st}} Z_i^{(3)}$, then $\hat{C}_{X_i^{(1)},X_i^{(2)}}(t,s) \le \hat{C}_{Y_i^{(1)},Y_i^{(2)}}(t,s)$ for all $t, s \in [0,1]$. So, for any positive discrete random variable $N \in \{1,2,\ldots\}$ and for all $t, s \in (0,1)$, we have

$$\begin{aligned}
\bar{F}_{J_1,J_2}(t,s) &= \sum_{n=1}^{\infty} (\bar{F}_{X_1^{(1)},X_2^{(2)}}(t,s))^n P(N=n) \\
&= \sum_{n=1}^{\infty} [\hat{C}_{X_1^{(1)},X_2^{(2)}}(\bar{F}_{X_1^{(1)}}(t), \bar{F}_{X_1^{(2)}}(s))]^n P(N=n) \\
&\le \sum_{n=1}^{\infty} [\hat{C}_{Y_1^{(1)},Y_1^{(2)}}(\bar{F}_{Y_1^{(1)}}(t), \bar{F}_{Y_1^{(2)}}(s))]^n P(N=n) \\
&\le \bar{F}_{K_1,K_2}(t,s).
\end{aligned}$$

So, this completes the proof. ■

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## 3. DEPENDENCE PROPERTIES

In this section, we discuss some properties according to the TMO$(\alpha,\beta;g)$ model which is the copula of the extended MO model in (5). Some properties, such as copula density, Kendall distribution, an invariance property and dependence measures, are investigated.

### 3.1. Copula Density and Singular Component

For the TMO$(\alpha,\beta;g)$ model, the first- and second-order derivatives are

$$\frac{\partial C_g(u,v)}{\partial u}=\begin{cases}\dfrac{\alpha g^{\alpha-1}(u)g'(u)g(v)}{g'[C_g(u,v)]}, & g^{1-\beta}(v)<g^{1-\alpha}(u)\\ \dfrac{g^{\beta}(v)g'(u)}{g'[C_g(u,v)]}, & g^{1-\beta}(v)>g^{1-\alpha}(u)\end{cases}, \tag{8}$$

and

$$\begin{aligned}\frac{\partial^2 C_g(u,v)}{\partial u\partial v}&=\frac{\alpha g^{\alpha-1}(u)g'(u)g'(v)}{[g'(C_g(u,v))]^2}\{[g'(C_g(u,v))]^2-g^{\alpha}(u)g(v)g''(C_g(u,v))\}I_A\\&+\frac{\beta g^{\beta-1}(v)g'(u)g'(v)}{[g'(C_g(u,v))]^2}\{[g'(C_g(u,v))]^2-g^{\beta}(v)g(u)g''(C_g(u,v))\}I_B,\end{aligned}$$

respectively, where $A=\{(u,v)|g^{1-\beta}(v)<g^{1-\alpha}(u)\}$ and $B=\{(u,v)|g^{1-\beta}(v)>g^{1-\alpha}(u)\}$. So, the copula density of TMO$(\alpha,\beta;g)$ is $c_g(u,v)=(\partial^2 C_g(u,v)/\partial v\partial u)I_{[u\neq v]}+h(u)I_{[u=v]}$, where $h(u)=\lim_{v\to u^+}(\partial/\partial u)C_g(u,v)-\lim_{v\to u^-}(\partial/\partial u)C_g(u,v)$. For the TMO$(\alpha,\beta;g)$ model, $h(\cdot)$ has a complex form, but in special case, we have the following example.

Example 2: *The first- and second-order derivatives of* ECA$(\alpha;g)$ *are*

$$\frac{\partial C_g(u,v)}{\partial u}=\frac{g^{\alpha}(v)g'(u)}{g'[g^{-1}(g(u)g^{\alpha}(v))]}I_{[u<v]}+\frac{\alpha g^{\alpha-1}(u)g'(u)g(v)}{g'[g^{-1}(g^{\alpha}(u)g(v))]}I_{[u>v]},$$

*and*

$$\begin{aligned}\frac{\partial^2 C_g(u,v)}{\partial u\partial v}&=\frac{\alpha g^{\alpha-1}(\max(u,v))g'(u)g'(v)}{(g'(C_g(u,v)))^3}[(g'(C_g(u,v)))^2\\&-g^{\alpha}(\max(u,v))g(\min(u,v))g''(C_g(u,v))]I_{[u\neq v]},\end{aligned}$$

*respectively. So, the copula density of* ECA$(\alpha;g)$ *is*

$$c_g(u,v)=\frac{\partial^2 C_g(u,v)}{\partial v\partial u}I_{[u\neq v]}+h(u)I_{[u=v]},$$

*where* $h(u)=(1-\alpha)g^{\alpha}(u)g'(u)/g'[g^{-1}(g^{\alpha+1}(u))]$.

### 3.2. Kendall Distribution

The Kendall distribution of copula $C(\cdot,\cdot)$ is defined as $K(t)=P(C(U,V)\leq t)$ where its name is due to its relation with the Kendall measure of dependence $\tau=4\int_0^1 t\mathrm{d}K_C(t)-1$ (see [22]). The Kendall distribution for MO$(\alpha,\beta)$ is given by $K_{\mathrm{MO}}(t)=t-((\alpha+\beta-2\alpha\beta)/(1-\alpha\beta))t\ln(t)$. Using Proposition 1 in Genest and Rivest [10], we obtain

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the Kendall distribution of $\mathrm{TMO}(\alpha,\beta;g)$ as $K_{\mathrm{TMO}}(t) = t - ((\alpha+\beta-2\alpha\beta)/(1-\alpha\beta)) \cdot (g(t)\ln[g(t)]/g'(t))$, $0 < t < 1$. So, for instance, the Kendall distribution of $\mathrm{ECA}(\alpha;g)$ is given by $K_{\mathrm{ECA}}(t) = t - (2\alpha/(1+\alpha))(g(t)\ln[g(t)]/g'(t))$, $0 < t < 1$. Finding a functional relation between the distortion function and the Kendall distribution can be helpful in estimating the distortion function. Based on this idea, one can easily solve $K_{\mathrm{TMO}}(t)$ with respect to $g$. Thus, we have $\ln(g(t)) = \exp\{-\int_{x_0}^{t}(\gamma/(K_{\mathrm{TMO}}(x)-x))\mathrm{d}x\}$, where $\gamma = (\alpha+\beta-2\alpha\beta)/(1-\alpha\beta)$, and so the empirical version of $g(\cdot)$, denoted by $g_n(\cdot)$, can be obtained via $g_n(t) = \exp\{\exp\{-\int_{x_0}^{t}(\gamma/(\hat{K}_n(x)-x))\mathrm{d}x\}\}$. Note that the function $K_{\mathrm{TMO}}(t)$ is estimated by means of empirical distribution function $\hat{K}_n(t_i)$, where $T_i = C(U_i,V_i)$ is the proportion of the pairs $(X_j,Y_j)$, which are less than or equal to the pair $(X_i,Y_i)$ for all $i \neq j$. Hence, the function $g(\cdot)$ can be estimated with this method for applications.

### 3.3. Invariance Property

Consider the case where $P(N = 1/\gamma) = 1$, where $\gamma = 1/([1/\zeta]+1)$ such that $\zeta > 0$. So, $g(t) = t^{\gamma}$. The following property states that when $g(t) = t^{\gamma}$ for $\gamma > 0$, $\mathrm{TMO}(\alpha,\beta;g)$ and $\mathrm{MO}(\alpha,\beta)$ are equal.

Proposition 3: *The copula* $\mathrm{TMO}(\alpha,\beta;g)$ *is* $\mathrm{MO}(\alpha,\beta)$*-invariant if and only if* $g(t) = t^{\gamma}$ *for some* $\gamma > 0$.

Proof: If $g(t) = t^{\gamma}$ for any $\gamma > 0$, then for all $u,v \in [0,1]$, we have

$$\begin{aligned} C_g(u,v) &= \min\{(u^{\alpha\gamma}v^{\gamma})^{1/\gamma}, (u^{\gamma}v^{\beta\gamma})^{1/\gamma}\} \\ &= \min\{u^{\alpha}v, uv^{\beta}\} \\ &= C_{\mathrm{MO}}(u,v). \end{aligned}$$

On the other hand, if $C_g(u,v) = C_{\mathrm{MO}}(u,v)$ for all $u,v \in [0,1]$, then

$$g(u^{\alpha}v) = g^{\alpha}(u)g(v) \quad \text{or} \quad g(u^{\alpha}v) = g(u)g^{\beta}(v). \tag{9}$$

Let $g(u^{\alpha}v) = g^{\alpha}(u)g(v)$. By putting $u = v = 1$, we can conclude that $g(1) = 1$. Thus, if we put $v = 1$, then for every $u, \alpha \in [0,1]$,

$$g(u^{\alpha}) = g^{\alpha}(u). \tag{10}$$

By letting $T(u) = \ln(g(u))$ and applying (10), we have

$$\begin{aligned} T(u) &= \ln[g(e^{-(-\ln(u))})] \\ &= \ln[g^{-\ln(u)}(e^{-1})] \\ &= -\ln(u)\ln(g(e^{-1})) \\ &= \ln(u^{-\ln(g(e^{-1}))}) \end{aligned}$$

Hence, $g(u) = u^{\gamma}$ for all $\gamma > 0$, where $\gamma = -\ln[g(e^{-1})]$. For $g(u^{\alpha}v) = g(u)g^{\alpha}(v)$, if we put $u = v$, we have $g(u^{\alpha+1}) = g^{\beta+1}(u)$ and since this is a functional equation of the same form, $g(u) = u^{\gamma}$ for some $\gamma > 0$. So for every equations in (9), we conclude that there exists $\gamma > 0$ such that $g(u) = u^{\gamma}$ and the proof is complete. ■

Remark 2: *It is worth noting that when* $g(t) = t^{\gamma}$ *for* $\gamma > 0$*, we have* $K_{\mathrm{TMO}}(t) = K_{\mathrm{MO}}(t)$ *for all* $t$.

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### 3.4. Dependence Concepts

Let $(X,Y)$ be a pair of continuous random variable whose copula is $C$. The joint random vector $(X,Y)$ is positive (negative) quadrant dependent shown as PQD (NQD) if $C(u,v) \geq (\leq)\ uv$ for all $u,v \in (0,1)$. The random variable $Y$ is left tail decreasing (increasing) in $X$, denoted by LTD$(Y|X)$ (LTI$(Y|X)$), whenever $C(u,v)/u$ is nonincreasing (nondecreasing) in $u$ for all $v \in (0,1)$. The random variable $Y$ is stochastically increasing (decreasing) in $X$ denoted by SI$(Y|X)$ SD$(Y|X)$, if $\partial C(u,v)/\partial u$ is a nonincreasing (nondecreasing) in $u$ for all $v \in (0,1)$ (see [22]). Let $\mathscr{G}$ be the class of all TMO$(\alpha,\beta;g)$ and $\mathscr{G}_i,\ i=1,\ldots,5$ be the sub-classes of $\mathscr{G}$ defined for $u,v,\alpha \in (0,1)$ as

$$\mathscr{G}_1 = \{g \in \mathscr{G} : \alpha \ln(g(u)) + \ln(g(v)) \geq \ln(g(uv)), \ln(g(u)) + \beta \ln(g(v)) \geq \ln(g(uv))\},$$

$$\mathscr{G}_2 = \left\{g \in \mathscr{G} : \frac{g'(u)g^{\beta}(v)}{g'[g^{-1}(g(u)g^{\beta}(v))]} \leq \frac{1}{u} g^{-1}(g(u)g^{\beta}(v)), g^{1-\alpha}(u) \leq g^{1-\beta}(v)\right\},$$

$$\mathscr{G}_3 = \left\{g \in \mathscr{G} : \frac{\alpha g^{\alpha-1}(u)g(v)g'(u)}{g'[g^{-1}(g^{\alpha}(u)g(v))]} \leq \frac{1}{u} g^{-1}(g^{\alpha}(u)g(v)), g^{1-\alpha}(u) \geq g^{1-\beta}(v)\right\},$$

$$\mathscr{G}_4 = \left\{g \in \mathscr{G} : \frac{g''(u)}{g'(u)} \leq \frac{g''[g^{-1}(g(u)g^{\beta}(v))]}{g'[g^{-1}(g(u)g^{\beta}(v))]}, g^{1-\alpha}(u) \leq g^{1-\beta}(v)\right\}$$

and

$$\mathscr{G}_5 = \left\{g \in \mathscr{G} : \frac{g''(u)}{g'(u)} + (\alpha-1)\frac{g'(u)}{g(u)} \leq \alpha^2 g^2(v) \frac{g''[g^{-1}(g^{\alpha}(u)g(v))]}{g'[g^{-1}(g^{\alpha}(u)g(v))]}, g^{1-\alpha}(u) \geq g^{1-\beta}(v)\right\}.$$

The next proposition defines different classes of positive dependence structures. Noting that for a negative dependence structure, we have to consider the complement of the mentioned sets.

PROPOSITION 4: *Let $N \in \{1,2,\ldots\}$ such that $E(t^N) = g^{-1}(t)$ and $(X,Y) \sim$ TMO$(\alpha,\beta;g)$. Then*

- $(X,Y)$ *is* PQD *whenever* $g \in \mathscr{G}_1$,
- $(X,Y)$ *is* LTD$(Y|X)$ *whenever* $g \in \mathscr{G}_2$ *or* $g \in \mathscr{G}_3$ *and*
- $(X,Y)$ *is* SI$(Y|X)$ *whenever* $g \in \mathscr{G}_4$ *or* $g \in \mathscr{G}_5$.

PROOF: The proof for PQD is trivial. For LTD$(Y|X)$ using (8) and splitting the answer into two parts $g^{1-\alpha}(u) \leq g^{1-\beta}(v)$ and $g^{1-\alpha}(u) \geq g^{1-\beta}(v)$, we complete the proof. For SI$(Y|X)$, we know that SI$(Y|X) \Leftrightarrow \partial^2 C_g(u,v)/\partial u^2 < 0$, thus

$$\begin{aligned}\frac{\partial^2 C_g(u,v)}{\partial u^2} &= \frac{[g''(u)g^{\alpha-1}(u) + (\alpha-1)g^{\alpha-2}(u)(g'(u))^2]g'[C_g(u,v)]}{(g'[C_g(u,v)])^2} I_A \\ &\quad - \frac{\alpha^2 g^2(u) g^{\alpha-1}(u) g'(u) g''[C_g(u,v)]}{(g'[C_g(u,v)])^2} I_A \\ &\quad + \frac{g''(u)g^{\beta}(v)g'[C_g(u,v)] - g'(u)g^{\beta}(v)g^{"}[C_g(u,v)]}{(g'[C_g(u,v)])^2} I_B,\end{aligned}$$

where $A = \{(u,v)|g^{1-\alpha}(u) > g^{1-\beta}(v)\}$ and $B = \{(u,v)|g^{1-\alpha}(u) < g^{1-\beta}(v)\}$. The condition $\partial^2 C_g(u,v)/\partial u^2 < 0$ for $g^{1-\alpha}(u) < g^{1-\beta}(v)$ reduces to the class $\mathscr{G}_4$ and for $g^{1-\alpha}(u) > g^{1-\beta}(v)$ implies the class $\mathscr{G}_5$ that completes the proof. ■

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TABLE 1. Distortion functions constructed from life distributions

| Distribution | Survival function | DFR | Distortion function |
|---|---|---|---|
| Log-logistic | $[1+(x/a)^b]^{-1},\ a,b>0$ | $b\le 1$ | $[1+(\frac{-\ln(x)}{a})^b]^{-1}$ |
| Exp-Logarithm | $\frac{1}{\ln(p)}\ln[1-(1-p)e^{-bx}], p\in(0,1)\&b>1$ | All values | $\frac{1}{\ln(p)}[1-(1-p)x^b]$ |
| Weibull | $\exp\{-x^a\}, a>0$ | $a\le 1$ | $\exp\{-(-\ln(x))^a\}$ |

REMARK 3: *According to Durante et al. [8], if $g^{-1}(e^x)$ is a class of log-convex distortion functions, then* $\mathrm{TMO}(\alpha,\beta;g)$ *is TP2 and has a positive dependence structure. Based on their idea, one can construct bivariate distorted copula models with a positive dependence structure using any base copula (in this case, the MO copula model). Let $\bar{F}$ has a decreasing failure rate (DFR), then by setting $g^{-1}(t)=\bar{F}(-\ln(t))$, we can construct some vareity of distortion functions. Note that the inverse of a distortion function is also a distortion function. Some examples of these distortion functions can be constructed using univariate ageing models given in Table 1.*

### 3.5. Dependence Measures

For every pair $(X,Y)$, four famous measures of association are Kendall's tau ($\tau = 4E[C(U,V)]-1$), Spearman's rho ($\rho = 12\iint_{[0,1]^2} C(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3$), Gini's gamma ($\gamma = 4\int_{[0,1]} C(u,1-u)\,\mathrm{d}u - 4\int_{[0,1]}(u-C(u,u))\,\mathrm{d}u$) and Blomqvist's beta ($\mathrm{Beta} = 4C(0.5,0.5)-1$), respectively (see [22]).

PROPOSITION 5: *Let $(X,Y)\sim \mathrm{TMO}(\alpha,\beta;g)$. Then, we have*

$$Beta_g = 4\min\left\{g^{-1}\left(g^{\alpha+1}\left(\frac{1}{2}\right)\right), g^{-1}\left(g^{\beta+1}\left(\frac{1}{2}\right)\right)\right\} - 1$$

*and*

$$\tau_g = 1 - 4\gamma\int_0^1 t(\ln(g(t))+1)\,\mathrm{d}t + 4\gamma\int_0^1 \frac{tg''(t)g(t)\ln(g(t))\,\mathrm{d}t}{(g'(t))^2},$$

*respectively, where $\gamma=(\alpha+\beta-2\alpha\beta)/(1-\alpha\beta)$.*

PROOF: The result for Blomqvist's beta is trivial. For Kendall's tau, we use $\tau = 4\int_0^1 t\,\mathrm{d}(K_C(t)) - 1$. So using some simple calculus computations, the proof is completed. ■

It is clear that the direct computation of these measures for $\mathrm{TMO}(\alpha,\beta;g)$ is complicated. As an example, if we choose the Poisson–MO model with $\alpha=\beta$, then these measures can be computed and their values are given in Table 2.

As shown in Table 2, we have strong and weak positive dependence for different values of $r$ and $\alpha$.

*3.5.1. Tail dependence* For every copula $C$, the lower and upper tail dependence coefficients are defined by $\lambda_{\mathrm{L}}(C)=\lim_{u\to 0^+} C(u,u)/u$ and $\lambda_{\mathrm{U}}(C)=\lim_{u\to 1^-}(1-2u+C(u,u))/(1-u)$. For $\mathrm{MO}(\alpha,\beta)$, we have $\lambda_{\mathrm{L}}(C)=0$ and $\lambda_{\mathrm{U}}(C)=\min\{1-\alpha,1-\beta\}$. The proceeding proposition gives the lower and upper tail dependence for $\mathrm{TMO}(\alpha,\beta;g)$.

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TABLE 2. The values of some dependence measures for the Poisson–MO model with $\alpha = \beta$ and $r = 0.8,10$

| $\alpha$ | $r=0.8$ | | | | $r=10$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\tau$ | *Beta* | $\gamma$ | $\rho$ | $\tau$ | *Beta* | $\gamma$ | $\rho$ |
| 0.1 | 0.816 | 0.875 | 0.876 | 0.886 | 0.766 | 0.887 | 0.872 | 0.882 |
| 0.3 | 0.538 | 0.652 | 0.761 | 0.678 | 0.459 | 0.640 | 0.645 | 0.667 |
| 0.5 | 0.340 | 0.459 | 0.556 | 0.493 | 0.276 | 0.440 | 0.449 | 0.476 |
| 0.7 | 0.193 | 0.291 | 0.379 | 0.327 | 0.159 | 0.267 | 0.278 | 0.305 |
| 0.9 | 0.079 | 0.145 | 0.152 | 0.177 | 0.082 | 0.117 | 0.128 | 0.150 |

PROPOSITION 6: *Let* $g'(1) \neq 0$ *and* $(X,Y) \sim \mathrm{TMO}(\alpha,\beta;g)$. *Then, we have*

$$\lambda_{\mathrm{L}}(C_g) = 0, \quad \lambda_{\mathrm{U}}(C_g) = \min\{1-\alpha, 1-\beta\}.$$

*In particular, for* $\mathrm{ECA}(\alpha;g)$, *we have* $\lambda_{\mathrm{L}}(C_g) = 0$ *and* $\lambda_{\mathrm{U}}(C_g) = 1-\alpha$.

PROOF: Let $N \in \{1,2,\ldots\}$ such that $E(t^N) = g^{-1}(t)$. According to $\mathrm{TMO}(\alpha,\beta;g)$, we have

$$\begin{aligned}
\lambda_{\mathrm{L}}(C_g) &= \lim_{u\to 0^+} \frac{g^{-1}(\min\{g^{\alpha+1}(u), g^{\beta+1}(u)\})}{u} \\
&= \lim_{u\to 0^+} \min\left\{ \frac{(\alpha+1)g^{\alpha}(u)g'(u)}{g'(g^{-1}(g^{\alpha+1}(u)))}, \frac{(\beta+1)g^{\beta}(u)g'(u)}{g'(g^{-1}(g^{\beta+1}(u)))} \right\}, \\
&= \min\left\{ \frac{(\alpha+1)g^{\alpha}(0)g'(0)}{g'(g^{-1}(g^{\alpha+1}(0)))}, \frac{(\beta+1)g^{\beta}(0)g'(0)}{g'(g^{-1}(g^{\beta+1}(0)))} \right\}, \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{\mathrm{U}}(C_g) &= \lim_{u\to 1^-} \frac{1-2u+\min\{g^{-1}(g^{\alpha+1}(u)), g^{-1}(g^{\beta+1}(u))\}}{1-u} \\
&= 2 - \lim_{u\to 1^-} \min\left\{ \frac{(\alpha+1)g^{\alpha}(u)g'(u)}{g'(g^{-1}(g^{\alpha+1}(u)))}, \frac{(\beta+1)g^{\beta}(u)g'(u)}{g'(g^{-1}(g^{\beta+1}(u)))} \right\} \\
&= \min\{1-\alpha, 1-\beta\}.
\end{aligned}$$

■

## 4. RANDOM DATA GENERATION AND SIMULATION STUDY

Simulating random numbers is essential in Monte Carlo studies and is an effective way to understand the behavior of the dependence structure of distributions for modeling dependent data. The following algorithm is used to generate random numbers from $\mathrm{TMO}(\alpha,\beta;g)$ in Theorem 1.

Considering this algorithm, scatter plots of 1,000 random samples from $\mathrm{TMO}(\alpha,\beta;g)$ when $g^{-1}(\cdot)$ is the probability generating function of truncated Poisson and logarithmic series with parameter $r$ and different values of $\alpha$ and $\beta$ are presented in Figures 2–6. Figure 2 shows the MO model with high dependency ($\tau = 0.789$) on the left side and the effect of Poisson distortion. Clearly, as the parameter of the Poisson–MO distorted model increases, the dependency decreases. Figure 3 shows the effect of the logarithmic series distortion on

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**Algorithm 1.** Random number generation from $\mathrm{TMO}(\alpha, \beta; g)$

1. Generate $N$ from a known probability generating function $g^{-1}(\cdot)$.
2. Generate $N$ pairs of $(x_i, y_i)$ from $\mathrm{MO}(\alpha, \beta)$ distribution where $\alpha = \frac{\lambda_1}{\lambda_1+\lambda_3}, \beta = \frac{\lambda_2}{\lambda_2+\lambda_3}$.
3. Put $x' = \min\{x_1, \ldots, x_N\}$ and $y' = \min\{y_1, \ldots, y_N\}$.
4. Set $u = g^{-1}(\exp\{-(\lambda_1 + \lambda_3)x'\})$ and $v = g^{-1}(\exp\{-(\lambda_2 + \lambda_3)y'\})$.
5. The desired pair is $(u, v)$.

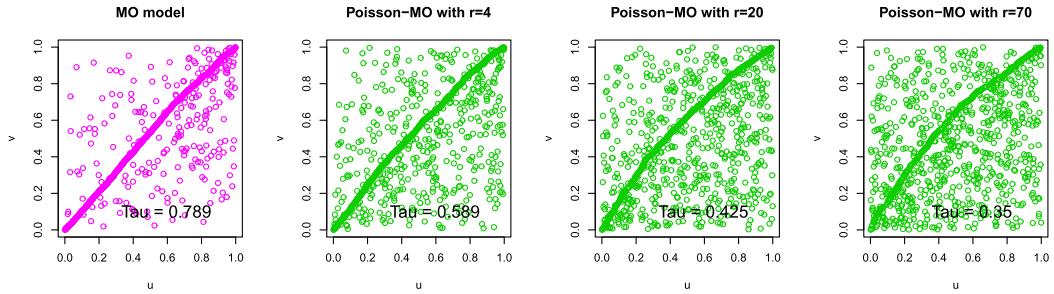



FIGURE **2.** Uniform scaled generated data based on Algorithm 1 for the (first from left) MO model with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 9$ ($\alpha = 0.1, \beta = 0.18$) and the (second, third and fourth from left) Poisson–MO model given in (6) with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 9$ ($\alpha = 0.1, \beta = 0.18$).

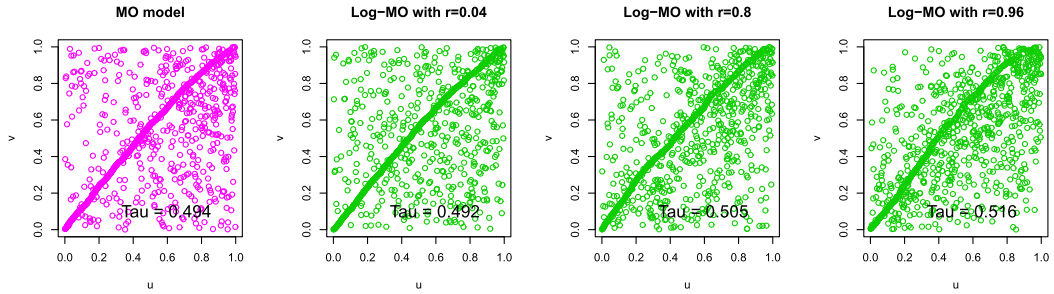



FIGURE **3.** Uniform scaled generated data based on Algorithm 1 for the (first from left) MO model with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ ($\alpha = 0.33, \beta = 0.40$) and the (second, third and fourth from left) Logarithmic–MO model given in (7) with $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

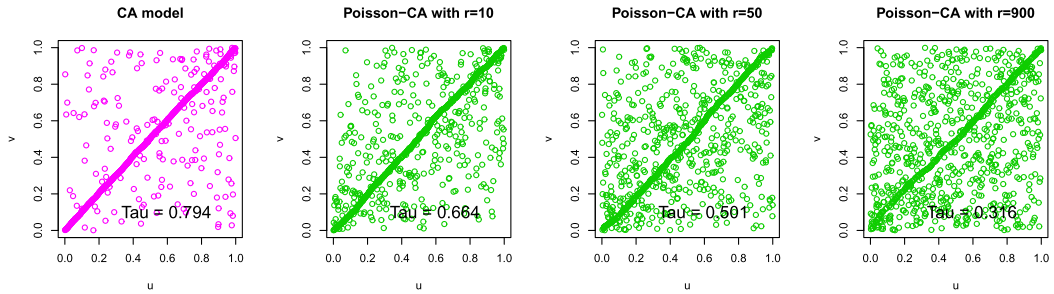



FIGURE **4.** Uniform scaled generated data based on Algorithm 1 for the (first from left) Cuadras–Augé model with $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 9$ ($\alpha = \beta = 0.1$) and the (second, third and fourth from left) Poisson–CA model given in (6) with $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 9$ ($\alpha = \beta = 0.1$).

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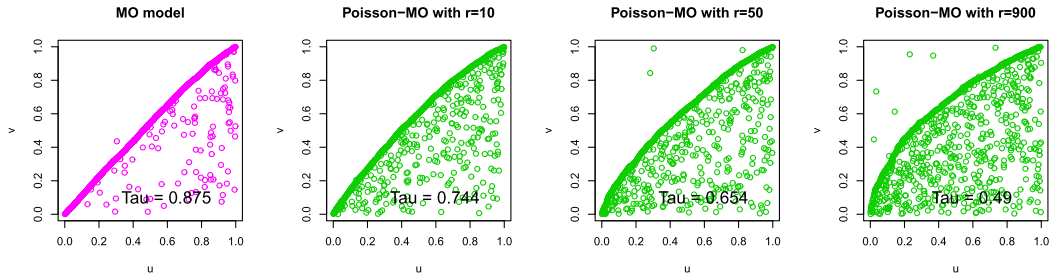



FIGURE **5.** Uniform scaled generated data based on Algorithm 1 for the (first from left) MO model with $\lambda_1 = 0.01, \lambda_2 = 2, \lambda_3 = 9$ $(\alpha = 0.99, \beta = 0.18)$ and the (second, third and fourth from left) Poisson–MO model given in (6) with $\lambda_1 = 0.01, \lambda_2 = 2, \lambda_3 = 9$ $(\alpha = 0.99, \beta = 0.18)$.

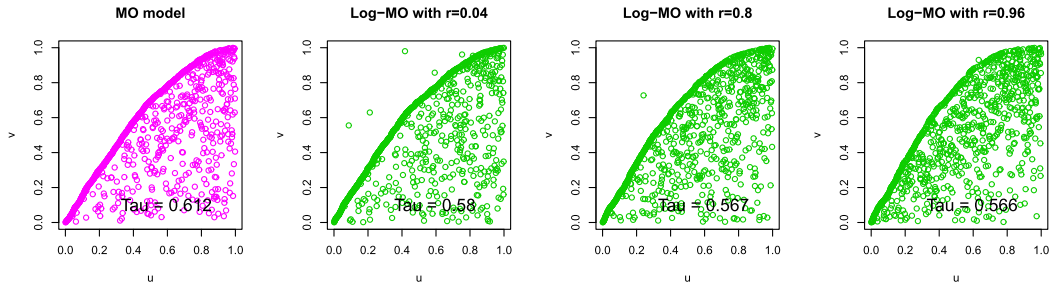



FIGURE **6.** Uniform scaled generated data based on Algorithm 1 for the (first from left) MO model with $\lambda_1 = 0.01, \lambda_2 = 2, \lambda_3 = 3$ $(\alpha = 0.99, \beta = 0.40)$ and the (second, third and fourth from left) Logarithmic–MO model given in (7) with $\lambda_1 = 0.01, \lambda_2 = 2, \lambda_3 = 3$ $(\alpha = 0.99, \beta = 0.40)$.

the MO model with intermediate dependency ($\tau = 0.494$). As the parameter of logarithmic series distribution ($r$) increases, dependence grows mildly. Figure 4 illustrates the impact of the Poisson distortion on the Cuadras–Augé model with high dependency ($\tau = 0.794$). As shown, the dependence between the $u$-data decreases as the value of distortion parameter increases. Figure 5 displays the MO model when most of the generated data are beneath the diagonal ($\lambda_1 \to 0$). In this case, the Poisson distortion decreases the dependence measure $\tau$ as the value of $r$ increases. Figure 6 shows the MO model when most of the generated data are beneath the diagonal ($\lambda_1 \to 0$). It is clear that the logarithmic series distortion mildly decreases the dependency as $r \to 1$.

Now, we want to discuss the maximum likelihood estimator (MLE) of parameters for the model given in Proposition 1 when $g^{-1}$ is the probability generating function distributed from the truncated Poisson with parameter $\theta$ (we will focus on this model here and in the next section). Based on Proposition 1 for any $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, we have

$$\bar{F}(t,s;\underline{\lambda},\theta) = \begin{cases} \dfrac{1}{e^{\theta}-1}[\exp\{\theta\{-\lambda_1 t - (\lambda_2+\lambda_3)s\}\} - 1], & t < s \\ \dfrac{1}{e^{\theta}-1}[\exp\{\theta\{-(\lambda_1+\lambda_3)t - \lambda_2 s\}\} - 1], & t > s \\ \dfrac{1}{e^{\theta}-1}[\exp\{\theta\{-(\lambda_1+\lambda_2+\lambda_3)t\}\} - 1], & t = s \end{cases} . \tag{11}$$

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Let $h(t,s;\underline{\lambda}) = \exp\{-\lambda_1 t - \lambda_2 s - \lambda_3 \max(t,s)\}$, then the probability density function of (11) is given as follows:

$$f(t,s;\underline{\lambda},\theta) = \begin{cases} \dfrac{(\lambda_2+\lambda_3)\lambda_1\theta}{e^\theta-1} h(t,s;\underline{\lambda}) \exp\{\theta h(t,s;\underline{\lambda})\}[1+\theta h(t,s;\underline{\lambda})], & t<s \\ \dfrac{(\lambda_1+\lambda_3)\lambda_2\theta}{e^\theta-1} h(t,s;\underline{\lambda}) \exp\{\theta h(t,s;\underline{\lambda})\}[1+\theta h(t,s;\underline{\lambda})], & t>s \\ \dfrac{\theta\lambda_3}{e^\theta-1} h(t,s;\underline{\lambda}) \exp\{\theta h(t,s;\underline{\lambda})\}, & t=s \end{cases} . \quad \textbf{(12)}$$

Consider a random sample of size $m$, say $\{(t_1,s_1),\ldots,(t_m,s_m)\}$ from the Poisson–MO model given in (12). Let $m_1$, $m_2$ and $m_3$ denote the number of observations for which $t<s$, $t>s$ and $t=s$, respectively, such that $m = m_1+m_2+m_3$. The log-likelihood function for a given sample can be written as follow:

$$\begin{aligned} l(\underline{\lambda},\theta) = &-\lambda_1 \sum_{i=1}^{m} t_i - \lambda_2 \sum_{i=1}^{m} s_i - \lambda_3 \sum_{i=1}^{m} \max(t_i,s_i) + \theta \sum_{i=1}^{m} h(t_i,s_i;\underline{\lambda}) \\ &+ \sum_{i=1}^{m_1} \ln[1+\theta h(t_i,s_i;\underline{\lambda})] + \sum_{i=m_1+1}^{m_2} \ln[1+\theta h(t_i,s_i;\underline{\lambda})] \\ &+ m_1 \ln\left[\frac{(\lambda_2+\lambda_3)\lambda_1\theta}{e^\theta-1}\right] + m_2 \ln\left[\frac{(\lambda_1+\lambda_3)\lambda_2\theta}{e^\theta-1}\right] + m_3 \ln\left[\frac{\lambda_3\theta}{e^\theta-1}\right]. \end{aligned} \quad \textbf{(13)}$$

The MLEs can be obtained by maximizing (13) with respect to the unknown parameters $(\underline{\lambda},\theta)$. The first partial derivatives and elements of the Fisher information matrix are given in Appendix. As expected, we cannot obtain the solution of the likelihood equations. They cannot be obtained in explicit forms and numerical techniques are required. For the case $m_1,m_2,m_3>0$, the maximization can be performed using the routine`optim` with the Broyden–Fletcher–Goldfarb–Shanno algorithm known as the "BFGS" method in the R software. Initial values for optimization are derived based on the global nonlinear optimization package "Rsolnp" in the R software version 3.6.1. While maximizing the log-likelihood function, we put the constraints $\theta,\lambda_1,\lambda_2,\lambda_3>0$. Local maximums are identified after having different values for the parameters. In order to achieve the global maximum, we select $(\hat{\theta},\hat{\underline{\lambda}})$ that has the most likelihood, that is,

$$(\hat{\theta},\hat{\underline{\lambda}}) = \arg\max_{\theta,\underline{\lambda}\in\Theta} L(\theta,\underline{\lambda}).$$

In the simulation process, the global maximum was unique and did not converge to the boundary of the parameter space.

Balakrishnan and Basu [4, p. 334], discussed the unique existence of MLE for the case where at least one of $m_1$ or $m_2$ or $m_3$ are zero for the MO model. We give the following remark for the log-likelihood function given in (13) for the same specified situation.

REMARK 4: *As in Example 1, for any random variable $N \in \{1,2,\ldots\}$, the joint vector $(J_1,J_2)|N=n$ is distributed from a Marshall–Olkin model with parameters $(n\lambda_1, n\lambda_2, n\lambda_3)$.*

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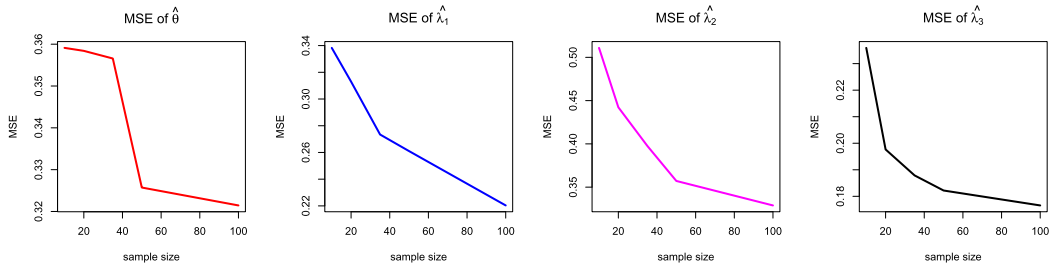



FIGURE **7.** MSE versus sample size for parameters (first from left) $\theta$, (second from left) $\lambda_1$, (third from left) $\lambda_2$ and (fourth from left) $\lambda_3$ based on the MLE estimator with 10,000 iterations.

*So,*

$$\begin{aligned} P(J_1 < J_2) &= \sum_n P(J_1 < J_2 | N = n) P(N = n) \\ &= \sum_n \frac{n\lambda_1}{n\lambda_1 + n\lambda_2 = n\lambda_3} P(N = n) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned} \tag{14}$$

*Similarly,* $P(J_1 > J_2) = \lambda_2/(\lambda_1 + \lambda_2 + \lambda_3)$ *and* $P(J_1 = J_2) = \lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$. *Thus,*

- $P(m_1 = 0) = [1 - P(J_1 < J_2)]^n \to 0$ *as* $n \to \infty$,
- $P(m_2 = 0) = [1 - P(J_1 > J_2)]^n \to 0$ *as* $n \to \infty$ *and*
- $P(m_3 = 0) = [1 - P(J_1 = J_2)]^n \to 0$ *as* $n \to \infty$.

*These imply that the case where at least one of* $m_1$ *or* $m_2$ *or* $m_3$ *are zero is very rare, and in practice, if happening, the MLE for* $(\theta, \underline{\lambda})$ *may not be unique on the parameter space. For instance, when* $m_1 = m_2 = 0$, *the system of normal equations (given in Appendix) reduces to three equations with four unknown parameters, and this implies that MLE is not unique.*

Now, we want to asses the finite sample performance of MLE based on the log-likelihood function in (13). The assessment is based on mean squared error (MSE) with true values $(\underline{\lambda}, \theta) = ((1, 2, 3), 1)$ which is given in Figure 7. Clearly, MSE converges to zero as sample size increases.

## 5. AN APPLICATION IN ENGINEERING

For illustration purposes, we use the data given in Balakrishnan and Basu [4]. The data consisting of failure times of an insulating fluid contingent upon various constant elevated voltage stresses. The number of failure times was observed at each stress level. The resulting data are categorized in 32 and 36 kV voltages. To make the data suitable for the MO model and its transformations, the data are rounded. Let $X_1 := 32$ kV and $X_2 := 36$ kV, one can realize that each of $X_1 < X_2$, $X_1 > X_2$ or $X_1 = X_2$ is possible. A summary of the data is given in Table 3. Based on the results in Table 4, both marginals (32 and 36 kV) are exponentially distributed. The scatterplot of observations is presented in Figure 8. The concentration on the data suggests high dependency and $\tau_n = 0.94$ supports this point.

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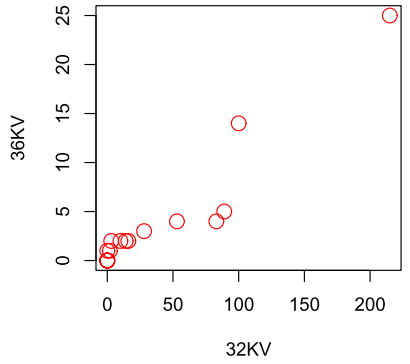



FIGURE **8.** Scatterplot of 32 versus 36 kV.

**Table 3.** A summary of the data

| Statistics | 32 kV | 36 kV |
|---|---|---|
| Min | 0 | 0 |
| $Q1$ | 1 | 1 |
| $Q2$ | 14 | 2 |
| Mean | 40.87 | 4.33 |
| $Q3$ | 68 | 4 |
| Max | 215 | 25 |
| SD | 59.84 | 6.67 |
| Kendall's tau | 0.9433 | |
| Spearman's rho | 0.9808 | |
| Rho/Tau | 1.03 | |

**Table 4.** Goodness-of-fit test results for marginal distributions

| Var. | Fitted distribution | Parameter estimation | Log-likelihood | KS test |
|---|---|---|---|---|
| 32 kV | Exponential | $\hat{\theta} = 0.024$ | −70.65472 | $p$-value = 0.07 |
| 36 kV | Exponential | $\hat{\theta} = 0.23$ | −36.99506 | $p$-value = 0.41 |

**Table 5.** Goodness-of-fit test results for joint distribution

| Model | Parameter estimation (MLE) | AIC | $C$–$V$ $p$-value |
|---|---|---|---|
| MO | $\hat{\lambda}_1 = 0.015$, $\hat{\lambda}_2 = 0.166$, $\hat{\lambda}_3 = 0.045$ | 128.27 | 0.161 |
| Poisson–MO | $\hat{\theta} = 22.8$, $\hat{\lambda}_1 = 27.01$, $\hat{\lambda}_2 = 26.3$, $\hat{\lambda}_3 = 27.3$ | 79.901 | 0.386 |

Now, we choose the Poisson–MO model in (3) with parameters $(\underline{\lambda}, \theta)$ (as a competitor of the MO model). In order to estimate the parameters based on the method mentioned in the previous section, we maximize the log-likelihood function in (13). The results are given in Table 5. Based on the values of AIC, Poisson–MO is a better model than MO. Also, the Cramer Von-Mises goodness-of-fit test suggests that both models are suitable for data.

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## 6. THE MULTIVARIATE CASE

In this section, we give the multivariate version of the model in (4). As given in Li [14], let $(X_i^{(1)},\ldots,X_i^{(m)})$ be *i.i.d.* distributed from the multivariate MO survival model with parameters $\{\lambda_S|S\subset\{1,\ldots,m\}\}$ as follows:

$$\bar{F}_{\mathrm{MO}}(x_1,\ldots,x_m)=\exp\left\{-\sum_{i=1}^{m}\lambda_i x_i-\sum\sum_{i<j}\lambda_{ij}\max\{x_i,x_j\}\right.$$
$$\left.-\sum\sum\sum_{i<j<k}\lambda_{ijk}\max\{r_i,r_j,r_k\}-\ldots-\lambda_{1\ldots m}\bigvee_{i=1}^{m}x_i\right\},\qquad(15)$$

where $\bigvee_{i\in S}x_i=\max_{i\in S}\{x_i\}$. Let $N$ be a discrete random variable taking the values $\{1,2,\ldots\}$. Also, suppose that $N$ is independent of $\{(X_i^{(1)},\ldots,X_i^{(m)})|i\geq 1\}$ and $E[t^N]=g^{-1}(t)$, where $g$ is a continuous increasing and concave function with $g(0)=0$ and $g(1)=1$. Then, the multivariate transformed MO model can be constructed with the following vector:

$$(J_1,\ldots,J_m)=(\min\{X_1^{(1)},\ldots,X_N^{(1)}\},\ldots,\min\{X_1^{(m)},\ldots,X_N^{(m)}\}).$$

The survival function of $(J_1,\ldots,J_m)$ can be derived by conditioning w.r.t $N$:

$$\bar{F}_{\mathbf{J}}(x_1,\ldots,x_m)=\sum_{n=1}^{\infty}[\bar{F}_{\mathrm{MO}}(x_1,\ldots,x_m)]^n P(N=n).$$

So, the survival function of the multivariate transformed MO (denoted by MTMO) model is derived as follows:

$$\bar{F}(x_1,\ldots,x_m)=g^{-1}\left(\left\{\exp\left\{-\sum_{i=1}^{n}\lambda_i x_i-\sum\sum_{i<j}\lambda_{ij}\max\{x_i,x_j\}\right.\right.\right.$$
$$\left.\left.\left.-\sum\sum\sum_{i<j<k}\lambda_{ijk}\max\{x_i,x_j,x_k\}-\cdots-\lambda_{1\ldots m}\bigvee_{i=1}^{m}x_i\right\}\right\}\right).\qquad(16)$$

The marginal distributions of the model in (16) is given as follows:

$$\bar{F}_i(x_i)=g^{-1}\left(\exp\left\{-\left(\lambda_i+\sum\sum_{i<j}\lambda_{ij}+\sum\sum\sum_{i<j<k}\lambda_{ijk}+\cdots+\lambda_{1\ldots m}\right)x_i\right\}\right).\quad(17)$$

To represent the survival function in (16) according to its marginal survival functions, we first give the consecutive notations, for any $S\subseteq\{1,\ldots,m\}$,

$$\bigvee_{i\in S}x_i=\max\{x_i:i\in S\},\quad \bigwedge_{i\in S}x_i=\min\{x_i:i\in S\}.$$

Clearly, for any $I\subseteq\{1,\ldots,m\}$,

$$\bigwedge_{i\in S}x_i=\sum_{k\in I}x_k+\cdots+(-1)^{s-1}\sum_{S\subseteq I,|S|=s}\bigvee_{k\in S}x_k+\cdots+(-1)^{|I|-1}\bigvee_{k\in I}x_k.$$

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By using the mentioned terms in (16), we have

$$\begin{aligned}
&\bar{F}(x_1,\dots,x_m)\\
&\quad= g^{-1}\left(\exp\left\{-\sum_{i=1}^{m}\left(\sum_{i\in S}\lambda_S\right)x_i+\cdots+(-1)^k\sum_{|I|=k}\left(\sum_{S:I\subseteq S}\lambda_S\right)\right.\right.\\
&\qquad\left.\left.\times\bigwedge_{i\in I}x_i+\cdots+(-1)^m\lambda_{12\dots m}\bigwedge_{i=1}^{m}x_i\right\}\right)\\
&\quad= g^{-1}\left(\prod_{i=1}^{m}\exp\left[-\left(\sum_{i\in S}\lambda_S\right)x_i\right]\cdots\prod_{|I|=k}\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)\bigwedge_{i\in I}x_i\right]\right.\\
&\qquad\left.\cdots\exp\left[(-1)^m\lambda_{12\dots m}\bigwedge_{i=1}^{m}x_i\right]\right)\\
&\quad= g^{-1}\left(\prod_{k=1}^{m}\prod_{|I|=k}\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)\bigwedge_{i\in I}x_i\right]\right).
\end{aligned}$$

Set

$$^{|I|}\Delta_{i\in I}t_i=\begin{cases}\bigvee_{i\in I}t_i, & \text{if } |I| \text{ is odd}\\ \bigwedge_{i\in I}t_i, & \text{if } |I| \text{ is even}\end{cases}. \tag{18}$$

Due to the fact that $\exp\{(-1)^k(\sum_{S:I\subseteq S}\lambda_S)\bigwedge_{i\in I}x_i\}$ is nondecreasing (nonincreasing) in $x_i$ when $k$ is even (odd), we have, for any $i\in I$ with $|I|=k$,

$$\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)\bigwedge_{i\in I}x_i\right]={}^{k}\Delta_{i\in I}\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)x_i\right].$$

So,

$$\bar{F}(x_1,\dots,x_m)=g^{-1}\left(\prod_{k=1}^{m}\prod_{|I|=k}{}^{k}\Delta_{i\in I}\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)x_i\right]\right). \tag{19}$$

For each i and $I\subseteq\{1,\dots,m\}$, set $\alpha_i^I=\sum_{I\subseteq S}\lambda_S/\sum_{i\in S}\lambda_S$. Clearly, $\alpha_i^I\le 1$ for any $i\in I$. It follows from (17) that for any $1\le i\le n$, $i\in I$,

$$\exp\left[(-1)^k\left(\sum_{S:I\subseteq S}\lambda_S\right)x_i\right]=[g(\bar{F}_i(x_i))]^{(-1)^{k+1}\alpha_i^I},$$

which, together with (19), concludes that

$$\bar{F}(x_1,\dots,x_m)=g^{-1}\left(\prod_{k=1}^{m}\prod_{|I|=k}{}^{k}\Delta_{i\in I}[g(\bar{F}_i(x_i))]^{(-1)^{k+1}\alpha_i^I}\right). \tag{20}$$

The following statement gives the survival copula associated with (20) based on the well-known Sklar's theorem.

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PROPOSITION 7: *Let $(J_1,\ldots,J_m)$ be from the MTMO model in (16). Then, its corresponding survival copula is*

$$\hat{C}_{J_1,\ldots,J_m}(u_1,\ldots,u_m)=g^{-1}\left(\prod_{k=1}^{m}\prod_{|I|=k}{}^{I}\Delta_{i\in I}[g(u_i)]^{(-1)^{k+1}\alpha_i^I}\right), \tag{21}$$

*for $I\subseteq\{1,\ldots,m\}$, where $\alpha_i^I=\sum_{I\subset S}\lambda_S/\sum_{I\in S}\lambda_S$ and ${}^{k}\Delta$ is the min–max operator given in (18) and $\bigwedge_{i\in S}x_i=\min_{i\in S}\{x_i\}$.*

Based on the transformations $g$ introduced in Remark 1, new multivariate sub-models, such as Poisson–MO, Logarithmic–MO, and extended Cuadras–Augé, can be constructed. For instance, for any Archimedean generator $\phi(\cdot)$ when $g(t)=e^{-\phi(t)}$, the survival copula associated with (16) is a model from the multivariate Archimax survival copula as given in Charpentier et al. [5]. Hence, the model in (16) has some intersection with the class of Archimax copulas.

Regarding to Ahmed et al. [1], a sequence of random variables $X_1,\ldots,X_m$ is said to be right tail increasing in sequence (RTIS), if $P[X_{i+1}>x_{i+1}|\bigcap_{j=1}^{i}(X_j>x_j)]$ is increasing in $x_1,\ldots,x_m$, for $i=1,2,\ldots,n-1$. The concept RTIS concludes the positive dependence structure. Note that a function $f$ is TP2, if for every $x_1<x_2$ and $y_1<y_2$,

$$\begin{vmatrix} f(x_1,y_1) & f(x_1,y_2)\\ f(x_2,y_1) & f(x_2,y_2)\end{vmatrix}\geq 0.$$

which is equivalent to $\ln(f(x,y))/\partial x\partial y\geq 0$. The following statement shows that the introduced model in (16) has the positive dependence structure under some regular condition.

PROPOSITION 8: *Let $(J_1,\ldots,J_m)$ be from the MTMO model in (16). Then, the vector $(J_1,\ldots,J_m)$ is RTIS, if $g$ is log-convex.*

PROOF: Let $\bar{G}_{ij}(x_i,x_j)$ be the survival function in (4) where $\bar{G}_{ij}(x_i,x_j)=g(\bar{F}(x_i,x_j))$, $\forall i\neq j$ and $\bar{F}(x_i,x_j)$ is defined as in (2). Then, $\bar{G}_{ij}(x_i,x_j)$ is TP2, if $\ln(\bar{G}_{ij}(x_i,x_j))/\partial x_i\partial x_j\geq 0$. So,

$$\begin{aligned}\frac{\ln(\bar{G}_{ij}(x_i,x_j))}{\partial x_i\partial x_j}\geq 0\Leftrightarrow\frac{1}{[g(\bar{F}(x_i,x_j))]^2}\Bigg\{&f(x_i,x_j)g'[\bar{F}(x_i,x_j)]g[\bar{F}(x_i,x_j)]\\&+\frac{\partial\bar{F}(x_i,x_j)}{\partial x_i}\frac{\partial\bar{F}(x_i,x_j)}{\partial x_j}g(\bar{F}(x_i,x_j))g''[\bar{F}(x_i,x_j)]\\&-\frac{\partial\bar{F}(x_i,x_j)}{\partial x_i}\frac{\partial\bar{F}(x_i,x_j)}{\partial x_j}(g'(\bar{F}(x_i,x_j)))^2\Bigg\}\geq 0.\end{aligned} \tag{22}$$

The inequality in (22) holds, if $g(\bar{F}(x_i,x_j))g''[\bar{F}(x_i,x_j)]-(g'(\bar{F}(x_i,x_j)))^2\geq 0$. Thus, $\bar{G}_{ij}(x_i,x_j)$ is TP2, if $g$ is log-convex. On the other hand, according to Theorem 4.4 in Ahmed et al. [1], the vector $(J_1,\ldots,J_m)$ is RTIS, if the corresponding multivariate survival function is TP2 in each pair of arguments for fixed values of the remaining arguments. Since $g$ is log-convex, we can say $\bar{G}_{ij}(x_i,x_j)$ is TP2 for all $i\neq j$, and hence, the vector $(J_1,\ldots,J_m)$ is RTIS. ■

Based on Proposition 8 and distortion functions in Table 1, we can construct multivariate models with the positive dependence structure.

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## 7. CONCLUSION

In this paper, a new shock model is constructed that extends the well-known MO distribution by transformation. This extension preserves the lower and upper tail dependence of the MO model, making it more flexible for modeling by adding a distortion function to the MO model. Based on this shock model, we have given a stochastic comparison. Several dependence properties, such as dependence structure, association measures, tail dependence measures, singular component and Kendall's distribution, are given. For application, we have fitted this model to specific engineering data. The multivariate case of the proposed model is given. A closed form for its corresponding copula is derived and also its dependence structure is studied.

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## APPENDIX: NORMAL EQUATIONS AND THE OBSERVED FISHER INFORMATION MATRIX

The normal equations are as follows:

$$\begin{aligned}
\frac{\partial l}{\partial \lambda_1} &= -\sum_{i=1}^{m} t_i + \theta \sum_{i=1}^{m} -t_i h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{-\theta t_i h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} \\
&\quad + \sum_{i=m_1+1}^{m_2} \frac{-\theta t_i h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} + \frac{m_1}{\lambda_1} + \frac{m_2 \lambda_1}{\lambda_1 + \lambda_3} = 0 \\
\frac{\partial l}{\partial \lambda_2} &= -\sum_{i=1}^{m} s_i + \theta \sum_{i=1}^{m} -s_i h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{-\theta s_i h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} \\
&\quad + \sum_{i=m_1+1}^{m_2} \frac{-\theta s_i h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} + \frac{m_2}{\lambda_2} + \frac{m_1 \lambda_2}{\lambda_2 + \lambda_3} = 0 \\
\frac{\partial l}{\partial \lambda_3} &= -\sum_{i=1}^{m} h(t_i, s_i; \underline{\lambda}) + \theta \sum_{i=1}^{m} -\max(t_i, s_i) h(t_i, s_i; \underline{\lambda}) \\
&\quad + \sum_{i=1}^{m_1} \frac{-\theta \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} + \sum_{i=m_1+1}^{m_2} \frac{-\theta \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} \\
&\quad + \frac{m_3}{\lambda_3} + \frac{m_1 \lambda_3}{\lambda_2 + \lambda_3} + \frac{m_2 \lambda_3}{\lambda_1 + \lambda_3} = 0 \\
\frac{\partial l}{\partial \theta} &= \sum_{i=1}^{m} h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} + \sum_{i=m_1+1}^{m_2} \frac{h(t_i, s_i; \underline{\lambda})}{1 + \theta h(t_i, s_i; \underline{\lambda})} \\
&\quad + m \frac{e^{\theta} - 1 - \theta e^{\theta}}{\theta(e^{\theta} - 1)}
\end{aligned}$$

The observed Fisher information matrix is given as follows:

$$I = -E \begin{bmatrix}
\frac{\partial^2 l}{\partial \lambda_1^2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 l}{\partial \lambda_1 \partial \theta} \\
\frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_2^2} & \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 l}{\partial \lambda_2 \partial \theta} \\
\frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 l}{\partial \lambda_3^2} & \frac{\partial^2 l}{\partial \lambda_3 \partial \theta} \\
\frac{\partial^2 l}{\partial \theta \partial \lambda_1} & \frac{\partial^2 l}{\partial \theta \partial \lambda_2} & \frac{\partial^2 l}{\partial \theta \partial \lambda_3} & \frac{\partial^2 l}{\partial \theta^2}
\end{bmatrix}$$

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such that

$$\frac{\partial^2 l}{\partial \lambda_1^2} = \theta \sum_{i=1}^{m} t_i^2 h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{\theta t_i^2 h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta t_i^2 h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \frac{m_1}{\lambda_1^2} + \frac{m_2 \lambda_3}{(\lambda_1+\lambda_3)^2}$$

$$\frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} = \theta \sum_{i=1}^{m} t_i s_i h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{\theta t_i s_i h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta t_i s_i h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2}$$

$$\frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_1} = \theta \sum_{i=1}^{m} t_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{\theta t_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta t_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \frac{m_2 \lambda_1}{(\lambda_1+\lambda_3)^2}$$

$$\frac{\partial^2 l}{\partial \theta \partial \lambda_1} = -\sum_{i=1}^{m_1} \frac{\theta t_i h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \sum_{i=m_1+1}^{m_2} \frac{\theta t_i^2 h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2}$$

$$\frac{\partial^2 l}{\partial \lambda_2^2} = \theta \sum_{i=1}^{m} s_i^2 h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{\theta s_i^2 h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta s_i^2 h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \frac{m_2}{\lambda_2^2} + \frac{m_1 \lambda_3}{(\lambda_2+\lambda_3)^2}$$

$$\frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_2} = \theta \sum_{i=1}^{m} s_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \frac{\theta s_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta s_i \max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \frac{m_1 \lambda_2}{(\lambda_2+\lambda_3)^2}$$

$$\frac{\partial^2 l}{\partial \theta \partial \lambda_2} = -\sum_{i=1}^{m} s_i h(t_i, s_i; \underline{\lambda}) - \sum_{i=1}^{m_1} \frac{\theta s_i h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \sum_{i=m_1+1}^{m_2} \frac{\theta s_i h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2}$$

$$\frac{\partial^2 l}{\partial \lambda_3^2} = \sum_{i=1}^{m} \max(t_i, s_i) h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m} \theta [\max(t_i, s_i)]^2 h(t_i, s_i; \underline{\lambda}) + \sum_{i=1}^{m_1} \theta [\max(t_i, s_i)]^2 h(t_i, s_i; \underline{\lambda}) + \sum_{i=m_1+1}^{m_2} \theta [\max(t_i, s_i)]^2 h(t_i, s_i; \underline{\lambda}) - \frac{\lambda_2 m_1}{(\lambda_2+\lambda_3)^2} - \frac{\lambda_1 m_2}{(\lambda_1+\lambda_3)^2} - \frac{m_3}{\lambda_3^2}$$

$$\frac{\partial^2 l}{\partial \theta \partial \lambda_3} = -\theta \sum_{i=1}^{m} \max(t_i, s_i) h(t_i, s_i; \underline{\lambda}) - \sum_{i=1}^{m_1} \frac{\max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \sum_{i=m_1+1}^{m_2} \frac{\max(t_i, s_i) h(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2}$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\sum_{i=1}^{m_1} \frac{h^2(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} - \sum_{i=m_1+1}^{m_2} \frac{h^2(t_i, s_i; \underline{\lambda})}{[1+\theta h(t_i, s_i; \underline{\lambda})]^2} + \frac{\theta(e^\theta - 1) - (e^\theta - 1)^2 + \theta^2 e^\theta}{\theta^2 (e^\theta - 1)^2}$$

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