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Sifting short intervals

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1. *Introduction*

Let $\phi(X)$ denote a function satisfying $\lim_{X\to\infty}\phi(X)=\infty$. Let $\mathscr{B}$ be a sequence of positive integers and let $T(X)=T(X,\mathscr{B},\phi)$ be the set of real numbers y, $1\leqslant y\leqslant X$, such that the interval $(y,y+\phi(y)]$ contains no member of $\mathscr{B}$.

We consider, for certain natural sequences $\mathscr{B}$, the problem of finding functions ϕ which grow relatively slowly, yet are such that the Lebesgue measure $\mu(T)$ satisfies

$$(*)\qquad \lim_{X\to\infty} X^{-1}\mu(T(X))=0.$$

Selberg (10) showed, in the case where $\mathscr{B}$ is the set of primes, that, if one assumes the Riemann Hypothesis, then (*) holds with $\phi(X)=g(X)\mathscr{L}^2$, where here, as throughout the paper, $\mathscr{L}=\log X$ and $g(X)$ denotes an arbitrary function satisfying $\lim_{X\to\infty}g(X)=\infty$. Recently, Heath-Brown (2) has shown that if one assumes, in addition, the pair correlation conjecture of Montgomery, then one may take $\phi(X)=g(X)\mathscr{L}$, a result which seems likely to be best possible.

Selberg (10) also introduced a method which produces unconditional results and which, when combined with the Huxley density theorem, allows $\phi(X)=X^A$ for any fixed $A>\frac{1}{6}$ (see (9)).

Heath-Brown (1) seems to have been the first to consider this problem for almost-primes P_r, using sieve methods to obtain $\phi(X)=X^{\frac{1}{11}}$ for $\mathscr{B}=P_2$ and $\phi(X)=\mathscr{L}^{35+\varepsilon}$ for $\mathscr{B}=P_3$. (Here P_r denotes the set of integers having no more than r prime factors, multiplicity included.) Subsequently Wolke (11), building on an analytic argument of Motohashi (8), obtained P_2 with $\phi(X)=\mathscr{L}^A$ for a rather large value of A. (His paper mentions $A=5\cdot10^9$.) An interesting feature of this method is that, in contrast to the sieve, it produces integers with *precisely* two prime factors.

This paper was motivated by the following question. If one is willing to admit P_r for larger values of r, how far can one reduce the exponent of $\mathscr{L}$? In this direction we prove:

Theorem 1. *There exists a positive constant c such that, for any $A>5$, and all $y\leqslant x$ apart from a set of measure $o(x)$, the number of P_4 in the interval $(y,y+\log^A y]$ exceeds $c\log^{A-1}y$.*

Remarks. We give the result for P_4 because the method yields this with a modest amount of effort. We expect that it should be possible to obtain P_3 with greater care and that even P_2 is not hopelessly out of reach. This would be of interest, in part because the corresponding problem for arithmetic progressions (7) seems inaccessible

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to the Motohashi–Wolke argument. On the other hand, we are not able to lower the exponent 5 for P_r for any fixed r. Such a reduction seems to depend on a detailed analysis of the cancellation in the sifting weights. (See the remarks at the conclusion of § 3.)

As is often the case, a more satisfactory answer can be obtained for the corresponding problem where $\mathscr{B}$ consists of those integers which can be written as the sum of two squares.

THEOREM 2. *Let $g(y) \to \infty$ as $y \to \infty$. For all $y \leqslant X$ apart from a set of measure $o(X)$, the interval $(y, y+g(y)\log^{\frac{1}{2}} y]$ contains an integer which can be written as the sum of two squares.*

This result, which seems likely to be best possible, is in fact a fairly simple consequence of a result due to Hooley (3), namely:

THEOREM (Hooley). *Let $0 \leqslant \gamma < \frac{5}{3}$ be fixed. Let $\{s_n \mid n = 1, 2, \ldots\}$ denote the set of integers which can be represented as the sum of 2 squares. Then*

$$\sum_{s_n \leqslant X} (s_{n+1} - s_n)^\gamma \ll X \mathscr{L}^{\frac{1}{2}(\gamma-1)}.$$

Remark. The proof of Theorem 2 requires merely the existence of a $\gamma > 1$ with the stated property.

The arguments used to prove Theorem 1 give also an analogous result, a Brun–Titchmarsh theorem for 'most' short intervals. We state this as

THEOREM 3. *Let $A > 5$ and $\epsilon > 0$. We have for all $y \leqslant X$, apart from a set of measure $o(X)$, the upper bound*

$$\pi(y + \log^A y) - \pi(y) \leqslant (4+\epsilon)\log^{A-1} y.$$

We omit the proof which is almost identical to that for Theorem 1, but relies on the upper bound Rosser sieve. Here again one expects that the constant 4 may be improved by the same ideas that would lead to P_3 in Theorem 1.

Without further information one cannot use Hooley's result to derive a corresponding theorem for sums of two squares (or a lower bound of the correct order of magnitude). One can prove, however, by employing the $\frac{1}{2}$-residue sieve in conjunction with the method for Theorem 3:

THEOREM 4. *There exists a positive constant c such that, for $A > \frac{5}{2}$, we have for $y \leqslant X$, apart from a set of measure $o(X)$, the upper bound*

$$\sum_{y < s_n \leqslant y + \log^A y} 1 < c \log^{A-\frac{1}{2}} y.$$

2. *Relevant tools*

The proof of Theorem 1 rests on an application of the lower bound Rosser sieve. The first lemma, which borrows from that theory only what is needed for this purpose, is contained in (6). Although the error term here is given in somewhat different form, the validity of this formulation is implicit in the proof given in (6). An analogous statement applies to the Jurkat–Richert sieve; see the remark in ((1), p. 359).

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Let $\mathscr{A}$ denote the set of integers in the interval $(y, y(1+1/V)]$. Let $S(\mathscr{A}, Z)$ denote the number of integers in $\mathscr{A}$, having no prime factor $< Z$.

Let $\psi(t) = t - [t] - \frac{1}{2}$ and let

$$r_y(d) = \psi\left(\frac{y}{d}\right) - \psi\left(\frac{y}{d}\left(1+\frac{1}{V}\right)\right).$$

LEMMA 1. *Let $2 \leqslant Z \leqslant D$. There exists a function $f(u)$ which is positive for $u > 2$, and a sequence λ_d of real numbers, this sequence depending only on D and Z, and satisfying $|\lambda_d| \leqslant 1$, such that*

$$S(\mathscr{A}, Z) > \frac{y}{V}\prod_{p<Z}\left(1-\frac{1}{p}\right)\left\{f\left(\frac{\log D}{\log Z}\right)+O(\log^{-\frac{1}{3}} D)\right\} - |R_y(D)|,$$

where

$$R_y(D) = \sum_{d<D}\lambda_d r_y(d).$$

For real t, let $e(t) = e^{2\pi i t}$ and denote by $\|t\|$, the distance from t to the nearest integer.

LEMMA 2. *For $H > 0$ we have*

$$\psi(t) = \frac{i}{2\pi}\sum_{0<|h|<H}\frac{1}{h}e(ht) + O\left(\min\left\{1, \frac{1}{H\|t\|}\right\}\right)$$

Proof. See ((4), p. 15) and note that, for $\|t\| < H^{-1}$,

$$\sum_{0<|h|<H}\frac{1}{h}|e(ht)| = 2\sum_{1\leqslant h<H}\frac{|\sin(2\pi ht)|}{h} = 2\sum_{1\leqslant h<H}\frac{|\sin(2\pi h\|t\|)|}{h} \ll H\|t\|.$$

LEMMA 3. *For τ the usual divisor function, we have*

$$\sum_{k\leqslant 2z}\frac{1}{k}\sum_{b\leqslant z}\tau(b)\tau(b+k) \ll z\log^3 z.$$

Proof. Following Ingham's proof of (1) in (5), we have, uniformly for $k \leqslant 2z$,

$$\sum_{b\leqslant z}\tau(b)\tau(b+k) \ll z\log^2 z\sum_{d|k}\frac{1}{d}.$$

Insertion of this in the above sum and an interchange of the order of summation yield the result.

3. *Error term estimates*

In this section we obtain an upper bound for $|R_y(D)|$ on average which allows, for most y, a non-trivial application of Lemma 1.

LEMMA 4. *Let $4 \leqslant D^2 \leqslant V/2 \leqslant X/2\mathscr{L}$, and*

$$J = \int_X^{2X}|R_y(D)|\,dy.$$

For some positive constant c_1, we have

$$J < c_1 X^{\frac{3}{2}}\mathscr{L}^{\frac{3}{2}}V^{-\frac{1}{2}}.$$

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Proof. We begin by replacing in J, each $r_y(d)$ by series of the type described in Lemma 2, with $H = d$. This gives

$$J = \frac{1}{2\pi} S + O(T),$$

where

$$S = \int_X^{2X} \left| \sum_{d<D} \lambda_d \sum_{0<|h|<d} \frac{1}{h} \{e(hyd^{-1}) - e(hyd^{-1}(1+V^{-1}))\} \right| dy,$$

and

$$T = \sum_{d<D} \sum_{m<d} \frac{1}{m} I(d,m).$$

Here $I(d,m)$ is the measure of the set of y in $[X, 2X]$ for which either

$$\frac{m-1}{d} \leqslant \left\| \frac{y}{d} \right\| < \frac{m}{d} \quad \text{or} \quad \frac{m-1}{d} \leqslant \left\| \frac{y(1+1/V)}{d} \right\| < \frac{m}{d}.$$

Since, for each m, we have $I(d,m) \ll X/d$, it follows that

$$J = \frac{1}{2\pi} S + O(X\mathscr{L}^2).$$

By the Cauchy–Schwarz inequality,

$$S \leqslant X^{\frac{1}{2}} I^{\frac{1}{2}},$$

where

$$I = \sum_{d_i<D} \lambda_{d_1} \lambda_{d_2} \sum_{0<|h_i|<d_i} \frac{1}{h_1 h_2} \int_X^{2X} \theta(y)\, dy,$$

and, in turn,

$$\theta(y) = \left\{ e\left(\frac{h_1 y}{d_1}(1+V^{-1})\right) - e\left(\frac{h_1 y}{d_1}\right) \right\} \left\{ e\left(\frac{-h_2 y}{d_2}(1+V^{-1})\right) - e\left(\frac{-h_2 y}{d_2}\right) \right\}.$$

Let I' denote the contribution to I from terms for which $h_1/d_1 = h_2/d_2$. Given such a pair, we write $h_1 = mh_0$, $d_1 = md_0$, $h_2 = nh_0$, $d_2 = nd_0$ where $(h_0, d_0) = 1$. Since

$$|\theta(y)| = 4 \left| \sin\left(\frac{\pi h_1 y}{V d_1}\right) \sin\left(\frac{\pi h_2 y}{V d_2}\right) \right|,$$

this gives

$$I' \ll \sum_{d_0<D} \sum_{\substack{0<|h_0|<d_0 \\ (h_0, d_0)=1}} \frac{1}{h_0^2} \sum_{\substack{m<D \\ n<D}} \frac{1}{mn} \int_X^{2X} \sin^2\left(\frac{\pi h_0 y}{V d_0}\right) dy.$$

Using the estimate $|\sin t| \leqslant |t|$ for those terms with $|h_0| < d_0 V/X$, and the estimate $|\sin t| \leqslant 1$ for the others, we obtain

$$I' \ll X^2 V^{-1} \mathscr{L}^3.$$

It remains to estimate the contribution I'' to I from those pairs for which $h_1/d_1 \neq h_2/d_2$

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and by symmetry it may be assumed that the former fraction is the larger. Inserting into each factor of $\theta(y)$ the formula

$$e^{au_2} - e^{au_1} = \int_{u_1}^{u_2} a\, e^{au}\, du,$$

we obtain

$$I'' \ll \Sigma^* \frac{1}{d_1 d_2} \int_1^{1+V^{-1}} du \int_1^{1+V^{-1}} dv \left| \int_X^{2X} y^2 e((h_1 u d_1^{-1} - h_2 v d_2^{-1})\, y)\, dy \right|,$$

where Σ^* denotes summation over h_1, d_1, h_2, d_2 subject to the conditions $0 < |h_i| < d_i < D$ and $h_2 d_1 < h_1 d_2$. For any such term we have $h_1 d_1^{-1} - h_2 d_2^{-1} \geqslant 1/d_1 d_2 > D^{-2}$ and, for any reals u, v in the interval $[1, 1 + V^{-1}]$, we have

$$h_1 u d_1^{-1} - h_2 v d_2^{-1} \geqslant h_1 d_1^{-1} - h_2 d_2^{-1} - V^{-1} > \tfrac{1}{2}(h_1 d_1^{-1} - h_2 d_2^{-1}).$$

Integrating by parts twice

$$\int_X^{2X} y^2 e(\alpha y)\, dy \ll X^2 |\alpha|^{-1},$$

and so

$$I'' \ll X^2 V^{-2} \Sigma^* \frac{1}{h_1 d_2 - h_2 d_1} \ll X^2 V^{-2} \sum_{k < 2D^2} \frac{1}{k} \sum_{b < D^2} \tau(b)\, \tau(b+k).$$

By Lemma 3,

$$I'' \ll X^2 V^{-2} D^2 \mathscr{L}^3 \ll X^2 V^{-1} \mathscr{L}^3.$$

This completes the proof of Lemma 4.

Remarks. An examination of the above proof shows that the sum I' exerts a limitation on the length of the intervals whereas the sum I'' controls the level D of distribution.

The former sum can be estimated more carefully as

$$I' \ll \frac{X^2}{V} \sum_{d_0 < D} \phi(d_0) \prod_{\substack{p \mid d_0 \\ p > d_0 V/X}} \left(1 - \frac{1}{p}\right)^{-1} \left(\sum_{\substack{m < D \\ m \equiv 0 (d_0)}} \frac{\lambda_m}{m} \right)^2.$$

It seems to us fairly reasonable to conjecture that

$$\sum_{d_0 < D} \phi(d_0) \left(\sum_{\substack{m < D \\ m \equiv 0 (d_0)}} \frac{\lambda_m}{m} \right)^2 \ll (\log Z)^{-1}.$$

This conjecture if true would allow one to replace the constant 5 by 1 in Theorem 1. (The choices of H and D in the last lemma would have to be slightly modified.) Similar remarks apply to Theorems 3 and 4. They also apply to other sieves.

Our choice of the combinatorial sieve is motivated by the trivial estimate $|\lambda_d| \leqslant 1$, whereas for the Selberg sieve the corresponding trivial estimate is

$$|\rho_d| = |\Sigma_{[d_1, d_2] = d} \lambda_{d_1} \lambda_{d_2}| \leqslant 3^{\nu(d)}.$$

However, it may well be easier to detect the relevant cancellation in this case.

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4. *Proof of Theorem* 1

We estimate trivially the exceptional set of $y \leqslant x^{\frac{1}{2}}$ and dissect the interval $(x^{\frac{1}{2}}, x]$ into disjoint sub-intervals of the form $(X, 2X]$ (where the smallest one may pass below $x^{\frac{1}{2}}$).

For a sub-interval of this form we let $V = 2X\mathscr{L}^{-A}$ and note that, for any y in $(X, 2X]$, we have $y/V \leqslant \log^A y$.

Let $D = \frac{1}{2}V^{\frac{1}{2}}$ and, with fixed $\epsilon > 0$, let $Z = D^{\frac{1}{2}-\epsilon}$. Then, provided $\epsilon < 1/10$ and x is sufficiently large, Lemma 1 gives, for each y in any of the sub-intervals,

$$S(\mathscr{A}, Z) > c(\epsilon)\mathscr{L}^{A-1} - |R_y(D)|.$$

Moreover $S(\mathscr{A}, Z)$ counts only integers with four or fewer prime factors.

Let $\delta > 0$ and suppose, if possible, that in $(X, 2X]$ the set of y for which the number of P_4 in $(y, y+\log^A y]$ is less than $\frac{1}{2}c(\epsilon)$ has measure $> \delta X/2$. The same is then true, a fortiori, if $\log^A y$ is replaced by y/V. Thus

$$\int_X^{2X} |R_y(D)|\, dy > \frac{\delta c(\epsilon)}{4} X\mathscr{L}^{A-1}.$$

By Lemma 4,

$$\frac{\delta c(\epsilon)}{4} X\mathscr{L}^{A-1} < c_1 X^{\frac{3}{2}}\mathscr{L}^{\frac{3}{2}}(2X\mathscr{L}^{-A})^{-\frac{1}{2}},$$

which, for large X, is a contradiction if $A > 5$.

Summation over the sub-intervals shows that the set of y in $[1, x]$ for which $(y, y+\log^A y]$ contains fewer than $\frac{1}{2}c(\epsilon)\mathscr{L}^{A-1}$ numbers P_4, has measure

$$< \frac{\delta}{2}x + x^{\frac{1}{2}} < \delta x \quad \text{for large } x.$$

5. *Proof of Theorem* 2

As before, it suffices to estimate the exceptional set in $(X^{\frac{1}{2}}, X]$. Let $M = M(X)$ denote the infimum of $g(y)$ for y in $(X^{\frac{1}{2}}, X]$ and take $X > 1$ sufficiently large that $M > 0$. Since, if the result is true, it holds a fortiori when the gaps are lengthened, we may assume $M < \mathscr{L}$.

Let $\{s_j\}$ denote the integers in $(X^{\frac{1}{2}}, X]$ that are representable as the sum of two squares and let $d_j = s_{j+1} - s_j$. Partition the interval $(X^{\frac{1}{2}}, X]$ into disjoint sub-intervals of length $M\mathscr{L}^{\frac{1}{2}}$ (the smallest one may pass below $X^{\frac{1}{2}}$), and refine the partition by including the s_j. Let $\{\sigma_j\}$ be the points of this partition and let $\delta_j = \sigma_{j+1} - \sigma_j$.

We associate to each exceptional y a gap in this partition. If y in $(X^{\frac{1}{2}}, X]$ is such that the interval $(y, y+g(y)\log^{\frac{1}{2}} y]$ contains no s_j, then the interval $(y, y+M\sqrt{(\mathscr{L}/2)}]$ contains at most one σ_j and hence intersects a gap whose length δ_k satisfies $\delta_k \geqslant M\sqrt{(\mathscr{L}/8)} > \frac{1}{3}M\mathscr{L}^{\frac{1}{2}}$.

Also, this gap can be associated to exceptional y only from the interval $[\sigma_k - M\sqrt{(\mathscr{L}/2)}, \sigma_{k+1}]$ of measure $< 2M\mathscr{L}^{\frac{1}{2}}$. Assume now that the measure of the exceptional y in $(X^{\frac{1}{2}}, X]$ is αX, and let γ be fixed subject to $1 < \gamma < \frac{5}{3}$.

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We have,

$$\sum_j \delta_j^\gamma \geqslant \tfrac{1}{2}\alpha X 3^{-\gamma}(M\mathscr{L}^{\frac{1}{2}})^{\gamma-1}.$$

On the other hand, since $\gamma > 1$ and $\{\sigma_j\}$ is a refinement of $\{s_j\}$, we have by Hooley's result,

$$\sum_j \delta_j^\gamma \leqslant \sum_j d_j^\gamma \ll X\mathscr{L}^{\frac{1}{2}(\gamma-1)}.$$

Thus $\alpha \ll M^{1-\gamma}$, and, since $M \to \infty$ as $X \to \infty$, we must have $\alpha \to 0$.

REFERENCES

(1) Heath-Brown, D. R. Almost-primes in arithmetic progressions and short intervals. *Math. Proc. Cambridge Philos. Soc.* **83** (1978), 357–375.

(2) Heath-Brown, D. R. Gaps between primes, and the pair correlation of zeros of the zeta-function. *Acta Arith.* to appear.

(3) Hooley, C. On the intervals between numbers that are sums of two squares. *Acta Math.* **127** (1971), 279–297.

(4) Huxley, M. N. The distribution of prime numbers. Oxford, London, 1972.

(5) Ingham, A. E. Some asymptotic formulae in the theory of numbers. *J.L.M.S.* **2** (1927), 202–208.

(6) Iwaniec, H. Rosser's sieve. *Acta Arith.* **36** (1980), 171–202.

(7) Motohashi, Y. On almost primes in arithmetic progressions. *J. Math. Soc. Japan* **28** (1976), 363–383.

(8) Motohashi, Y. A note on almost-primes in short intervals. *Proc. Japan Acad. Ser.* A, **55** (1979), 225–226.

(9) Nicolas, J. L. Répartition des nombres largements composés. *Acta Arith.* **34** (1979), 379–390.

(10) Selberg, A. On the normal density of primes in small intervals and the difference between consecutive primes. *Arch Math. Naturvid.* **47** (1943), 87–105.

(11) Wolke, D. Fast-Primzahlen in kurzen Intervallen. *Math. Ann.* **244** (1979), 233–242.

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