ADDITIONAL NOTE ON TWO MOMENT PROBLEMS FOR BOUNDED FUNCTIONS

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1. This note is a sequel to one by the author (*Proc. Cambridge Phil. Soc.* 42 (1946), 189–96; referred to as I), and should be read in conjunction with it. Our object is to prove I, Theorem II, when $n-1$ is odd (see I, § 13). We need the following additional clause to I, Theorem I.

Lemma 1. *If $f(x)$ is a step function with a finite set of discontinuities, and equal to* 0 *or* 1, *then $\sigma(x)$ is a step function with a finite set of discontinuities; and conversely.*

In I, § 4 we proved that if f is zero in an interval, then σ is constant in that interval. The same proof shows that the result is true when $f = 1$ in an interval.

To prove the converse, let $\sigma(x)$ be a step function with discontinuities at x_k $(k = 1, \ldots, r)$. Then

$$1 + \int \frac{d\sigma}{\zeta - x} = 1 + \Sigma \frac{d_k}{\zeta - x_k},$$

where the d_k are positive; and

$$\Im \log \left(1 + \int \frac{d\sigma}{\zeta - x}\right) \to \arg u(\xi) \pmod{2\pi}$$

as $\eta \to 0$, where

$$u(\xi) = 1 + \Sigma \frac{d_k}{\xi - x_k}.$$

By I, Theorem I,

$$-\int \frac{\eta f(x)}{\eta^2 + (x-\xi)^2} dx \to \arg u(\xi) \pmod{2\pi}. \tag{1}$$

We observe that $u(\xi)$ is a decreasing function of ξ in any interval (a, b) or Δ, which does not include a point x_k. Hence, either $u > 0$ in Δ, or $u < 0$ in Δ, or there is a number c such that $u > 0$ in (a, c) and $u < 0$ in (c, b). Then, in the respective cases, the second member of (1) is, mod 2π, 0 in Δ, or $-\pi$ in Δ, or 0 in (a, c) and $-\pi$ in (c, b). Since the first member tends to $-\pi f(\xi)$ for almost all ξ, and $0 \leqslant f \leqslant 1$, it follows that f is 0 in Δ, or 1 in Δ, or 0 in (a, c) and 1 in (c, b) respectively. This proves the lemma.

It is important to notice that since f is integrable, it is zero outside some finite interval.

It will be convenient to have a short expression for 'a step function with a finite number of discontinuities'. We shall write 's-function'.

In proving I, Theorem II with $n-1$ odd, we observe that if we are given equations I (2), and f is an s-function equal to 0 or 1, then the proof of I, § 11 is valid, since all infinite integrals are absolutely convergent. And if we are given the equations I (3) with σ an s-function, then the same is true. Therefore, we may exclude these possibilities. But, then, what we have to prove is the case $n-1$ odd of the following theorem.

Theorem I. *Given the numbers $s_0, \ldots, s_{n-1}$, define $t_0, \ldots, t_{n-1}$ by the equation*

$$\exp\left(\sum_0^{n-1} \frac{s_k}{z^{k+1}}\right) = 1 + \sum_0^{n-1} \frac{t_k}{z^{k+1}} + \ldots. \tag{2}$$

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In order that there shall be a function $f(x)$, $0 \leqslant f \leqslant 1$, *which is not an s-function equal to* 0 *or* 1, *and such that*

$$s_k = \int x^k f(x)\,dx \quad (k = 0, \ldots, n-1), \tag{3}$$

it is necessary and sufficient that there shall be a non-decreasing function $\sigma(x)$, *which is not an s-function and such that*

$$t_k = \int x^k\,d\sigma \quad (k = 0, \ldots, n-1). \tag{4}$$

The case $n-1$ even (case E) of this theorem follows from I, Theorem II together with Lemma 1, if we remember that the functions f and σ are connected as in I, Theorem I. The case $n-1$ odd (case O) will be deduced from the following lemma.

2. Lemma 2. *If* $t_0, \ldots, t_m$ *are numbers which satisfy*

$$t_i = \int x^i\,d\phi \quad (i = 0, \ldots, m), \tag{5}$$

where ϕ *is non-decreasing, and not an s-function, then*

(i) *if* m *is even, and* t_{m+1} *denotes any number, there is a function* σ, *non-decreasing and not an s-function, such that*

$$t_i = \int x^i\,d\sigma \quad (i = 0, \ldots, m+1);$$

(ii) *if* m *is odd, and* t_{m+1} *denotes any sufficiently large number, the same conclusion holds.*

The lemma will be proved if we can show that there are numbers $t_{m+1}, t_{m+2}, \ldots, t_{m+1}$ having the required degree of arbitrariness, such that all the determinants

$$D_r = |\, t_{i+j} \,|_0^r \quad (r = 0, 1, \ldots)$$

are positive (see Widder, *The Laplace Transform* (1941), p. 135).

Let $2s$ denote the greatest even integer which does not exceed m. Then

$$\sum_{i,j=0}^{s} a_i a_j t_{i+j} = \int (\Sigma a_i x^i)^2\,d\phi > 0,$$

unless all the a's are zero, since ϕ is not an s-function. Thus, the quadratic form is positive-definite. Hence $\qquad D_r > 0 \quad (r = 0, \ldots, s).$

If m is odd, we can choose t_{m+1} so large as to make $D_{s+1} > 0$, since the coefficient of t_{m+1} in D_{s+1} is D_s. If m is even, we give t_{m+1} any value, and then choose t_{m+2} so large that $D_{s+1} > 0$. This process can be continued, and gives the desired sequence.

3. We can now deduce the case O of Theorem I from case E. Given the n equations (3), the first $n-1$ of them imply

$$t_k = \int x^k\,d\phi \quad (k = 0, \ldots, n-2), \tag{6}$$

where ϕ is non-decreasing and not an s-function, and

$$\exp\left(\sum_0^{n-1} \frac{s_k}{z^{k+1}}\right) = 1 + \sum_0^{n-2} \frac{t_k}{z^{k+1}} + \ldots. \tag{7}$$

This is an application of case E. The t's of (7) are the same as in (2). By Lemma 2 (i) ($m = n-2$), whatever number t_{n-1} we choose, and in particular if we choose it as the

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number so denoted in (2), then there is a non-decreasing function σ, which is not an s-function, and which satisfies (4).

Conversely, given the n equations (4), then by Lemma 2 (ii) we can choose t_n sufficiently large so that

$$t_k = \int x^k dg \quad (k = 0, \ldots, n), \tag{8}$$

where g is non-decreasing and is not an s-function. By case E, (8) implies

$$s_k = \int x^k f(x)\, dx \quad (k = 0, \ldots, n),$$

where f is not an s-function equal to 0 or 1, $0 \leqslant f \leqslant 1$, and the s_k are defined by

$$\exp\left(\sum_0^n \frac{s_k}{z^{k+1}}\right) = 1 + \sum_0^n \frac{t_k}{z^{k+1}} + \ldots.$$

But then $s_0, \ldots, s_{n-1}$ are the same as in (2). Hence equations (3) are true.

4. *Correction to* I. There is a passage in I § 1 which, though it does not affect what is subsequently proved, may give rise to a misunderstanding. It is there implied that in order that the equations I (3) shall be satisfied, it is necessary and sufficient that

$$\sum_{i,k=0}^{m} t_{i+k} x_i x_k$$

shall be positive definite or semi-definite, where $2m$ is the greatest even integer which does not exceed $n-1$. This implication is incorrect; and the mistake arose from making small changes in the enunciation of Achyèser and Krein's results.

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RANK CORRELATION AND PERMUTATION DISTRIBUTIONS

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Suppose that we have n objects denoted by $1, \ldots, n$, and suppose them to be rearranged so that the rank of the object i is $p(i)$. Let s be the least number of interchanges of neighbours required to restore the permutation to the normal order. Now let $u(n, s)$ be the number of distinct arrangements which require s interchanges. Then Haden (1) has proved that

$$u(n, s) = \frac{6n!}{\{\pi n(n-1)(2n+5)\}^{\frac{1}{2}}}\left[\exp\left\{\frac{-36t^2}{n(n-1)(2n+5)}\right\} + O\left(\frac{1}{n}\right)\right],$$

where $t = s - \frac{1}{4}n(n-1)$, and we have used the symbol t where Haden uses σ, since the latter has statistical associations. This theorem is very similar to the theorem established by Kendall (2, 3) (see also Feller (4), Dantzig (5)), which states that his coefficient of rank correlation, τ, between the order $1, 2, \ldots, n$ and $\{p(i)\}$, is such that

$$S = \tfrac{1}{2}n(n-1)\tau$$

is distributed about a zero mean with variance $\frac{1}{18}n(n-1)(2n+5)$ in a distribution

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