Canad. Math. Bull. 2026, pp. 1–19
http://dx.doi.org/10.4153/S0008439526102598
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

Monomial ideals whose all matching powers are Cohen–Macaulay

Antonino Ficarra and Somayeh Moradi

Abstract. In the present article, we aim to classify monomial ideals whose all matching powers are Cohen–Macaulay. We especially focus our attention on edge ideals. The Cohen–Macaulayness of the last matching power of an edge ideal is characterized, providing an algebraic analog of the famous Tutte theorem regarding graphs having a perfect matching. For chordal graphs, very well-covered graphs, and Cameron–Walker graphs, we completely solve our problem.

1 Introduction

Let S be either a regular local ring, or the polynomial ring $S = K[x_1, \dots, x_n]$ over a field K, and let $I \subset S$ be an ideal. We say that I is Cohen–Macaulay if S/I is a Cohen–Macaulay ring. Assume, moreover, that I is radical. Then, a classical result in commutative algebra guarantees that I is a complete intersection if and only if I^k is Cohen–Macaulay for infinitely many k. This result was proved in the late seventies, more or less independently, by Achilles and Vogel in [1], by Cowsik and Nori in [7], and by Waldi in [43].

In particular, let $I = I(G)$ be the edge ideal of a finite simple graph G on the vertex set $\{x_1, \dots, x_n\}$, that is, the ideal in the polynomial ring $S = K[x_1, \dots, x_n]$ generated by those monomials $x_i x_j$ for which $\{x_i, x_j\}$ is an edge. Hereafter, we will say that G is a Cohen–Macaulay graph, if $I(G)$ is Cohen–Macaulay. We have that $I(G)$ is a radical ideal, and the aforementioned result implies that all powers $I(G)^k$ are Cohen–Macaulay if and only if $E(G)$ consists of disjoint edges.

The situation becomes more interesting if one turns its attention to other kind of powers. The first paper in this direction [34], written by Minh and Trung, characterizes those unmixed two-dimensional monomial ideals $I \subset S$ whose (almost) all symbolic powers $I^{(k)}$ are Cohen–Macaulay. After the appearance of this article, many researchers tried to characterize all those unmixed monomial ideals $I \subset S$ whose (almost) all symbolic powers are Cohen–Macaulay, culminating with the beautiful solution due, independently, to Minh and Trung [35], and Varbaro [42]. Writing any squarefree monomial ideal I as the Stanley–Reisner ideal I_Δ of a simplicial complex

Received by the editors August 18, 2025 UTC; revised August 9, 2026 UTC; accepted August 31, 2026 UTC.

A.F. was partly supported by the Istituto Nazionale di Alta Matematica (INDAM), and also by the Grant JDC2023-051705-I funded by MICIU/AEI/10.13039/501100011033 and by the FSE+. S.M. is supported by the Alexander von Humboldt Foundation.

AMS subject classification: **13C05**, **13C14**, **13C15**, **05E40**.

Keywords: Cohen–Macaulay ring, matching power, edge ideal.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

Δ, the aforementioned authors proved, independently, that $I_\Delta^{(k)}$ is Cohen–Macaulay for all $k \gg 0$ if and only if Δ is a matroid.

Very recently, in [3, 17], the concept of squarefree power was introduced. Subsequently, this concept was extended to all monomial ideals with the introduction of *matching powers* [16] (see also [11, 13, 15, 18, 19, 21, 22, 30, 37–40]). Hereafter, we always refer to this generalized concept. Let $I \subset S$ be a monomial ideal, and let $\mathcal{G}(I)$ be its minimal monomial generating set. The kth matching power of I is the monomial ideal generated by the products $u_1 \dots u_k$, where the $u_i \in \mathcal{G}(I)$ form a regular sequence. In other words, $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing$ for all $1 \le i < j \le k$, where the *support* of a monomial $w \in S$ is defined as the set $\operatorname{supp}(w) = \{x_i : x_i \text{ divides } w\}$. Moreover, the *monomial grade* of I is defined as $\nu(I) = \max\{k : I^{[k]} \neq 0\}$. A *matching* of G is a family of pairwise disjoint edges of G. Let $\nu(G)$ be the maximum size of a matching of G. Then, it is easy to see that $\nu(I(G)) = \nu(G)$. Furthermore, the generators of $I(G)^{[k]}$ correspond to the vertex sets of the k-matchings of G, which motivates the choice to name the ideals $I^{[k]}$ the matching powers of I. Our goal is to identify and classify those monomial ideals $I \subset S$ whose matching powers $I^{[k]}$ are Cohen–Macaulay, for all $1 \le k \le \nu(I)$.

Recently, in [13], Das et al. proved that $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \le k \le \nu(G)$, if G is a Cohen–Macaulay forest. Moreover, it is proven in [16] and [15] that $I(G)^{[\nu(G)]}$ is a polymatroidal ideal for any graph G. By a result due to Herzog and Hibi [26], the Cohen–Macaulay polymatroidal ideals are of very special nature. These two facts suggest that there should be a pleasant theory regarding the Cohen–Macaulayness of the matching powers of an edge ideal. This is what we consider in this article. More precisely, we study when all matching powers of $I(G)$ are Cohen–Macaulay. We characterize such graphs G when G has a perfect matching and the $(\nu(G)-1)$th matching power of $I(G)$ has height 2. These conditions are fulfilled when G is a Cohen–Macaulay very well-covered graph. Chordal graphs and Cameron–Walker graphs are other families for which we give such characterization. We will also discuss the situation for the graphs having up to seven vertices.

The article is structured as follows. In Section 2, we investigate the hereditary nature of the Cohen–Macaulay property of $I(G)^{[k]}$. It is shown that if $I(G)^{[k]}$ possesses the Cohen–Macaulay property, then so does $I(G')^{[k]}$, where G' is either the subgraph $G\backslash N_G[x]$ for any vertex x of G or a connected component of G (see Theorem 2.5 and Corollary 2.6). Another main result in this section is Theorem 2.8, which gives a purely combinatorial characterization for the edge ideals whose last non-vanishing matching power is Cohen–Macaulay. More precisely, it is shown that $I(G)^{[\nu(G)]}$ is Cohen–Macaulay if and only if either G has a perfect matching, or $G\backslash\{i\}$ has a perfect matching of size $|V(G)|-1$ for all $i \in V(G)$. This result can be seen as an algebraic analog of the famous Tutte theorem in classical graph theory [41]. Recall that Tutte's theorem gives necessary and sufficient graph-theoretic conditions for a graph to admit a perfect matching. Similarly, Theorem 2.8 characterizes the existence of a perfect (or near-perfect) matching through an algebraic property of the squarefree power $I(G)^{[\nu(G)]}$. The characterization given in Theorem 2.8 plays an important role in the study of edge ideals of graphs whose all matching powers are Cohen–Macaulay.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

In Section 3, we concentrate our attention on the family of graphs which admit a perfect matching. In Theorem 3.2, we characterize the graphs G from this family for which all matching powers $I(G)^{[k]}$ are Cohen–Macaulay, provided that $\dim S/I(G)^{[\nu(G)-1]} = |V(G)| - 2$. This dimension can take only the two values $|V(G)| - 2$ and $|V(G)| - 3$, as is shown in Lemma 3.4. To prove Theorem 3.2, we use the Hilbert–Burch theorem and a well-known result from graph theory on the characterization of trees. An interesting family of graphs which meets the conditions of Theorem 3.2 is the family of the Cohen–Macaulay very well-covered graphs. Using a characterization of this family due to Crupi et al. [12], in Proposition 3.7, we give a formula for the dimension of $S/I(G)^{[k]}$ for all k. In particular, when $k = \nu(G) - 1$, this dimension is $|V(G)| - 2$. So, applying our main theorem to this family we get Corollary 3.8, which shows that the only very well-covered graphs for which all matching powers of their edge ideals are Cohen–Macaulay are only the Cohen–Macaulay forests. Bipartite graphs and whisker graphs are subfamilies of very well-covered graphs. Applying our result to these families, we obtain the analogous characterizations (see Corollaries 3.9 and 3.10). We close this section by obtaining the normalized depth function of edge ideals of very well-covered graphs G such that $I(G)^{[k]}$ is Cohen–Macaulay, for all $1 \leq k \leq \nu(G)$ (see Corollary 3.11).

In Section 4, we answer our question on Cohen–Macaulay matching powers for the family of chordal graphs and Cameron–Walker graphs. For chordal graphs, we use a result by Herzog et al. [28] which characterizes Cohen–Macaulay chordal graphs. In Theorem 4.1, we prove that all matching powers of the edge ideal of a chordal graph are Cohen–Macaulay if and only if G is either a complete graph or a Cohen–Macaulay forest. Using the characterizations given for chordal graphs and bipartite graphs, in Theorem 4.5, we answer our question for the family of Cameron–Walker graphs.

In the final Section 5, using the *Macaulay2* [25] packages, `NautyGraphs` [33], and `MatchingPowers` [21], we determined, up to isomorphism, all graphs G with a small number of vertices for which $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq \nu(G)$. Our experiments would indicate that among the graphs on $n \geq 5$ non-isolated vertices, besides the complete graphs and the Cohen–Macaulay forests, there are only two extra graphs, up to isomorphism, for which the matching powers of their edge ideal is Cohen–Macaulay for all k. Furthermore, we expect that such a classification does not depend on the underlying field K of the polynomial ring $S = K[x_i : i \in V(G)]$.

2 Cohen–Macaulayness of matching powers of edge ideals

Throughout this article, G is a finite simple graph, K is a fixed field, and $S = K[x_i : x_i \in V(G)]$ is a polynomial ring over K. Since the edge ideal of G does not change by removing isolated vertices, we may extra assume that G has no isolated vertices. We denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. In this section, we show that the Cohen–Macaulayness of a matching power of $I(G)$ is inherited by its connected components and by its subgraphs $G\backslash N_G[x]$ for any vertex x of G. Moreover, we characterize those edge ideals whose last non-vanishing matching power is Cohen–Macaulay.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

Recall that for a vertex $x_i \in V(G)$, the sets

$$N_G(x_i) = \{x_j \in V(G) : \{x_i, x_j\} \in E(G)\},$$
$$N_G[x_i] = N_G(x_i) \cup \{x_i\},$$

are called the *open neighborhood* and the *closed neighborhood* of x_i in G.

The following lemma is needed for the proof of Theorem 2.5.

Lemma 2.1 *Let $I \subset S$ be a monomial ideal and $u \in S$ be a monomial with $u \notin I$. If S/I is Cohen–Macaulay, then $S/(I : u)$ is Cohen–Macaulay and moreover depth S/I = depth $S/(I : u)$.*

Proof By [6, Lemma 4.1], depth $S/I \leq$ depth $S/(I : u)$. Combining this with the inequalities depth $S/(I : u) \leq \dim S/(I : u) \leq \dim S/I$ implies the assertion. ■

For what follows, we need to recall some concepts and results from [14, 23]. Let $I \subset S$ be a monomial ideal. By $\mathcal{G}(I)$, we denote the unique minimal set of monomial generators of I. Let $J, L \subset S$ be monomial ideals such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(L)$. Following [23], we say that $I = J + L$ is a *Betti splitting* if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(L) + \beta_{i-1,j}(J \cap L), \quad \text{for all } i, j \geq 0. \tag{2.1}$$

Consider the natural short exact sequence

$$0 \to J \cap L \to J \oplus L \to I \to 0. \tag{2.2}$$

By [23, Proposition 2.1], Equation (2.1) holds if and only if the induced maps

$$\operatorname{Tor}_i^S(K, J \cap L) \to \operatorname{Tor}_i^S(K, J) \oplus \operatorname{Tor}_i^S(K, L)$$

in the Tor exact sequence arising from the above sequence are zero for all $i \geq 0$. In this case, we say that the inclusion map $J \cap L \to J \oplus L$ is Tor-*vanishing*. The next result appeared implicitly firstly in [14, Proposition 3.1] (see also [36, Lemma 4.2]).

Proposition 2.2 *Let $J, L \subset S$ be nonzero monomial ideals with $J \subset L$. Suppose there exists a map $\varphi : \mathcal{G}(J) \to \mathcal{G}(L)$ such that for any $\varnothing \neq \Omega \subseteq \mathcal{G}(J)$ we have*

$$\operatorname{lcm}(u : u \in \Omega) \in \mathfrak{m}(\operatorname{lcm}(\varphi(u) : u \in \Omega)),$$

where $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$. Then, the inclusion map $J \to L$ is Tor-vanishing.

For a monomial ideal $I \subset S$, we denote by $\partial^* I$ the ideal generated by the elements of the form u/x_i, where $u \in \mathcal{G}(I)$, and x_i is a variable dividing u. The following result was proved in [36, Proposition 4.4].

Lemma 2.3 *Let $J, L \subset S$ be nonzero monomial ideals. Suppose that $\partial^* J \subseteq L$. Then, $J \subseteq \mathfrak{m}L$ and there exists a map $\varphi : \mathcal{G}(J) \to \mathcal{G}(L)$ satisfying the assumption of Proposition 2.2. In particular, the inclusion map $J \to L$ is Tor-vanishing.*

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

Lemma 2.4 *Let G be a graph, $1 \leq k \leq \nu(G)$ be an integer, and $x \in V(G)$ be a vertex. Then,*

$$(I(G)^{[k]} : x) = \sum_{y \in N_G(x)} yI(G\backslash\{x,y\})^{[k-1]} + I(G\backslash N_G[x])^{[k]}. \tag{2.3}$$

If $I(G\backslash N_G[x])^{[k]} \neq 0$, then this decomposition is a Betti splitting, and in particular,

$$\operatorname{depth} S/I(G\backslash N_G[x])^{[k]} - |N_G(x)| \geq \operatorname{depth} S/(I(G)^{[k]} : x).$$

Proof Equation (2.3) was proved in [13, Lemma 2.9(ii)]. To simplify our discussion, we set $I = (I(G)^{[k]} : x)$, $J = \sum_{y \in N_G(x)} yI(G\backslash\{x,y\})^{[k-1]}$, $L = I(G\backslash N_G[x])^{[k]}$, and we assume, up to a relabeling, that $N_G(x) = \{y_1, \ldots, y_m\}$.

Firstly, we notice that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(L)$. Indeed, all the minimal generators of J are monomials of degree $2k-1$ and are divided by some variable y_i, whereas each minimal generator of L is a monomial of degree $2k$ and is not divided by y_i, for all $1 \leq i \leq m$.

Now, we compute $J \cap L$. We have

$$\begin{aligned} J \cap L &= \Big(\sum_{i=1}^{m} y_i I(G\backslash\{x,y_i\})^{[k-1]}\Big) \cap I(G\backslash N_G[x])^{[k]} \\ &= \sum_{i=1}^{m} y_i [I(G\backslash\{x,y_i\})^{[k-1]} \cap I(G\backslash N_G[x])^{[k]}] \\ &= \sum_{i=1}^{m} y_i I(G\backslash N_G[x])^{[k]} = (y_1, \ldots, y_m) I(G\backslash N_G[x])^{[k]}. \end{aligned}$$

Here, in the second equality, we used that y_i does not divide any generator of $I(G\backslash N_G[x])^{[k]}$, and in the third equality we used, for $1 \leq i \leq m$, the chain of inclusions $I(G\backslash N_G[x])^{[k]} \subset I(G\backslash N_G[x])^{[k-1]} \subseteq I(G\backslash\{x,y_i\})^{[k-1]}$.

To prove that the inclusion map $J \cap L \to J \oplus L$ is Tor-vanishing, it is enough to show that the inclusion maps $J \cap L \to J$ and $J \cap L \to L$ are Tor-vanishing.

Notice that $\partial^* I(G\backslash N_G[x])^{[k]} \subset I(G\backslash N_G[x])^{[k-1]} \subseteq \sum_{i=1}^{m} I(G\backslash\{x,y_i\})^{[k-1]}$. Therefore, by Lemma 2.3, there exists a map

$$\varphi : \mathcal{G}(I(G\backslash N_G[x])^{[k]}) \to \mathcal{G}\Big(\sum_{i=1}^{m} I(G\backslash\{x,y_i\})^{[k-1]}\Big)$$

such that for every non-empty subset $\Omega \subseteq \mathcal{G}(I(G\backslash N_G[x])^{[k]})$, we have

$$\operatorname{lcm}(u : u \in \Omega) \in \mathfrak{m}(\operatorname{lcm}(\varphi(u) : u \in \Omega)),$$

where $\mathfrak{m}$ is the graded maximal ideal of S.

Each minimal generator of $J \cap L = (y_1, \ldots, y_m)L$ is of the form $y_i u$ with $1 \leq i \leq m$ and $u \in \mathcal{G}(I(G\backslash N_G[x])^{[k]})$. Now, we define a map $\Phi : \mathcal{G}(J \cap L) \to \mathcal{G}(J)$ by setting $\Phi(y_i u) = y_i \varphi(u)$. It is clear that $\Phi(y_i u) \in \mathcal{G}(J)$. Let $\Omega = \{y_{i_\ell} u_\ell\}_\ell \subseteq \mathcal{G}(J \cap L)$ be non-empty. Then,

$$\begin{aligned} \operatorname{lcm}(\{y_{i_\ell} u_\ell\}_\ell) = \operatorname{lcm}(\{y_{i_\ell}\}_\ell)\operatorname{lcm}(\{u_\ell\}_\ell) &\in \operatorname{lcm}(\{y_{i_\ell}\}_\ell) \cdot \mathfrak{m}(\operatorname{lcm}(\{\varphi(u_\ell)\}_\ell)) \\ &= \mathfrak{m}(\operatorname{lcm}(\{\Phi(y_{i_\ell} u_\ell)\}_\ell)). \end{aligned}$$

Thus, Proposition 2.2 implies that $J \cap L \to J$ is Tor-vanishing.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

To prove that $J \cap L \to L$ is Tor-vanishing, notice that $J \cap L = (y_1, \dots, y_m)L$ and that the variables y_i do not divide any minimal generator of L. Therefore, we can define a map $\Psi : \mathcal{G}(J \cap L) \to \mathcal{G}(L)$ by setting $\Psi(y_i u) = u$ for all $1 \le i \le m$ and all $u \in \mathcal{G}(L)$. It is clear that this map satisfies the condition in Proposition 2.2, and so the map $J \cap L \to L$ is Tor-vanishing.

Hence, Equation (2.3) is indeed a Betti splitting. It follows from Equation (2.1) that

$$\operatorname{depth} S/I = \min\{\operatorname{depth} S/J, \operatorname{depth} S/L, \operatorname{depth} S/(J \cap L) - 1\}.$$

Since $J \cap L = (y_1, \dots, y_m)L$, we have $\operatorname{depth} S/(J \cap L) = \operatorname{depth} S/L - (m-1)$. Therefore, it follows that

$$\operatorname{depth} S/L - m = \operatorname{depth} S/(J \cap L) - 1 \ge \operatorname{depth} S/I.$$

Since $m = |N_G(x)|$, the proof is complete. ∎

As a consequence, we obtain the following useful permanence property.

Theorem 2.5 *Let G be a graph and $1 \le k \le \nu(G)$ be an integer. If $I(G)^{[k]}$ is Cohen–Macaulay, then $I(G\backslash N_G[x])^{[k]}$ is Cohen–Macaulay, for any vertex $x \in V(G)$.*

Proof Consider a vertex $x \in V(G)$. We may assume that $I(G\backslash N_G[x])^{[k]} \ne 0$. Then, by Lemma 2.4, we have

$$\operatorname{depth} S/I(G\backslash N_G[x])^{[k]} - |N_G(x)| \ge \operatorname{depth} S/(I(G)^{[k]} : x).$$

Since $I(G)^{[k]}$ is Cohen–Macaulay, by Lemma 2.1, we have

$$\operatorname{depth} S/(I(G)^{[k]} : x) = \operatorname{depth} S/I(G)^{[k]}.$$

From this fact and the assumption that $I(G)^{[k]}$ is Cohen–Macaulay, we get that $\dim S/I(G\backslash N_G[x])^{[k]} - |N_G(x)| \ge \dim S/I(G)^{[k]}$. We show that indeed

$$\dim S/I(G\backslash N_G[x])^{[k]} - |N_G(x)| = \dim S/I(G)^{[k]}. \tag{2.4}$$

Let $N_G(x) = \{y_1, \dots, y_m\}$, and let $P \subset S$ be a monomial prime ideal such that $I(G\backslash N_G[x])^{[k]} \subseteq P$. Then, one can see that $I(G)^{[k]} \subset (y_1, \dots, y_m, P)$. Since any minimal prime ideal of $I(G\backslash N_G[x])^{[k]}$ is a monomial ideal generated by variables, it follows that height $I(G)^{[k]} \le$ height $I(G\backslash N_G[x])^{[k]} + |N_G(x)|$. This shows the equality (2.4). Thus, we obtain

$$\begin{aligned}\operatorname{depth} S/I(G\backslash N_G[x])^{[k]} &\ge \operatorname{depth} S/I(G)^{[k]} + |N_G(x)| = \dim S/I(G)^{[k]} + |N_G(x)| \\ &= \dim S/I(G\backslash N_G[x])^{[k]} \ge \operatorname{depth} S/I(G\backslash N_G[x])^{[k]}.\end{aligned}$$

Hence, $I(G\backslash N_G[x])^{[k]}$ is Cohen–Macaulay, as desired. ∎

The following result is an immediate consequence of Theorem 2.5.

Corollary 2.6 *Let G be a graph with the connected components $G_1, \dots, G_m$, and let $1 \le k \le \nu(G)$ be an integer. If $I(G)^{[k]}$ is Cohen–Macaulay, then $I(G_i)^{[k]}$ is Cohen–Macaulay for all i.*

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

The converse of Corollary 2.6 does not hold in general.

Example 2.7 Let G be a graph with the connected components K_2 and K_3, where K_n denotes the complete graph on n vertices. Then, $I(G)$ is Cohen–Macaulay, but $I(G)^{[2]}$ is not Cohen–Macaulay since $\dim S/I(G)^{[2]} = 4$ and $\operatorname{depth} S/I(G)^{[2]} = 3$.

Next, we characterize those edge ideals $I(G)$ having the property that their last non-vanishing matching power is Cohen–Macaulay.

It is clear that for any graph G on n non-isolated vertices we have $\nu(G) \leq \lfloor \frac{n}{2} \rfloor$.

A *perfect matching* in G is a matching M of G such that each vertex of G is an endpoint of some edge in M. So, G has a perfect matching if and only if $n = 2\nu(G)$.

Theorem 2.8 *Let G be a graph on n non-isolated vertices. The following conditions are equivalent:*

(a) $I(G)^{[\nu(G)]}$ *is Cohen–Macaulay.*
(b) $I(G)^{[\nu(G)]} = \mathfrak{m}^{[2\nu(G)]}$ *and* $\nu(G) = \lfloor \frac{n}{2} \rfloor$.
(c) *Either G has a perfect matching, or $G\backslash\{i\}$ has a perfect matching of size $|V(G)| - 1$ for all $i \in V(G)$.*

Proof (a) $\Rightarrow$ (b) By [16, Theorem 1.7] (or also [15, Theorem 1.1]), $I(G)^{[\nu(G)]}$ is a squarefree polymatroidal ideal. By Herzog and Hibi [26, Theorem 4.2] (see also [10, Corollary 3]), the only squarefree Cohen–Macaulay polymatroidal ideals are the squarefree principal ideals and the squarefree Veronese ideals $\mathfrak{m}^{[s]}$ for $1 \leq s \leq n$. Then, our assumption implies that $I(G)^{[\nu(G)]}$ is either a squarefree principal ideal or a squarefree Veronese ideal generated in degree $2\nu(G)$.

Let $k = \nu(G)$. We claim that $k = \lfloor \frac{n}{2} \rfloor$ and $I(G)^{[k]} = \mathfrak{m}^{[2k]}$.

To this end, we first show that any variable divides some generator of $I(G)^{[k]}$. Up to relabeling, we may assume that $M = \{e_i : 1 \leq i \leq k\}$, with $e_i = \{2i-1, 2i\}$ for all i, is a k-matching of G. If $2k = n$, the assertion is proved. Suppose $2k < n$. Thus, it remains to show that x_v divides some generator of $I(G)^{[k]}$ for all $2k < v \leq n$. There exists $w \in V(G)$ with $\{v, w\} \in E(G)$. We have $w \in V(M)$; otherwise, $M \cup \{\{v, w\}\}$ would be a $(k+1)$-matching of G, which is absurd because $\nu(G) = k$. Thus, $w \in e_i$ for some i. Hence, $M' = (M\backslash e_i) \cup \{\{v, w\}\}$ is a k-matching of G, and the generator of $I(G)^{[k]}$ corresponding to this matching is divided by x_v.

Now, if $2k = n$, then $I(G)^{[k]} = (x_1 x_2 \dots x_n) = \mathfrak{m}^{[n]} = \mathfrak{m}^{[2k]}$ and $k = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$. Suppose now $2k < n$. Then, since all variables divide some generator of $I(G)^{[k]}$, it follows that $I(G)^{[k]}$ is not principal. Thus, by the first paragraph of the proof, $I(G)^{[k]}$ is the squarefree Veronese ideal $\mathfrak{m}^{[2k]}$ of degree $2k$ in the variables $x_1, \dots, x_n$. It remains to show that $k = \lfloor \frac{n}{2} \rfloor$. Let K_n be the complete graph on n vertices. Then, $I(G) \subseteq I(K_n)$ and $I(G)^{[k]} = \mathfrak{m}^{[2k]} = I(K_n)^{[k]}$. Thus, [18, Proposition 1.3] implies that $I(G)^{[\ell]} = I(K_n)^{[\ell]} = \mathfrak{m}^{[2\ell]}$ for all $\ell \geq k$. Since $k = \nu(G)$, we have $I(G)^{[\ell]} = 0$ for all $\ell > k$. Hence, $\mathfrak{m}^{[2\ell]} = 0$ for all $\ell > k$, and so $k = \lfloor \frac{\nu(\mathfrak{m})}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$.

(b) $\Rightarrow$ (a) Conversely, if $\nu(G) = \lfloor \frac{n}{2} \rfloor$ and $I(G)^{[\nu(G)]} = \mathfrak{m}^{[2\nu(G)]}$, then $I(G)^{[\nu(G)]}$ is Cohen–Macaulay by the result of Herzog and Hibi [26, Theorem 4.2].

(b) $\Leftrightarrow$ (c) Let $k = \lfloor \frac{n}{2} \rfloor$. Then, either $n = 2k$ or $n = 2k+1$. Depending on these cases, we see that $I(G)^{[\nu(G)]} = \mathfrak{m}^{[2k]}$ if and only if statement (c) holds. ∎

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

3 Graphs with perfect matchings and very well-covered graphs

To study our question on Cohen–Macaulay matching powers and taking Theorem 2.8 into account, it makes sense to consider the family of graphs which have a perfect matching. This is our focus in this section. It turns out that the Cohen–Macaulayness of the $(\nu(G)-1)$th matching power for this class of graphs is crucial in order to get the answer for all matching powers. Theorem 3.2 shows this fact. In order to prove our result, we use the Hilbert–Burch theorem and the next elementary result of graph theory.

Lemma 3.1 *Let G be a connected graph on n non-isolated vertices. Then,*

$$|E(G)| \geq n-1,$$

and equality holds if and only if G is a tree.

We now state and prove our main result in the section.

Theorem 3.2 *Let G be a graph with a perfect matching of size $n \geq 2$ such that $\dim S/I(G)^{[n-1]} = 2n-2$. Then, the following conditions are equivalent:*

(a) *$I(G)^{[k]}$ is Cohen–Macaulay, for all $1 \leq k \leq \nu(G) = n$.*
(b) *G is a Cohen–Macaulay forest.*

Proof The implication (b) $\Rightarrow$ (a) is a particular case of [13, Corollary 3.6].

(a) $\Rightarrow$ (b) First, we prove the assertion for the case that G is a connected graph. By assumption on the dimension, we get height $I(G)^{[n-1]} = 2$. Thus, $I(G)^{[n-1]}$ is a perfect ideal of grade 2, and by the Hilbert–Burch theorem [4, Theorem 1.4.17], the minimal (multi)graded free resolution of $I(G)^{[n-1]}$ is of the form

$$\mathbb{F} \ : \ 0 \to S^{m-1} \xrightarrow{X} S^m \to I(G)^{[n-1]} \to 0,$$

where m is the minimal number of generators of $I(G)^{[n-1]}$, the matrix X represents the homomorphism $S^{m-1} \to S^m$, and $I_{m-1}(X) = I(G)^{[n-1]}$. Here, for an integer t and a matrix A, we denote by $I_t(A)$ the ideal generated by all t-minors of A.

We claim that $m \geq 2n-1$. To this end, we show that there is an injective map

$$\Phi \ : \ E(G) \to \mathcal{G}(I(G)^{[n-1]}).$$

By Lemma 3.1, $|E(G)| \geq |V(G)| - 1 = 2n-1$. Thus, if we show the existence of the map Φ, then $m \geq |E(G)| \geq 2n-1$ as desired.

Let $\{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ be a perfect matching of G. We set $\mathbf{x} = x_1x_2\ldots x_n$ and $\mathbf{y} = y_1y_2\ldots y_n$ and define Φ as follows:

(i) $\Phi(\{x_i, y_i\}) = (\mathbf{xy})/(x_iy_i) = \prod_{\ell \neq i}(x_\ell y_\ell) \in \mathcal{G}(I(G)^{[n-1]})$;

(ii) $\Phi(\{x_i, y_j\}) = (\mathbf{xy})/(y_ix_j) = (x_iy_j)\prod_{\ell \neq i,j}(x_\ell y_\ell) \in \mathcal{G}(I(G)^{[n-1]})$;

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

$$\text{(iii)} \quad \Phi(\{x_i, x_j\}) = (\mathbf{xy})/(y_i y_j) = (x_i x_j) \prod_{\ell \neq i,j} (x_\ell y_\ell) \in \mathcal{G}(I(G)^{[n-1]});$$

$$\text{(iv)} \quad \Phi(\{y_i, y_j\}) = (\mathbf{xy})/(x_i x_j) = (y_i y_j) \prod_{\ell \neq i,j} (x_\ell y_\ell) \in \mathcal{G}(I(G)^{[n-1]}).$$

It is clear that the above monomials are pairwise distinct. Hence, Φ is injective.

Since $m \geq 2n-1$ and each nonzero entry of X is an homogeneous element of degree at least one, we see that each nonzero $(m-1)$-minor of X has degree at least $m-1 \geq 2n-2$. Since $I(G)^{[n-1]}$ is equigenerated in degree $2(n-1)$, we then see that the equality $I_{m-1}(X) = I(G)^{[n-1]}$ implies that $m = 2n-1$ and that each entry of X is linear (i.e., of degree one). By the inequalities $m \geq |E(G)| \geq 2n-1$, we obtain $|E(G)| = 2n-1$. So, Lemma 3.1 implies that G is a Cohen–Macaulay tree.

Now, suppose that G has connected components $G_1, \ldots, G_r$. We show that each G_i is a tree. Since G has a perfect matching, each G_i has a perfect matching. Let $|V(G_i)| = 2n_i$ for all i. We fix an integer $1 \leq i \leq r$. If $n_i = 1$, then G_i is a tree, and we are done. So, we may assume that $n_i \geq 2$. Set $J = \prod_{j=1, j\neq i}^{r} I(G_j)^{[n_j]}$. Then, $J = (u)$ for some monomial u of degree $2n-2n_i$ and we have the inclusion $uI(G_i)^{[n_i-1]} \subseteq I(G)^{[n-1]}$. Since $I(G)^{[n-1]}$ is Cohen–Macaulay with height $I(G)^{[n-1]} = 2$, we conclude that $I(G)^{[n-1]} = \bigcap_{\ell=1}^{t} P_\ell$, where each P_ℓ is a monomial prime ideal of height 2. If $u \in P_\ell$ for all ℓ, then $u \in I(G)^{[n-1]}$. This is not possible since $I(G)^{[n-1]}$ is generated in degree $2n-2$ and $\deg u = 2n-2n_i < 2n-2$. So, $u \notin P_\ell$ for some ℓ. Since $uI(G_i)^{[n_i-1]} \subseteq P_\ell$, we obtain $I(G_i)^{[n_i-1]} \subseteq P_\ell$. This shows that height $I(G_i)^{[n_i-1]} \leq 2$, which implies that $\dim S_i/I(G_i)^{[n_i-1]} \geq 2n_i - 2$, where $S_i = K[V(G_i)]$. Since $n_i \geq 2$, by Lemma 3.4, it follows that $\dim S_i/I(G_i)^{[n_i-1]} = 2n_i - 2$. Moreover, by Corollary 2.6, we conclude that $I(G_i)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq n_i$. Therefore, by the first part of the proof on the connected case, it follows that G_i is a Cohen–Macaulay tree. Hence, G is a Cohen–Macaulay forest. ∎

In the proof of Theorem 3.2, it is shown that for a connected graph G with a perfect matching such that $S/I(G)^{[n-1]}$ is Cohen–Macaulay of dimension $2n-2$, the Hilbert–Burch matrix X has only variables as nonzero entries. This gives the following corollary.

Corollary 3.3 *Let G be a connected graph with a perfect matching of size n such that $S/I(G)^{[n-1]}$ is Cohen–Macaulay of dimension $2n-2$. Then, $I(G)^{[n-1]}$ has a linear resolution.*

The dimension of $S/I(G)^{[\nu(G)-1]}$ which is referred in the statement of Theorem 3.2 could get only two values as the following lemma shows.

Lemma 3.4 *Let G be a graph with a perfect matching on $2n$ vertices. Then,*

$$2k-1 \leq \dim S/I(G)^{[k]} \leq n+k-1,$$

for all $1 \leq k \leq \nu(G) = n$. In particular, if $n \geq 2$, then

$$\dim S/I(G)^{[n-1]} \in \{2n-3, 2n-2\}.$$

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

Proof Let $M = \{e_1, \dots, e_n\}$ be a perfect matching of G with $e_i = \{x_i, y_i\}$ for all i. For each non-empty subset $F \subseteq [n]$, we set $u_F = \prod_{i \in F}(x_i y_i)$. Let $1 \leq k \leq n$ be an integer and let $J = (u_F : F \subseteq [n], |F| = k)$. Since $J \subseteq I(G)^{[k]}$, to prove the right-hand side inequality, it is enough to show that height $J \geq n - k + 1$. If this was not the case, there would exist a monomial prime ideal P such that $J \subseteq P$ and height $P \leq n - k$. Set $A = \{i \in [n] : x_i \in P \text{ or } y_i \in P\}$. Without loss of generality, assume that $A \subseteq [n - k]$. Then, $u_F \notin P$ for $F = [n]\backslash[n - k]$, which is a contradiction. Finally, by [18, Proposition 1.1] we have depth $S/I(G)^{[k]} \geq 2k - 1$. ■

Remark 3.5 The inequality $2n - 3 \leq \dim S/I(G)^{[n-1]}$ in the previous lemma can also be obtained as follows. Recall that the *big cosize* of a monomial ideal $J \subset S$ with minimal generating set $\mathcal{G}(J) = \{u_1, \dots, u_m\}$ is denoted by big-cosize J, and it is defined as the smallest integer s such that

$$\operatorname{lcm}(u_{j_1}, u_{j_2}, \dots, u_{j_s}) = \operatorname{lcm}(u_1, u_2, \dots, u_m),$$

for all integers $1 \leq j_1 < j_2 < \cdots < j_s \leq m$. It is shown in [2, Theorem 3.3(a)] that proj.dim $S/J \leq$ big-cosize J. Notice that $I(G)^{[n-1]}$ is a monomial ideal equigenerated in degree $2n - 2$ in the polynomial ring S having $2n$ variables. From this observation, it follows immediately that big-cosize $I(G)^{[n-1]} \leq 3$. Hence, proj.dim $S/I(G)^{[n-1]} \leq 3$. Then, the Auslander–Buchsbaum formula implies that $2n - 3 \leq$ depth $S/I(G)^{[n-1]} \leq \dim S/I(G)^{[n-1]}$, as desired.

An interesting family of graphs G which satisfy the assumption that G has a perfect matching of size n and $\dim S/I(G)^{[n-1]} = 2n - 2$ are Cohen–Macaulay very well-covered graphs. Before showing this fact, we recall some definitions.

A graph G is called *unmixed* or *well–covered* if all the minimal vertex covers of G have the same size. Equivalently, $I(G)$ is an unmixed ideal (see [27]). In [24, Corollary 3.4], Gitler and Valencia proved that for an unmixed graph G, we have

$$\operatorname{height} I(G) \geq |V(G)|/2.$$

We say that G is *very well-covered* if G is unmixed and height $I(G) = |V(G)|/2$.

By a result of Favaron [20, Theorem 1.2], very well-covered graphs have always perfect matchings. Thus, by our Theorem 2.8, $I(G)^{[\nu(G)]}$ is Cohen–Macaulay, and we have $|V(G)| = 2n$ for some $n \geq 1$, for any very well-covered graph G.

To classify those very well-covered graphs G such that $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq \nu(G)$, we can use Theorem 3.2, once we show that $\dim S/I(G)^{[n-1]} = 2n - 2$, where n is the matching number of G. This equality is proved in Proposition 3.7. For the proof of this result, we will need the following fundamental classification due to Crupi et al. [12]. Here, we use a formulation of this result given in [32, Lemma 3.1] (see also [8, Characterization 1.12] and [9]). Hereafter, we identify each vertex of the graph with the corresponding variable.

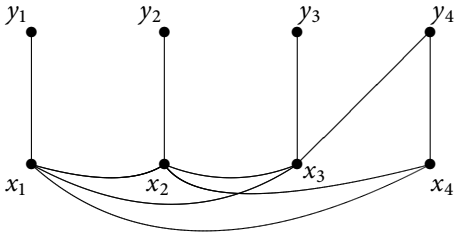
Theorem 3.6 *Let G be a very well-covered graph. Then, $|V(G)| = 2n$ for some $n \geq 1$ and the following conditions are equivalent:*

(a) *$I(G) \subset S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ is Cohen–Macaulay.*
(b) *There exists a relabeling of $V(G) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ such that:*

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

(i) $X = \{x_1, \ldots, x_n\}$ *is a minimal vertex cover of* G, *and* $Y = \{y_1, \ldots, y_n\}$ *is a maximal independent set of* G;
(ii) $x_i y_i \in I(G)$ *for all* $1 \le i \le n$;
(iii) *if* $x_i y_j \in I(G)$, *then* $i \le j$;
(iv) *if* $x_i y_j \in I(G)$, *then* $x_i x_j \notin I(G)$;
(v) *if* $z_i x_j, y_j x_k \in I(G)$, *then* $z_i x_k \in I(G)$ *for any distinct* i, j, k, *and* $z_i \in \{x_i, y_i\}$.

Here is a typical example of a Cohen–Macaulay very well-covered graph. This graph has eight vertices, and it satisfies the conditions (i)–(v) of the previous result.



G

Proposition 3.7 *Let* G *be a Cohen–Macaulay very well-covered graph on* $2n$ *vertices. Then,*

$$\dim S/I(G)^{[k]} = n + k - 1 \text{ for all } 1 \le k \le \nu(G).$$

Proof We have $|V(G)| = 2n$ and $\nu(G) = n$, for some $n \ge 1$. By Lemma 3.4, the result follows once we show that $\dim S/I(G)^{[k]} \ge n + k - 1$ for all k. Equivalently, we must prove that height $I(G)^{[k]} \le n - k + 1$ for all k. Consider the prime ideal $P = (x_1, x_2, \ldots, x_{n-k+1})$ for all $1 \le k \le n$. It is enough to show that $I(G)^{[k]} \subseteq P$. Suppose the contrary. Then, we could find a monomial generator $u \in \mathcal{G}(I(G)^{[k]})$ which is not divided by any of the variables $x_1, x_2, \ldots, x_{n-k+1}$. Such an element u, if it exists, would be a squarefree monomial of degree $2k$ in the set of variables

$$\{y_i : 1 \le i \le n - k + 1\} \cup \{x_j, y_j : n - k + 2 \le j \le n\}.$$

Note that none of the variables y_i, with $1 \le i \le n - k + 1$, can divide u because by Theorem 3.6(iii), any edge containing y_i must contain also a vertex from the set

$$\{x_i : 1 \le i \le n - k + 1\},$$

and then the corresponding variable $x_i \in P$ would divide u, which is forbidden. Hence, u would be a squarefree monomial in the set of variables

$$\{x_j, y_j : n - k + 2 \le j \le n\}.$$

This would imply that $\deg(u) \le 2(k-1)$, which is a contradiction since $\deg(u) = 2k$. Hence, $I(G)^{[k]} \subseteq P$ and height $I(G)^{[k]} \le n - k + 1$. ∎

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

Using Theorem 3.2 and Proposition 3.7, we are able to classify the edge ideals of very well-covered graphs whose all matching powers are Cohen–Macaulay.

Corollary 3.8 *Let G be a very well-covered graph. Then, the following conditions are equivalent:*

(a) $I(G)^{[k]}$ *is Cohen–Macaulay, for all* $1 \leq k \leq \nu(G)$.
(b) G *is a Cohen–Macaulay forest.*

Unmixed bipartite graphs and whisker graphs are examples of very well-covered graphs. Recall that a graph G is called *bipartite* if there exists a partition $X \cup Y$ of the vertex set $V(G)$ such that any edge of G is of the form $\{x, y\}$ for some $x \in X$ and some $y \in Y$. A *whisker graph* is a graph obtained by attaching a pendant edge to each vertex of a given graph.

Corollary 3.8 implies immediately as follows.

Corollary 3.9 *The only bipartite graphs G for which $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq \nu(G)$ are the Cohen–Macaulay forests.*

Corollary 3.10 *The only whisker graphs G for which $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq \nu(G)$ are the Cohen–Macaulay forests.*

Recall that the *normalized depth function* g_I of squarefree monomial ideal $I \subset S$ is defined by setting $g_I(k) = \operatorname{depth} S/I^{[k]} - (\alpha(I^{[k]}) - 1)$, for all $1 \leq k \leq \nu(I)$, where $\alpha(I^{[k]})$ is the initial degree of $I^{[k]}$. Using Proposition 3.7, we obtain the following corollary.

Corollary 3.11 *Let G be a very well-covered graph on $2n$ vertices such that $I(G)^{[k]}$ is Cohen–Macaulay, for all $1 \leq k \leq \nu(G)$. Then,*

$$g_{I(G)}(k) = n - k, \quad \text{for all} \ \ 1 \leq k \leq \nu(G).$$

4 Chordal graphs and Cameron–Walker graphs

We characterize chordal graphs and Cameron–Walker graphs whose matching powers of edge ideals are all Cohen–Macaulay. For chordal graphs, we use a result by Herzog et al. [28] which characterizes the Cohen–Macaulay chordal graphs. A *clique* of a graph G is a subset $F \subseteq V(G)$ such that the induced subgraph of G on F is a complete graph. The *clique complex* of G is the simplicial complex on the vertex set $V(G)$ whose faces are the cliques of G, and we denote it by $\Delta(G)$. A *free vertex* of $\Delta(G)$ is a vertex which belongs to precisely one facet of $\Delta(G)$. The main theorem in [28] shows that G is a Cohen–Macaulay chordal graph if and only if $V(G)$ is the disjoint union of the facets of $\Delta(G)$ which admit a free vertex.

We set r_G to be the number of facets of $\Delta(G)$ which admit a free vertex.

Theorem 4.1 *Let G be a chordal graph. The following conditions are equivalent:*

(a) $I(G)^{[k]}$ *is Cohen–Macaulay, for all* $1 \leq k \leq \nu(G)$.
(b) G *is a Cohen–Macaulay forest or a complete graph.*

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

For the proof of this theorem, we need some auxiliary lemmas.

Lemma 4.2 *Let $S = K[x_1, \ldots, x_n]$ and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal homogeneous ideal. Then, depth $S/\mathfrak{m}^{[k]} = \dim S/\mathfrak{m}^{[k]} = k - 1$ for all $1 \leq k \leq n$.*

Proof Notice that $\mathfrak{m}^{[k]}$ is the so-called squarefree Veronese ideal in the variables $x_1, \ldots, x_n$ generated in degree k. This ideal is also denoted by $I_{n,k}$ in [10]. The assertion follows from [10, Lemma 2] and the Auslander–Buchsbaum formula. ■

Lemma 4.3 *Let $R = K[x_1, \ldots, x_m]$, $T = K[y_1, \ldots, y_n]$, and $S = R \otimes_K T = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Let $I_1 \subset R$ be the edge ideal of the complete graph on the vertex set $\{x_1, \ldots, x_m\}$, $I_2 \subset T$ be the edge ideal of the complete graph on the vertex set $\{y_1, \ldots, y_n\}$, and set $I(G) = I_1 + I_2$. Suppose that $m + n \geq 5$. Then, there exists an integer $k \in \{\nu(G) - 1, \nu(G)\}$ such that $I(G)^{[k]}$ is not Cohen–Macaulay.*

Proof Let $\mathfrak{m} = (x_1, \ldots, x_m)$ and $\mathfrak{n} = (y_1, \ldots, y_n)$. Then, $I_1 = \mathfrak{m}^{[2]}$ and $I_2 = \mathfrak{n}^{[2]}$.

If at least one of m and n is an odd integer, say m, then $I(G)^{[\nu(G)]}$ is not Cohen–Macaulay. Indeed, let $x_i \in V(K_n)$. Then, neither G nor $G\backslash\{x_i\}$ have perfect matchings because K_m is a component of both graphs with an odd number of vertices. Then, by Theorem 2.8(c), $I(G)^{[\nu(G)]}$ is not Cohen–Macaulay.

Suppose now that both m and n are even integers, say $m = 2r$ and $n = 2t$. Then, $m + n \geq 6$ and we have

$$I(G)^{[\nu(G)-1]} = \mathfrak{m}^{[2r]}\mathfrak{n}^{[2t-2]} + \mathfrak{m}^{[2r-2]}\mathfrak{n}^{[2t]}, \tag{4.1}$$

with the convention that $J^{[0]} = S$ for a monomial ideal $J \subset S$. We claim that (4.1) is a Betti splitting. Indeed,

$$(\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t-2]}) \cap (\mathfrak{m}^{[2r-2]}\mathfrak{n}^{[2t]}) = \mathfrak{m}^{[2r]}\mathfrak{n}^{[2t]}.$$

Since $\mathfrak{m}$ and $\mathfrak{n}$ are monomial ideals in disjoint sets of variables, it is clear that the inclusion maps $\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t]} \to \mathfrak{m}^{[2r]}\mathfrak{n}^{[2t-2]}$ and $\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t]} \to \mathfrak{m}^{[2r-2]}\mathfrak{n}^{[2t]}$ are Tor-vanishing, if and only if the inclusion maps $\mathfrak{n}^{[2t]} \to \mathfrak{n}^{[2t-2]}$ and $\mathfrak{m}^{[2r]} \to \mathfrak{m}^{[2r-2]}$ are Tor-vanishing, which is clearly the case. Therefore, Equation (2.1) yields

$$\operatorname{depth} S/I(G)^{[\nu(G)-1]} = \min\left\{\begin{matrix}\operatorname{depth} S/\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t-2]}, \ \operatorname{depth} S/\mathfrak{m}^{[2r-2]}\mathfrak{n}^{[2t]}, \\ \operatorname{depth} S/\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t]} - 1\end{matrix}\right\}. \tag{4.2}$$

Again, because $\mathfrak{m}$ and $\mathfrak{n}$ are monomial ideals in disjoint sets of variables, using Lemma 4.2, we obtain that

$$\operatorname{depth} S/\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t-2]} = \begin{cases}\operatorname{depth} R/\mathfrak{m}^{[2r]} + \operatorname{depth} T/\mathfrak{n}^{[2t-2]} + 1 = m + n - 3, & \text{if } t > 1,\\ \operatorname{depth} R/\mathfrak{m}^{[2r]} + \operatorname{depth} T = m + n - 1, & \text{if } t = 1.\end{cases}$$

Similarly,

$$\operatorname{depth} S/\mathfrak{m}^{[2r-2]}\mathfrak{n}^{[2t]} = \begin{cases} m + n - 3, & \text{if } r > 1,\\ m + n - 1, & \text{if } r = 1.\end{cases}$$

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

and

$$\operatorname{depth} S/\mathfrak{m}^{[2r]}\mathfrak{n}^{[2t]} = \operatorname{depth} R/\mathfrak{m}^{[2r]} + \operatorname{depth} T/\mathfrak{n}^{[2s]} + 1 = m + n - 1.$$

Since $m + n \geq 6$, we have $r + t \geq 3$. Thus, at least one of the integers r or t is strictly bigger than one, and so, the previous three formulas combined with (4.2) yield that

$$\operatorname{depth} S/I(G)^{[v(G)-1]} = m + n - 3.$$

We claim that $\dim S/I(G)^{[v(G)-1]} = m + n - 2$, which finishes the proof. For this aim, it is enough to show that $\operatorname{height} I(G)^{[v(G)-1]} = 2$. We claim that $I(G)^{[v(G)-1]}$ is contained in (x_1, y_1). This is clear, as the following computation shows

$$I(G)^{[v(G)-1]} = \begin{cases} (x_1 \dots x_m)\mathfrak{n}^{[2t-2]} + (y_1 \dots y_n)\mathfrak{m}^{[2r-2]}, & \text{if } r > 1 \text{ and } t > 1, \\ (x_1 \dots x_m) + (y_1 y_2)\mathfrak{m}^{[2r-2]}, & \text{if } r > 1 \text{ and } t = 1, \\ (x_1 x_2)\mathfrak{n}^{[2t-2]} + (y_1 \dots y_n), & \text{if } r = 1 \text{ and } t > 1. \end{cases}$$

Since at least one of r and t is strictly bigger than one, the previous computation also shows that no principal monomial prime ideal can be an associated prime of $I(G)^{[v(G)-1]}$. Hence, $\operatorname{height} I(G)^{[v(G)-1]} > 1$. Consequently, $(x_1, y_1) \in \operatorname{Ass} I(G)^{[v(G)-1]}$. This shows that $\operatorname{height} I(G)^{[v(G)-1]} = 2$, which concludes the proof. ∎

Proof of Theorem 4.1 (a) ⇒ (b) Let G be a Cohen–Macaulay chordal graph. If G is a forest, we are done. So, we may assume that G has a clique of size at least 3. We need to prove that $r_G = 1$. Indeed, we show by induction on r_G that if $r_G \geq 2$, then $I(G)^{[k]}$ is not Cohen–Macaulay for some integer k. For the first step of the induction assume that $r_G = 2$. First, we show that if G is connected with $r_G = 2$, then $I(G)^{[2]}$ is not Cohen–Macaulay. Let F_1, F_2 be the facets of $\Delta(G)$ which admit a free vertex. Since $I(G)$ is Cohen–Macaulay, by the characterization of Cohen–Macaulay chordal graphs [28] mentioned in the beginning of this section, we have $V(G)$ as the disjoint union of F_1 and F_2. Since by our assumption G is not a forest, we may assume that $|F_1| \geq 3$. Let $x_1 \in F_1$ and $x_2 \in F_2$ be free vertices. Since G is connected by our assumption, we may choose $x_i, x_j \in F\backslash\{x_1\}$ and $x_k \in F_2\backslash\{x_2\}$ so that $\{x_i, x_k\} \in E(G)$. Then, the prime ideals $P = (x_\ell : x_\ell \in V(G)\backslash\{x_1, x_i, x_j, x_2\})$ and $Q = (x_\ell : x_\ell \in V(G)\backslash\{x_1, x_i, x_k\})$ are minimal prime ideals of $I(G)^{[2]}$. This shows that $I(G)^{[2]}$ is not unmixed and hence not Cohen–Macaulay.

Suppose now that $r_G = 2$ but G is not connected. Then, G is the disjoint union of two complete graphs K_n and K_m. We have $n + m \geq 5$; otherwise, G would be a forest. Then, Lemma 4.3 implies that $I(G)^{[k]}$ is not Cohen–Macaulay for some $k \in \{v(G) - 1, v(G)\}$.

Now, let $r_G > 2$. Assume inductively that for any Cohen–Macaulay chordal graph H which is not a forest and with $2 \leq r_H < r_G$, the ideal $I(H)^{[k]}$ is not Cohen–Macaulay for some k. Let $F_1, \dots, F_{r_G}$ be the facets of $\Delta(G)$ which admit a free vertex. Since by our assumption G is not a forest, we may assume that $|F_1| \geq 3$. Choose a free vertex $x \in F_{r_G}$ and set $H = G\backslash N_G[x]$. Since $V(G)$ is the disjoint union of $F_1, \dots, F_{r_G}$, we have $V(H) = V(G)\backslash F_{r_G}$ and the facets of $\Delta(H)$ which admit a free vertex are $F_1, \dots, F_{r_G-1}$. Therefore, H is a Cohen–Macaulay chordal graph and

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

$2 \leq r_H = r_G - 1 < r_G$. Moreover, H it is not a forest since $|F_1| \geq 3$. So, by our induction hypothesis, $I(H)^{[k]}$ is not Cohen–Macaulay for some k. Hence, by Theorem 2.5, $I(G)^{[k]}$ is not Cohen–Macaulay for some k.

(b) $\Rightarrow$ (a) If G is a Cohen–Macaulay forest, the claim holds by [13, Corollary 3.6]. Now, let G be a complete graph K_n. For any integer $1 \leq k \leq \nu(G)$, the ideal $I(G)^{[k]}$ is the squarefree Veronese ideal $\mathfrak{m}^{[2k]}$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S. So, it is Cohen–Macaulay by [26, Theorem 4.2]. ■

Next, we study the class of Cameron–Walker graphs. We recall some concepts which are used in the sequel. An *induced matching* of G is a matching M of G such that for any two edges $e = \{a, b\}$ and $e' = \{c, d\}$ belonging to M, the induced subgraph of G on $\{a, b, c, d\}$ consists of two disjoint edges e and e'. We denote by $\operatorname{im}(G)$ the maximum size of an induced matching of G.

A graph G is called a *Cameron–Walker graph* if G is connected and $\nu(G) = \operatorname{im}(G)$. Cameron–Walker graphs are classified in [5, Theorem 1] (see also [29]), where it is shown that G is a Cameron–Walker graph if and only if G is either a star graph, or a star triangle, or it consists of a connected bipartite graph with vertex partition $[n] \sqcup [m]$ such that there is at least one leaf edge attached to each vertex $i \in [n]$ and that there may be possibly some pendant triangles attached to each vertex $j \in [m]$. By a star triangle, we mean a graph joining some triangles at one common vertex.

The following result is crucial for the characterization of Cameron–Walker graphs whose all matching powers are Cohen–Macaulay.

Proposition 4.4 *Let G be a Cameron–Walker graph such that $I(G)^{[\nu(G)]}$ is Cohen–Macaulay. Then, G is either a star triangle or a bipartite graph.*

Proof Suppose that G is neither a star triangle nor a bipartite graph. Then, G is a graph consisting of a connected bipartite graph with vertex partition $[n] \sqcup [m]$ such that there is at least one leaf attached to each vertex in $[n]$ and that there exists at least one pendant triangle T attached to some vertex in $[m]$. Suppose that G has a perfect matching. Since $\nu(G) = \operatorname{im}(G)$, we conclude that G has a perfect matching which is an induced matching M. Thus, any vertex of the triangle T is an endpoint of an edge in the matching M, which is not possible since M is an induced matching of G. So, G does not have a perfect matching. Consider some vertex $i \in [n]$. Then, the pendant vertices of G which are adjacent to i are isolated vertices in the graph $G \setminus \{i\}$. Since there exists at least one of them, we conclude that $G \setminus \{i\}$ does not have a perfect matching. So, it follows from Theorem 2.8 that $I(G)^{[\nu(G)]}$ is not Cohen–Macaulay. ■

Theorem 4.5 *Let G be a Cameron–Walker graph. The following conditions are equivalent:*

(a) *$I(G)^{[k]}$ is Cohen–Macaulay, for all $1 \leq k \leq \nu(G)$.*
(b) *G is either K_2 or K_3.*

Proof (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b) By Corollary 4.4, we have that G is either a star triangle or a bipartite graph. Since a star triangle is a chordal graph, by Theorem 4.1 and the assumption

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

(a), we conclude that G is a triangle K_3. Now, suppose that G is bipartite. Then, by Corollary 3.9, we have that G is a Cohen–Macaulay forest. Suppose that G is not an edge. By [29, Theorem 1.3], G should consist of a connected bipartite graph with vertex partition $[n] \sqcup [m]$ such that there is exactly one leaf edge attached to each vertex $i \in [n]$ and that there is exactly one pendant triangle attached to each vertex $j \in [m]$. This is not possible since G is a forest. ■

5 Edge ideals of small graphs whose all matching powers are Cohen–Macaulay

In this final section, we provide the classification of the edge ideals of graphs with a small number of vertices ($n \in \{2, 3, 4, 5, 6, 7\}$) whose all matching powers are Cohen–Macaulay. This classification was obtained using *Macaulay2* [25], and the packages `NautyGraphs` [33] and `MatchingPowers` [21]. We run our experiments in the polynomial ring $S = \mathbb{Q}[x_i : i \in V(G)]$ over the rational numbers $\mathbb{Q}$.

Proposition 5.1 *Let G be a graph on $n \leq 7$ non-isolated vertices. Assume that $I(G)^{[k]}$ is Cohen–Macaulay over any field for all $1 \leq k \leq \nu(G)$. Then, the following hold:*

(i) *If $n \in \{2, 3, 4\}$, then G is either a complete graph or a Cohen–Macaulay forest.*
(ii) *If $n = 5$, then $G \in \{K_5, C_5, H\}$, where H and C_5 are the graphs depicted in Figure 1.*
(iii) *If $n = 6$, then G is either a Cohen–Macaulay forest or $G \in \{K_6, G_1, G_2\}$, where G_1 and G_2 are the graphs depicted in Figure 2.*
(iv) *If $n = 7$, then $G \in \{K_7, H_1, H_2\}$, where H_1 and H_2 are the graphs depicted in Figure 3.*

Conversely, for all graphs listed above, $I(G)^{[k]}$ is Cohen–Macaulay over any field for all $1 \leq \ k \leq \nu(G)$.

Proof Assume that $I(G)^{[k]}$ is Cohen–Macaulay over any field for all $1 \leq k \leq \nu(G)$. Computations by *Macaulay2* show that over the field $\mathbb{Q}$ we have the given characterization (i)–(iv) for a graph G on $n \leq 7$ vertices.

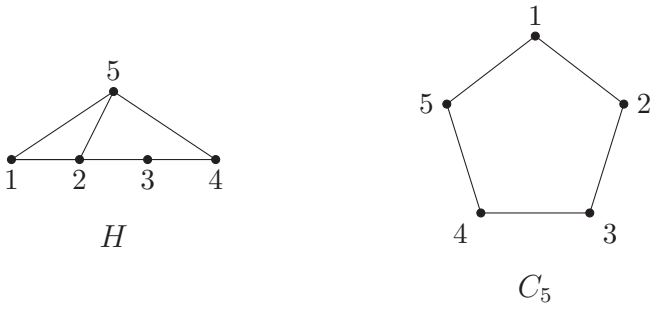


Figure 1: Graphs H and C_5.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

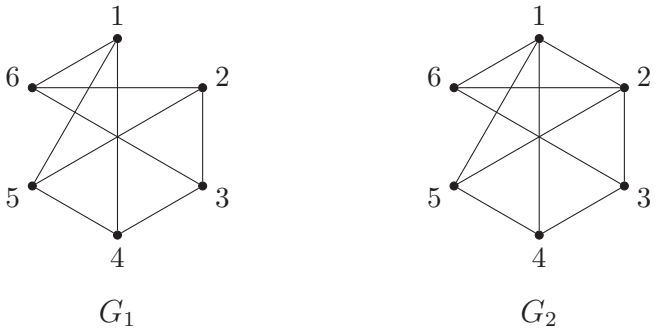


Figure 2: Graphs G_1 and G_2.

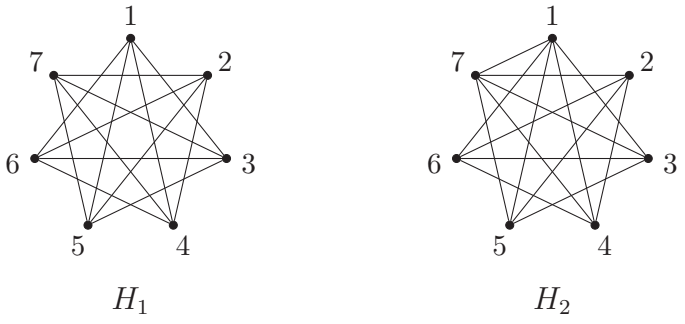


Figure 3: Graphs H_1 and H_2.

Conversely, consider a graph G in the above list. If G is a complete graph or a Cohen–Macaulay forest, then $I(G)^{[k]}$ is Cohen–Macaulay over any field for all $1 \leq k \leq \nu(G)$ (see Theorem 4.1). Moreover, for any graph $G \in \{H, C_5, G_1, G_2, H_1, H_2\}$ discovered above, it turns out that $I(G)^{[k]} = \mathfrak{m}^{[2k]}$ for all $2 \leq k \leq \nu(G)$, where $\mathfrak{m}$ is the graded maximal ideal of the polynomial ring. So, $I(G)^{[k]}$ is a squarefree Veronese ideal. Since the Cohen–Macaulay property of a squarefree Veronese ideal does not depend on the base field K, it follows that for the extra graphs $G \in \{H, C_5, G_1, G_2, H_1, H_2\}$, we have that $I(G)^{[k]}$ is Cohen–Macaulay for all k and every field K. ∎

It is well-known that up to 10 vertices, the Cohen–Macaulay property of $I(G)$ is independent of the field K of $S = K[x_i : i \in V(G)]$ (see [31]). Note that the Cohen–Macaulay property of the matching powers of complete graphs and Cohen–Macaulay forest is also independent of K.

Conjecture 5.2 *The family of graphs G for which $I(G)^{[k]}$ is Cohen–Macaulay for all $1 \leq k \leq \nu(G)$ does not depend on the characteristic of K.*

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

The above experiments also suggest that the following question could have a positive answer.

Question 5.3 *Is it true that for all $n \geq 5$, up to isomorphism, there are only two graphs on n non-isolated vertices, non-isomorphic to the complete graph K_n, and non-isomorphic to a Cohen–Macaulay forest, for which all the matching powers of their edge ideals are Cohen–Macaulay?*

Acknowledgments We thank the anonymous referee for their helpful suggestions and careful reading of the article.

References

[1] R. Achilles and W. Vogel, *Über vollständige Durchschnitte in lokalen Ringen*. Math. Nachr. **89**(1979), 285–298.
[2] L. Amata, M. Crupi, and A. Ficarra, *Projective dimension and Castelnuovo–Mumford regularity of t-spread ideals*. Internat. J. Algebra Comput. **32**(2022), no. 4, 837–858.
[3] M. Bigdeli, J. Herzog, and R. Zaare-Nahandi, *On the index of powers of edge ideals*. Comm. Algebra **46**(2018), 1080–1095.
[4] W. Bruns and J. Herzog, Cohen–Macaulay rings, Cambridge University Press, Cambridge, 1998.
[5] K. Cameron and T. Walker, *The graphs with maximum induced matching and maximum matching the same size*. Discrete Math. **299**(2005), 49–55.
[6] G. Caviglia, H. T. Hà, J. Herzog, M. Kummini, N. Terai, and N. V. Trung, *Depth and regularity modulo a principal ideal*. J. Algebraic Combin. **49**(2019), no. 1, 1–20.
[7] R. C. Cowsik and M. V. Nori, *On the fibers of blowing up*. J. Indian Math. Soc. (N.S.) **40**(1976), 217–222.
[8] M. Crupi and A. Ficarra, *Very well–covered graphs by Betti splittings*. J. Algebra **629**(2023), 76–108. https://doi.org/10.1016/j.jalgebra.2023.03.033
[9] M. Crupi and A. Ficarra, *Very well-covered graphs via the Rees algebra*, Mediterr. J. Math. **21**(2024), no. 4, 135.
[10] M. Crupi and A. Ficarra, *Cohen–Macaulayness of vertex splittable monomial ideals*. Mathematics **12**(2024), no. 6, 912. https://doi.org/10.3390/math12060912
[11] M. Crupi, A. Ficarra, and E. Lax, *Matchings, Squarefree Powers and Betti splittings*. Illinois J. Math. **69** (2025), no. 2, 353–372.
[12] M. Crupi, G. Rinaldo, and N. Terai, *Cohen–Macaulay edge ideals whose height is half of the number of vertices*. Nagoya Math. J. **201**(2011), 116–130.
[13] K. K. Das, A. Roy, and K. Saha, *Square-free powers of Cohen-Macaulay forests, cycles, and whiskered cycles*. Proc. Amer. Math. Soc. **154**(2026), no. 2, 549–565.
[14] S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*. J. Algebra **129**(1990), no. 1, 1–25.
[15] N. Erey and A. Ficarra, *Forests whose matching powers are linear*. Preprint, 2024. https://arxiv.org/abs/2403.17797
[16] N. Erey and A. Ficarra, *Matching powers of monomial ideals and edge ideals of weighted oriented graphs*. J. Algebra Appl. **25**(2026), no. 11, Article no. 2650118.
[17] N. Erey, J. Herzog, T. Hibi, and S. Saeedi Madani, *Matchings and squarefree powers of edge ideals*. J. Comb. Theory Ser. A **188**(2022), Paper No. 105585, 24 pp.
[18] N. Erey, J. Herzog, T. Hibi, and S. Saeedi Madani, *The normalized depth function of squarefree powers*. Collect. Math. (2023). https://doi.org/10.1007/s13348-023-00392-x
[19] N. Erey and T. Hibi, *Squarefree powers of edge ideals of forests*. Electron. J. Combin. **28**(2021), no. 2, P2.32.
[20] O. Favaron, *Very well covered graphs*. Discret. Math. **42**(1982), nos. 2–3, 177–187.
[21] A. Ficarra, *Matching powers: Macaulay2 package*. Preprint, 2023. https://arxiv.org/abs/2312.13007
[22] A. Ficarra, J. Herzog, and T. Hibi, *Behaviour of the normalized depth function*. Electron. J. Combin. **30**(2023), no. 2, P2.31.
[23] C. A. Francisco, H. T. Ha, and A. Van Tuyl, *Splittings of monomial ideals*. Proc. Amer. Math. Soc. **137**(2009), no. 10, 3271–3282.

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press

[24] I. Gitler and C. E. Valencia, *Bounds for invariants of edge-rings*. Comm. Algebra **33**(2005), no. 5, 1603–1616.
[25] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. http://www.math.uiuc.edu/Macaulay2
[26] J. Herzog and T. Hibi, *Cohen-Macaulay polymatroidal ideals*. European J. Combin. **27**(2006), no. 4, 513–517.
[27] J. Herzog and T. Hibi, Monomial ideals, Graduate Texts in Mathematics, 260, Springer, 2011.
[28] J. Herzog, T. Hibi, and X. Zheng, *Cohen-Macaulay chordal graphs*. J. Combin. Theory Ser. A **113**(2006), no. 5, 911–916. Elsevier, San Diego, CA.
[29] T. Hibi, A. Higashitani, K. Kimura, and A. O'Keefe, *Algebraic study on Cameron-Walker graphs*. J. Algebra **422**(2015), 257–269.
[30] E. Kamberi, F. Navarra, and A. A. Qureshi, *On squarefree powers of simplicial trees*. J. Algebra **693**(2026), 240–276.
[31] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*. J. Combin. Theory Ser. A **113**(2006), 435–454.
[32] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai, and S. Yassemi, *Vertex decomposability and regularity of very well–covered graphs*. J. Pure Appl. Algebra **215**(2011), 2473–2480.
[33] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112. The software package Nauty. http://cs.anu.edu.au/~bdm/nauty/
[34] N. C. Minh and N. V. Trung, *Cohen–Macaulayness of powers of two-dimensional squarefree monomial ideals*. J. Algebra **322**(2009), 4219–4227.
[35] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley-Reisner ideals*. Adv. Math. **226**(2011), no. 2, 1285–1306.
[36] H. D. Nguyen and T. Vu, *Powers of sums and their homological invariants*. J. Pure Appl. Algebra **223**(2019), no. 7, 3081–3111.
[37] S. A. Seyed Fakhari, *On the Regularity of squarefree part of symbolic powers of edge ideals*. J. Algebra **665**(2025), 103–130.
[38] S. A. Seyed Fakhari, *An increasing normalized depth function*. J. Commut. Algebra **16**(2024), no. 4, 497–499.
[39] S. A. Seyed Fakhari, *On the Castelnuovo-Mumford regularity of squarefree powers of edge ideals*. J. Pure Appl. Algebra **228**(2024), no. 3.
[40] S. A. Seyed Fakhari, *Matchings and Castelnuovo-Mumford regularity of squarefree powers of edge ideals*. J. Commut. Algebra **17**(2025), no. 2, 203–207.
[41] W. T. Tutte, *The factorization of locally finite graphs*. Canad. J. Math. **2**(1950), 44–49.
[42] M. Varbaro, *Symbolic powers and matroids*. Proc. Amer. Math. Soc. **139**(2011), 2357–2366.
[43] R. Waldi, *Vollständige Durchschnitte in Cohen–Macaulay–Ringen*. Arch. Math. **31**(1978/1979), 439–442.

Universidade de Evora, Portugal
e-mail: antficarra@unime.it

Ilam University, Islamic Republic of Iran
e-mail: so.moradi@ilam.ac.ir

https://doi.org/10.4153/S0008439526102598 Published online by Cambridge University Press