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DRAWING MULTISETS OF BALLS FROM TENABLE BALANCED LINEAR URNS

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We investigate the evolution of an urn of balls of two colors, where one chooses a pair of balls and observes rules of ball addition according to the outcome. A nonsquare ball addition matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \\ e & f \end{smallmatrix}\right)$ corresponds to such a scheme, in contrast to Pólya urn models that possess a square ball addition matrix. We look into the case of constant row sum (the so-called balanced urns) and identify a linear case therein. Two cases arise in linear urns: the nondegenerate and the degenerate. Via martingales, in the nondegenerate case one gets an asymptotic normal distribution for the number of balls of any color. In the degenerate case, a simpler probability structure underlies the process. We mention in passing a heuristic for the average-case analysis for the general case of constant row sum.

1. HISTORICAL PERSPECTIVE

Classical Pólya urn theory considers the evolution of an urn containing balls of k different colors. At each stage *one* ball is drawn at random and its color is observed. If a ball of color i, $i = 1, \ldots, k$, is drawn at a stage, it is replaced into the urn together with a_{ij} balls of color j, $j = 1, 2, \ldots, k$.

An associated $k \times k$ matrix, called *the ball addition scheme*, or simply the scheme, is an arrangement of the numbers a_{ij} in the form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{pmatrix},$$

a matrix the rows of which are indexed by the color of the ball selected, and the columns of which are indexed by the color of the ball added. The chief interest lies usually in the composition of the urn after a large number of draws.

Naturally, progress in Pólya urn theory came in increments. The first study goes back to Eggenberger and Pólya [9], who studied an urn with two colors with the addition scheme

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

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Throughout the 20th century several generalizations were considered, including:

(i) Bernard Friedman's urn [13], with the symmetric scheme

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are fixed numbers. Bernstein [7], Savkevich [32], and Friedman [13] studied this scheme. Freedman [12] discusses a normal asymptotic theory of these urns.

(ii) Urns with k colors (see Athreya and Karlin [1]).

(iii) Schemes allowing taking out balls—negative entries in the addition scheme (see Bagchi and Pal [3], Mahmoud [23], Mahmoud and Smythe [25,26], Mahmoud, Smythe, and Szymański [27], and Janson [18] for applications involving random trees, or Smythe [33] and Janson [17] for the general theory).

(iv) Schemes with random replacement—the entries a_{ij} are random variables (see Athreya and Ney [2]).

(v) Adaptive schemes where the number of extra balls of each color to be added depends on time (see Pemantle [29], and Bai and Hu [4]).

(vi) Analytic urns amenable to generating function methods, functional equations, and singularity analysis (see Flajolet, Gabarró, and Pekari [11], and Flajolet, Dumas, and Puyhaunbert [10]).

(vii) Poissonized models embedded in real time (see Balaji and Mahmoud [5] and Balaji, Mahmoud, and Watanabe [6]).

Johnson and Kotz [19] provides a good survey for the classical material with a focus on exact distributions. Kotz and Balakrishnan [21], Mahmoud [24], Pemantle [30] are modern surveys that go beyond the topics of Johnson and Kotz [19]. Most investigations focused on the case of a *balanced* urn (with a scheme with constant row sum) as it gives rise to certain eigenvalue properties that add mathematical convenience. For example, the investigations of Gouet [14,15], Flajolet, Gabarró, and Pekari [11], Flajolet, Dumas, and Puyhaunbert [10] assume this condition. An early attempt on a very special case of the 2×2 schemes with different row sum was made by Rosenblatt [31]. The case of nonconstant row sum lacks the mathematical elegance as was demonstrated in Athreya and Karlin [1], who studied the average behavior of such general urns. More recently Kotz, Mahmoud, and Robert [22] have looked at the implication of relaxing the balance condition of constant row sum on the exact distribution.

Yet another generalization in a different direction has been suggested by a study in random graphs (Mahmoud and Tsukiji [28]). Instead of picking one ball at random, consider picking a number of balls. This line was further considered with additional contributions in Johnson, Kotz, and Mahmoud [20] and Chen and Wei [8]. Depending on the multiset that comes out, we replace the drawn multiset in the urn and add a corresponding number of balls of each color. In the case of k colors and multisets of size $m \geq 2$, the ball addition scheme will thus be a nonsquare matrix, with $\binom{k+m-1}{m}$ rows indexed with multisets of size m, and k columns are indexed by numbers a_{ij}, which can be positive or negative, or even random in the most general case.

We make a modest attempt on the class of urns with $k = m = 2$, which will give us clues on the flavor of this nonclassic urn. We confine our attention to the balanced case,

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with a fixed total number of balls in each addition, and where the number of balls added of each color is fixed. We anticipate the unbalanced case of nonconstant row sum to be significantly harder than the balanced case.

We shall see shortly that a certain quadratic indicial equation underlies this urn. We identify a case for which a complete asymptotic theory is possible. This case is that when the indicial equation is reduced to a linear equation. Two cases arise within, a nondegenerate case and a degenerate one. We show that in the nondegenerate case a normal asymptotic theory dominates; the case is amenable to martingale theory. Otherwise, an urn with a linear indicial equation degenerates into a simpler law. The same route seems fraught with complication in the case of quadratic indicial equation.

2. URNS EVOLVING BY DRAWING PAIRS OF BALLS

We have delineated the scope of the investigation to be a study of an urn with two colors, where one draws two balls at a time. We assume our two colors to be white W and red R. The possible outcome multisets of any draw are $\{W, W\}, \{W, R\}$, or $\{R, R\}$. If the pair $\{W, W\}$ is drawn, we replace the pair into the urn and add a white balls and b red balls. Likewise, we add, respectively, c white and d red balls upon drawing $\{W, R\}$, and we add, respectively, e white and f red balls upon drawing $\{R, R\}$. Organized as a matrix, the rows of which are labeled with the multisets drawn, and the columns of which are labeled with the color of balls added, we have the associated rectangular urn scheme

$$\begin{array}{c} \\ WW \\ WR \\ RR \end{array} \begin{array}{c} \begin{array}{cc} W & R \end{array} \\ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array}.$$

We confine our attention to the balanced case, where the total number of balls added of all colors is a fixed number $K \geq 1$ (the balance factor).

An urn is *tenable* if it is always possible to draw pairs from it and apply the rules indefinitely. If all the entries in the scheme are nonnegative, no further constraints are necessary for tenability. However, if some entries are negative, the starting numbers of balls of the two colors have to satisfy some divisibility conditions, or else the urn may "get stuck." That is, there may come a time when drawing balls according to the rules is no longer feasible. For example, the scheme $\left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is clearly tenable no matter how we start with $s \geq 2$ balls, whereas the scheme $\left(\begin{smallmatrix} -3 & 5 \\ 0 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ may get stuck if $W_0 \equiv 2 \ \mathbf{mod}\ 3$, when drawing $\{W, W\}$ persists. We shall assume we are dealing only with tenable urns.

Assume a tenable urn starts out with $s \geq 2$ balls in total. The total number of balls after n draws is obviously

$$t_n = Kn + s.$$

Let W_n and R_n be the number of white and red balls, respectively, after n random draws (a *random draw* refers to one step of drawing two balls without replacement, where all *distinct* pairs of balls are equally likely). For the rest of the paper, the term *urn* refers specifically to a tenable balanced urn with $k = m = 2$ and random pair drawing at each stage.

Three indicator random variables influence the evolution. These are I_n^{WW}, I_n^{WR}, and I_n^{RR}, where I_n^{XY} generally indicates a draw at the nth step of the multiset $\{X, Y\}$, that is,

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I_n^{XY} assumes the value 1, when the multiset $\{X, Y\}$ is drawn at the nth stage, and assumes the value 0 otherwise. Of course, the three indicators I_n^{WW}, I_n^{WR}, and I_n^{RR} are mutually exclusive, only one of them must be 1, the rest are 0. In view of these definitions we have the stochastic recurrence

$$W_n = W_{n-1} + aI_n^{WW} + cI_n^{WR} + eI_n^{RR}. \tag{1}$$

Let $\mathcal{F}_n$ be the sigma field generated by the first n draws. Conditioning the last relation on the sigma field $\mathcal{F}_{n-1}$,

$$\begin{aligned}\mathbf{E}\big[W_n \,|\, \mathcal{F}_{n-1}\big] = W_{n-1} &+ a\mathbf{E}\big[I_n^{WW} \,|\, \mathcal{F}_{n-1}\big] + c\mathbf{E}\big[I_n^{WR} \,|\, \mathcal{F}_{n-1}\big] \\ &+ e\mathbf{E}\big[I_n^{RR} \,|\, \mathcal{F}_{n-1}\big].\end{aligned} \tag{2}$$

According to the drawing rules, the conditional expectations of the indicators are

$$\mathbf{E}\big[I_n^{WW} \,|\, \mathcal{F}_{n-1}\big] = \frac{\binom{W_{n-1}}{2}}{\binom{t_{n-1}}{2}}; \tag{3}$$

$$\mathbf{E}\big[I_n^{WR} \,|\, \mathcal{F}_{n-1}\big] = \frac{\binom{W_{n-1}}{1}\binom{R_{n-1}}{1}}{\binom{t_{n-1}}{2}}; \tag{4}$$

$$\mathbf{E}\big[I_n^{RR} \,|\, \mathcal{F}_{n-1}\big] = \frac{\binom{R_{n-1}}{2}}{\binom{t_{n-1}}{2}}. \tag{5}$$

Using the relation $R_n = t_n - W_n$, the recurrence becomes

$$\mathbf{E}\big[W_n \,|\, \mathcal{F}_{n-1}\big] = \frac{a - 2c + e}{t_{n-1}(t_{n-1} - 1)}\mathbf{E}\big[W_{n-1}^2 \,|\, \mathcal{F}_{n-1}\big] + g_n W_{n-1} + e,$$

where

$$g_n = \frac{Kn + s - K - (e - a)}{Kn + s - K}.$$

To develop an intuition in the cases where "equilibrium" is attained (i.e., when the number of balls of each color reaches a steady proportion over time), we consider the case where $W_n/(Kn)$ converges in some sense to some constant value, say w. Surely, the average sense is a goal and, if possible, a probabilistic sense (such as convergence in probability) is preferable. If a state of equilibrium is attained, then of course $R_n/(Kn)$ also approaches a limit (say r) in the same sense, and necessarily

$$w + r = 1. \tag{6}$$

This intuitive approach helps us identify cases with normal behavior, an intuition that will be rigorized later on.

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Plugging the conditional expectations (3)–(5) into (2) then taking expectation, one obtains

$$\begin{aligned}\mathbf{E}[W_n] = \mathbf{E}[W_{n-1}] &+ \frac{a}{t_{n-1}(t_{n-1}-1)}\mathbf{E}\big[W_{n-1}(W_{n-1}-1)\big] \\ &+ \frac{2c}{t_{n-1}(t_{n-1}-1)}\mathbf{E}\big[W_{n-1}R_{n-1}\big] \\ &+ \frac{e}{t_{n-1}(t_{n-1}-1)}\mathbf{E}\big[R_{n-1}(R_{n-1}-1)\big].\end{aligned}$$

If a state of equilibrium is attained, one can interpret this recurrence as

$$wKn \approx wK(n-1) + aw^2 + 2cwr + er^2,$$

which in view of (6) simplifies in the limit to the quadratic equation

$$(a-2c+e)w^2 + (2c-2e-K)w + e = 0,$$

the solutions of which are

$$w = \frac{K + 2(e-c) \pm \sqrt{K^2 + 4\big[K(e-c) + c^2 - ae\big]}}{2(a-2c+e)},$$

if $a - 2c + e \neq 0$. We call this quadratic equation the *indicial equation*. It clearly suggests that a linear urn with

$$a - 2c + e = 0$$

is "easier" to handle; its indicial equation is only linear, with the unique solution

$$w = \frac{e}{K + e - a}.$$

Indeed, we shall see in a rigorous way that this linear case possesses a linear recurrence equation for average and is approachable with the theory of martingales. A complete asymptotic theory will be derived for this linear case. The case of a fully quadratic indicial equation (a reference to the case $a - 2c + e \neq 0$) appears to be significantly harder to approach with these tools. Nevertheless, we shall be able to say a few words about a heuristic analysis of the behavior of the urn on average, but not about its asymptotic distribution.

The rest of the exposition is organized as follows. The brief Section 3 introduces some notation. In Section 4 we study urns with a linear indicial equation. The section is partitioned into three subsections. In Subsection 4.1 we study the exact and asymptotic mean value of the number of white balls. Equipped with sharp asymptotics for the mean, in Subsection 4.2 we take the computation one step further and find the asymptotic variance of the number of white balls. In Subsection 4.3 we show that, unless the process degenerates, it satisfies the martingale central limit theorem. A heuristic word about urns with fully quadratic indicial equations is mentioned in passing in Section 5, which gives a few concluding remarks as well.

3. NOTATION

The difference $\theta \stackrel{\text{def}}{=} e - a$ will appear often. We assume throughout that $\theta \geq 1$. (If the urn parameters are such that $e < a$, we can always study the behavior of red balls instead,

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and the case $\theta = 0$ is of no interest; it involves no randomness because $\theta = 0$ means that $a = c = e$, and $W_n \equiv an + W_0$.)

The symbol $\bigtriangledown$ denotes the backward difference operator, that is, for any function h, $\bigtriangledown h_n = h_n - h_{n-1}$. The number $H_n^{(k)} = \sum_{i=1}^{n} 1/i^k$ is the nth harmonic number of order k. The following are well-known asymptotic representations for the harmonic numbers of the first three orders:

$$H_n^{(1)} = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n} + O\Big(\frac{1}{n^3}\Big);$$
$$H_n^{(2)} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} + O\Big(\frac{1}{n^2}\Big);$$
$$H_n^{(3)} = \zeta(3) - \frac{1}{2n^2} + O\Big(\frac{1}{n^3}\Big),$$

where $\gamma = 0.5772\ldots$ is Euler's constant and ζ is Riemann's Zeta function.

The notations $\xrightarrow{\mathcal{D}}$, $\xrightarrow{P}$, and $\xrightarrow{\mathcal{L}_j}$, respectively, indicate convergence in distribution, in probability, and in $\mathcal{L}_j$. The normal variate with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$. The notation $O_{\mathcal{L}_j}\big(h(n)\big)$ expresses a random variable that is bounded in the $\mathcal{L}_j$ norm by $h(n)$; thus a sequence X_n of random variables is $O_{\mathcal{L}_j}\big(h(n)\big)$ if there exist a positive constants C, and a positive integer n_0, such that $\mathbf{E}\big[|X_n^j|\big] \le C|h^j(n)|$, for all $n \ge n_0$.

Rather long algebraic expressions appear in this investigation. For convenience, it will be our general strategy to always come up with names for long algebraic expressions.

4. LINEAR BALANCED URNS

We call an urn with a linear indicial equation a *linear urn*.

4.1. The Mean Number of White Balls

In a linear urn, the recurrence for the conditional expected value of the number of white balls is purely linear:

$$\mathbf{E}\big[W_n \,|\, \mathcal{F}_{n-1}\big] = g_n W_{n-1} + e,$$

and W_n is almost a martingale, but not quite so. We shall consider in Subsection 4.3 a martingalization procedure. We need sharp asymptotics, at least down to the third leading order term in the average, otherwise we cannot recognize the degenerate case in the variance.

Proposition 1: *In a tenable balanced linear urn, the average number of white balls after n draws is*

$$\mathbf{E}[W_n] = \frac{eK}{K+\theta}\, n + \frac{es}{K+\theta} + \frac{\rho}{n^{\theta/K}} + O\Big(\frac{1}{n^{\theta/K+1}}\Big),$$

as $n \to \infty$, where

$$\rho = \begin{cases} \Big[W_0 - \dfrac{es}{K+\theta}\Big] \dfrac{\Gamma\big(s/K\big)}{\Gamma\big(s - \theta/K\big)}, & \text{if } (s-\theta)/K \text{ is not a negative integer;} \\ \Big[\mathbf{E}[W_{1+(\theta-s)/K}] - e\Big]\, \Gamma\Big(\dfrac{\theta}{K}\Big), & \text{otherwise.} \end{cases}$$

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PROOF: The recursive formulas of the Gamma function can be employed to show by induction that the exact solution for average is:

$$\mathbf{E}[W_n] = \frac{eK}{K+\theta}\, n + \frac{es}{K+\theta} + \rho \frac{\Gamma\big(n + (s-\theta)/K\big)}{\Gamma\big(n + s/K\big)},$$

for $n \geq 0$, where ρ is as defined in the proposition in the case $(s-\theta)/K$ is not a negative integer. To avoid singularities of the Gamma function in the case $(s-\theta)/K$ is a negative integer, say $-\ell$, we start the induction at $\ell + 1$. The resulting formula is well defined for all $n \geq \ell + 1$. The proposition as stated follows from Stirling's approximation to the Gamma function. ■

4.2. The Variance

Variance calculation in random structures tends to be very lengthy. Here we only sketch this calculation. Squaring the stochastic recurrence (1) we obtain a recurrence for the conditional second moments:

$$\begin{aligned}\mathbf{E}\big[W_n^2 \,|\, \mathcal{F}_{n-1}\big] &= W_{n-1}^2 + (a^2 + 2aW_{n-1})\mathbf{E}\big[I_n^{WW} \,|\, \mathcal{F}_{n-1}\big] \\ &\quad + (c^2 + 2cW_{n-1})\mathbf{E}\big[I_n^{WR} \,|\, \mathcal{F}_{n-1}\big] \\ &\quad + (e^2 + 2eW_{n-1})\mathbf{E}\big[I_n^{RR} \,|\, \mathcal{F}_{n-1}\big].\end{aligned}$$

When we plug in the conditional expectations of the three indicators as in (3)–(5), we see that the third-order W_{n-1}^3 vanishes and the coefficient of the second-order term W_{n-1}^2 is $(a - 2c + e)/(t_{n-1}(t_{n-1} - 1))$, which cancels out in a linear urn. The rest of the calculation is only tedious algebra to put the equation in a wieldy form. One obtains

$$\begin{aligned}\mathbf{E}\big[W_n^2 \,|\, \mathcal{F}_{n-1}\big] &= \frac{2t_{n-1}^2 - 2(1+2\theta)t_{n-1} + \theta(\theta+4)}{2t_{n-1}(t_{n-1}-1)} W_{n-1}^2 \\ &\quad + \Big[2e + \frac{2(c^2 - e^2)}{t_{n-1} - 1} + \frac{e^2 - a^2}{t_{n-1}(t_{n-1}-1)}\Big] W_{n-1} + e^2.\end{aligned}$$

Taking expectations, we put the unconditional second moment recurrence in the form

$$\mathbf{E}[W_n^2] = A_n \mathbf{E}[W_{n-1}^2] + B_n \tag{7}$$

(valid for $n \geq 1$), where

$$A_n = \frac{2t_{n-1}^2 - 2(1+2\theta)t_{n-1} + \theta(\theta+4)}{2t_{n-1}(t_{n-1}-1)},$$

and

$$B_n = \Big[2e + \frac{2(c^2 - e^2)}{t_{n-1} - 1} + \frac{e^2 - a^2}{t_{n-1}(t_{n-1}-1)}\Big]\mathbf{E}[W_{n-1}] + e^2.$$

The form of (7) is iteratable; it unwinds into

$$\mathbf{E}[W_n^2] = \sum_{j=1}^{n} A_n A_{n-1} \ldots A_{j+1} B_j + A_n A_{n-1} \ldots A_1 W_0^2.$$

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It is evident that we need to develop sharp asymptotics for products of the form $A_{j+1}\dots A_n$. The ingredients on the right-hand side include

$$A_i = 1 - \frac{2\theta}{Ki} + \frac{\theta(\theta+4s-4K)}{2K^2i^2} - \frac{\theta(2(s-K)(\theta+2s-2K)-\theta)}{2K^3i^3} + O\Big(\frac{1}{i^4}\Big), \qquad \text{for } i = j+1,\dots,n;$$

$$B_j = \Big[2e + \frac{2(c^2-e^2)}{Kj} - \frac{\theta^2-4(s-K)(c^2-e^2)}{2K^2j^2} + O\Big(\frac{1}{j^3}\Big)\Big]\mathbf{E}[W_{j-1}] + e^2.$$

For $\mathbf{E}[W_{j-1}]$ we use the asymptotic result of Proposition 1 to obtain

$$\begin{aligned} B_j &= \Big[2e + \frac{2(c^2-e^2)}{Kj} - \frac{\theta^2-4(s-K)(c^2-e^2)}{2K^2j^2} + O\Big(\frac{1}{j^3}\Big)\Big] \\ &\quad \times \Big[\frac{eK}{K+\theta}(j-1) + \frac{es}{K+\theta} + \frac{\rho}{j^{\theta/K}} + O\Big(\frac{1}{j^{\theta/K+1}}\Big)\Big] + e^2 \\ &= \frac{2e^2K}{K+\theta}j + e\Big[\frac{e(\theta-K+2s)+2(c^2-e^2)}{K+\theta}\Big] - \frac{e\theta^2}{2K(K+\theta)j} \\ &\quad + \frac{2e\rho}{j^{\theta/K}} + O\Big(\frac{1}{j^{\theta/K+1}}\Big). \end{aligned}$$

Write

$$B_j = \beta_1 j + \beta_2 + \frac{\beta_3}{j} + \frac{2e\rho}{j^{\theta/K}} + O\Big(\frac{1}{j^{\theta/K+1}}\Big),$$

with

$$\beta_1 = \frac{2e^2K}{K+\theta}, \qquad \beta_2 = e\Big[\frac{e(\theta-K+2s)+2(c^2-e^2)}{K+\theta}\Big], \qquad \beta_3 = -\frac{e\theta^2}{2K(K+\theta)}.$$

Further, we write

$$A_i = 1 - \frac{\alpha_1}{i} + \frac{\alpha_2}{i^2} + \frac{\alpha_3}{i^3} + O\Big(\frac{1}{i^4}\Big),$$

with

$$\alpha_1 = \frac{2\theta}{K}, \quad \alpha_2 = \frac{\theta(\theta+4s-4K)}{2K^2}, \quad \alpha_3 = -\frac{\theta(2(s-K)(\theta+2s-2K)-\theta)}{2K^2}.$$

So,

$$\begin{aligned} \ln(A_{j+1}\dots A_n) &= \sum_{i=j+1}^{n} \ln\Big[1 - \frac{\alpha_1}{i} + \frac{\alpha_2}{i^2} + \frac{\alpha_3}{i^3} + O\Big(\frac{1}{i^3}\Big)\Big] \\ &= -\alpha_1 \sum_{i=j+1}^{n} \frac{1}{i} + \Big(\alpha_2 - \frac{\alpha_1^2}{2}\Big) \sum_{i=j+1}^{n} \frac{1}{i^2} \\ &\quad + \Big(\alpha_1\alpha_2 + \alpha_3 - \frac{\alpha_1^3}{3}\Big) \sum_{i=j+1}^{n} \frac{1}{i^3} + O\Big(\frac{1}{j^3}\Big) \end{aligned}$$

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$$
\begin{aligned}
&= \alpha_1 \ln \frac{j}{n} + \Big(\frac{\alpha_1}{2} - \frac{\alpha_1^2}{2} + \alpha_2\Big)\Big(\frac{1}{j} - \frac{1}{n}\Big) \\
&\quad + \Big[-\frac{\alpha_1}{12} - \frac{1}{2}\Big(\alpha_2 - \frac{\alpha_1^2}{2}\Big) \\
&\quad + \frac{1}{2}\Big(\alpha_1\alpha_2 + \alpha_3 - \frac{\alpha_1^3}{3}\Big)\Big]\Big(\frac{1}{j^2} - \frac{1}{n^2}\Big) \\
&\quad + O\Big(\frac{1}{j^3}\Big),
\end{aligned}
$$

where we went via harmonic numbers of the first three orders and their asymptotic approximation. Let $\gamma_1 = \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_1^2 + \alpha_2$, and $\gamma_2 = -\frac{1}{12}\alpha_1 - \frac{1}{2}(\alpha_2 - \frac{1}{2}\alpha_1^2) + \frac{1}{2}(\alpha_1\alpha_2 + \alpha_3 - \frac{1}{3}\alpha_1^3)$. Working with expansions, we can thus write

$$
\begin{aligned}
\mathbf{E}[W_n^2] = \sum_{j=1}^{n} &\Big[\beta_1 j + \beta_2 + \frac{\beta_3}{j} + \frac{2e\rho}{j^{\theta/K}} + O\Big(\frac{1}{j^{\theta/K+1}}\Big)\Big] \\
&\times \Big[1 + \gamma_1\Big(\frac{1}{j} - \frac{1}{n}\Big) + \gamma_2\Big(\frac{1}{j^2} - \frac{1}{n^2}\Big) + \frac{1}{2}\gamma_1^2\Big(\frac{1}{j} - \frac{1}{n}\Big)^2 \\
&+ O\Big(\frac{1}{j^3}\Big)\Big]\Big(\frac{j}{n}\Big)^{\alpha_1}.
\end{aligned}
$$

Expanding products, then going through sums of powers of integers and simplifying, we obtain

$$
\begin{aligned}
\mathbf{E}[W_n^2] = &\frac{\beta_1}{\alpha_1 + 2}\, n^2 + \Big(\frac{\beta_1}{2} + \frac{\beta_1\gamma_1 + \beta_2}{\alpha_1 + 1} - \frac{\beta_1\gamma_1}{\alpha_1 + 2}\Big)\, n \\
&+ \Big[\frac{\beta_1}{12}(1 + \alpha_1) + \frac{\beta_2}{2} + \frac{2\beta_1\gamma_2 + \beta_1\gamma_1^2 + 2\beta_2\gamma_1 + 2\beta_3}{2\alpha_1} \\
&- \gamma_1\frac{\beta_1\gamma_1 + \beta_2}{\alpha_1 + 1} + \beta_1\frac{\gamma_1^2 - 2\gamma_2}{2(\alpha 1 + 2)}\Big] + \frac{2eK\rho}{K + \theta}\, n^{1-\theta/K} \\
&+ A_n A_{n-1} \dots A_1 W_0^2 + o(1).
\end{aligned}
$$

Note that the leading term matches that in the square of $\mathbf{E}[W_n]$. The effect of the initial condition W_0^2 dies out asymptotically at the rate $O(n^{-2\theta/K})$. The variance now follows by subtracting the square of the mean from the last expression, where quadratic orders cancel out leaving at most a linear asymptotic equivalent. The exact cancelation that occur in the $n^{1-\theta/K}$ term is quite remarkable.

PROPOSITION 2: *In a tenable balanced linear urn the variance of the number of white balls after n draws is*

$$
\mathrm{Var}[W_n] = \sigma_1^2 n + \sigma_2^2 + o(1),
$$

as $n \to \infty$, with

$$
\begin{aligned}
\sigma_1^2 &= \frac{eK\big[e\theta(4K + 3\theta) + 4(c^2 - e^2)(K + \theta)\big]}{2(K + \theta)^2(K + 2\theta)} \\
\sigma_2^2 &= \frac{e\theta(K - a)(\theta^2 + 4(s - 1)\theta - 2K)}{8(K + \theta)^2(K + 2\theta)}.
\end{aligned}
$$

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There are instances of linear urns with $\sigma_1^2 = 0$. For example, in the linear scheme $\left(\begin{smallmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{smallmatrix}\right)$, the number of white balls has the variance

$$\begin{aligned}\mathrm{Var}[W_n^2] =& \frac{(s-1)(s-2)(s-W_0)\big[n^2+3(s-2)n-3s+2W_0(s-2)+5\big]n}{(n+s-1)^2(n+s-3)(n+s-2)^2}\\ =& \frac{(s-1)(s-2)(s-W_0)}{n^2}+O\Big(\frac{1}{n^3}\Big)\\ \to& 0.\end{aligned}$$

It is worthy of note that whenever $\sigma_1 = 0$, the constant σ_2 is also 0. This follows from the algebraic structure of a linear urn. Let $\delta = K/\theta$. Taking into account the linearity $2c = a + e$, one sees that σ_1^2 simplifies to

$$\sigma_1^2 = \frac{eK}{2\theta(\delta+1)^2(\delta+2)}\big[\theta(\delta+1)-e\big].$$

If $\sigma_1 = 0$, then $K = \delta e/(\delta+1)$. But then

$$\delta K = (\delta+1)K - K = \delta e - \delta\theta = \delta a,$$

or $K = a$, and $\sigma_2 \equiv 0$, too.

In schemes with $\sigma_1^2 \equiv 0$ the linear and constant terms in the variance are annulled, and the variance is $o(1)$. One expects sharper concentration and degeneracy in the probabilistic structure in comparison with the case $\sigma_1^2 > 0$. We shall refer to linear schemes with $\sigma_1^2 = 0$ as *degenerate schemes*, otherwise $\sigma_1^2 > 0$ and the scheme will be called *nondegenerate.* In all linear cases, an application of Chebyshev's inequality gives us the following.

Proposition 3: *In a tenable balanced linear urn, the proportion of white balls converges in probability:*

$$\frac{W_n}{Kn} \xrightarrow{P} w \stackrel{\text{def}}{=} \frac{e}{K+\theta},$$

as $n \to \infty$.

4.3. Limit Laws for Balanced Urns Via Martingales

We introduce a martingale transform. Let

$$M_n = \mu_n W_n + \nu_n,$$

for nonrandom correction factors μ_n and ν_n that will render M_n a true centered martingale. The correction factors are then obtained from recurrence equations they must satisfy.

Lemma 1: *In a tenable balanced linear urn, the random variable*

$$M_n = \frac{W_n}{g_2\cdots g_n} - e\sum_{j=2}^{n}\frac{1}{g_2\cdots g_j} - \mathbf{E}[W_1]$$

is a centered martingale.

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PROOF: Let $M_n = \mu_n W_n + \nu_n$. For M_n to be a martingale, it must satisfy

$$\begin{aligned}\mathbf{E}\big[M_n \,|\, \mathcal{F}_{n-1}\big] &= \mu_n \mathbf{E}\big[W_n \,|\, \mathcal{F}_{n-1}\big] + \nu_n \\ &= \mu_n [g_n W_{n-1} + e] + \nu_n \\ &= M_{n-1}.\end{aligned}$$

So, we want to enforce that

$$\mu_n g_n W_{n-1} + e\mu_n + \nu_n = \mu_{n-1} W_{n-1} + \nu_{n-1},$$

for all $n \geq 1$. This is possible if we let

$$\mu_n = \frac{\mu_{n-1}}{g_n} = \frac{\mu_{n-2}}{g_n g_{n-1}} = \cdots = \frac{\mu_1}{g_n g_{n-1} \cdots g_2}.$$

We also want to enforce $e\mu_n + \nu_n = \nu_{n-1}$. This linear recurrence unfolds as

$$\begin{aligned}\nu_n &= \nu_{n-1} - e\mu_n \\ &= \nu_{n-2} - e\mu_{n-1} - e\mu_n \\ &\vdots \\ &= \nu_1 - e(\mu_2 + \cdots + \mu_n).\end{aligned}$$

We also wish to have M_n centered, that is, $\mathbf{E}[M_n] = \mathbf{E}[M_1] = 0$, or $\mu_1 \mathbf{E}[W_1] + \nu_1 = 0$. Hence

$$M_n = \frac{\mu_1 W_n}{g_2 \cdots g_n} - \mu_1 e \sum_{j=2}^{n} \frac{1}{g_2 \cdots g_j} - \mu_1 \mathbf{E}[W_1]$$

is a centered martingale, for any arbitrary constant μ_1; let us take $\mu_1 = 1$. ■

For any fixed $t > 0$, the random variable $n^{-t} \bigtriangledown M_k$ is a martingale difference sequence (with respect to the increasing sigma-field sequence $\mathcal{F}_k$). The orders of magnitude for mean and variance will help us verify the martingale central limit theorem for an appropriate choice of t. It suffices to check the conditional Lindeberg condition and the conditional variance condition on the martingale differences (see Hall and Hyde [16]).

LEMMA 2: *Let $t > \theta/K$, and $I_{\mathcal{E}}$ be the indicator of the event $\mathcal{E}$. For any fixed $\varepsilon > 0$,*

$$\sum_{j=1}^{n} \mathbf{E}\Big[\Big(\frac{\bigtriangledown M_j}{n^t}\Big)^2 I_{\{|n^{-t} \bigtriangledown M_j| > \varepsilon\}} \,|\, \mathcal{F}_{j-1}\Big] \xrightarrow{P} 0,$$

as $n \to \infty$.

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PROOF: We have

$$\bigtriangledown M_j = \frac{\bigtriangledown W_j}{g_2 \dots g_{j-1}} + \frac{\theta W_j}{g_2 \dots g_{j-1}(Kj + s - K - \theta)} - \frac{e}{g_2 \dots g_j}.$$

Hence

$$|\bigtriangledown M_j| \le \frac{|\bigtriangledown W_j|}{g_2 \dots g_{j-1}} + \frac{\theta W_j}{g_2 \dots g_{j-1}(Kj + s - K - \theta)} + \frac{e}{g_2 \dots g_j}. \tag{8}$$

The differential in the number of white balls is bounded from above by the constant $\max\{|a|, |c|, |e|\}$, and

$$\prod_{i=2}^{j-1} g_i = \frac{\Gamma(j - 1 + (s-\theta)/K)\,\Gamma(1 + s/K)}{\Gamma(j - 1 + s/K)\,\Gamma(1 + (s-\theta)/K)}$$

is of the order $j^{-\theta/K}$ by Stirling's approximation of the Gamma function (in the case $(s-\theta)/K$ is a negative integer, say $-\ell$, the product is identically zero when the index i passes through $\ell + 1$). Hence, the first term on the right-hand side of (8) is $O(j^{\theta/K})$. The number of white balls after j draws cannot exceed $Kj + s$, the total number of balls in the urn. So, the second term on the right-hand side of (8) is also $O(j^{\theta/K})$. Likewise, the third term on the right-hand side of (8) is $O(j^{\theta/K})$. Hence, $(\bigtriangledown M_j)^2 = O(j^{2\theta/K})$, and if $t > \theta/K$ the sets $\{(\bigtriangledown M_j)^2 > \varepsilon n^{2t}\}$ are all empty for sufficiently large n, and for all $j = 1, \dots n$. Lindeberg's conditional condition has been verified for any choice $t > \theta/K$. ■

A Z-conditional variance condition requires that, for some value of t,

$$\sum_{j=1}^{n} \mathbf{E}\Big[\Big(\frac{\bigtriangledown M_j}{n^t}\Big)^2 \,|\, \mathcal{F}_{j-1}\Big] \xrightarrow{P} Z,$$

for the random variable Z.

LEMMA 3: *For any fixed $\varepsilon > 0$,*

$$\sum_{j=1}^{n} \mathbf{E}\Big[\Big(\frac{\bigtriangledown M_j}{n^{\theta/K + \frac{1}{2}}}\Big)^2 \,|\, \mathcal{F}_{j-1}\Big] \xrightarrow{P} \sigma_1^2,$$

as $n \to \infty$.

According to the martingale formulation of Lemma 1, we have

$$\mathbf{E}\big[(\bigtriangledown M_j)^2 \,|\, \mathcal{F}_{j-1}\big] = \frac{1}{(g_2 \dots g_{j-1})^2} \mathbf{E}\Big[\Big(\bigtriangledown W_j + \frac{\theta W_j}{Kj + s - K - \theta} - \frac{e}{g_j}\Big)^2 \,|\, \mathcal{F}_{j-1}\Big].$$

Upon squaring the variable in the conditional expectation, we get six terms. One term is the nonrandom quantity e^2/g_j^2. Each of the other random quantities has a conditional expectation that can be specified in terms of the number of white balls and the conditional first two moments as follows:

$$\begin{aligned}\mathbf{E}\big[(\bigtriangledown W_j)^2 \,|\, \mathcal{F}_{j-1}\big] &= W_{j-1}^2 - 2W_{j-1}\mathbf{E}\big[W_j \,|\, \mathcal{F}_{j-1}\big] \\ &\quad + \mathbf{E}\big[W_j^2 \,|\, \mathcal{F}_{j-1}\big];\end{aligned}$$

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$$\mathbf{E}\Big[\frac{\theta^2 W_j^2}{(Kj+s-K-\theta)^2}\,|\,\mathcal{F}_{j-1}\Big] = \frac{\theta^2}{(Kj+s-K-\theta)^2}\mathbf{E}[W_j^2\,|\,\mathcal{F}_{n-1}];$$

$$\mathbf{E}\Big[2\frac{\theta W_j \bigtriangledown W_j}{Kj+s-K-\theta}\,|\,\mathcal{F}_{j-1}\Big] = \frac{2\theta}{Kj+s-K-\theta}\Big(\mathbf{E}\big[W_j^2\,|\,\mathcal{F}_{j-1}\big] - W_{j-1}\mathbf{E}\big[W_j\,|\,\mathcal{F}_{j-1}\big]\Big);$$

$$\mathbf{E}\Big[-2\frac{e\bigtriangledown W_j}{g_j}\,|\,\mathcal{F}_{j-1}\Big] = \frac{2e}{g_j}W_{j-1} - \frac{2e}{g_j}\mathbf{E}\big[W_j\,|\,\mathcal{F}_{j-1}\big];$$

$$\mathbf{E}\Big[-2\frac{e\theta}{g_j(Kj+s-K-\theta)}W_j\,|\,\mathcal{F}_{j-1}\Big] = -\frac{2e\theta}{g_j(Kj+s-K-\theta)}\mathbf{E}[W_j\,|\,\mathcal{F}_{j-1}\big].$$

In turn, each of these is expressible in terms of conditional expectations of the indicators of the chosen multisets. We illustrate this on only one of these relations:

$$\mathbf{E}\big[(\bigtriangledown W_j)^2\,|\,\mathcal{F}_{j-1}\big] = a^2\mathbf{E}\Big[I_n^{WW}\,|\,\mathcal{F}_{n-1}\Big] + c^2\mathbf{E}\Big[I_n^{WR}\,|\,\mathcal{F}_{n-1}\Big] + e^2\mathbf{E}\Big[I_n^{RR}\,|\,\mathcal{F}_{n-1}\Big];$$

the rest of the relations follow suit.

Again what is left is lengthy and will only be sketched. Adding up the six terms, we have a relation of the form

$$\mathbf{E}\Big[(\bigtriangledown M_j)^2\,|\,\mathcal{F}_{j-1}\Big] = \xi_1(j)W_{j-1}^2 + \xi_2(j)W_{j-1} + \xi_3(j),$$

where $\xi_i(j)$, for $i = 1, 2, 3$, are functions of j and the urn parameters. In this latter form, we can approximate both W_{j-1} and W_{j-1}^2 in $\mathcal{L}_1$ as follows. The Cauchy–Schwartz inequality gives us

$$\mathbf{E}\big[|W_j - wKj|\big] \le \sqrt{\mathbf{E}\big[(W_j - wKj)^2\big]} \tag{9}$$

$$= \sqrt{\mathbf{E}\big[\big(W_j - \mathbf{E}[W_j]\big)^2\big] + \big(\mathbf{E}[W_j] - wKj\big)^2}$$

$$= O(\sqrt{j}\,), \tag{10}$$

following from Propositions 1 and 2. This implies that

$$W_j = wKj + O_{\mathcal{L}_1}(\sqrt{j}\,).$$

Further, by the Cauchy–Schwartz inequality again, we have

$$\mathbf{E}\big[|W_j^2 - w^2K^2j^2|\big] \le \sqrt{\mathbf{E}\big[(W_j - wKj)^2\big]\,\mathbf{E}\big[(W_j + wKj)^2\big]}.$$

One has the bound $W_j = O(j)$, which together with (9) and (10) gives

$$W_j^2 = w^2K^2j^2 + O_{\mathcal{L}_1}\big(j^{3/2}\big).$$

Now one has

$$\mathbf{E}\big[(\bigtriangledown M_j)^2\,|\,\mathcal{F}_{j-1}\big] = \xi_1(j)\big[w^2K^2(j-1)^2 + O_{\mathcal{L}_1}\big(j^{3/2}\big)\big] + \xi_2(j)\big[wK(j-1) + O_{\mathcal{L}_1}(\sqrt{j}\,)\big] + \xi_3(j).$$

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Working out the details, one reassembles the elements of σ_1^2:

$$\mathbf{E}\Big[(\triangledown M_j)^2 \,|\, \mathcal{F}_{j-1}\Big] = \Big(\frac{2\theta}{K} + 1\Big)\,\sigma_1^2 j^{2\theta/K} + O_{\mathcal{L}_1}\big(j^{2\theta/K-\frac{1}{2}}\big).$$

Therefore,

$$\mathbf{E}\big[(n^{-t} \triangledown M_j)^2 \,|\, \mathcal{F}_{j-1}\big] = \frac{1}{n^{2t}} \sum_{j=1}^{n} \Big[\Big(\frac{2\theta}{K} + 1\Big)\,\sigma_1^2 j^{2\theta/K} + O_{\mathcal{L}_1}\big(j^{2\theta/K-\frac{1}{2}}\big)\Big],$$

which, if $t = \theta/K + \frac{1}{2}$, converges to σ_1^2 (of course, what else?), in $\mathcal{L}_1$, and hence in probability too. The σ_1^2-conditional variance condition has been verified. □

Lemmas 2–3 lead us to a main result of this paper.

THEOREM 1: *In a tenable balanced linear nondegenerate urn, the number of white balls has the central limit tendency*

$$\frac{W_n - wn}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_1^2),$$

as $n \to \infty$; if the linear tenable balanced urn scheme is degenerate, instead one has

$$W_n - wn \xrightarrow{\mathcal{L}_2} 0,$$

where σ_1^2 and w are effectively computable constants (given in Propositions 2 and 3) in terms of the urn parameters.

5. CONCLUDING REMARKS

Pólya urn models and several of their generalizations mentioned in the Introduction possess a square ball addition matrix. Such schemes have been studied via eigenvalue theory.

We studied a class of urn models that grow by choosing pairs of balls at random at the various stages of the evolution. The corresponding urn scheme is a 3×2 matrix, not a square matrix. The classical eigenvalue theory may then be a difficult path to follow. We identified a linear subclass of these urn schemes, and found therein two cases of interest, the degenerate and the nondegenerate. The nondegenerate case is approached via martingales and manifests central limit tendency after a large number of draws. The degenerate case is simpler in structure, owing to its $o(1)$ variance.

We are already aware of one application in a random graph evolving in a fashion that emulates the growth of a random binary circuit of gates (Mahmoud and Tsukiji [28]).

Several extensions readily suggest themselves, like for example, quadratic schemes, urns with more ball colors, or urns evolving by drawing larger multisets of balls. The methods used in the investigation of 3×2 linear schemes seem promising for urns with more ball colors or urns where the rule is to draw a multiset of more than two balls. It is plausible that one may be able to identify linear cases in these urns where martingales remain a valid tool.

Average-case analysis of 3×2 quadratic schemes may be within reach. We have a heuristic argument suggesting that

$$\mathbf{E}[W_n] \sim wn,$$

as $n \to \infty$, where w is the positive root of the indicial equation. For example, for the scheme $\left(\begin{smallmatrix} 3 & 3 \\ 2 & 4 \\ 4 & 2 \end{smallmatrix}\right)$, the heuristic approach suggests that one may be able to rigorize an argument

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showing that $\mathbf{E}[W_n] \sim \left[\frac{5}{3} + \frac{1}{3}\sqrt{13}\right] n \approx 2.868517092\, n$, as $n \to \infty$. Simulation supports this hypothesis.

Ultimately one may want to attack the problem of rectangular schemes with nonconstant row sum. A lot remains to be done!

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