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Iterating exponential functions with cyclic exponents

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1. *Introduction*

This paper is concerned with expressions of the form

$$[a_1, a_2, \dots, a_n] = f_{a_1} \circ f_{a_2} \circ \dots \circ f_{a_n}(1) \quad (a_1, a_2, \dots, a_n > 0),$$

where $f_a(x) = a^x$. Thus we have

$$[a_1] = a_1, \quad [a_1, a_2] = a_1^{a_2}, \quad [a_1, a_2, a_3] = a_1^{(a_2^{a_3})}, \dots.$$

Such expressions have been described variously as continued exponentials, iterated exponentials, towers of exponents or hyperpowers. We shall frequently make use of their elementary properties, as for example

$$[a_1, \dots, a_k, [a_{k+1}, \dots, a_n]] = [a_1, \dots, a_n],$$

$$[a_1, \dots, a_n, 0] = [a_1, \dots, a_{n-1}], \quad [a_1, \dots, a_n, 1] = [a_1, \dots, a_n],$$

and

$$\frac{d}{dx}[a_1, \dots, a_n, x] = \prod_{k=1}^{n} [a_k, \dots, a_n, x] \log a_k.$$

A well-known result, due to Euler [**4**], states that the sequence

$$[a], [a, a], [a, a, a], \dots \quad (a > 0)$$

is convergent if and only if $\log a \in \{te^{-t} : |t| \leqslant 1\} = [-e, 1/e]$.

Recently in [**1**] we obtained a generalization of Euler's result to two cyclic exponents a_1, a_2. However, the method of [**1**] does not readily extend to the case of n cyclic exponents $a_1, a_2, \dots, a_n$, for $n \geqslant 3$. We now provide such a generalization.

Theorem 1. *If* $a_1, a_2, \dots, a_n > 0$, *then the sequence*

$$[a_1], [a_1, a_2], \dots, [a_1, a_2, \dots, a_n], [a_1, a_2, \dots, a_n, a_1], \dots \tag{1·1}$$

is convergent if and only if there is a unique representation of the numbers

$$b_k = \log a_k \quad (k = 1, 2, \dots, n) \tag{1·2}$$

in the form

$$b_1 = t_1 e^{-t_2}, b_2 = t_2 e^{-t_3}, \dots, b_n = t_n e^{-t_1}, \tag{1·3}$$

where

$$|t_1 t_2 \dots t_n| \leqslant 1.$$

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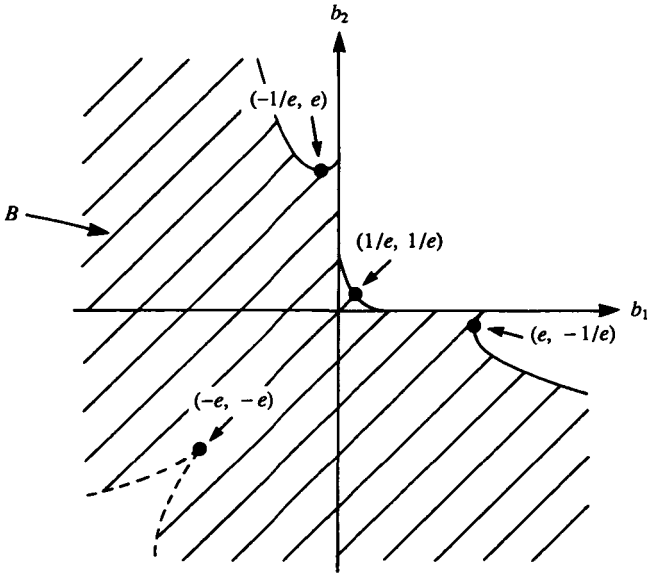


Fig. 1

We assume throughout that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are related by (1·2). According to Theorem 1, the set

$$B = \{\mathbf{b} \in \mathbb{R}^n : \text{the sequence (1·1) is convergent}\}$$

consists of those points covered exactly once by the restriction of the mapping

$$\Phi : (t_1, \ldots, t_n) \mapsto (t_1 e^{-t_2}, \ldots, t_n e^{-t_1}) \tag{1·4}$$

to

$$H = \{(t_1, \ldots, t_n) : |t_1 \ldots t_n| \leqslant 1\}. \tag{1·5}$$

For $n = 2$, the set B is shaded in the figure above; it was discussed in detail in [1]. Notice, in particular, that the part of ∂B in the third quadrant does not belong to B, except for the single point $(-e, -e)$. The mapping Φ covers the omitted cusp three times.

In order to give a similar description of B for $n \geqslant 3$, we introduce the following notation. Put

$$S = \{\mathbf{s} = (s_1, \ldots, s_n) : s_k = \pm 1 \quad \text{for} \quad k = 1, \ldots, n\}$$

and, for each $\mathbf{s} \in S$, define the orthant

$$Q_{\mathbf{s}} = \{(t_1, \ldots, t_n) : \operatorname{sgn}(t_k) = \operatorname{sgn}(s_k), \quad \text{for} \quad k = 1, \ldots, n\}. \tag{1·6}$$

Notice that, for each $\mathbf{s} \in S$, we have $\Phi(Q_{\mathbf{s}}) \subseteq Q_{\mathbf{s}}$, and also that

$$\mathbb{R}^n \setminus \bigcup_{\mathbf{s} \in S} Q_{\mathbf{s}} \subseteq B.$$

THEOREM 2. *Let* $\mathbf{s} = (s_1, \ldots, s_n) \in S$.

(i) *If* $s_k = 1$ *for* $k = 1, \ldots, n$, *then the restriction of* Φ *to* $H \cap Q_{\mathbf{s}}$ *is one-one and so*

$$B \cap Q_{\mathbf{s}} = \Phi(H \cap Q_{\mathbf{s}}).$$

If $\mathbf{b} \in Q_{\mathbf{s}} \setminus B$, *then the sequence* (1·1) *tends to infinity.*

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(ii) *If an odd number of* s_k *equals* -1, *then the restriction of* Φ *to* $H \cap Q_s$ *is one-one and so*

$$B \cap Q_s = \Phi(H \cap Q_s).$$

If $\mathbf{b} \in Q_s \backslash B$, *then the sequence* (1·1) *has exactly two limit points.*

(iii) *If an even (positive) number of* s_k *equals* -1, *then* $B \cap Q_s$ *is that component of* $Q_s \backslash \Phi(\partial H)$ *whose boundary includes the origin, together with a subset of* $Q_s \cap \Phi(\partial H)$ *of dimension* $n-2$.

If $b \in Q_s \backslash B$, *then the sequence* (1·1) *has exactly two limit points.*

Theorem 2 enables us to give several explicit conditions on $\mathbf{b}$ in order that the sequence (1·1) be either convergent or divergent.

THEOREM 3. (i) *If* $b_k \in [-e, 1/e]$ *for* $k = 1, \ldots, n$, *then the sequence* (1·1) *is convergent.*

(ii) *If* $b_k < 0$ *for* $k = 1, \ldots, n$, *and*

$$\left(\prod_{k=1}^{n} |b_k| \right)^{1/n} \leqslant e,$$

then the sequence (1·1) *is convergent.*

(iii) *If* $b_k > 0$ *for* $k = 1, \ldots, n$, *with some* $b_k \neq e^{-1}$, *and*

$$\left(\prod_{k=1}^{n} b_k \right)^{1/n} \geqslant 1/e,$$

then the sequence (1·1) *tends to infinity.*

Fig. 1 suggests various other possible sufficient conditions for the convergence of (1·1). For example, the part of the boundary of B which lies in the first quadrant is a convex curve (as can easily be checked) and so lies above the line $b_1 + b_2 = 2/e$. Thus, for $n = 2$ and $b_1, b_2 \geqslant 0$, the condition $\frac{1}{2}(b_1 + b_2) \leqslant 1/e$ is sufficient for the convergence of (1·1). It seems natural therefore to conjecture that for $b_1, \ldots, b_n \geqslant 0$ the condition

$$\frac{1}{n}(b_1 + \ldots + b_n) \leqslant \frac{1}{e}, \tag{1·7}$$

is sufficient for the convergence of (1·1). Our calculations suggest that this is true for $n = 3$, 4, 5 and 6 but, rather surprisingly, it is false for $n \geqslant 7$.

THEOREM 4. *For* $n \geqslant 7$, *there are numbers* $b_1, b_2, \ldots, b_n \geqslant 0$, *such that* (1·7) *holds but the sequence* (1·1) *tends to infinity.*

Finally we point out that part (i) of Theorem 3 can also be deduced from the following result of Barrow [3].

THEOREM 5. *If*

$$b_n \in [-e, 1/e] \quad \text{for} \quad n = 1, 2, \ldots, \tag{1·8}$$

then the sequence $[a_1], [a_1, a_2], [a_1, a_2, a_3], \ldots$, *is convergent.*

We include, in the final section of this paper, a detailed proof of Barrow's remarkable result, which presents some difficulties in the original.

Finally we remark that Thron [7] has given a version of Theorem 5 for complex exponentials. He showed that if $b_n \in \{|z| \leqslant 1/e\}$, for $n = 1, 2, \ldots$, then the sequence

$$[e^{b_1}], [e^{b_1}, e^{b_2}], [e^{b_1}, e^{b_2}, e^{b_3}], \ldots,$$

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is convergent. It is natural to conjecture that a version of Theorem 5 holds with (1·8) replaced by $b_n \in \Omega$, where $\Omega \supseteq [-e, 1/e] \cup \{|z| \leqslant 1/e\}$. We present some partial results in this direction in a forthcoming paper [**2**].

2. *The Schwarzian derivative*

Our proof of Theorem 1 relies on a lemma about functions with negative Schwarzian derivative. Recall that the Schwarzian derivative of a C^3 function f is given by

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2·1}$$

This operator, introduced by Schwarz in 1873, is important in complex analysis because it annihilates any fractional linear transformation T. Moreover, since

$$SF(z) = Sg(f(z))\,(f'(z))^2 + Sf(z), \tag{2·2}$$

if $F = g \circ f$, the Schwarzian derivatives of f and $T \circ f$ are identical.

D. Singer[**6**] discovered the importance of negative Schwarzian derivative to the iteration of real functions. In particular, two key facts follow from (2·1) and (2·2):

(1) if $f' \neq 0$ and f has strictly negative Schwarzian derivative, then each stationary value of $|f'|$ is a local maximum of $|f'|$;

(2) if f and g both have strictly negative Schwarzian derivative then $g \circ f$ has strictly negative Schwarzian derivative.

These two facts are used to prove the next lemma, which is needed for Theorem 1.

LEMMA 1. *Let f be a C^3 function satisfying*

(i) $f'(x) < 0$ *for* $x \in \mathbb{R}$,

(ii) $Sf(x) < 0$ *for* $x \in \mathbb{R}$ *and*

(iii) $|f'(c)| \leqslant 1$,

where c is the unique fixed point of f. Then

$$\lim_{k\to\infty} f^k(x) = c \quad \textit{for} \quad x \in \mathbb{R}.$$

(*Here f^k denotes the k-fold composition of f.*)

Proof. It is sufficient to prove that

$$\lim_{k\to\infty} f^{2k}(x) = c \quad \text{for} \quad x \in \mathbb{R},$$

and this will follow if we prove that

$$f^2(x) > x \quad \text{for} \quad x < c,$$

$$f^2(x) < x \quad \text{for} \quad x > c,$$

since f^2 is evidently strictly increasing.

First we claim that there is a number $r > 0$ such that

$$\begin{aligned} f^2(x) &> x \quad \text{for} \quad c - r < x < c, \\ f^2(x) &< x \quad \text{for} \quad c < x < c + r. \end{aligned} \tag{2·3}$$

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This is clear if $|f'(c)| < 1$, since $0 < (f^2)'(c) < 1$. On the other hand, if $f'(c) = -1$, then $(f^2)'(c) = 1$ and $(f^2)''(c) = 0$, so that $Sf^2 < 0$ implies that $(f^2)'''(c) < 0$. Hence (2·3) holds in either case.

To complete the proof, we show that f^2 has the unique fixed point c. Indeed, if d is a fixed point of f^2 and $d \neq c$, then $f(d)$ is also a fixed point of f^2 and we may assume that

$$d < c < f(d).$$

By the Mean Value Theorem, therefore, there exist a, b such that $a < c < b$ and

$$(f^2)'(a) = (f^2)'(b) = 1.$$

Since $(f^2)'(c) \leqslant 1$, we deduce that $(f^2)'$ has a local minimum on $[a, b]$, and this is impossible because $Sf^2 < 0$ and $(f^2)' \neq 0$.

3. *Proof of Theorem 1*

First recall that $\mathbf{a}$ and $\mathbf{b}$ are related by (1·2), and put

$$\phi_{\mathbf{b}}(x) = [a_1, a_2, \ldots, a_n, x] \quad (x \in \mathbb{R}). \tag{3·1}$$

The sequence (1·1) consists of the n subsequences $\phi_{\mathbf{b}}^k(x)$, where x belongs to the set of initial values

$$\mathscr{I} = \{0, 1, [a_1], [a_1, a_2], \ldots, [a_1, a_2, \ldots, a_{n-2}]\}, \tag{3·2}$$

and the only possible limits of these subsequences are the fixed points of $\phi_{\mathbf{b}}$. We remark that, if $a_k \neq 1$ for $k = 1, \ldots, n$, then

$$[a_k, a_{k+1}, \ldots, a_n, a_1, \ldots, a_{k-1}] \neq 1, \text{ for } k = 1, \ldots, n,$$

and so none of the initial values in $\mathscr{I}$ is a fixed point of $\phi_{\mathbf{b}}$.

The proof of Theorem 1 involves detailed consideration of the nature of the fixed points of $\phi_{\mathbf{b}}$. First we explain how the representation (1·3) arises.

LEMMA 2. *Let Φ be given by* (1·4). *There is a one-one correspondence between the fixed points c of $\phi_{\mathbf{b}}$ and the solutions $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ of the equation $\Phi(\mathbf{t}) = \mathbf{b}$, such that $c = e^{t_1}$.*

Under this correspondence we have $\phi_{\mathbf{b}}'(c) = t_1 t_2 \ldots t_n$.

Proof. First note that any fixed point c of $\phi_{\mathbf{b}}$ must be positive. If $c = e^{t_1}$ is a fixed point of $\phi_{\mathbf{b}}$, then

$$t_1 = \log \phi_{\mathbf{b}}(c) = b_1[e^{b_2}, \ldots, e^{b_n c}].$$

Hence

$$b_1 = t_1 e^{-t_2} \quad \text{where} \quad t_2 = b_2[e^{b_3}, \ldots, e^{b_n c}],$$

$$b_2 = t_2 e^{-t_3} \quad \text{where} \quad t_3 = b_3[e^{b_4}, \ldots, e^{b_n c}],$$

$$\vdots$$

and

$$b_n = t_n e^{-t_1},$$

so that $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is the unique solution of $\Phi(\mathbf{t}) = \mathbf{b}$ such that $c = e^{t_1}$. On the other hand, it is clear that if $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is a solution to $\Phi(\mathbf{t}) = \mathbf{b}$, then $c = e^{t_1}$ is a fixed point of $\phi_{\mathbf{b}}$.

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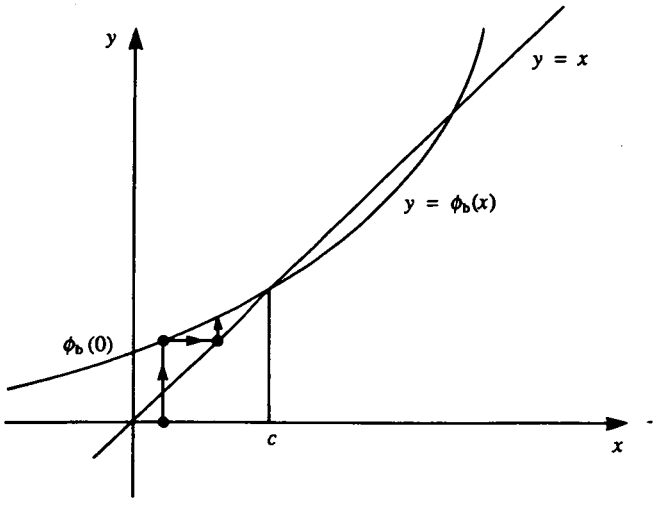


Fig. 2

Also

$$\phi_{\mathfrak{b}}'(c) = \prod_{k=1}^{n} [a_k, \ldots, a_n, c] \log a_k$$

$$= \prod_{k=1}^{n} e^{t_k} \log a_k = t_1 t_2 \ldots t_n,$$

and so the proof of Lemma 2 is complete.

Theorem 1 now follows from the next lemma.

LEMMA 3. *The sequence* (1·1) *is convergent if and only if* $\phi_{\mathfrak{b}}$ *has a unique non-repulsive fixed point.*

Proof. If some $a_k = 1$, then the lemma is a triviality and so we assume that $a_k \neq 1$ for $k = 1, \ldots, n$. The function $\phi_{\mathfrak{b}}$ is then strictly monotonic and lies between upper and lower limits M and m, where

$$m, M \in \mathscr{I} \cup \{\infty\}. \tag{3·3}$$

This is easily checked by induction.

We discuss three possible cases. Suppose first that

$$a_k > 1 \quad \text{for} \quad k = 1, \ldots, n.$$

Then $\phi_{\mathfrak{b}}$ is increasing and convex (see Fig. 2), since

$$\phi_{\mathfrak{b}}'(x) = \prod_{k=1}^{n} [a_k, \ldots, a_n, x] \log a_k$$

is increasing, and also $M = \infty$.

We deduce that if $\phi_{\mathfrak{b}}$ has at least one fixed point, then it has a unique non-repulsive fixed point c, which exceeds $\phi_{\mathfrak{b}}(0)$. Also

$$0 < 1 < [a_1] < [a_1, a_2] < \ldots < [a_1, a_2, \ldots, a_{n-2}] < \phi_{\mathfrak{b}}(0) < c,$$

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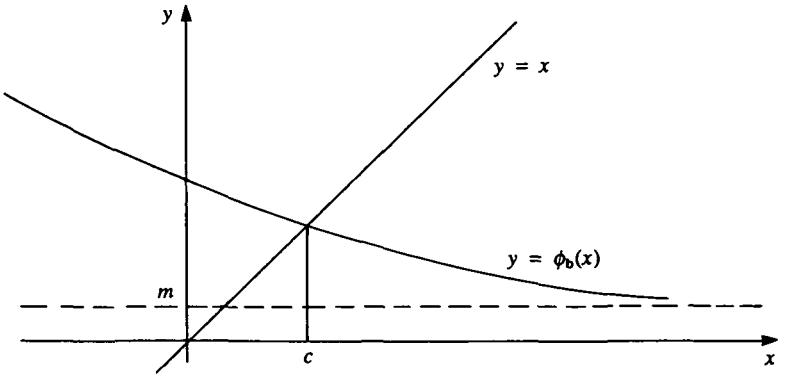


Fig. 3

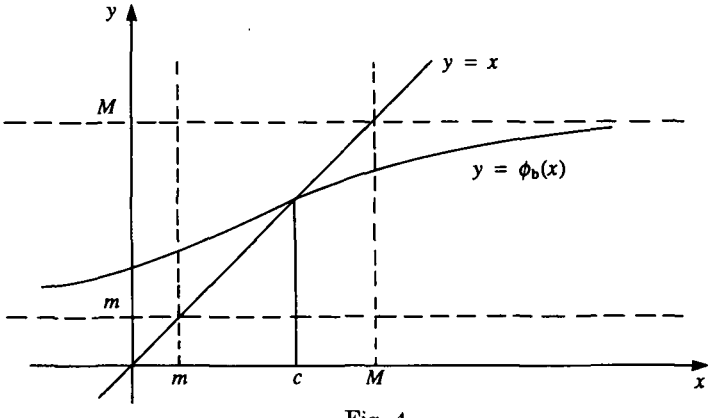


Fig. 4

and so the subsequences (3·2) have common limit c if and only if ϕ_b has such a fixed point.

Next suppose that $a_k < 1$ for an odd number of k. Then ϕ_b is decreasing and so has a unique fixed point c. If the subsequences (3·2) are convergent, then their common limit must be c and $|\phi_b'(c)| \leqslant 1$ (see Fig. 3).

On the other hand, if $|\phi_b'(c)| \leqslant 1$, then we deduce from Lemma 1 that

$$\lim_{k\to\infty} \phi_b^k(x) = c \quad \text{for} \quad x \in \mathbb{R},$$

because ϕ_b has negative Schwarzian derivative. Indeed, if $f_a(x) = a^x$, for $a > 0$, $a \neq 1$, then

$$Sf_a(x) = -\tfrac{1}{2}(\log a)^2 < 0, \quad \text{and so} \quad S\phi_b(x) < 0,$$

since the negative Schwarzian derivative property is preserved under composition.

Finally, suppose that $a_k < 1$ for an even (positive) number of k. Then ϕ_b is increasing but $M < \infty$, and the fixed points of ϕ_b lie in $[m, M]$ (see Fig. 4).

If the subsequences (3·2) are convergent with a common limit, c say, then

$$\lim_{k\to\infty} \phi_b^k(m) = \lim_{k\to\infty} \phi_b^k(M) = c,$$

in view of (3·3). Hence c must be the unique fixed point of ϕ_b and also must be non-repulsive.

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On the other hand, if $\phi_{\mathbf{b}}$ has a unique non-repulsive fixed point c, then $\phi_{\mathbf{b}}$ has exactly one fixed point, because the infimum and supremum of

$$\{x \in [m, M] : \phi_{\mathbf{b}}(x) = x\}$$

are both non-repulsive fixed points. Hence

$$\lim_{k \to \infty} \phi_{\mathbf{b}}^k(x) = c \quad \text{for} \quad x \in \mathbb{R},$$

and so the subsequences (3·2) all converge to c.

This completes the proof of Theorem 1.

4. *Proof of Theorem 2*

To prove Theorem 2 we need the following lemma which describes in more detail the possible fixed points of $\phi_{\mathbf{b}}$ when $\phi_{\mathbf{b}}$ is increasing and bounded.

LEMMA 4. *If $a_k < 1$ for an even (positive) number of k, then $\phi_{\mathbf{b}}$ has a unique point of inflection $\xi_{\mathbf{b}}$. Further, precisely one of the following holds (see Fig. 5):*

(i) *$\phi_{\mathbf{b}}$ has a unique fixed point c_1, and*

$$\phi_{\mathbf{b}}'(c_1) \leqslant 1;$$

(ii) *$\phi_{\mathbf{b}}$ has two fixed points $c_1 < c_2$, and either*

$$\phi_{\mathbf{b}}'(c_1) < 1, \quad \phi_{\mathbf{b}}'(c_2) = 1,$$

or

$$\phi_{\mathbf{b}}'(c_1) = 1, \quad \phi_{\mathbf{b}}'(c_2) < 1;$$

(iii) *$\phi_{\mathbf{b}}$ has three fixed points $c_1 < c_2 < c_3$, and*

$$\phi_{\mathbf{b}}'(c_1) < 1, \quad \phi_{\mathbf{b}}'(c_2) > 1, \quad \phi_{\mathbf{b}}'(c_3) < 1.$$

Proof. Since $\phi_{\mathbf{b}}$ is increasing and lies between finite limits, $\phi_{\mathbf{b}}$ must have at least one point of inflection. Any such point of inflection must be a local maximum point of $\phi_{\mathbf{b}}'$, because $\phi_{\mathbf{b}}$ has negative Schwarzian derivative. Hence $\phi_{\mathbf{b}}$ has a unique point of inflection $\xi_{\mathbf{b}}$, with

$$\phi_{\mathbf{b}}'(\xi_{\mathbf{b}}) = \max\{\phi_{\mathbf{b}}'(x) : x \in \mathbb{R}\}.$$

Next, note that since $\phi_{\mathbf{b}}$ has just one point of inflection $\phi_{\mathbf{b}}'$ can take the value 1 at most twice and so $\phi_{\mathbf{b}}$ can have at most three fixed points. It is now easy to check that the only possibilities which can occur are (i), (ii) and (iii).

To prove part (i) of Theorem 2, notice that if $a_k > 1$ for $k = 1, \ldots, n$, then $\phi_{\mathbf{b}}$ has at most one non-repulsive fixed point and so the restriction of Φ to $H \cap Q_s$ is indeed one-one. Hence

$$B \cap Q_s = \Phi(H \cap Q_s).$$

If $\mathbf{b} \in Q_s \backslash B$, then the proof of Lemma 3 shows that $\phi_{\mathbf{b}}$ has no fixed points and

$$\phi_{\mathbf{b}}^n(x) \to \infty \quad \text{as} \quad n \to \infty,$$

for all $x \in \mathbb{R}$. Hence the sequence (1·1) tends to infinity.

To prove part (ii) of Theorem 2, notice that if $a_k < 1$ for an odd number of k, then

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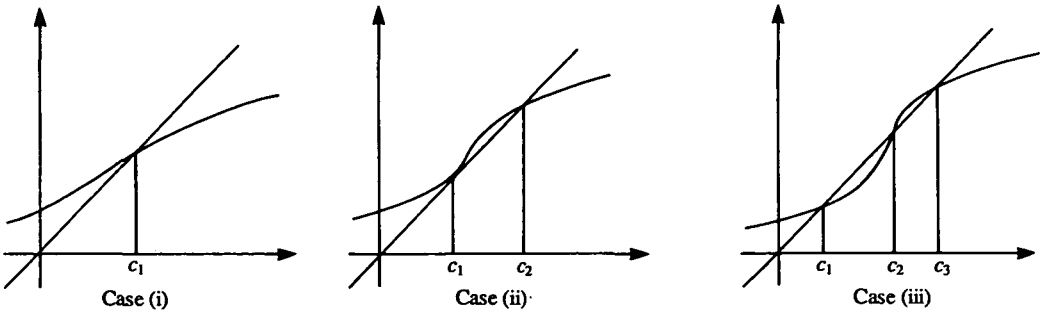


Fig. 5

$\phi_{\mathbf{b}}$ is decreasing and has exactly one fixed point. Hence the restriction of Φ to $H \cap Q_s$ is again one-one and so

$$B \cap Q_s = \Phi(H \cap Q_s).$$

If $\mathbf{b} \in Q_s \backslash B$, then the unique fixed point of $\phi_{\mathbf{b}}$ is repulsive. Hence, by Lemma 4, the function $\phi_{\mathbf{b}}^2$ has a repulsive fixed point and two attractive fixed points. Since none of the initial values in $\mathscr{I}$ (see (3·2)) can be a fixed point of $\phi_{\mathbf{b}}$ and since the upper and lower limits m, M of $\phi_{\mathbf{b}}$ both belong to $\mathscr{I}$, we deduce that the sequence (1·1) has exactly two limit points.

The proof of part (iii) of Theorem 2 is rather more involved. If $a_k < 1$ for an even (positive) number of k, then $\phi_{\mathbf{b}}$ has at most one indifferent fixed point, by Lemma 4. Hence the restriction of the mapping Φ to $\partial H \cap Q_s$ is one-one. Also, it is easy to check that

$$\Phi(\mathbf{t}) \to \infty \quad (\mathbf{t} \to \infty, \mathbf{t} \in \partial H). \tag{4·1}$$

Indeed if $|\Phi(\mathbf{t})| \leqslant M$ for some $M < \infty$ and some $\mathbf{t} \in \partial H$, then

$$|t_k e^{-t_{k+1}}| \leqslant M \quad \text{for} \quad k = 1, \ldots, n,$$

and so

$$t_{k+1} \geqslant \log(|t_k|/M) \quad \text{for} \quad k = 1, \ldots, n. \tag{4·2}$$

Thus, if $L(x) = \log(x/M)$, $x > 0$, and K is so large that

$$L^1(x) > L^2(x) > \ldots > L^n(x) > 1 \quad \text{for} \quad x \geqslant K, \tag{4·3}$$

then, by (4·2) and (4·3),

$$|t_k| \leqslant K \quad \text{for} \quad k = 1, \ldots, n,$$

since $|t_1 \ldots t_n| = 1$. Hence (4·1) follows and we deduce that $Q_s \backslash \Phi(\partial H)$ consists of two unbounded components.

Next note that if $\mathbf{b} \in Q_s$ and $\phi_{\mathbf{b}}$ has a fixed point c with $\phi_{\mathbf{b}}'(c) \neq 1$, then, by the Implicit Function Theorem, the fixed point $c = c(\mathbf{b})$ can be continued real analytically along any path in Q_s starting from $\mathbf{b}$, provided that

$$\frac{\partial}{\partial x}(\phi_{\mathbf{b}}(x) - x)|_{x=c} \tag{4·4}$$

does not become zero. Notice that if $\mathbf{b}$ remains bounded then c also remains bounded.

Now the expression (4·4) can only become zero if $\mathbf{b} \in \Phi(\partial H)$. Since $\phi_{\mathbf{0}}(x) \equiv 1$, it follows that the component of $Q_s \backslash \Phi(\partial H)$, whose boundary contains $\mathbf{0}$, consists of

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those $\mathbf{b}$ in Q_s for which case (i) of Lemma 4 holds with $\phi_{\mathbf{b}}'(c_1) < 1$. The other component of $Q_s \backslash \Phi(\partial H)$ consists of those $\mathbf{b}$ in Q_s for which case (iii) of Lemma 4 holds.

The remaining points of B in Q_s are those $\mathbf{b}$ in Q_s for which case (i) of Lemma 4 holds with

$$\phi_{\mathbf{b}}'(c_1) = 1.$$

For such points $\mathbf{b}$, the fixed point c_1 is equal to the unique point of inflection $\xi_{\mathbf{b}}$ of $\phi_{\mathbf{b}}$. Thus, if $\mathbf{b} = \Phi(\mathbf{t})$ where $\mathbf{t} \in \partial H$, then $\xi_{\mathbf{b}} = e^{t_1}$. We now write the condition

$$\phi_{\mathbf{b}}''(\xi_{\mathbf{b}}) = 0 \tag{4·5}$$

in terms of $\mathbf{t}$.

As in the proof of Lemma 2, we have

$$[a_k, \ldots, a_n, \xi_{\mathbf{b}}] = e^{t_k} \quad (k = 1, \ldots, n)$$

and so, for $k = 1, \ldots, n$,

$$\frac{\partial}{\partial x}[a_k, \ldots, a_n, x]|_{x=\xi_{\mathbf{b}}} = \prod_{j=k}^{n} e^{t_j} \log a_j = e^{t_k - t_1} \prod_{j=1}^{n} t_j.$$

Hence

$$\phi_{\mathbf{b}}''(\xi_{\mathbf{b}}) = \frac{\partial}{\partial x}\left\{\prod_{k=1}^{n} [a_k, \ldots, a_n, x] \log a_k\right\}\Bigg|_{x=\xi_{\mathbf{b}}},$$

$$= t_1 t_2 \ldots t_n e^{-t_1} \sum_{k=1}^{n} t_k \ldots t_n,$$

and so condition (4·5) reduces to

$$\sum_{k=1}^{n} t_k \ldots t_n = 0. \tag{4·6}$$

To complete the proof we show that

$$J = \left\{\mathbf{t} \in Q_s : t_1 \ldots t_n = 1, \quad \sum_{k=1}^{n} t_k \ldots t_n = 0\right\}$$

is non-empty, by noting that $\Sigma_{k=1}^{n} t_k \ldots t_n$ takes both positive and negative values for $\mathbf{t} \in Q_s \cap \partial H$. This holds because

$$\sum_{k=1}^{n} t_k \ldots t_n \to 1 \quad \text{as} \quad t_n \to 0,$$

with $\mathbf{t} \in Q_s \cap \partial H$ and $t_1 t_n, t_2, \ldots, t_{n-1}$ held constant, whereas if we choose $m \geqslant 2$ so that $t_m \ldots t_n$ is negative, then

$$\sum_{k=1}^{n} t_k \ldots t_n \to -\infty \quad \text{as} \quad |t_m| \to \infty,$$

with $\mathbf{t} \in Q_s \cap \partial H$ and $t_{m-1} t_m$, t_k $(k \neq m-1, m)$ held constant.

It is now easy to check that at any point of intersection of

$$t_1 \ldots t_n = 1 \quad \text{and} \quad \sum_{k=1}^{n} t_k \ldots t_n = 0,$$

the tangent hyperplanes fail to coincide. Hence J has dimension $n-2$. Since the restriction of the mapping Φ to $\partial H \cap Q_s$ is one-one and continuous, we deduce that the topological dimension of $\Phi(J)$ is also $n-2$.

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More precisely, if $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and

$$U_{\mathbf{s}} = \{(u_3, \ldots, u_n) : \operatorname{sgn}(u_k) = \operatorname{sgn}(s_k), k = 3, \ldots, n\},$$

then there is a smooth mapping $F : U_{\mathbf{s}} \to \Phi(J)$ whose Jacobian matrix F' has rank $n-2$ at all points of $U_{\mathbf{s}}$. To prove this, define $\Psi : U_{\mathbf{s}} \to \mathbb{R}^n$ by

$$t_k = u_k \quad (k = 3, \ldots, n)$$

$$t_2 = -G(u_3, \ldots, u_n)/(u_3 \ldots u_n),$$

$$t_1 = -1/G(u_3, \ldots, u_n),$$

where

$$G(u_3, \ldots, u_n) = 1 + \sum_{k=3}^{n} u_k \ldots u_n.$$

Evidently $\Psi : U_{\mathbf{s}} \to J$ is a homeomorphism, and we now define $F = \Phi \circ \Psi$.

To prove that $F' = \Phi' \Psi'$ has rank $n-2$, we note that Ψ' has rank $n-2$ and show that if $\mathbf{t} = \Psi(\mathbf{u})$, then $\operatorname{Ker} \Phi'(\mathbf{t})$ and the column space of $\Psi'(\mathbf{u})$ are disjoint. In fact, if $\mathbf{t} = \Psi(\mathbf{u})$ then

$$\frac{\partial t_k}{\partial u_j} = \delta_{kj} \quad (3 \leqslant k, j \leqslant n) \tag{4·7}$$

so that Ψ' has rank $n-2$. Now

$$\Phi'(\mathbf{t}) = \begin{pmatrix} e^{-t_2} & -t_1 e^{-t_2} & 0 & \ldots & 0 \\ 0 & e^{-t_3} & -t_2 e^{-t_3} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & e^{-t_n} & -t_{n-1} e^{-t_n} \\ -t_n e^{-t_1} & 0 & \ldots & 0 & e^{-t_1} \end{pmatrix}$$

has determinant $(1 - t_1 \ldots t_n) e^{-(t_1 + \ldots + t_n)}$, so that Φ' is singular on J. Evidently Φ' has rank $n-1$ on J, with $\operatorname{Ker} \Phi'$ spanned by the vector

$$(1, t_2 \ldots t_n, t_3 \ldots t_n, \ldots, t_n).$$

In view of (4·7), $\operatorname{Ker} \Phi'$ and the column space of Ψ' only fail to be disjoint if

$$1 = \sum_{j=3}^{n} u_j \ldots u_n \frac{\partial t_1}{\partial u_j} = \frac{1}{G^2} \sum_{j=3}^{n} u_j \ldots u_n \sum_{k=3}^{j} \left(\frac{u_k \ldots u_n}{u_j} \right),$$

that is,

$$\left(1 + \sum_{k=3}^{n} u_k \ldots u_n\right)^2 = \sum_{j=3}^{n} u_j \ldots u_n \sum_{k=3}^{j} \left(\frac{u_k \ldots u_n}{u_j} \right). \tag{4·8}$$

Now

$$\sum_{j=3}^{n} u_j \ldots u_n \sum_{k=3}^{j} \left(\frac{u_k \ldots u_n}{u_j} \right) = \sum_{k=3}^{n} u_k \ldots u_n \sum_{j=k}^{n} \left(\frac{u_j \ldots u_n}{u_j} \right)$$

$$= \sum_{k=3}^{n} u_k \ldots u_n \left(1 + \sum_{j=k}^{n-1} u_{j+1} \ldots u_n\right),$$

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which is precisely half the sum of the cross-terms of the left-hand side of (4·8). Hence the difference between the left-hand side and the right-hand side of (4·8) is

$$1+\sum_{k=3}^{n}(u_k\dots u_n)^2+\sum_{k=3}^{n}u_k\dots u_n\left(1+\sum_{j=k}^{n-1}u_{j+1}\dots u_n\right)$$
$$=\frac{1}{2}\left(1+\sum_{k=3}^{n}u_k\dots u_n\right)^2+\tfrac{1}{2}+\tfrac{1}{2}\sum_{k=3}^{n}(u_k\dots u_n)^2$$
$$\geqslant \tfrac{1}{2}.$$

Thus (4·8) has no solutions, and so F' has rank $n-2$, as required.

Finally if $\mathbf{b}\in Q_s\backslash B$, then either case (ii) or case (iii) of Lemma 4 holds. Once again, none of the initial values in $\mathscr{F}$ (see (3·2)) can be a fixed point of $\phi_{\mathbf{b}}$ and so the sequence (1·1) has exactly two limit points.

This completes the proof of Theorem 2.

5. *Proof of Theorem 3*

It follows from Theorem 2 that if $\mathbf{b}$ lies in the component Ω of $\mathbb{R}^n\backslash\Phi(\partial H)$ which contains the origin, then the sequence (1·1) is convergent. The following lemma shows that

$$\prod_{k=1}^{n}[-e,1/e]\subseteq\Omega\cup\{(-e,\dots,-e),(1/e,\dots,1/e)\}.$$

Since we know that (1·1) is convergent if $\mathbf{b}=(-e,\dots,-e)$ or $\mathbf{b}=(1/e,\dots,1/e)$ this will prove part (i) of Theorem 3.

LEMMA 5. $\Phi(\partial H)\cap\prod_{k=1}^{n}[-e,1/e]=\{(-e,\dots,-e),(1/e,\dots,1/e)\}$.

Proof. If the lemma is false, then there exist $t_1,\dots,t_n$, not all equal to 1 and not all equal to -1, such that

$$|t_1\dots t_n|=1, \tag{5·1}$$

and

$$t_k e^{-t_{k+1}}\begin{cases}\leqslant 1/e & \text{if} \quad t_k>0,\\ \geqslant -e & \text{if} \quad t_k<0.\end{cases}$$

Hence

$$t_{k+1}\geqslant\begin{cases}1+\log t_k & \text{if} \quad t_k>0, \qquad (5·2)\\ \log(-t_k)-1 & \text{if} \quad t_k<0. \qquad (5·3)\end{cases}$$

Since the t_k are not all equal to 1, it follows from (5·2) and (5·1) that

$$t_k<1 \quad \text{for} \quad k=1,\dots,n. \tag{5·4}$$

Hence not all the t_k are positive.

Now, if we add the inequalities in (5·2) and (5·3), and then use (5·1), we obtain

$$t_1+\dots+t_n\geqslant p-q,$$

where p,q are the number of positive t_k and negative t_k, respectively. Thus, if $t_{k_1},\dots,t_{k_q}$ denote the negative t_k, then

$$t_{k_1}+\dots+t_{k_q}\geqslant -q, \tag{5·5}$$

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by (5·4). Therefore, by the Theorem of the Means,

$$1 \geqslant \frac{(-t_{k_1})+\ldots+(-t_{k_q})}{q} \geqslant ((-t_{k_1})\ldots(-t_{k_q}))^{1/q}. \tag{5·6}$$

If $q < n$, then (5·4) and (5·6) together contradict (5·1). On the other hand if $q = n$ and there is equality in (5·5) then not all the t_k are equal, since we have excluded $t_1 = \ldots = t_n = -1$. Hence at least one of the inequalities in (5·6) is strict, and so (5·1) is contradicted once again. This completes the proof of Lemma 5.

To prove part (ii) of Theorem 3, we again use the fact that if $\mathbf{b}$ lies in the component Ω of $\mathbb{R}^n \backslash \Phi(\partial H)$ which contains the origin, then the sequence (1·1) is convergent. We show in fact that if $\mathbf{s} = (-1, \ldots, -1)$, then

$$\Phi(\partial H) \cap \left\{\mathbf{b} \in Q_{\mathbf{s}} : \left(\prod_{k=1}^{n} |b_k|\right)^{1/n} \leqslant e\right\} = \{(-e, \ldots, -e)\}. \tag{5·7}$$

If $\mathbf{b} = \Phi(\mathbf{t})$ where $\mathbf{t} = (t_1, \ldots, t_n) \in Q_{\mathbf{s}} \cap \partial H$, then

$$\left(\prod_{k=1}^{n} |b_k|\right)^{1/n} = e^{(-t_1-\ldots-t_n)/n} \geqslant e,$$

by the Theorem of the Means, with equality only if $t_1 = \ldots = t_n = -1$. This proves (5·7).

Finally we prove part (iii) of Theorem 3. Here it is sufficient to show that if $\mathbf{s} = (1, \ldots, 1)$, then

$$\Phi(\partial H) \cap \left\{\mathbf{b} \in Q_{\mathbf{s}} : \left(\prod_{k=1}^{n} b_k\right)^{1/n} \geqslant 1/e\right\} = \{(1/e, \ldots, 1/e)\}. \tag{5·8}$$

This implies that if $\mathbf{b} \in Q_{\mathbf{s}}, \mathbf{b} \neq (1/e, \ldots, 1/e)$ and

$$\left(\prod_{k=1}^{n} b_k\right)^{1/n} \geqslant 1/e,$$

then $\mathbf{b}$ lies in $Q_{\mathbf{s}} \backslash B$ and so the sequence (1·1) tends to infinity. In fact, if $\mathbf{b} = \Phi(\mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_n) \in Q_{\mathbf{s}} \cap \partial H$, then

$$\left(\prod_{k=1}^{n} b_k\right)^{1/n} = e^{-(t_1+\ldots+t_n)/n} \leqslant e^{-1},$$

by the Theorem of the Means, with equality only if $t_1 = \ldots = t_n = 1$. This proves (5·8).

6. *Proof of Theorem 4*

To prove that condition (1·7) is not sufficient for the convergence of (1·1) when $b_1, b_2, \ldots, b_n > 0$ and $n \geqslant 7$, we shall find $\mathbf{t} = (t_1, \ldots, t_n)$ with $t_1, \ldots, t_n > 0$ and $n \geqslant 7$, such that $t_1 t_2 \ldots t_n = 1$ and

$$\frac{1}{n}(t_1 e^{-t_2} + t_2 e^{-t_3} + \ldots + t_n e^{-t_1}) < e^{-1}. \tag{6·1}$$

If $\mathbf{b} = (b_1, \ldots, b_n) = \Phi(\mathbf{t})$, then $\phi_{\mathbf{b}}$ is convex increasing and has a unique fixed point which is indifferent. Thus if we choose $\mathbf{b}' = (b_1', \ldots, b_n')$ such that

$$b_k' > b_k \quad \text{for} \quad k = 1, \ldots, n, \tag{6·2}$$

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then

$$\phi_{\mathbf{b}'}(x) > \phi_{\mathbf{b}}(x) \geqslant x \quad \text{for} \quad x \in \mathbb{R},$$

so that (1·1) tends to infinity. We can clearly choose $\mathbf{b}'$ so that

$$\frac{1}{n}(b_1' + b_2' + \ldots + b_n') < e^{-1},$$

also holds, in view of (6·1).

The following lemma shows how to choose $t_1, \ldots, t_n > 0$ with $t_1 t_2 \ldots t_n = 1$ and $n \geqslant 7$ so that (6·1) holds. In fact it shows rather more; namely, that for each $p \geqslant 1$ the statement

$$\left\{\mathbf{b}: b_k > 0, k = 1, \ldots, n, \left(\frac{1}{n}\sum_{k=1}^{n} b_k^p\right)^{1/p} \leqslant e^{-1}\right\} \subseteq B$$

is false for n large enough. This shows in a sense that Theorem 3, part (i), is best possible near $\mathbf{b} = (1/e, \ldots, 1/e)$.

LEMMA 6. *For each $p \geqslant 1$, and $n > 5p + 3/2$ there exist $t_1, t_2, \ldots, t_n > 0$ such that $t_1 t_2 \ldots t_n = 1$ and*

$$\left[\frac{1}{n}(t_1^p e^{-pt_2} + t_2^p e^{-pt_3} + \ldots + t_n^p e^{-pt_1})\right]^{1/p} < e^{-1}.$$

Proof. If $t_k = 1$ for $k = 1, \ldots, n$, then both sides are equal. The idea of the proof is to make suitable small variations around this case of equality. As a first step, put

$$\tau_k = \tau_k(\epsilon) = 1 + \lambda_k \epsilon \quad (k = 1, \ldots, n), \tag{6·3}$$

with

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0, \tag{6·4}$$

so that

$$\tau_1 \tau_2 \ldots \tau_n = 1 + \alpha(\epsilon),$$

where

$$\alpha(\epsilon) = O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

Then put

$$t_k = \begin{cases} \tau_k & (k = 1, 2, \ldots, n-1), \\ \tau_n/(1+\alpha(\epsilon)) & (k = n), \end{cases} \tag{6·5}$$

so that $t_1 t_2 \ldots t_n = 1$ and $t_n = \tau_n(1 - \alpha(\epsilon) + O(\epsilon^4))$ as $\epsilon \to 0$.

Now, after a tedious calculation, we find that

$$t_1^p e^{-pt_2} + t_2^p e^{-pt_3} + \ldots + t_n^p e^{-pt_1}$$
$$= e^{-p}[n + \tfrac{1}{2}\epsilon^2 p((2p-1)(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) - 2p(\lambda_1\lambda_2 + \ldots + \lambda_n\lambda_1))] + O(\epsilon^3).$$

So to obtain the example, for a given $p \geqslant 1$, we need numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that (6·4) holds, and

$$(2p-1)(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) < 2p(\lambda_1\lambda_2 + \ldots + \lambda_n\lambda_1). \tag{6·6}$$

The required numbers $t_1, \ldots, t_n$ are then obtained by choosing $\epsilon > 0$ small enough and using (6·3) and (6·5).

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We define the numbers λ_k as follows. The definition is slightly different for n even or odd. If $n = 2m$, then we put

$$\lambda_k = \begin{cases} 1 & (1 \leqslant k \leqslant m-1), \\ \frac{1}{2} & (k = m), \\ -1 & (m+1 \leqslant k \leqslant 2m-1), \\ -\frac{1}{2} & (k = 2m), \end{cases}$$

whereas if $n = 2m+1$, then we put

$$\lambda_k = \begin{cases} 1 & (1 \leqslant k \leqslant m-1), \\ \frac{1}{2} & (k = m), \\ -1 & (m+1 \leqslant k \leqslant 2m), \\ \frac{1}{2} & (k = 2m+1). \end{cases}$$

In each case (6·4) is satisfied and also

$$\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 = n - \tfrac{3}{2},$$

$$\lambda_1\lambda_2 + \ldots + \lambda_n\lambda_1 = n-4,$$

so that

$$\frac{\lambda_1^2 + \ldots + \lambda_n^2}{\lambda_1\lambda_2 + \ldots + \lambda_n\lambda_1} = 1 + \frac{5}{2(n-4)}.$$

Hence (6·6) holds if we choose n so large that

$$\frac{5}{2(n-4)} < \frac{1}{2p-1},$$

that is,

$$n > 5p + \tfrac{3}{2},$$

as required.

Remarks. 1. The proof of Lemma 6 is inspired by a counter-example given by M. J. Lighthill to disprove the intriguing cyclic inequality

$$\frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \ldots + \frac{x_n}{x_1+x_2} \geqslant \frac{n}{2} \quad (x_1, \ldots, x_n > 0),$$

first proposed by H. S. Shapiro. This inequality is discussed in some detail in [5] on p. 132. We are grateful to Professor J. Clunie for our knowledge of Lighthill's counter-example.

2. We mentioned in §1 that condition (1·7) appears to be sufficient for the convergence of (1·1) when $b_1, \ldots, b_n \geqslant 0$ and $n = 3, 4, 5, 6$. It is interesting to note that the construction used in Lemma 6 does not work for these values of n because if $\lambda_1 + \ldots + \lambda_n = 0$ and $n \leqslant 6$, then

$$\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 \geqslant 2(\lambda_1\lambda_2 + \ldots + \lambda_n\lambda_1).$$

For example, if $n = 6$ then

$$\lambda_1 + \lambda_2 + \ldots + \lambda_6 = 0$$

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implies that

$$0 \geqslant (\lambda_1+\lambda_3+\lambda_5)(\lambda_2+\lambda_4+\lambda_6) = (\lambda_1\lambda_2+\lambda_2\lambda_3+\ldots+\lambda_6\lambda_1)+(\lambda_1\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_6).$$

Thus

$$(\lambda_1^2+\ldots+\lambda_6^2)-2(\lambda_1\lambda_2+\ldots+\lambda_6\lambda_1) \geqslant (\lambda_1+\lambda_4)^2+(\lambda_2+\lambda_5)^2+(\lambda_3+\lambda_6)^2 \geqslant 0.$$

The cases $n < 6$ can also be checked directly, or by a simple backwards induction argument.

7. *Proof of Theorem 5*

In this section we again use the notation

$$f_a(x) = a^x \quad (a > 0, x \in \mathbb{R}).$$

First note that if $e^{-e} \leqslant a \leqslant e^{1/e}$ then f_a maps the interval $[0, e]$ into itself (see Fig. 6).

Suppose now that $e^{-e} \leqslant a_n \leqslant e^{1/e}$, for $n = 1, 2, \ldots$. The idea of the proof of Theorem 5 is to put

$$I_n = f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}([0, e]),$$

so that

$$[0, e] \supseteq I_1 \supseteq I_2 \supseteq \ldots.$$

If we can show that

$$|I_n| \to 0 \quad \text{as} \quad n \to \infty, \tag{7·1}$$

then $\bigcap_{n=1}^{\infty} I_n = \{c\}$ for some $c \in [0, e]$, and so

$$\lim_{n\to\infty} [a_1, a_2, \ldots, a_n] = \lim_{n\to\infty} f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(1) = c.$$

We prove (7·1) by a careful analysis of the effects of the maps f_a, where $e^{-e} \leqslant a \leqslant e^{1/e}$, on closed intervals $I \subseteq [0, e]$. Notice that we do not need to consider $a = 1$, since the sequence (1·1) is automatically convergent if any of the terms $a_n = 1$.

LEMMA 7. *If* $1 < a \leqslant e^{1/e}$ *and* $I \subseteq [0, e]$ *then*

$$|f_a(I)| \leqslant \alpha_1(|I|),$$

where

$$\alpha_1(t) = e(1-e^{-t/e}) < t \quad (t > 0)$$

is concave increasing.

Proof. Let $t = |I|$. Then we may assume that $a = e^{1/e}$ and that

$$I = [e-t, e].$$

Hence

$$|f_a(I)| = e - e^{(e-t)/e} = e(1-e^{-t/e}) < t,$$

as required.

It is harder to obtain satisfactory estimates for the distortion of intervals by maps f_a, where $e^{-e} \leqslant a < 1$. This is because, for $e^{-e} \leqslant a < 1/e$, these maps are expanding, for small values of x.

It is a simple matter to check that if (x, a^x) lies in the shaded set in Fig. 7 bounded by the y-axis and $x = -y \log y$ then

$$|f_a'(x)| = -a^x \log a \geqslant 1.$$

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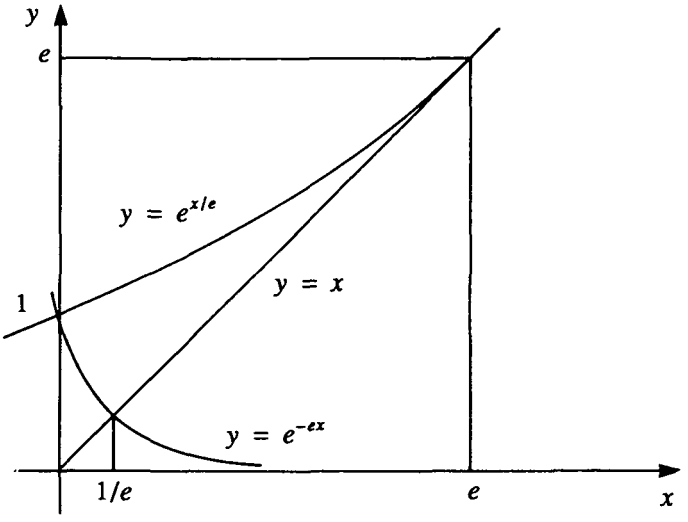


Fig. 6

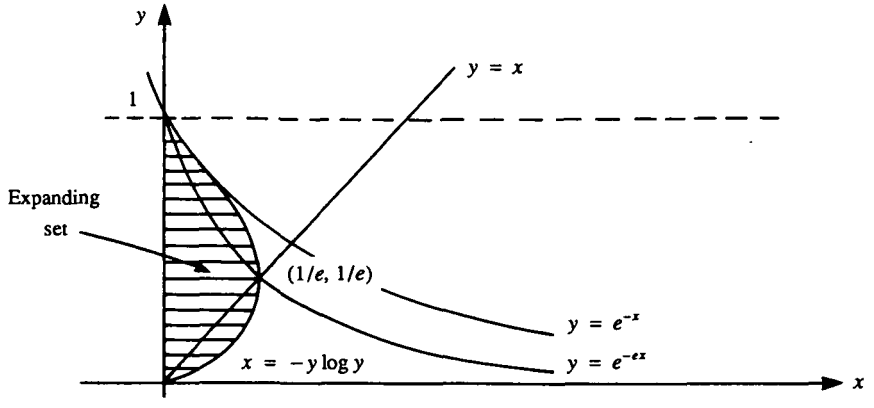


Fig. 7

Note that the graphs

$$x = -y\log y, \quad y = x \quad \text{and} \quad y = e^{-ex}$$

meet at $(1/e, 1/e)$, which is the right-most point of the expanding set.

We say that $f_a|_I$ is non-expanding if

$$|f_a'(x)| \leqslant 1 \quad (x \in I).$$

LEMMA 8. *If $e^{-e} \leqslant a < 1$ and $f_a|_I$ is non-expanding, then*

$$|f_a(I)| \leqslant \alpha_2(|I|),$$

where

$$\alpha_2(t) = 1 - e^{-t} < t \quad (t > 0)$$

is concave increasing.

Proof. Again let $t = |I|$. Then we may assume that

$$I = [b, b+t],$$

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where $b = 0$ if $1/e \leqslant a < 1$, and where b satisfies

$$a^b \log a = -1$$

if $e^{-e} \leqslant a < e^{-1}$. In the case $1/e \leqslant a < 1$, we deduce that

$$|f_a(I)| = 1 - a^t \leqslant 1 - e^{-t},$$

as required. In the case $e^{-e} \leqslant a < e^{-1}$, we have

$$|f_a(I)| = -\int_b^{b+t} a^x \log a \, dx = -\int_0^t a^{x+b} \log a \, dx$$

$$= \int_0^t a^x \, dx < \int_0^t e^{-x} \, dx = 1 - e^{-t},$$

as required.

To deal with the expanding set we make use of a key observation on p. 153 of Barrow [3]. If $|f_a'(x)| > 1$ for some x in the closed interval I, then I cannot be the image of a closed interval under a mapping f_b, for $b \geqslant 1$.

Thus we need to estimate the effect of $f_a \circ f_b$ on a closed interval I, when $e^{-e} \leqslant a, b < 1$.

LEMMA 9. *There is a concave increasing function α_3 with*

$$0 < \alpha_3(t) < t \quad \text{for} \quad t > 0,$$

such that, if $e^{-e} \leqslant a, b < 1$ then

$$|f_a \circ f_b(I)| \leqslant \alpha_3(|I|). \tag{7·2}$$

Proof. First note that if ξ is the unique point of inflection of $f_a \circ f_b$, then

$$b^\xi \log a + 1 = 0,$$

and so $a^{b^\xi} = e^{-1}$.

So if I lies entirely to the right of ξ and $t = |I|$ then

$$|f_a \circ f_b(I)| \leqslant a^{b^{\xi+t}} - a^{b^\xi} = e^{-b^t} - e^{-1} \leqslant e^{-e^{-et}} - e^{-1},$$

$$= \beta_1(t),$$

say, since $b^t \geqslant e^{-et}$ for $t \geqslant 0$.

Also, if I lies entirely to the left of ξ and $t = |I|$, then

$$|f_a \circ f_b(I)| \leqslant a^{b^\xi} - a^{b^{\xi-t}} = e^{-1} - e^{-b^{-t}} \leqslant e^{-1} - e^{-e^{et}},$$

$$= \beta_2(t),$$

say, since $b^t \geqslant e^{-et}$ for $t \geqslant 0$.

It is straightforward to check that β_1, β_2 are concave increasing with

$$0 < \beta_k(t) < t \quad (k = 1, 2, \quad t > 0).$$

For example, we need only note that

$$\beta_1(t) = a^{a^{(1/e+t)}} - a^{a^{(1/e)}}$$

and

$$\beta_2(t) = a^{a^{(1/e)}} - a^{a^{(1/e-t)}},$$

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where $a = e^{-e}$. Hence we can choose β_3 concave increasing with

$$0 < \beta_k(t) \leqslant \beta_3(t) < t \quad (k = 1, 2, \quad t > 0).$$

Now note that if $u, v \geqslant 0$ and $u+v = t$, then

$$\beta_3(u) + \beta_3(v) \leqslant 2\beta_3(t/2) = \alpha_3(t), \tag{7·3}$$

say, where α_3 is also concave increasing with

$$0 < \alpha_3(t) < t \quad (t > 0).$$

It is now clear from (7·3) that (7·2) holds with this function α_3.

To complete the proof of Theorem 5, we choose a concave, increasing function α such that

$$0 \leqslant \max\{\alpha_1(t), \alpha_2(t), \alpha_3(t)\} \leqslant \alpha(t) < t \quad (t > 0).$$

We than put

$$I_{n,k} = f_{a_k} \circ f_{a_{k+1}} \circ \ldots \circ f_{a_n}([0, e]) \quad (k = 1, 2, \ldots, n)$$

so that

$$I_{n,k} = f_{a_k}(I_{n,k+1}) \quad (k = 1, 2, \ldots, n-1).$$

By Lemmas 7, 8 and 9 we have

$$|I_{n,k}| \leqslant \begin{cases} \alpha(|I_{n,k+1}|) & \text{if } f_{a_k}|_{I_{n,k+1}} \text{ is non-expanding,} \\ \alpha(|I_{n,k+2}|) & \text{otherwise.} \end{cases} \tag{7·4}$$

Also it is clear that $|f_{a_n}([0, e])| \leqslant e$, and so, by repeated application of (7·4), we obtain

$$|I_n| = |I_{n,1}| \leqslant \alpha^{[n/2]}(e).$$

Since $\alpha^{[n/2]}(e) \to 0$ as $n \to \infty$, we deduce that (7·1) holds and so the proof of Theorem 5 is complete.

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