SOLUTION OF SOME ENGINEERING PARTIAL DIFFERENTIAL EQUATIONS GOVERNED BY THE MINIMAL OF A FUNCTIONAL BY GLOBAL OPTIMIZATION METHOD

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ABSTRACT

Many engineering problems are governed by partial differential equations which can be solved by analytical as well as numerical methods, and examples include the plasticity problem of a geotechnical system, seepage problem and elasticity problem. Although the governing differential equations can be solved by either iterative finite difference method or finite element, there are however limitations to these methods in some special cases which will be discussed in the present paper. The solutions of these governing differential equations can all be viewed as the stationary value of a functional. Using an approximate solution as the initial solution, the stationary value of the functional can be obtained easily by modern global optimization method. Through the comparisons between analytical solutions and fine mesh finite element analysis, the use of global optimization method will be demonstrated to be equivalent to the solutions of the governing partial differential equations. The use of global optimization method can be an alternative to the finite difference/ finite element method in solving an engineering problem, and it is particularly attractive when an approximate solution is available or can be estimated easily.

Keywords: Differential equation, Ultimate limit state, Optimization method, Variational principle.

1. INTRODUCTION

There are many engineering problems which are governed by a system of partial differential equations. Many of these equations can be cast into a form of functional (energy) minimization by using the variational principle. The weak form of the governing equations can then be solved by the finite element and finite difference method which are used by many engineers in various disciplines. Examples of such differential equation systems include the static equilibrium and seepage problems for which many computer programs (usually finite element based) are available for the engineers. Beside the static equilibrium and seepage problems, the ultimate limit state of a geotechnical system is also important in geotechnical analysis. Since the complete information on the insitu stresses, loading history and constitutive relation are usually not known with good accuracy, the ultimate conditions of many geotechnical problems are determined and used for the analysis and design. The ultimate condition for two-dimensional problem is usually governed by the plasticity hyperbolic partial differential equations for which analytical solutions are available only for some simple problems. In general, the solution of such hyperbolic slip line equations are solved by an iterative finite difference method along the characteristic lines as the problem is by nature a nonlinear problem.

For many geotechnical problems governed by the ultimate limit state, the limit equilibrium method (LEM) is actually more commonly adopted instead of the plasticity solutions, as only part of the solution domain is governed by the plasticity based equation and the geometry under consideration is usually highly irregular. Classical differential equations based plasticity formulation is impossible to be solved for most of the real problems and are seldom adopted in practice. The use of numerical method to solve the plasticity differential equations for slope stability problem is also very difficult as practical problems are usually nonhomogeneous with various internal and external forces. Limited successes to the solution of slip line equation have been achieved by Sokolovskii [1] for simple problems, but the extension of the partial differential equation analyt-

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ical solution to slightly more complicated problem has not been possible up to the present. Since LEM is adopted instead of the plasticity equations, the factor of safety which is related to the ultimate state is usually determined. The basic assumptions of the plasticity formulation and the LEM are however actually similar, as both formulations are governed by equilibrium and yield condition. In general, the simple LEM formulation is a poor approximation to the hyperbolic plasticity differential equations except for simple problems.

The scales of many civil engineering projects are usually large. The authors have come across many projects in Hong Kong and other countries where the solution domain exceed 100m in size. When the finite element method is used for the analysis, there is a major technical problem to be solved. 32bit finite element program can access less than 4G Ram (exclude the system requirement), and many three-dimensional commercial programs cannot accept more than 500000 three-dimensional elements in the analysis. Engineers are forced to use either 64bit program (if available) or use a coarse mesh for the analysis. For example, for the Port Said East container terminal in Egypt where the diaphragm wall is about 200m in length, the authors have tried different 32bit finite element programs in vain because of the memory limitation of 32bit computer program.

In views of the difficulty in solving the hyperbolic partial differential equation for plasticity problem as well as the computer memory problem for three-dimensional elliptical partial differential equation problem, the authors propose the use of optimization method as an alternative to the classical approach, and the present work is based on the extension of the works by Cheng *et al*. [2]. The variational principle which forms the basis of the finite element method is demonstrated by Cheng *et al*. [2] to be equivalent to the global optimum solution of a system. In the previous work, Cheng *et al*. [2] have considered only the equivalence of the ultimate limit equilibrium solution and the plasticity solution for a very special case, while the general solution from plasticity equations will be demonstrated to be equivalent to the solution from global optimization in the present paper. Furthermore, the authors have extended the application of global optimization to plane strain elasticity problem and seepage problem in the present work. The use of an approximate solution from a coarse mesh to evaluate more accurate solution through global optimization analysis is also a new piece of work which is adopted by the authors for the solution of a very large scale engineering project. It will be demonstrated that this new approach can be useful and effective for practical large scale and difficult problems.

2. CLASSICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

Denn [3] has given a detailed application of the variational principle to optimization problems which will not be repeated here, and it is demonstrated that the variational principle is an equivalent form of the optimization problem. On the other hand, Cheng *et al*. [2] have demonstrated the use of the modern global optimization method (with application to LEM) as a replacement of the variational principle which is the reverse of the works by Denn [3]. The works by Cheng *et al*. [2] will be further extended to the solution of plasticity hyperbolic partial differential equation in the present study.

Unlike the plasticity formulation, LEM is well known to be a statically indeterminate problem (Abramson *et al*. [4]) which relies on the assumptions on the internal force distributions $f(x)$. The Morgenstern-Price's (MP) method [5]) and the Janbu's rigorous method ([6,7]) are the most common 'rigorous' formulations while other existing formulations are just variants of these two basic methods. In the MP method [5], the inclination of the total internal force (see Fig. 1 for illustration) which is given by X/E is usually expressed as $\lambda f(x)$, where λ is a mobilization factor while $f(x)$ is a function specifying the normalized inclination of the interslice forces. The function $f(x)$ takes a value between 0 to 1 and x is a normalized distance in the range 0 to 1.0. In the Janbu's rigorous method ([6,7]), the distance between the thrust line and the base of the slip surface (h_t in Fig. 1) is assumed to be known while the local moment equilibrium is used in the stability formulation. From the plasticity lower bound theorem, as long as a statically admissible stress field is defined over a domain, the solution will be a lower bound of the ultimate limit state/failure. In this respect, LEM can be considered as an approximate but not an exact lower bound solution (Chen [8]), as force (lumped the stresses over a finite length) instead of stress is considered in the classical LEM.

Plasticity solution for the bearing capacity problem has been provided by Prandtl [9], Hill [10], Sokolovskii [1], Cheng and Au [11], Cheng *et al*. [12] and many others. Lateral earth pressure is another classical plasticity problem which has been considered by Sokolovskii [1], Chen [8], Soubra [13], Cheng [14] and many others. Detailed numerical treatment to the analysis of the plasticity equations have been considered and extended to various conditions. All the above studies are practically limited to homogeneous problems, as numerical treatment to nonhomogeneous conditions is "practically impossible" for plasticity slip line equations. In views of these situations, many engineers are still employing the LEM for their routine

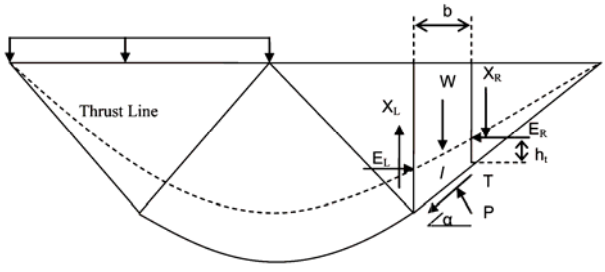


Fig. 1 Typical forces acting on a slice for LEM formulation

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works, and actually there are more LEM based computer programs than finite element based computer programs for slope stability analysis at present.

In this paper, the author will demonstrate that the use of LEM with an optimization analysis will yield solutions practically the same as those obtained from the plasticity slip line equations. Towards this goal, the slip line surface and $f(x)$ or the thrust line from the slip line solutions will be determined. The factors of safety will then be back-computed from LEM, and the equivalence between the LEM (using correct internal force distribution) and plasticity solutions will be illustrated. Based on that, the authors will demonstrate clearly that the classical limit equilibrium methods with a prescribed internal/external force assumption will be a lower bound to the ultimate condition. Finally, $f(x)$ will be taken a variable and the maximum extremum from the LEM will be determined, which will then be demonstrated to be equivalent to the slip line solution. As a further illustration, an example where there is no exact solution to the plasticity based slip line method will be solved by the equivalent limit equilibrium analysis using the extremum or equivalently variational method in this study.

$f(x)$ in classical LEM is commonly accepted to be indeterminate, and virtually all the engineers simply take $f(x)$ equal to 1.0 for engineering application. Even though this approach is generally acceptable for slope stability problem, it is a poor assumption for bearing capacity problem so that LEM is not adopted for bearing capacity determination. The bearing capacity of a simple footing is governed by the plasticity slip line Eqs. (1) and (2) under the Mohr-Coulomb yield criterion, and analytical solutions to Eqs. (1) and (2) are actually available for some simple cases. Sokolovskii [1], Booker and Zheng [15], Cheng [14], Cheng and Au [11], Cheng *et al*. [12] and many other have provided the solutions to this classical problem and the extension of this problem. In the present study, the slip line program SLIP developed by Cheng and Au [11] and program ABC by Martin [16] will be used for the slip line analysis of bearing capacity problem.

$$\alpha\ characteristics: -\frac{\partial p}{\partial S_\alpha}\sin 2\mu + 2R\frac{\partial \theta}{\partial S_\alpha} + \gamma\left(\sin(\varepsilon+2\mu)\frac{\partial y}{\partial S_\alpha} + \cos(\varepsilon+2\mu)\frac{\partial x}{\partial S_\alpha}\right) = 0 \quad (1)$$

$$\beta\ characteristics: \frac{\partial p}{\partial S_\beta}\sin 2\mu + 2R\frac{\partial \theta}{\partial S_\beta} + \gamma\left(\sin(\varepsilon-2\mu)\frac{\partial y}{\partial S_\beta} + \cos(\varepsilon-2\mu)\frac{\partial x}{\partial S_\beta}\right) = 0 \quad (2)$$

In Eqs. (1) and (2), $p = \frac{\sigma_1+\sigma_3}{2}$, $R = \frac{\sigma_1-\sigma_3}{2} = p\sin\phi + c\cos\phi$, σ_1 and σ_3 are the major and minor principal stresses respectively, c and ϕ are the cohesive strength and friction angle of soil, $\mu = (\pi/4 - \phi/2)$, γ is the unit weight of soil, θ is the direction of the major principal stress to y axis and ε is the angle between the body force and y-axis. The numerical procedures to solve Eqs. (1) and (2) under various conditions have been given in details by Sokolovskii [1], Hill [10], Booker and Zheng [15], Cheng [14], Cheng and Au [11] and Cheng *et al*. [12] and will not be repeated here. The bearing capacity problem is then considered as a "horizontal slope problem", and the corresponding $f(x)$, thrust line and base normal forces will be determined from the iterative finite difference slip line solutions of Eqs. (1) and (2). Once the stress field (p, R, θ) for the slip line field as shown in Fig. 2(a) is determined by Eqs. (1) and (2) using program SLIP and counter-checked by program ABC, the interslice force function and the thrust line location can then be determined as follows:

STEP 1: Calculate the normal stress (σ_x, σ_y) and shear stress (τ_{xy}) at any grid point by SLIP using the following relations:

$$p = (\sigma_1+\sigma_3)/2\ ,\quad R = (\sigma_1-\sigma_3)/2 \quad (3)$$

$$\text{therefore,}\quad \sigma_1 = p+R\ ,\quad \sigma_3 = p-R \quad (4)$$

Using the Mohr-Coulomb relation

$$\sigma_x = p + R\cos 2\theta \quad \sigma_y = p - R\cos 2\theta$$
$$\tau_{xy} = R\sin 2\theta \quad (5)$$

STEP 2: For any specified section (with a given x-ordinate) as shown in Fig. 2(b), determine the normal stress and shear stress at equal vertical interval by bi-linear interpolation from the 4 grid points in the slip line field enclosing any given x and y coordinates;

STEP 3: Calculate the interslice normal force E and shear force X in each specified section by integrating the normal stress and shear stress at an vertical interval Δy in a vertical direction (as shown in Fig. 2(a)) by:

$$E = \sum \sigma_x\,\Delta y \quad X = \sum \tau_{xy}\,\Delta y \quad (6)$$

STEP 4: The maximum ratio of X/E across all section determined from STEP 3 is denoted as the mobilization factor λ.

STEP 5: $f(x)$ across the slip surface can be obtain by:

$$f(x) = \frac{X/E\ (at\ each\ location\ x)}{\lambda} \quad (7)$$

STEP 6: The average normal stress σ_{ar} at each element along the vertical direction can be determined by the stresses σ_{x1} at top and σ_{x2} at bottom of the element. The lever arm h_t of the normal stresses above the base of the slip surface is determined from

$$\sigma_{ar} = \frac{\sigma_{x1}+\sigma_{x2}}{2} \quad h_t = \frac{\sum[\sigma_{ar}\,\Delta y(h_i - y_i)]}{E} \quad (8)$$

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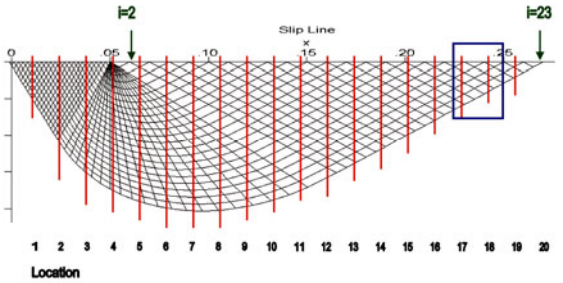


Fig. 2(a) Determination of $f(x)$ and thrust line from slip line analysis

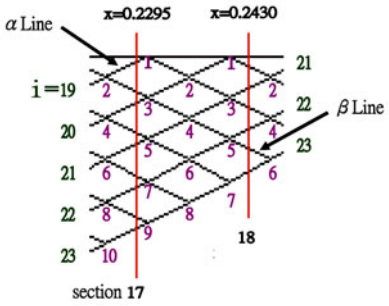


Fig. 2(b) Interpolation to get stresses at any point from the grid points in the slip line field

where h_i, y_i are the average y-ordinates of the element and the y-ordinate of the slip surface at section i respectively.

From the results of SLIP which gives the stresses at every point within the system, $f(x)$ for different ϕ is determined for cases associated, the bearing capacity factors from the use iterative finite difference solutions are obtained and are compared with the analytical bearing capacity factors N_c, N_q given in Eq. (9) and the limit analysis of N_γ in Eq. (10). A typical slip line field for the case of N_c where the weight of soil γ is zero is shown in Fig. 3, where the foundation load is applied on the ground surface at the left hand side. The slip line field from SLIP is the same as the classical solution with a central log-spiral zone (Hill [10], Booker and Zheng [15]). For the slip line field for N_γ as shown in Fig. 4 with $\phi = 30°$, the active zone is actually curved while the intermediate radial shear zone is not a true log-spiral zone with an inscribed angle less than 90°. The results in Figs. 3 and 4 are the same as the slip line solutions from Sokolovskii [1], Martin [16] and others, and the accuracy of SLIP and the iterative numerical procedures as proposed by Cheng and Au [11] is demonstrated.

$$N_q = e^{\pi\tan\phi}\tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right) \quad N_c = \cot\phi[N_q - 1] \tag{9}$$

$$N_\gamma = \frac{1}{4}\tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi\right)\left[\tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi\right)e^{\left(\frac{1}{2}3\pi\right)\tan\phi} - 1\right] + \frac{3\sin\phi}{1+9\sin^2\phi}\left\{\left[\tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi\right) - \frac{\cot\phi}{3}\right]e^{\left(\frac{1}{2}3\pi\right)\tan\phi} + \tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi\right)\frac{\cot\phi}{3} + 1\right\} \tag{10}$$

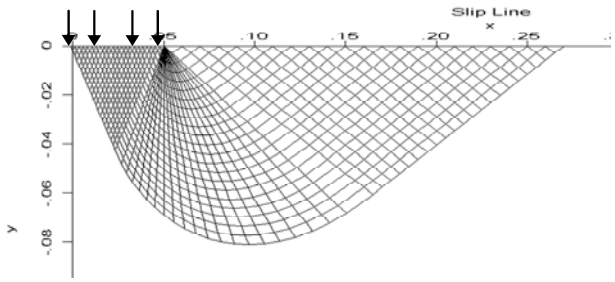


Fig. 3 Slip line for case of N_c when $\phi = 30°$, $N_c = 30.18$ from program SLIP and 30.14 from classical bearing capacity equation

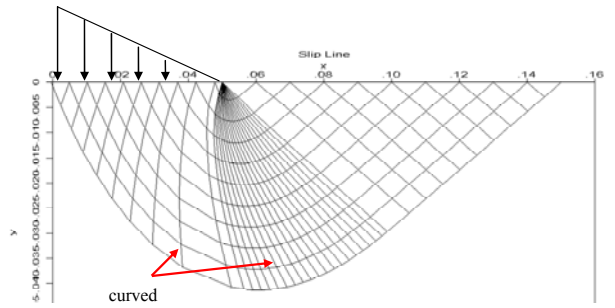


Fig. 4 Slip line for case of N_γ when $\phi = 30°$, $N_\gamma = 15.32$ from program SLIP and 15.3 from Sokolovskii [1]

It should be noted that while the ultimate bearing capacity factors from SLIP are relatively insensitive to the grid sizes used in the analysis, the thrust line is more sensitive to the size of the grid. This situation is particularly important for the case of N_γ, and a very fine grid is hence adopted for the case of N_γ or else there will be a larger fluctuation in the thrust line location. $f(x)$ for the cases associated with N_c, N_q and N_γ are given in Figs. 5, 6 and 7. The results of thrust line h_t for N_c (and N_q) at $\phi = 0$, 10°, 20°, 30° and 40° are given in Fig. 8. The horizontal axis is dimensionless distance x in the range of 0 to 1. The results of the thrust line ratio for N_q are the same as that for N_c and is also given in Fig. 8. This result is also in line with the results that $f(x)$ are the same for the cases of N_q and N_c. For the case of N_γ, the thrust line is given in Fig. 9.

When the correct $f(x)$ or the thrust line for the ultimate condition is defined, the problem can be back-analyzed in the following ways. For the case of N_c, a uniform pressure corresponding to cN_c is applied on ground surface without any surcharge outside the foundation. The unit weight of soil is set to zero in the slope stability analysis. For the case of N_q, a surcharge of 1 unit is applied outside the foundation, and the uniform foundation pressure is given by N_q unit while the unit weight of soil and cohesive strength are set to zero in the analysis. For the case N_γ, a triangular pressure (see Fig. 3) with the maximum equal to γBN_γ is applied on the ground surface (average pressure is $0.5\gamma BN_\gamma$), while the cohesive strength and surcharge outside the foundation are set to zero. Based on the $f(x)$ as shown in Figs. 5, 6 and 7 and the failure surfaces as given by the slip line solutions, the authors

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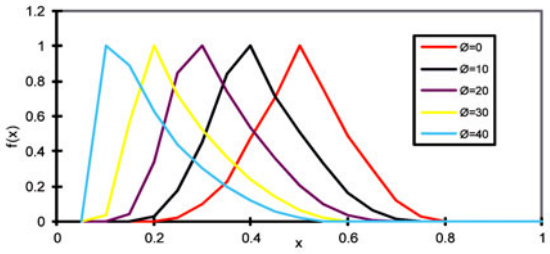


Fig. 5 $f(x)$ against different dimensionless distance x in case of N_c

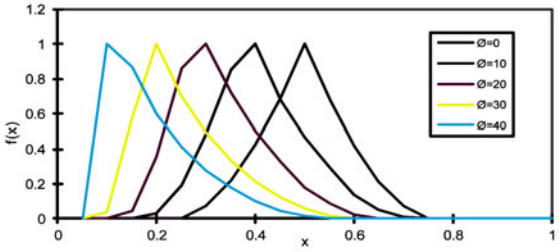


Fig. 6 $f(x)$ against different dimensionless distance x in case of N_q

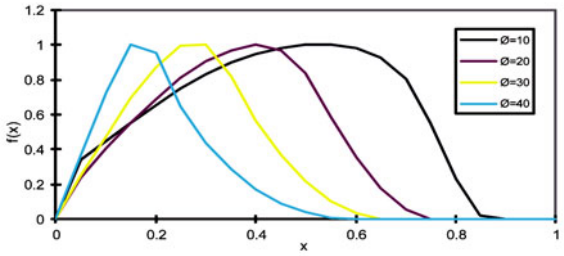


Fig. 7 $f(x)$ against different dimensionless distance x in case of N_γ

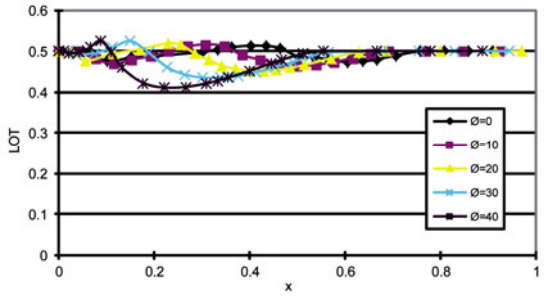


Fig. 8 The thrust line for different ϕ angles for case N_c (x coordinate is x ratio) (same result for N_q) (LOT = thrust line ratio)

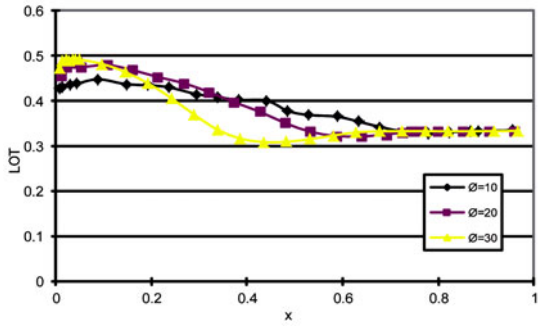


Fig. 9 The thrust line for different ϕ angles for case N_γ (x coordinate is x ratio)

have back computed the factors of safety to be nearly 1.0 for all the cases ! That means, if the correct internal forces relation $f(x)$ can be defined, the simple force/moment equilibrium LEM method can be equivalent to the plasticity based partial differential equations as given by Eqs. (1) and (2). For the ultimate condition, there is no difference between the uses of $f(x)$ or the thrust line in formulating a problem, provided that the $f(x)$ or thrust line corresponds to the ultimate condition. Other than the ultimate condition, the choice of $f(x)$ or thrust line will give different factors of safety (well known in limit equilibrium analysis) as the solutions are only typical lower bound solutions for every prescribed $f(x)$ or thrust line.

3. SOLUTION OF HYPERBOLIC PLASTICITY EQUATION BY GLOBAL OPTIMIZATION OF LIMIT EQUILIBRIUM ANALYSIS

Optimization process can be either unconstrained or constrained, and constrained analysis is actually more common in practice. The objective functions in many engineering or transportation problems are usually *N-P* (non-polynomial) hard type problem (difficult to be proved, but is commonly believed), and the objective function can be non-convex, non-smooth function with the presence of multiple local minima. For the maximum limit equilibrium method in the first part of the present paper, the objective function (factor of safety) is also not a continuous function while the stress analysis and seepage problem in the later part of the present paper are continuous problem. The continuity of the function can also greatly affect the optimization process, and different optimization algorithms are used for the continuous and discontinuous functions in the present paper.

For the solution of plasticity hyperbolic equation by means of global optimization of limit equilibrium method, since the objective safety factor function is discontinuous in nature, the use of modern heuristic optimization method will be the best approach. The modern heuristic global optimization methods that have evolved in recent years have attracted the attention and application of many engineers, economists and mathematicians. For example, the genetic algorithm; ant-colony method; simulated annealing method, particle swarm optimization, harmony search, Tabu search and fish swarm methods as well as variants of these methods are adopted and improved by Cheng *et al.* [2,17] and many others for engineering. Within the past 10 years, there are several hundred papers and more than 30 books devoted to the heuristic optimization methods, and many commercial engineering programs have also adopted heuristic optimization method in the programs.

For the previous problems where $f(x)$ or thrust line from the plasticity based slip line solutions are used, the factors of safety of the system from the LEM will be very close to 1.0. As discussed by the authors, whenever a $f(x)$ is prescribed, the solution will always be the

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lower bound which can be illustrated by the results in Table 1. The reverse of the problem is then considered in the following way. Taken the internal variables $f(x_i)$ to be a control variables, the factor of safety F of the system using the LEM is given by the MP method. Based on the lower bound concept, the factor of safety of such system must be less than 1.0 as shown in Table 1. If $f(x_i)$ (i = 1 to n, where n is the number of discretization of the problem) is set to 1.0 as the initial trial in the optimization analysis, a factor of safety F equal to 0.935 is obtained which is far from 1.0 (engineers adopt $f(x)$ = 1.0 is hence not appropriate for this case). The factors of safety are also far from 1.0 initially as the trial $f(x)$ is far from the rigorous solution. As the global optimization analysis proceeds according to Eq. (9) (simulated annealing method in the present study), the extremum (factor of safety) will tend to the theoretical value of 1.0 which is also the maximum of the functional, and no factor of safety exceeding 1.0 can be found. The lower bound concept can be visualize clearly by Fig. 10, which shows all the temporary factors of safety during simulated annealing analysis in searching for the maximum factor of safety using $f(x)$ as the variables in the extremum determination. It must be emphasized that the constraint in Eq. (11) is important in the optimization analysis, as violation of the constraints will give unreasonable F during the computation. In the present study, a very large value is assigned to F during the search if the constraint is violated. Interested readers may refer to references [17] and [2] for further details about the procedures for the global optimization analysis.

$$\text{Maximize } F \text{ subject to } 0. \le f(x_i) \le 1.0$$
$$\text{subject to no yielding along vertical faces} \qquad (11)$$

Although the lower bound principle is known for many years, the results as shown in Fig. 10 have not appear previously due to the lack of a stable optimization algorithm for this highly discontinuous problem. The modern heuristic optimization method has clearly illustrated the lower bound principle in plasticity theory, and has also demonstrated the equivalence between the plasticity formulation and the simple limit equilibrium method under the extremum condition. The use of simple equations in static mechanics with the control variables varied until the extremum condition is achieved is actually equivalently to the solution of the hyperbolic partial differential equations.

Based on the previous problems and the present problem, it is established that the maximum extremum from LEM is the same as the solutions from the hyperbolic differential equations. For a prescribed failure mechanism, the system will exercise its maximum strength before failure, and this is conceptually the lower bound theorem. Either $f(x)$ or the thrust line corresponding to the ultimate limit state will be sufficient to define the system, and the solution of the simple LEM problem by varying the internal variables is just equivalent to solving the plasticity hyperbolic slip line equations. This concept is important in that for highly irregular nonhomogeneous problem, LEM can still be applied without any difficulty, and the formulations for simple problem or nonhomogeneous highly irregular problem are the same under the LEM. On the other hand, up to the present, there is not any reported success in the solution of the present hyperbolic partial differential equation in nonhomogeneous highly irregular problem. The present approach can be viewed as the application of discretized variational principle to solve Eqs. (1) and (2). The actual mathematical tool is the modern heuristic optimization method which is an equivalent variational principle but can be a practical tool for generally complicated problem. Through global optimization analysis, the solution from the simple LEM extremum will be the solution of the complicated partial differential equation as given by Eqs. (1) and (2), hence solution of the partial differential equations as given by Eqs. (1) and (2) can be replaced by a simple global optimization analysis.

Table 1 Factor of safety corresponding to the three bearing capacity factors based on $f(x)$ = 1

ϕ (°)	N_c	N_q	N_γ
0	0.941	0.941	–
10	0.941	0.941	1.0
20	0.937	0.936	0.989
30	0.925	0.924	0.968
40	0.894	0.892	0.939

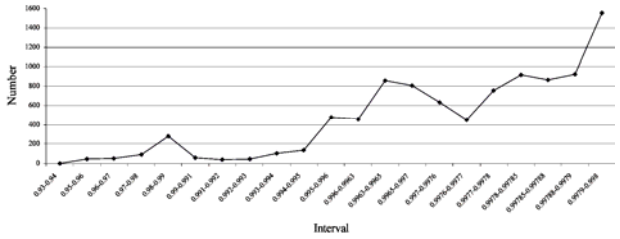
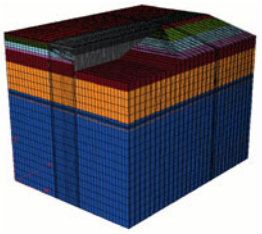


Fig. 10 Distribution of acceptable factor of safety during simulated annealing analysis for N_q with ϕ = 30° using $f(x)$ as the variable

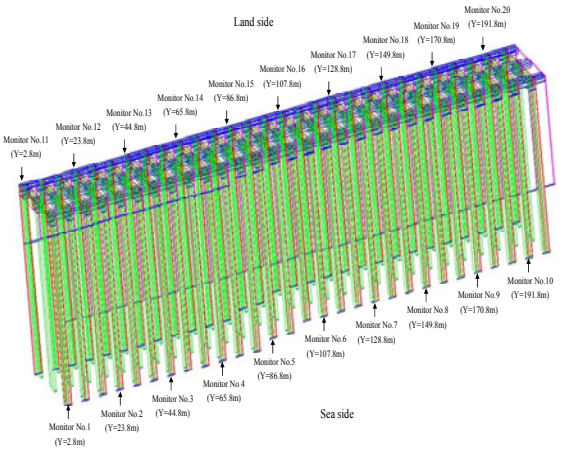
4. USE OF GLOBAL OPTIMIZATION METHOD AS A COMPLIMENT TO FINITE ELEMENT ANALYSIS

As a further illustration, the Port Said East container terminal is finally analyzed by using a 64bit computer program with the computer model as shown in Fig. 11. The computed wall displacement as shown in Fig. 12 is determined with a mesh of about 1000000 three-dimensional elements. The authors later adopt a 32bit computer program using 100000 elements for the analysis, and the maximum difference in the results amounts to more than 30% which is not acceptable. However, based on the approximate results from the coarse mesh analysis, a finer mesh with 800000 elements is constructed. The approximate displacements for the extra nodes are interpolated from the results for the coarse mesh. Based on the total potential energy E for the governing problem is then formulated as the function of the nodal displacement δ as Eq. (12).

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(a) Complete finite element mesh



(b) Enlarged view of the diaphragm walls and barettes

Fig. 11 Three-dimensional finite element mesh for Port Said East container terminal

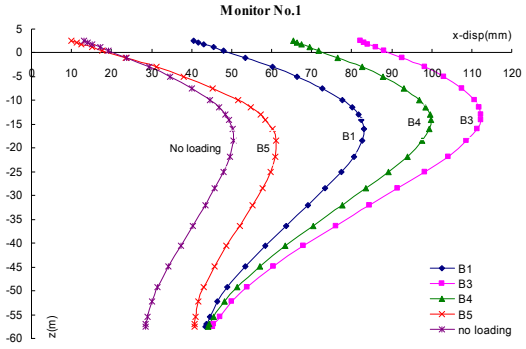


Fig. 12 Displacement of diaphragm wall at monitoring point 1 from 64bit calculation

$$E = \frac{1}{2}\int \sigma \cdot \xi \, dx \, dy - q\delta \tag{12}$$

$$\{\xi\} = [B]\{\delta\} \text{ and } \{\sigma\} = [D]\{\xi\} \tag{13}$$

where σ and ξ are the stress and strain within the elements, q is the external load, $[B]$ is the strain displacement matrix for the refined element, δ is the nodal displacement for the refined and $[D]$ is the elastic constitutive matrix. The strain and stress for refined elements are determined from Eq. (13) which are the standard finite element procedure. There are about 2 million of variables for the present three-dimensional problem, and it is extremely time-consuming to determine the minimum energy from scratch. Since an approximate trial displacement vector is available from the coarse mesh analysis and the energy function is a continuous function, the FSQP method based on Lawrence and Tits [18] is then used to minimize the error norm associated with the finer mesh without construction of the new global stiffness matrix. There are many variants of the SQP available in literature. For the method by Lawrence and Tits [18] which is a variation of the standard SQP, the intermediate solutions are generated within the feasible solution domain governed by the constraints (equalities and inequalities). A fast descent direction approaching the true SQP so as to ensure superlinear convergence, and the search direction is computed by solving QP and the tilting parameter. The generation of a new iterate requires only a single QP (against classical FSQP in solving 3 QPs) and two closely spaced linear least square problem which is not time consuming in the analysis. If a good initial trial is available, the convergence can be extremely fast, and this is the main reason why this method is adopted in the present analysis. The amount of computer memory required is greatly reduced using this strategy as the global stiffness matrix is not required to be formed. It should be noted that the determination of the extremum is actually equivalent to the use of variational principle which has been demonstrated by Cheng *et al*. [2]. There is thus a sound theoretical background in using global optimization as a replacement of the finite element method for the present problem.

After the minimization of the energy norm using the nodal displacements as the control variables, the difference of the wall displacement as determined from this 32bit computation as compared with a 64bit finite element analysis is less than 5% in general. In this respect, the use of global optimization method can be a useful tool for solving large scale engineering problem with limited computer resources. The use of the global optimization method can be a useful tool for the present problem as the finite element method is actually equivalent to the minimum energy formulation. Compared with the use of finite difference in solving an engineering system, the present strategy has the advantage that using an approximate solution from a coarse mesh, the solution time is greatly reduced. Using a 64 bit finite difference program, the present problem is solved in about 7 days using an Intel i7-920 with 8 processes under parallel processing, but only 2 days computer time is required for the present strategy (using 1 process in the computer code during computation).

Finally, the authors have also solved the Laplace seepage equation as given by Eq. (14) using the global optimization analysis. Instead of establishing the interpolation function and using the Galerkin finite element formulation for the minimum residual, the authors directly obtained the residual using the head values at each node and varies the nodal values until the minimum residual is obtained. A typical flow-net after the solution of Eq. (14) which is obtained by global optimization is shown in Fig. 14. It should be noted that the result as shown in Fig. 14 is virtually the same as that obtained from the finite element analysis or finite difference method except that there is no need to assemble the matrix for solution.

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} = 0 \tag{14}$$

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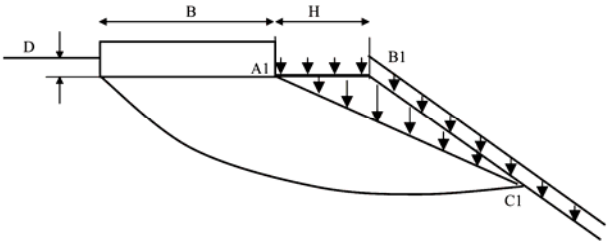


Fig. 13 Approximate modeling of footing with embedment and set back by slip line method

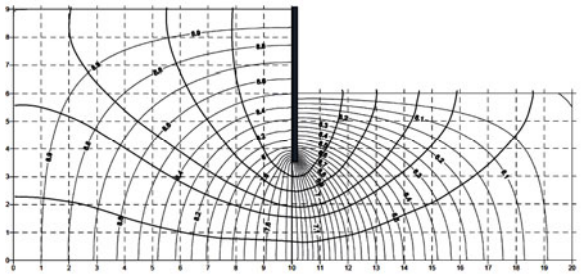


Fig. 14 Flow net by global optimization analysis

5. DISCUSSIONS

The classical plasticity solution which is a nonlinear hyperbolic problem is difficult or impossible to be solved for general problems. Although many analytical and numerical solutions to this plasticity equation have been obtained for various cases, all of these solutions are simple in the geometry and loading condition and the solution domain is completely governed by the plasticity equation. Beyond these limitations, only approximate solutions can be obtained. Most of the engineers are hence still applying the limit equilibrium method in their routine analysis and design works which are just approximation to the rigorous solutions. A typical slope stability analysis in Hong Kong takes only 15 ~ 30 minutes for the completion using the limit equilibrium method while finite element analysis may requires one to several days for the analysis. In views of the scale of works and the small difference between the results from LEM and finite element, it is not surprising that most of the engineers are still using the LEM in their daily routine design works.

In this paper, some simple and well studied plasticity problems are considered. The authors have demonstrated that as long as the ultimate limit equilibrium condition is considered, there is practically no difference between the plasticity based equation and the simple limit equilibrium formulation. The authors have also clearly illustrated that as long as an assumption is prescribed, the solution will be a lower bound of the ultimate condition which is shown in Fig. 10, hence the classical limit equilibrium methods are practically lower bounds to the ultimate condition. However, by taking $f(x)$ as a variable and optimizing the functional with respect to $f(x)$, the resulting solution which is still based on the simple force/moment equilibrium will be equivalent to the much more difficult plasticity solution. On the other hand, the use of limit equilibrium method is not limited to problems simple geometry and loading. In this respect, the use of extremum limit equilibrium method using global optimization method is equivalent to the solution of the partial differential equations system. Provided that the control variable and constraints are properly controlled during the optimization analysis, the simple limit equilibrium method will converge to the plasticity based solution.

As a further illustration, the bearing capacity problem of a buried foundation adjacent to a slope with a horizontal set-back which has been considered by Cheng and Au [11] and Graham *et al*. [19] is reconsidered in Fig. 13. This problem can only be considered by finite element analysis but not the plasticity based slip line equations, as part of the solution domain has not yielded and is not controlled by the plasticity slip line equations. Using 64 bit program Flac3D for this problem, it takes about 1 day for the computation (using Intel Xenon, with all 4 process running). Towards this problem, approximate slip line solution has been adopted by Cheng and Au [11] and Graham *et al*. [19]. Using the present approach, a fully yielded solution domain is not required, and optimized limit equilibrium method can be used to assess the bearing capacity of this footing. The results as shown in Table 2 have also demonstrated that the present approach has provided a good bearing capacity factor which is not possible for the classical Spencer's method using $f(x) = 1.0$ (always under-estimate N_γ). The results obtained in the present study can match the test results by Shields *et al*. [20] well while the approximate plasticity solutions by Graham *et al*. [19] are actually not good. This problem is interesting in that the hyperbolic slip line partial differential equation cannot be cast into a form of energy minimization, but it is still possible to use global optimization method to solve this type of partial differential equation.

Table 2 N_γ Use of slope stability method to N_γ determine as compared with slip line method by Cheng and Au [11] and Graham *et al.* [19] and model test by Shields *et al.* [20]

Geometry		Experimental			
H/B	D/B	Shields	Graham *et al.* [19]	Cheng and Au [11]	Present
		($\phi = 37^o$)			
0.0	0.0	24	64	27	26
0.0	0.5	45	140	48	47
0.0	1.0	70	176	77	74
1.0	0.0	30	100	46	42
1.0	0.5	55	148	73	66
1.0	1.0	75	196	98	85
		($\phi = 41^o$)			
0.0	0.0	100	140	54	76
0.0	0.5	120	200	93	110
0.0	1.0	150	320	146	146
1.0	0.0	120	200	91	108
1.0	0.5	150	300	130	144
1.0	1.0	180	380	173	167
2.0	0.0	150	260	148	149
2.0	0.5	175	350	190	186
2.0	1.0	200	450	232	221

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6. CONCLUSIONS

Based on the present study, the hyperbolic plasticity equation and the optimum limit equilibrium method using global optimization method are demonstrated to be the same in general under the ultimate condition, except for the ease of mathematically manipulation. The hyperbolic partial equations governing the α and β characteristic lines as given by Eqs. (1) and (2) can be well approximated by tuning $f(x)$ until the critical solution is obtained with simple force and moment equilibrium. This is an interesting and useful application of the lower bound concept to general irregular problem where classical slip line method fail to work.

The present work has demonstrated that the solution of some partial differential equations governed by the minimal of the functional can be replaced by the use of modern heuristic optimization method (an equivalent variational principle method), and this approach can be adopted as an alternative to the finite element method. The nodal variables (total head) are taken as the control variables, and the variables are varied by modern optimization method until the minimum residual from the Galerkin formulation has been achieved. Fundamentally, this alternative approach does not require the domain under consideration to be homogeneous. For problem where an initial approximate solution to a partial differential equation is known, the proposed method will be particularly efficient in the solution and offers an alternative to classical numerical method.

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