Math. Proc. Camb. Phil. Soc. (1984), **95**, 495
Printed in Great Britain

On truncation of the structural potential function

BY G. W. HUNT AND K. A. J. WILLIAMS
Department of Civil Engineering, Imperial College, London

(*Received* 6 *September* 1983; *revised* 7 *November* 1983)

Abstract

Two, essentially different, criteria for truncating a potential function of a conservative structural system are reviewed, one based on deflection considerations and the other on the mathematical concept of determinacy. Formulations in which they give different results are discussed; most importantly, these include Koiter's 1976 form for mode interaction in stiffened structure [**11**], which exhibits Thom's parabolic umbilic catastrophe [**15**]. For this case, two alternative schemes of perturbation analysis are presented, each describing asymptotically the post-buckling response. The more obvious approach turns out to be the less successful, being linked to an indeterminate, too severely truncated, form of the potential function.

The second, determinacy linked, approach generates as a first-order response a distinctive looping pattern of equilibrium paths, which has recently been identified in a simple model due to Budiansky and Hutchinson [**1**], by exact, closed-form, solution [**8**]. The convergence of each asymptotic scheme to the exact solution is briefly reviewed for this simple model, by plotting first and second order results.

1. *Introduction*

The formulation of an elastic structural model usually implies some form of consistent approximation applied to the deflection, with the effect of naturally truncating the corresponding potential energy function. For example, Koiter's 1945 analysis of cylinder buckling neglects all third and higher degree terms in the components of displacement and their derivatives, and this gives a potential function which is truncated after quartic terms of energy [**10**]. One notable exception, probably because of its inherent simplicity, is the pin-ended strut or elastica; Timoshenko and Gere [**20**] include a complete solution over gross deformations found by use of elliptic integrals, while the associated untruncated potential function is relatively easy to determine to as high an order as necessary and has been studied in some depth [**18**].

Meanwhile, modern differential topology has suggested a quite different criterion for potential function truncation, based on the concept of *determinacy*. Briefly, in structural mechanics terms, this concentrates on the most singular critical equilibrium state that a structure can experience, and pin-points the energy terms that are required for a correct local (or asymptotic) description of that state. Generally, more and higher order terms are required as the underlying singularity becomes more complex.

The concept of determinacy lies at the heart of catastrophe theory [**12**, **15**, **22**], which then goes on to 'unfold' the singularity so that 'structural stability'(in the topological sense) is assured; further perturbations can then only gently deform, but not fundamentally alter, the topology (see, for example, [**7**]).

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

Here we are concerned more with the concept of determinacy rather than full unfolding. We concentrate on the patterns of equilibrium paths generated by a compound bifurcation, which although not structurally-stable in the strict topological sense, nevertheless retain many stable properties. We note that path behaviour is now used as a basis for classification in the modern bifurcation theory of Golubitsky and Schaeffer [2, 3].

Frequently the two truncation criteria coincide, but sometimes they do not. For example, a naive formulation of the elastica might truncate the energy expansions after leading terms, it being supposed quite correctly that, for moderately large deflections, fourth powers of w and its derivatives are small compared with squared terms [18]. However, unlike other structural forms (cylinders, plates and arches for example), the approximation generates no cubic or quartic energy terms: it leaves only the quadratics, appropriate to linear eigenvalue study, and the critical state is thus seen as one of neutral equilibrium, represented by an entirely flat plateau of energy. Determinacy recognizes that this truncation is insufficient to describe completely the system behaviour.

The less trivial phenomenon of interest here involves the geometry of the parabolic umbilic castastrophe, the most complex of Thom's list of the initial seven archetypal stability phenomena [15]. This is a complicated problem, and one that might seem rather academic, were it not for the fact that the parabolic umbilic is emerging as the most important fundamental phenomenon underlying twofold interactive buckling in elastic structural systems [3, 5, 6]. Here, determinacy suggests truncation of the potential function after one of the quartic energy terms, but before the others. The consequences of this are explored in some depth, using perturbation methods of analysis.

2. *The parabolic umbilic catastrophe*

In most twofold mode interaction problems of elastic structures the potential function appears in the same very specific form, comprising a single cross-term cubic together with quartic energy terms. Examples abound in the mechanics literature. Most importantly, Koiter [11] developes this in his general theory of the interaction between local and overall buckling in stiffened plates and shells, partly by allowing for amplitude modulation of the local mode in the presence of the overall mode. Similarly, Budiansky and Hutchinson [1] arrive at the same form for a simple spring and link model, specifically designed to illustrate the phenomenon of mode interaction; this is fully analysed in a companion paper [8], a few results being presented here. All such problems can be seen as bifurcational manifestations of the parabolic umbilic catastrophe [12, 15, 22].

The relative importance of the form for elastic structures seems to be largely due to inherent symmetries [5, 6]. As an example, two of the many contributing modes of the axially-loaded cylinder are shown in Fig. 1. Here each of the modes is symmetric in its own right – if reversed it gives an identical shape and hence identical energy level. The instability can thus be seen as a balancing act, the structure not knowing which way to buckle. We note that Q_3 denotes uniform dilation of the cylinder, an important ingredient but one that can be ignored for the present argument.

But symmetry is broken if both modes occur simultaneously. There is a considerable release of circumferential stretching with q_1 and q_2 in the senses shown, but this is

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

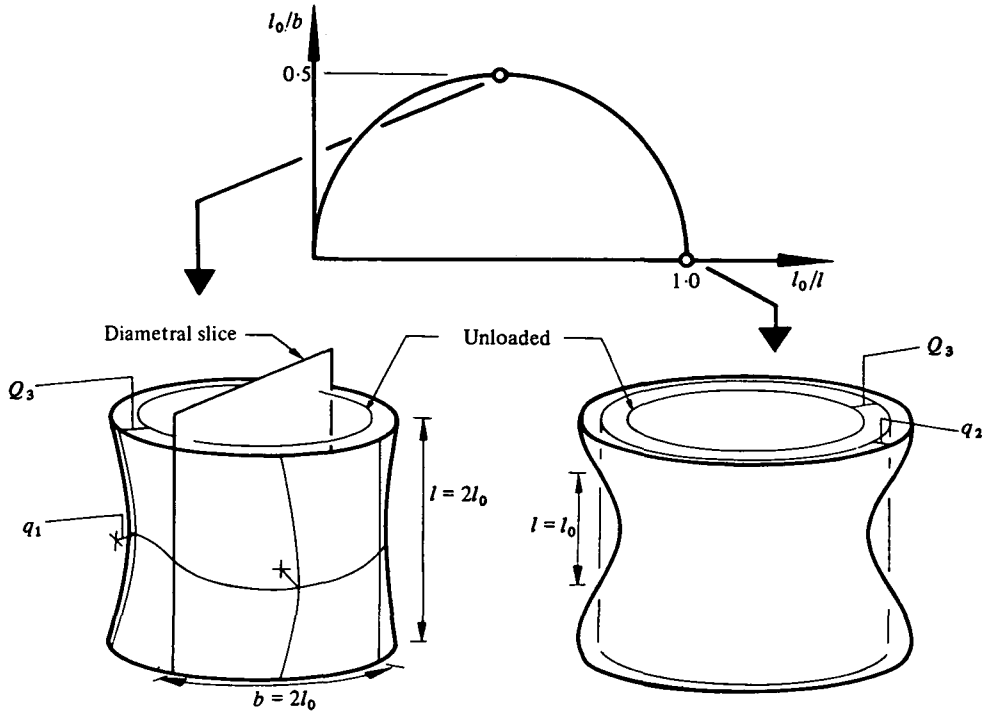


Fig. 1. Two of the contributing modes of the axially-compressed cylinder, exhibiting the typical symmetries of a parabolic umbilic interaction.

converted into extra stretching if q_2 is reversed. This gives a combination of modes which the system would like to adopt, and the result is both instability in the combined mode and imperfection-sensitivity. The fact that the system is also symmetric in q_1 in the presence of q_2 reduces the cubic energy contribution to a single cross-term [**5, 6**]. The exact nature of this complex interaction has only recently been recognised in the mathematics literature [**3**], where such problems are referred to as displaying 'hidden' symmetries.

We note that this mechanism is clearly of considerable significance, but we do not pretend that it fully explains the much more complex phenomenon of elastic cylinder buckling [**10**].

The potential function for such problems can be expressed as,

$$\begin{aligned}\mathscr{A} = {} & \tfrac{1}{2}\mathscr{A}^C_{112}u_1^2u_2 + \tfrac{1}{24}\mathscr{A}^C_{2222}u_2^4 + \tfrac{1}{4}\mathscr{A}^C_{1122}u_1^2u_2^2 + \tfrac{1}{24}\mathscr{A}^C_{1111}u_1^4 \\ & + \tfrac{1}{2}(\Lambda - \Lambda^C)(\mathscr{A}'^C_{11}u_1^2 + \mathscr{A}'^C_{22}u_2^2) + \tfrac{1}{2}\mathscr{A}'^C_{11}\sigma u_1^2 + \mathscr{A}^{1C}_1\epsilon^1u_1 + \mathscr{A}^{2C}_2\epsilon^2u_2 \\ & + \text{terms of higher order.} \end{aligned} \tag{1}$$

Some explanation is necessary. u_1 and u_2 represent the amplitudes of two buckling modes, the effect of higher modes being already fully accounted for by this stage, via the concept of the elimination of passive coordinates [**5**]. Λ is a loading parameter with critical value Λ^C, and σ is a splitting parameter introduced to separate the two contributing bifurcations on a fundamental equilibrium path [**5**]. The ϵ^i denote imperfections in the two critical modes, and will make no further appearance here, concentrating as

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

we do on the equilibrium paths of the perfect system. Setting all the control parameters, $\Lambda - \Lambda^C$, σ, and ϵ^i to zero describes the energy surface at the most singular possible state of the system, the twofold compound critical state C. The potential function is written as a Taylor expansion about C, subscripts denoting partial differentiation with respect to the u_i and primes denoting partial differentiation with respect to Λ. The form of (1) would provide a full unfolding of the parabolic umbilic catastrophe at C.

We include here the single cross-term cubic represented by the coefficient $\mathscr{A}^C_{112}$, together with the complete quartic contribution that would appear under the assumed conditions of symmetry. The concept of determinacy, however, relegates two of the quartics, $\mathscr{A}^C_{1122}$ and $\mathscr{A}^C_{1111}$, to the higher-order terms leaving $\mathscr{A}^C_{112}$ and $\mathscr{A}^C_{2222}$ at the core of the singularity [**12**, **15**, **22**]. This suggests that a cubic and a quartic energy term are, in an asymptotic sense, of the same order. We explore the consequences in some detail.

A mathematician will thus consider a potential function without $\mathscr{A}^C_{1122}$, $\mathscr{A}^C_{1111}$ and the rest of the higher-order terms, to obtain a correct local picture of the phenomenon. As an example, we plot here the equilibrium paths of a perfect system experiencing a *paraclinal point of bifurcation*, the bifurcational manifestation of the parabolic umbilic catastrophe. Truncating, and setting $\epsilon^i = 0$, we have,

$$\mathscr{A} = \tfrac{1}{2}\mathscr{A}^C_{112}u_1^2u_2 + \tfrac{1}{24}\mathscr{A}^C_{2222}u_2^4 + \tfrac{1}{2}\lambda(\mathscr{A}'^C_{11}u_1^2 + \mathscr{A}'^C_{22}u_2^2) + \tfrac{1}{2}\mathscr{A}'^C_{11}\sigma u_1^2 \tag{2}$$

where $\lambda = \Lambda - \Lambda^C$. We note that by removing the ϵ^i from the description we can no longer achieve universal unfolding, although the form of this bifurcational manifestation is nevertheless of considerable analytical and classificational significance [**3**, **8**].

The equilibrium equations are then simply,

$$\left.\begin{aligned} \frac{\partial \mathscr{A}}{\partial u_1} &= \mathscr{A}^C_{112}u_1u_2 + \mathscr{A}'^C_{11}u_1(\lambda+\sigma) = 0, \\ \frac{\partial \mathscr{A}}{\partial u_2} &= \tfrac{1}{2}\mathscr{A}^C_{112}u_1^2 + \tfrac{1}{6}\mathscr{A}^C_{2222}u_2^3 + \mathscr{A}'^C_{22}\lambda u_2 = 0. \end{aligned}\right\} \tag{3}$$

The first equation has two solutions,

$$u_1 = 0 \quad \text{and} \quad \mathscr{A}^C_{112}u_2 + \mathscr{A}'^C_{11}(\lambda+\sigma) = 0. \tag{4}$$

At constant σ, each defines a plane in the $\Lambda - u_i$ space of Fig. 2. The second equation picks out paths in the planes. We have first,

$$u_1 = 0, \quad \lambda = -\frac{1}{6}\frac{\mathscr{A}^C_{2222}}{\mathscr{A}'^C_{22}}u_2^2, \tag{5}$$

an *uncoupled* solution, identical to a distinct symmetric bifurcation critical with respect to u_2(16), lying in the $\Lambda - u_2$ plane. Secondly, the inclined plane carries the *coupled* solution,

$$u_1^2 = 2\frac{\mathscr{A}'^C_{22}}{\mathscr{A}^C_{112}}\sigma u_2 + 2\frac{\mathscr{A}'^C_{22}}{\mathscr{A}'^C_{11}}u_2^2 - \frac{1}{3}\frac{\mathscr{A}^C_{2222}}{\mathscr{A}^C_{112}}u_2^3, \quad \lambda = -\frac{\mathscr{A}^C_{112}}{\mathscr{A}'^C_{11}}u_2 - \sigma, \tag{6}$$

which becomes a single looping path at $\sigma = 0$ as shown. The inclination of the plane clearly results from the symmetry breaking properties of the combined mode.

The paths of Fig. 2 have been computer plotted in three dimensional form, at four different values of σ, for the particular case $\mathscr{A}^C_{112} = -2, \mathscr{A}^C_{2222} = -6, \mathscr{A}'^C_{11} = \mathscr{A}'^C_{22} = -2$.

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

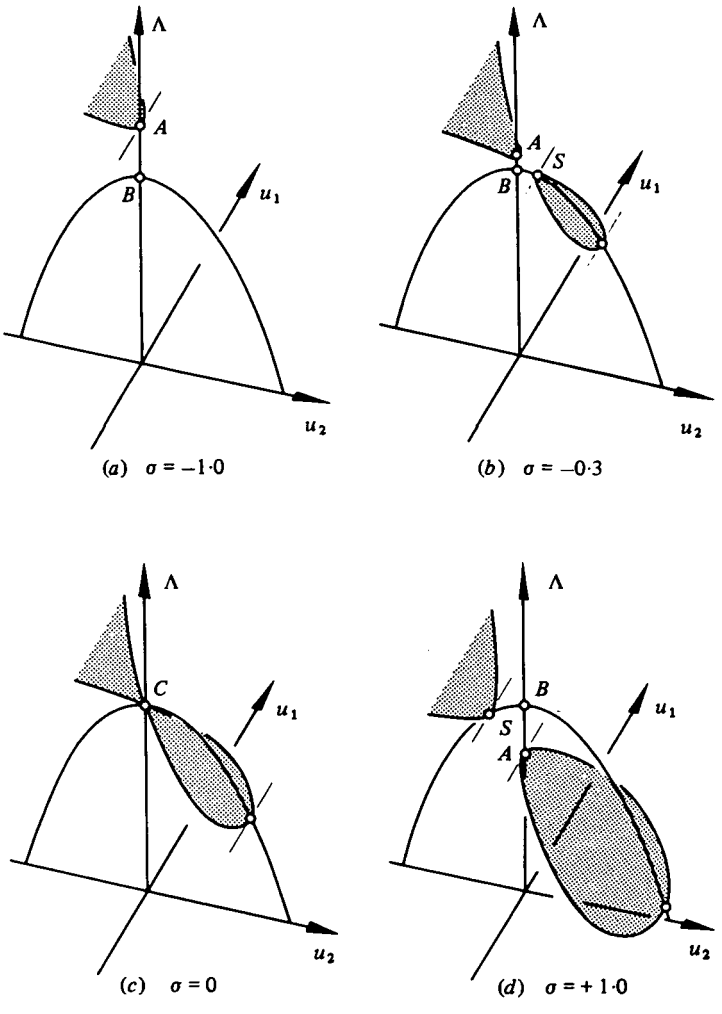


Fig. 2. Bifurcational manifestation of the parabolic umbilic catastrophe. Two contributing (distinct) bifurcations A and B, accompanied by a secondary bifurcation S, coincide at C. The inclined plane of the coupled solution is shown shaded.

The series clearly indicates how the splitting parameter σ allows for the approach and coincidence of two symmetric bifurcations A and B on the fundamental path $u_i = 0$.

Such plots are of course truncation dependent; truncating after just cubic terms, for example, gives straight lines for the paths at complete coincidence [4], these being the path tangents at C of Fig. 2. The fact that the patterns shown here do provide a correct local picture is borne out by the following perturbation analyses, where the effects of higher-order terms are investigated.

3. *The perturbation method*

As an alternative to truncation, it is sometimes useful to operate on the full potential function using perturbation techniques [8, 13, 14, 16, 18]. This versatile method has proved most useful for bifurcation analysis, and is presented in a general form by

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

Hunt [5]; it can be readily adapted for post-buckling, imperfection-sensitivity or secondary bifurcation study, for example.

For post-buckling analysis, the method sets up the variables – coordinates u_i and control parameters Λ^j – in Taylor series form as functions of s, a perturbation parameter,

$$\left.\begin{aligned} u_i &= u_i(s) = u_i^{(1)C}s + \tfrac{1}{2}u_i^{(2)C}s^2 + \ldots, \\ \Lambda^j &= \Lambda^j(s) = \Lambda^{j(1)C}s + \tfrac{1}{2}\Lambda^{j(2)C}s^2 + \ldots, \end{aligned}\right\} \tag{7}$$

expanded about the compound critical state C. Here bracketed superscripts denote full differentiation with respect to s. Substituting into the equilibrium equations, $\mathscr{A}_i \equiv \partial\mathscr{A}/\partial u_i = 0$, repeatedly differentiating with respect to s, and finally evaluating at C and solving, gives the coefficients of the expanded forms (7) to as high an order as necessary.

Usually, somewhere in the analysis, s is specified, often as one of the variables, u_2 say. This would allow us to write,

$$u_2^{(1)C} = 1, \quad u_2^{(2)C} = u_2^{(3)C} = \ldots = 0, \tag{8}$$

and simplify the evaluated equations. But it is sometimes more appropriate to specify s as the positive square root of a variable [4], giving for example

$$u_2^{(1)C} = 0, \quad u_2^{(2)C} = 2, \quad u_2^{(3)C} = u_2^{(4)C} = \ldots = 0, \tag{9}$$

or perhaps even the cube root, leading to

$$u_2^{(1)C} = u_2^{(2)C} = 0, \quad u_2^{(3)C} = 6, \quad u_2^{(4)C} = u_2^{(5)C} = \ldots = 0. \tag{10}$$

In this way we can set up alternative, fundamentally different, perturbation schemes, each of which can be thought of as being linked to some specific truncated form of the potential function, according to the terms that appear in the first order equations.

Two different schemes are used here to show the importance of determinacy if topologically-correct local behaviour is to be discovered. The first is associated with the parabolic umbilic form, truncated after the cubic term; this is found to be unable to cope very successfully with higher-order terms, in particular the quartic $\mathscr{A}^C_{2222}$, the effect of which continues up to infinite order in expansions (7). On the other hand, a scheme associated with the determinate parabolic umbilic form, including $\mathscr{A}^C_{2222}$ as a first-order effect, deals successfully with second-order terms in the second order equations.

4. *Scheme 1 – related to $\mathscr{A}^C_{112}$ truncation*

4·1. *First-order solution*

We start the analysis under a general fomulation, while implicitly assuming the parabolic umbilic form of equation (1). We thus have two active coordinates u_1 and u_2, the cubic contribution to the energy is limited to a single cross-term represented by the coefficient $\mathscr{A}^C_{112}$, and the control parameters are written in the general form Λ^j, although they are given by

$$\Lambda^1 = \lambda + \sigma, \quad \Lambda^2 = \lambda. \tag{11}$$

Substituting the variables written implicitly in terms of s (equations (7)), into the equilibrium equations $\mathscr{A}_i \equiv \partial\mathscr{A}/\partial u_i = 0$, we obtain the identity [5],

$$\mathscr{A}_i[u_j(s), \Lambda^k(s)] \equiv 0. \tag{12}$$

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

Repeated differentiation with respect to s now gives the series,

$$\left.\begin{aligned} \frac{d\mathscr{A}}{ds} &= \mathscr{A}_{ij}u_j^{(1)} + \mathscr{A}_i^j\Lambda^{j(1)} = 0 \quad \text{(E 1)} \\ \frac{d^2\mathscr{A}}{ds^2} &= \mathscr{A}_{ijk}u_j^{(1)}u_k^{(1)} + 2\mathscr{A}_{ij}^k u_j^{(1)}\Lambda^{k(1)} + \mathscr{A}_{ij}u_j^{(2)} \\ &\quad + \mathscr{A}_i^{jk}\Lambda^{j(1)}\Lambda^{k(1)} + \mathscr{A}_i^j\Lambda^{j(2)} = 0, \quad \text{(E 2)} \end{aligned}\right\} \tag{13}$$

continuing to (E 3), (E 4), etc. These do become rather lengthy and are not given explicitly here, but are presented in full up to (E 5) by Williams [**19**]. Subscripts on $\mathscr{A}$ are used to denote partial differentiation with respect to the u_i, superscripts for partial differentiation with respect to the Λ^j, and repeated suffices imply summation from 1 to 2.

On evaluating at C some of the coefficients vanish, as indicated by the original potential function expansion (1). (E 1) gives no information, while (E 2) written in full becomes

$$\left.\begin{aligned} \mathscr{A}_{112}u_1^{(1)}u_2^{(1)} + \mathscr{A}'_{11}u_1^{(1)}\Lambda^{1(1)}\big|^C &= 0, \\ \mathscr{A}_{112}(u_1^{(1)})^2 + 2\mathscr{A}'_{22}u_2^{(1)}\Lambda^{2(1)}\big|^C &= 0. \end{aligned}\right\} \tag{14}$$

These have a general solution,

$$u_i^{(1)C} \neq 0, \quad \Lambda^{j(1)C} \neq 0, \tag{15}$$

whether or not the potential function is truncated after just $\mathscr{A}_{112}^C$, $\mathscr{A}_{11}^{\prime C}$ and $\mathscr{A}_{22}^{\prime C}$. However, for the truncated form only, the higher-order equations (E 3), etc., would then give

$$u_i^{(k)C} = 0, \quad \Lambda^{j(k)C} = 0, \quad \text{for all} \quad k \neq 1. \tag{16}$$

Truncating the potential function thus truncates the series expansions (7) after the first term of each. We note also that, by truncating and differentiating directly we can obtain the same equations, only with variables u_i, Λ^j replacing the corresponding derivatives $u_i^{(1)C}$, $\Lambda^{j(1)C}$.

We see from equations (14) that the non-trivial solution specifically depends on $u_2^{(1)C}$ being non-zero. This is consistent with setting $s = u_2$, so substituting from (8) into the first of equations (14), we find two solutions,

$$u_1^{(1)C} = 0 \quad \text{and} \quad \Lambda^{1(1)C} = -\frac{\mathscr{A}_{112}}{\mathscr{A}'_{11}}\bigg|^C, \tag{17}$$

representing respectively the $\Lambda - u_2$ and inclined planes of Fig. 2. On substituting into the second of equations (14), the first solution becomes

$$u_1^{(1)C} = 0, \quad \Lambda^{2(1)C} = 0, \tag{18}$$

giving the first-order approximation to the uncoupled path of Fig. 2 as a horizontal straight line. Secondly, setting $\sigma = 0$ to study the example of complete coincidence, so $\Lambda^1 = \Lambda^2 = \lambda$, we have

$$u_1^{(1)C} = \pm\sqrt{\left(2\frac{\mathscr{A}'_{22}}{\mathscr{A}'_{11}}\right)}\bigg|^C, \quad \Lambda^{2(1)C} = -\frac{\mathscr{A}_{112}}{\mathscr{A}'_{11}}\bigg|^C, \tag{19}$$

giving straight line approximations to the coupled paths.

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

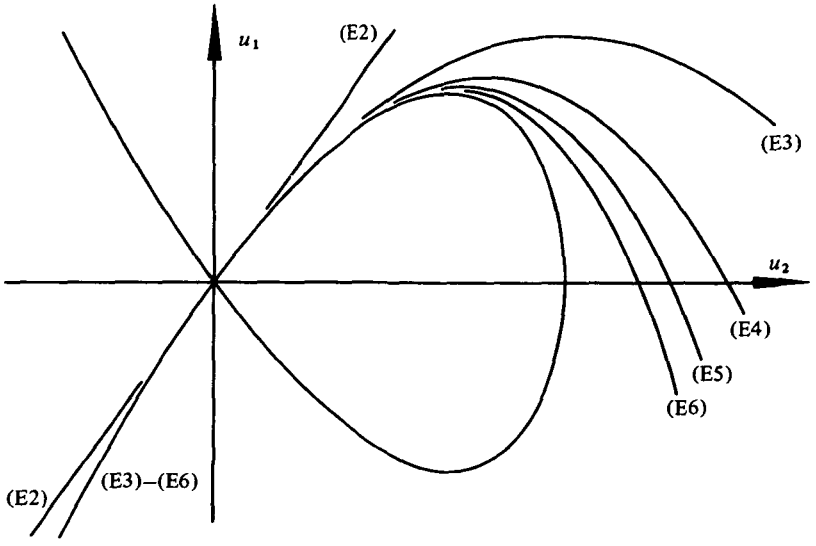


Fig. 3. The looping coupled solution of Fig. 2 (*c*) projected onto the base plane. Successive approximations of increasing accuracy from Scheme 1 are shown. Each solution has a mirror image about the u_2 axis, which is not shown.

It is thus clear that the analysis successfully picks up the tangents to the equilibrium paths of Fig. 2 (*c*). But the topological feature of the loop is lost, and cannot be retrieved with $s = u_2$ however many perturbation equations are considered, as seen below and in our companion paper [8].

4·2. *Higher-order terms*

The first-order results are hardly adequate to describe the post-buckling behaviour; the loop is missed, and they also give no indication of the uncoupled path curvature. Higher-order effects in higher-order equations are therefore significant. To simplify the ensuing discussion we shall restrict attention merely to the effect of $\mathscr{A}^C_{2222}$; we thus assume the determinate form of the potential function, truncating (1) after $\mathscr{A}^C_{112}$, $\mathscr{A}^C_{2222}$, $\mathscr{A}'^C_{11}$ and $\mathscr{A}'^C_{22}$.

If we now evaluate at C and set $\sigma = 0$ to study complete coincidence, (E 3) provides a curvature term for the uncoupled solution and gives for the coupled solutions

$$u_1^{(2)C} = \mp \frac{\mathscr{A}_{2222}}{3\mathscr{A}_{112}} \sqrt{\left(\frac{\mathscr{A}'_{11}}{2\mathscr{A}'_{22}}\right)}\Bigg|^C, \quad \Lambda^{2(2)C} = 0, \tag{20}$$

while (E 4) gives

$$u_1^{(3)C} = \pm \frac{1}{24} \frac{(\mathscr{A}_{2222})^2 \mathscr{A}'_{11}}{(\mathscr{A}_{112})^2 \mathscr{A}'_{22}} \sqrt{\left(\frac{\mathscr{A}'_{11}}{2\mathscr{A}'_{22}}\right)}\Bigg|^C, \quad \Lambda^{2(3)C} = 0. \tag{21}$$

Similar results would be expected from (E 5), etc., and it can be proved that the expanded form for u_1 of (7) continues to infinity.

It is thus clear that the scheme with $s = u_2$, linked to indeterminate truncation after the cubic $\mathscr{A}^C_{112}$ term, fails to cope successfully with $\mathscr{A}^C_{2222}$ as a higher-order term and

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

hence with the loop of Fig. 2. Fig. 3 shows a series of attempts to plot the loop, taking a successively increasing number of terms up to and including the result from (E 6).

5. *Scheme 2 – related to* $\mathscr{A}^C_{2222}$ *truncation*

5·1. *First-order solution*

An entirely different scheme of analysis is generated by setting s as the positive square root of u_2. The equilibrium equations (13) remain the same, but evaluation at C means that the u_2 derivatives of (9) now apply. This gives a different set of solutions from the first scheme, linked to truncation of the potential function (1) after $\mathscr{A}^C_{112}$, $\mathscr{A}^C_{2222}$, $\mathscr{A}'^C_{11}$ and $\mathscr{A}'^C_{22}$.

The analysis appears more complicated than the first, requiring equations of considerably higher order. In particular, complete perturbation equations up to (E 5) are required, as are the reduced equations up to (E 9) for a potential function truncated after quartic energy terms. These have been developed in full by Williams [**21**], but are not given explicitly here because of their length.

Substituting (9) into the second of equations (14), representing (E 2) with $i = 2$ and evaluated at C, we find

$$u_1^{(1)C} = 0. \tag{22}$$

Substituting this into (E 3) we get for $i = 2$,

$$\Lambda^{2(1)C} = 0, \tag{23}$$

while $i = 1$ gives

$$u_1^{(2)}\Lambda^{1(1)}\big|^C = 0. \tag{24}$$

We note that the solution set represented by $\Lambda^{1(1)C} = 0$, $u_1^{(2)C} \neq 0$, does exist, but can be ignored for our present purposes. It is found to lead to precisely the same results as the earlier Scheme 1, only with s^2 replacing s. We thus insist that

$$u_1^{(2)C} = 0. \tag{25}$$

Continuing, (E 4) with $i = 2$ gives

$$\Lambda^{2(2)C} = 0, \tag{26}$$

while with $i = 1$ it has two distinct solutions,

$$u_1^{(3)C} = 0 \text{ and } \quad \Lambda^{1(1)C} = 0, \tag{27}$$

which we will see represent the uncoupled and coupled paths, occupying the $\Lambda - u_2$ and inclined planes of Fig. 2, respectively. (E 5) with $i = 2$ next gives

$$\Lambda^{2(3)C} = 0, \tag{28}$$

while with $i = 1$ we obtain the first non-trivial equation of interest,

$$4\mathscr{A}_{112}u_1^{(3)} + 2\mathscr{A}'_{11}u_1^{(3)}\Lambda^{1(2)} + \mathscr{A}'_{11}u_1^{(4)}\Lambda^{1(1)}\big|^C = 0. \tag{29}$$

We note that the third term must vanish for both solution sets of (27) and so this can be immediately rewritten,

$$2\mathscr{A}_{112}u_1^{(3)} + \mathscr{A}'_{11}u_1^{(3)}\Lambda^{1(2)}\big|^C = 0. \tag{30}$$

Finally, (E 6) with $i = 2$ gives the second equation of interest,

$$\mathscr{A}_{112}(u_1^{(3)})^2 + 12\mathscr{A}_{2222} + 3\mathscr{A}'_{22}\Lambda^{2(4)}\big|^C = 0. \tag{31}$$

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

These equations allow non-zero derivatives $\Lambda^{1(2)C}$, $u_1^{(3)C}$ (for the coupled solution), and $\Lambda^{2(4)C}$, and the expanded forms (7) are thus all seen to start at different levels. The scheme is clearly linked to the determinate potential function, truncated after $\mathscr{A}^C_{2222}$.

The latter gives the same first-order results as the untruncated form, but with all higher derivatives than the above vanishing; this can be proved quite simply using induction techniques. Moreover, the earlier equilibrium equations (3), and the first-order perturbation equations (30) and (31) above, turn out to be the same, if we truncate each expansion (7) after the first non-vanishing term. We thus arrive at the same local representation of the paths as Fig. 2, including the loop.

The different orders of s involved in the leading terms of expansions (7) clearly result from adopting the truncated determinate form as the local description. This insists that a cubic energy term ($\mathscr{A}^C_{112}$) and a quartic ($\mathscr{A}^C_{2222}$) are of equal asymptotic, qualitative, significance, and hence appear initially of the same order in s. This also extends to the unfolding terms, represented by $\mathscr{A}'^C_{11}$ and $\mathscr{A}'^C_{22}$, and indeed to the imperfection terms of (7), which all appear of order s^8; on the other hand the higher $\mathscr{A}^C_{1122}$ term first appears of order s^{10}, while the $\mathscr{A}^C_{1111}$ term is of lowest order s^{12}. The study of imperfections is outside the scope of this paper, but we note in passing that ϵ^1 must be of order s^5, while ϵ^2 is of order s^6, by the same token. They will therefore each carry a slightly different asymptotic significance, in an interactive buckling problem involving the parabolic umbilic catastrophe. We observe finally that a term with coefficient $\mathscr{A}^{1C}_{222}$, the superscript implying partial differentiation with respect to Λ^1, could also appear of order s^8; this is unlikely to contribute to a structural potential function, however, because of symmetry considerations [6].

5·2. *Higher-order terms*

The difference between the two perturbation schemes only becomes truly apparent when higher-order effects are considered. We have seen that Scheme 1, related to truncation of the potential function after the cubic term, fails to cope successfully with the quartic $\mathscr{A}^C_{2222}$ term, which it sees as higher order. The present scheme includes this as first order, and we proceed now to examine the genuinely higher-order effects of the $\mathscr{A}^C_{1122}$ and $\mathscr{A}^C_{1111}$ terms. We find that, unlike the first scheme, each is neatly taken care of with the addition of single terms to expansions (7), and a potential function truncated after these quartics thus leads to a truncated form for each of the variables u_i and Λ^j. We note that to be strictly consistent all terms of order s^{10} and s^{12} should be included. However we have omitted them from this presentation for the sake of simplicity.

The potential function is therefore truncated after the complete cubic and quartic contribution of (1), reducing the perturbation equations to a manageable form, derived in full up to (E 9) [21]. We start by assuming that the coupled solution can be represented by $u_1^{(3)C} \neq 0$, $u_1^{(k)C} = 0$ for all $k \neq 3$; for the uncoupled solution, of course, all these derivatives vanish. This is somewhat akin to selecting the perturbation parameter by (8), (9) or (10), and could lead to solutions being missed, although in this case it becomes apparent that all the first-order solutions are covered. Such an assumption would manifestly break down in Scheme 1, because of the infinite number of terms of the u_1 expansion, represented by (20), (21), etc.

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

Continuing with the same solution routine as the first-order analysis, (E 6) with $i = 1$ gives the result,

$$\Lambda^{1(3)C} = 0, \tag{32}$$

while (E 7) with $i = 2$ gives

$$\Lambda^{2(5)C} = 0. \tag{33}$$

(E 7) with $i = 1$ generates the non-trivial equation,

$$12\mathscr{A}_{1122}u_1^{(3)} + \mathscr{A}'_{11}u_1^{(3)}\Lambda^{1(4)}\big|^C = 0, \tag{34}$$

which has the two solutions representing the uncoupled and coupled paths respectively,

$$u_1^{(3)} = 0 \quad \text{and} \quad \Lambda^{1(4)C} = -12\frac{\mathscr{A}_{1122}}{\mathscr{A}'_{11}}\bigg|^C. \tag{35}$$

In the same manner, (E 8) with $i = 2$ gives the result,

$$\Lambda^{2(6)C} = -10\frac{\mathscr{A}_{1122}}{\mathscr{A}'_{22}}(u_1^{(3)})^2\big|^C, \tag{36}$$

which is valid for both coupled and uncoupled solutions.

This takes care completely of the $\mathscr{A}^C_{1122}$ term. It will make no further appearance in any higher perturbation equation, since it would be bound to accompany some u_i derivative of order higher than $u_2^{(2)C}$ or $u_1^{(3)C}$.

Continuing, (E 8) with $i = 1$ now gives the result,

$$\Lambda^{1(5)C} = 0, \tag{37}$$

and finally (E 9) with $i = 1$ leads to

$$\Lambda^{1(6)C} = -\frac{10}{3}\frac{\mathscr{A}_{1111}}{\mathscr{A}'_{11}}(u_1^{(3)})^2\big|^C, \tag{38}$$

which takes care completely of the $\mathscr{A}^C_{1111}$ term, by the same reasoning as before.

It is now a simple matter to prove that all higher Λ^j derivatives are zero. We see that the full effect of $\mathscr{A}^C_{1122}$ is to introduce a single quartic term into the Λ^1 expansion, and a single sixth-power term into Λ^2. The full effect of $\mathscr{A}^C_{1111}$ is to introduce a single sixth-power term into Λ^1. We can thus conclude that no new topological features are introduced by these, or by implication any other, higher-order terms; they will serve merely to distort the paths of Fig. 2.

5·3. *An adaptation of Scheme* 1

We have found that it is possible to reproduce the results of Scheme 2 using an adaptation of Scheme 1. The catch is that the results from Scheme 2, or a related truncation study, are required before this can be attempted. This means that, as a method of solution, the adaptation is of little value, if any, and so we shall only present the first-order solution in any detail. The reason for this peculiarity is the inconsistent truncation of the perturbation expansions (7) that this scheme generates.

The first-order solution can be obtained by considering the pair of equilibrium

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

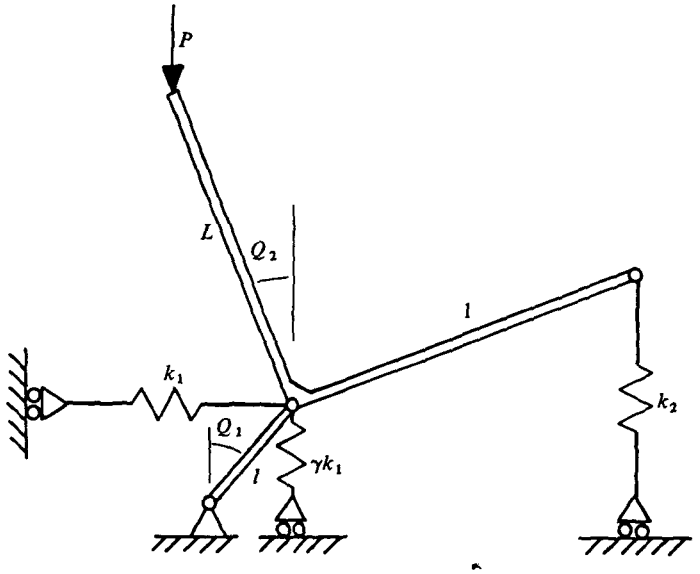


Fig. 4. Rigid-link interaction model (after Budiansky and Hutchinson[1]).

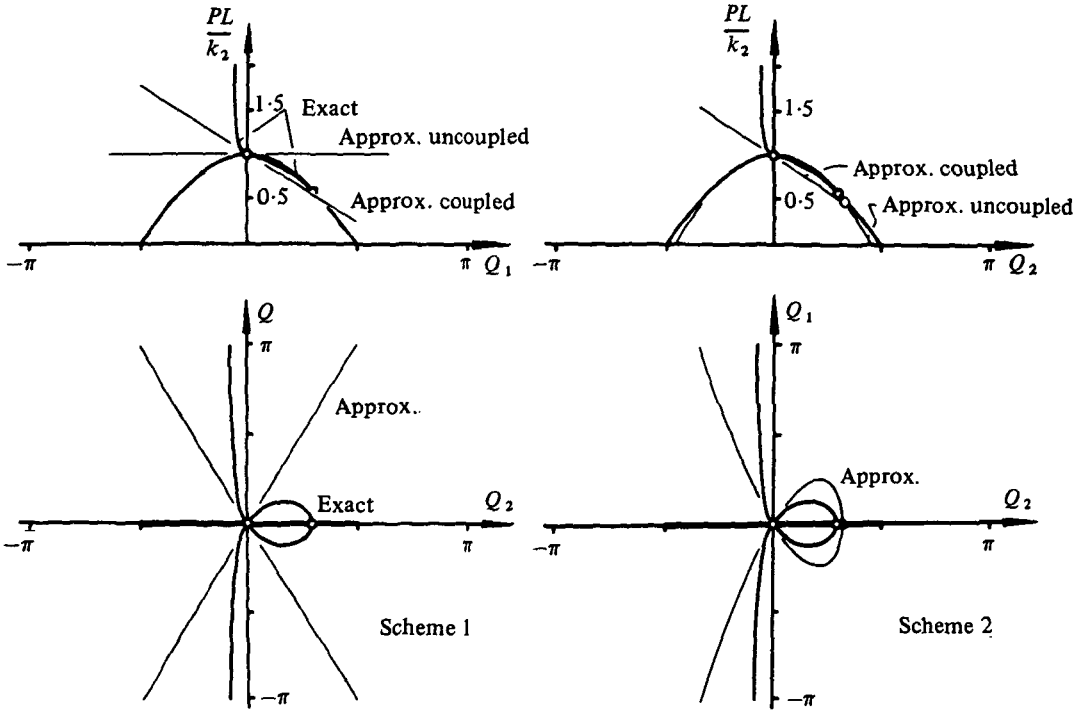


Fig. 5. First-order approximations to the exact equilibrium paths of the interaction model from each perturbation scheme, at complete coincidence of the contributing bifurcations.

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

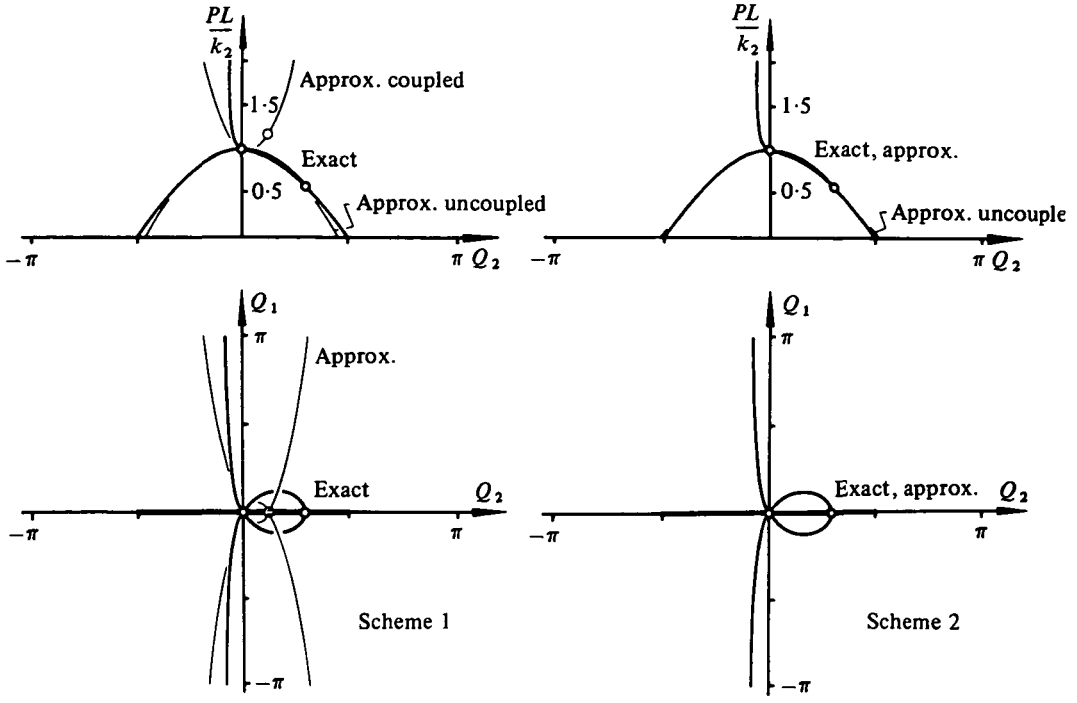


Fig. 6. Second-order approximations to the exact equilibrium paths of the interaction model from each perturbation scheme, at $\sigma = 0$. The results from Scheme 2 are nearly indistinquishable from the exact results.

equations (E 2), equations (14), with the equilibrium equation given by (E 3) with $i = 2$,

$$\mathscr{A}_{2222} + 3\mathscr{A}_{22}^2 \Lambda^{2(2)}|^C = 0, \tag{39}$$

but ignoring the equilibribrium equation given by (E 3) with $i = 1$. When solved these equations given the first-order solution of Scheme 2. However we find that the expansions of the variables u_1, u_2 and Λ^1 are truncated after their respective leading terms, whilst the expansion of Λ^2 is truncated after two terms. If we select a consistent truncation, this is one with all the variable expansions truncated after their leading terms, then we would have the first order solution of Scheme 1. Similarly it is possible to consider the higher order terms in this adaptation.

6. *An interactive buckling model*

A way of testing the validity of an asymptotic study is to compare it with a closed-form exact solution. This has been done for the two schemes presented in this paper [8], and we shall now briefly quote the results. The two degree of freedom spring and link model of Fig. 4 was considered, it being essentially the same as a model proposed by Budiansky and Hutchinson [1] to describe the phenomenon of interactive buckling. An exact, closed-form, solution was obtained, the bifurcating behaviour when free from imperfections being found to match the form of Fig. 2 precisely, for certain spring stiffness ratios. For other ratios a related form is obtained, which Scheme 2 predicts when higher order terms are considered.

In Fig. 5 the two sets of first-order solutions are compared with the exact results from

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

the model. The left-hand side shows the linear approximations given by Scheme 1; the right-hand side shows the results of the determinacy based approach, Scheme 2. We see that the loop of the exact solution is well predicted by the first order result from Scheme 2. It is perhaps not surprising that the determinacy based approach gives the better first-order approximation, since it inherently includes extra information in the extra term. The advantages of Scheme 2 only become truly apparent when we proceed to the higher-order solution.

The second-order results of the Schemes are plotted in Fig. 6. The benefits of Scheme 2 are now fully apparent. Scheme 1, although matching slopes and curvatures precisely, remains out of sympathy with the underlying topology; the second-order solution, for instance, represents barely an improvement over the first-order, if at all. Scheme 2, on the other hand, gives second-order results of very high accuracy; rapid convergence is clearly being obtained with increasing order of solution.

7. *Concluding remarks*

We have been concerned here mainly with the consequences of determinacy, barely mentioning the more obvious means of truncating via deflection considerations. A strut formulation shows that this too can be important, leading in effect to coefficients with different orders of magnitude in a complete potential energy expansion [**18**]. We do, however, suppose that such considerations are familiar to the structural mechanician, and note that they are also, by their very nature, problem dependent.

There must be some problems for which the notion of determinacy carries very little relevance, when it demands a potential function of such complexity that the associated equations become impossibly complex for instance. Among these we must include the buckling of the infinitely long, axially loaded, cylinder, exhibiting as it does a considerable number of interacting modes. The approach of Koiter [**10**], similar in many ways to our Scheme 1, of course still provides vital post buckling information, although the effects of higher terms remain unpredictable. There are other interactive buckling problems for which it does carry some relevance, perhaps most importantly those involving local and overall buckling in stiffened structures [**5, 11**]. The paper therefore clearly supports Thompson's call for the ideas of catastrophe theory to be recognized by the applied mechanics community [**17**].

One future area of study suggested by the present approach concerns finite separations of the two contributing bifurcations, A and B of Fig. 2, on the fundamental path F. With $\mathscr{A}^C_{1111}$ included, at some finite σ, this leads to a *butterfly catastrophe* on F [**5**]; the phenomenon may be of considerable practical significance, as discussed by Koiter [**10**], since it sometimes defines the limit of stability of the first critical point on F encountered during a normal loading sequence. The secondary bifurcation S of Fig. 2 may also be of practical importance, since for positive $\mathscr{A}^C_{2222}$ and a range of σ values, it defines the limit of stability of the uncoupled post-buckling equilibrium path. The locus of S can be traced using either pertubation scheme, but the results will differ; here, as elsewhere, we would expect Scheme 2 to be the more reliable.

We close with a brief comment on the mathematical background to determinacy and unfolding. In particular, diffeomorphisms (smooth, reversible, usually nonlinear, changes of variables) play a major role; direction preserving transformations can be used to test truncated forms of the potential function for determinacy (see, for example,

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

[11]), and they are also central to the topological classification of the catastrophes. With regard to the latter, it is interesting to note that apparently quite different forms of branching can be, topologically speaking, equivalent; monoclinal, homeoclinal and hill-top bifurcations are all manifestations of the hyperbolic umbilic catastrophe for instance [19]. On the other hand, apparently similar potential functions can give rise to very different forms of imperfection-sensitivity, as is seen in the case of the hyperbolic and elliptic umbilics [4].

It is certainly possible that these transformations, when known to exist, can be used directly in structural analysis [7]. Nevertheless, the perturbation approach remains a valuable tool, particularly in estimating the effects of high-order terms; here the nonlinear transformations necessary to remove them from an initial formulation might well be difficult to write explicitly.

It is interesting to note that, although in its most general form the parabolic umbilic catastrophe requires four control parameters for structural stability (see equation (1)), we are here restricting attention to perfect behaviour described by just two, λ and σ (equation (2)). This is, apparently, fully in the spirit of the modern work of Golubitsky, Marsden and Schaeffer on hidden symmetries [3], who, in referring to the reduced problem, state 'We expect that . . . is a universal unfolding for the parabolic umbilic in the context of hidden symmetries'. Outside this context a universal unfolding requires an imperfection parameter in each mode, and here the classical work on 'perturbed bifurcation theory', as described in conference proceedings [9, 13], could prove extremely useful.

REFERENCES

[1] B. Budiansky and J. W. Hutchinson. *Trends in Solid Mechanics 1979* (Proceedings of the Symposium dedicated to the 65th birthday of W. T. Koiter, Delft University of Technology, June 1979) (edited by J. F. Besseling and A. M. A. van der Heijden), p. 93 (Delft University Press, Holland, 1979).

[2] M. Golubitsky and D. Schaeffer. A theory for imperfect bifurcation via singularity theory. *Comm. Pure Appl. Math.* **32** (1979), 21–98.

[3] M. Golubitsky, J. Marsden and D. Schaeffer. Bifurcation problems with hidden symmetries. In *Partial Differential Equations and Dynamical Systems* (ed. W. Fitzgibbon) (Pitman, to be published).

[4] G. W. Hunt. Imperfection-sensitivity of semi-symmetric branching. *Proc. Roy. Soc. London Ser.* A **357** (1977), 193–211.

[5] G. W. Hunt. An algorithm for the nonlinear analysis of compound bifurcation. *Philos. Trans. Roy. Soc. London Ser.* A **300** (1981), 443–471.

[6] G. W. Hunt, Symmetries of elastic buckling. *Eng. Struct.* **4** (1982), 21–28.

[7] G. W. Hunt, N. Reay and T. Yoshimura. Local diffeomorphisms in the bifurcational manifestations of the umbilic catastrophes. *Proc. Roy. Soc. London Ser. A* **369** (1979), 47–65.

[8] G. W. Hunt and K. A. J. Williams. Closed form and asymptotic solutions for an interactive buckling model (to appear in *J. Mech. Phys. Solids*).

[9] J. B. Keller and S. Antman (eds.). *Bifurcation Theory and Non-linear Eigenvalue Problems* (Benjamin, 1969).

[10] W. T. Koiter. On the stability of elastic equilibrium. Dissertation. Delft, Holland, 1945 (English translation: NASA, Tech. Trans., F10 833, 1967).

(11) W. T. Koiter. General theory of mode interaction in stiffened plate and shell structures. Rep. WTHD 91, Delft Univ. of Tech., Holland (1976).

[12] T. Poston and I. Stewart. *Catastrophe Theory and its Applications* (Pitman, 1978).

[13] P. H. Rabinowitz (ed.). *Applications of Bifurcation Theory* (Academic Press, 1977).

[14] M. J. Sewell The static perturbation technique in buckling problems. *J. Mech. Phys. Solids* **13** (1965), 247.

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press

[15] R. THOM. *Structural Stability and Morphogenesis* (translated from the French by D. H. Fowler) (Benjamin, 1975).
[16] J. M. T. THOMPSON. Discrete branching points in the general theory of elastic stability. *J. Mech. Phys. Solids* **13** (1965), 295.
[17] J. M. T. THOMPSON. Catastrophe Theory in Mechanics: progess or digression. *J. Structural Mech.* **10** (2) (1982), 167–175.
[18] J. M. T. THOMPSON and G. W. HUNT. *A General Theory of Elastic Stability* (Wiley, 1973).
[19] J. M. T. THOMPSON and G. W. HUNT. The instability of evolving systems. *Interdisciplinary Sci. Rev.* **2** (1977), 240–262.
[20] S. TIMOSHENKO and J. GERE. *Theory of Elastic Stability* (McGraw-Hill, 1961).
[21] K. A. J. WILLIAMS. *Perturbation Equations for Post-buckling Study*. Interim report, Imperial College, London (1982).
[22] E. C. ZEEMAN. *Catastrophe Theory: Selected Papers*, 1972–1977 (Addison-Wesley, 1977).

https://doi.org/10.1017/S030500410006182X Published online by Cambridge University Press