But $\dfrac{(2n)!^2\, n\pi}{2^{4n}(n!)^4} = \dfrac{K_{2n}^2 (2n)^{4n+1} e^{-4n} n\pi}{2^{4n} K_n^4 n^{4n+2} e^{-4n}} = \dfrac{K_{2n}^2}{K_n^4} \cdot 2\pi$ so, letting $n \to \infty$ in (*), we thus see that $\dfrac{2\pi}{K^2} = 1$ and $K = \sqrt{2\pi}$.

Notice that (*) on its own provides a proof of Wallis's product:

$$\prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{2k \,.\, 2k} = \lim_{n\to\infty} \frac{(2n)!\,(2n+1)}{2^{4n}(n!)^4} = \frac{2}{\pi}.$$

Alternatively, we can leave $\Gamma\left(\frac{1}{2}\right)$ unevaluated. The line (*) and its sequel then show that Wallis's product is equal to $\dfrac{2}{\Gamma\left(\frac{1}{2}\right)^2}$ and that $\dfrac{2\Gamma\left(\frac{1}{2}\right)^2}{K^2} = 1$ or $K = \sqrt{2}\Gamma\left(\frac{1}{2}\right)$. Thus knowing that Wallis's product is $\dfrac{2}{\pi}$ or that $K = \sqrt{2\pi}$ in Stirling's formula suffices to show that

$$\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right) = 2\int_0^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$$

which in turn establishes the normalising factor for the standard normal distribution, $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$.

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99.19 When do the inradius and exradii form progressions?

Let ABC be a non-degenerate triangle with area Δ and side-lengths a, b, c where $a \leqslant b \leqslant c$ and $s = \frac{1}{2}(a+b+c)$. Then the inradius, r, and the exradii r_a, r_b, r_c are given by the standard formulae $r = \dfrac{\Delta}{s}$ and $r_a = \dfrac{\Delta}{s-a}$, etc. so that $r \leqslant r_a \leqslant r_b \leqslant r_c$. In this short note we

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investigate whether or not it is possible for r, r_a, r_b, r_c to form one of the standard progressions. The results are arguably more interesting than their derivations, so we will just summarise the routine algebraic steps involved.

1. Our first result is that r, r_a, r_b, r_c cannot form a harmonic progression. Otherwise, by definition, $\frac{1}{r}, \frac{1}{r_a}, \frac{1}{r_b}, \frac{1}{r_c}$ would form an arithmetic progression and so, from the formulae above, s, $s-a$, $s-b$, $s-c$ would form an arithmetic progression. This means that $2(s-a) = s + s - b$ and $2(s-b) = s - a + s - c$ leading to the triangle having to be the degenerate case with side-lengths a, $2a$, $3a$.

2. Our second result is that, up to similarity, there is a unique triangle for which r, r_a, r_b, r_c form a geometric progression.

   It is necessarily a right-angled triangle, so that $C = 90°$, and $A = \sin^{-1}\frac{1}{x} \approx 32.94°$ where $x \approx 1.8392868$ is the common ratio of the progression and is the unique real root of the 'tribonacci' equation $x^3 = x^2 + x + 1$.

   In this case, having $\frac{1}{s}, \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ in geometric progression necessitates that $\frac{1}{s(s-c)} = \frac{1}{(s-a)(s-b)}$ and $\frac{1}{(s-b)^2} = \frac{1}{(s-a)(s-c)}$. On substituting for s, the first of these equations rearranges to give $a^2 + b^2 = c^2$ so revealing the right-angle at C; the second gives $b = \frac{a^2 + c^2}{a + c}$. (It is worth noting here that, given any $0 < a \leqslant c$, the latter equation gives the value of b for which r_a, r_b, r_c are in geometric progression.) Eliminating b from our two equations leads to $c^3 - a^3 = ac^2 + a^2c$ so that $x = \frac{1}{\sin A} = \frac{c}{a}$ satisfies $x^3 = x^2 + x + 1$ with $x > 1$. Futhermore x is also the common ratio of the geometric progression r, r_a, r_b, r_c because $b = \frac{a^2 + c^2}{a + c}$ is a rearrangement of $\frac{b + c - a}{a + c - b} = \frac{c}{a}$ so the common ratio $\frac{\frac{1}{s-b}}{\frac{1}{s-a}}$ is equal to $\frac{c}{a}$.

3. Our final result is that, up to similarity, there is a unique triangle for which r, r_a, r_b, r_c form an arithmetic progression. Its sides a, b, c are proportional to the triple $\cos 40°$, 1, $\sqrt{3}\sin 40°$ so that its angles are $A \approx 42.08°, B \approx 61.02°, C \approx 76.90°$.

   This case involves heavier algebra. If $\frac{1}{s}, \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ are in arithmetic progression, the conditions $\frac{2}{s-a} = \frac{1}{s} + \frac{1}{s-b}$ and $\frac{2}{s-b} = \frac{1}{s-a} + \frac{1}{s-c}$ lead to the equations $2a^2 - b^2 = ab + bc - 2ac$ and

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$2b^2 - a^2 - c^2 = ab + bc - 2ac$. Together these imply that $c^2 = 3(b^2 - a^2)$ and then eliminating c leads via

$$3(b^2 - a^2) = c^2 = \frac{(2a^2 - b^2 - ab)^2}{(b - 2a)^2} = \frac{(b - a)^2(2a + b)^2}{(2a - b)^2}$$

to $3(a+b)(2a-b)^2 = (b-a)(2a+b)^2$ and $8a^3 - 6ab^2 + b^3 = 0$. Setting $\frac{a}{b} = \cos\theta$ for an auxiliary angle θ then gives $\cos 3\theta = -\frac{1}{2}$ so that $\theta = 40°$, $\frac{a}{b} = \cos 40°$ and $c^2 = 3(b^2 - a^2) = 3b^2 \sin^2 40°$. We thus have $a : b : c = \cos 40° : 1 : \sqrt{3} \sin 40°$. Moreover, since s is proportional to $1 + \cos 40° + \sqrt{3} \sin 40° = 1 + 2\cos 20°$, we see that the common difference d is characterised by

$$a : d = \frac{1}{s} : \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s}\right) = \frac{1}{2\cos 20° + 1} : \frac{1}{4\cos^2 20° - 1} = 2\cos 20° - 1 : 1.$$

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99.20 A projective Simson line

Introduction

Let ABC be a triangle with circumcircle Γ. Let P be a point in the plane of the triangle. Simson's theorem [1] is that the feet of the perpendiculars dropped from P to the side-lines of the triangle are collinear if, and only if, P is on Γ, in which case this line meets the line segment joining P to the orthocentre H of ABC at its midpoint. It seems that this result was misattributed to Robert Simson by Jean-Victor Poncelet, and the result was first published by William Wallace.

Instead of working with the feet of the perpendiculars, we can recast the statement as being that the three reflections of P in the sides of the triangle are collinear if, and only if, P lies on Γ, in which case this line passes through H.

It is known that the three reflections H_A, H_B and H_C of H in the sides of ABC are on Γ, so we can restate the result again as being that the three lines which are the reflection of PH_A in BC, the reflection of PH_B in CA and the reflection of PH_C in AB, are the same line if, and only if, P lies on Γ.

Result

We are now in a position to state a generalisation of this result which is a theorem of projective geometry. Note that we introduce the points A'', B'' and C'' to obviate the need for Euclidean reflections, and the arbitrary point Q plays the role of H. To see how this theorem specializes to the classical Simson configuration, first enlarge the Simson line with scale factor 2 from the distinguished point on the circumcircle, to obtain the doubled Simson line which passes through H, and then consider the reflections of the doubled Simson line in each of the three sides of triangle ABC. The points

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