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Weighted ergodic theorems and strong laws of large numbers

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Dedicated to Ulrich Krengel

Abstract. We investigate the convergence, in norm and almost everywhere (a.e.), of weighted ergodic averages as well as weighted sums of independent identically distributed (iid) random variables. The averages are true ones, normalized by the corresponding sums of weights, which are only assumed to be non-negative. The L_2-norm convergence in the mixing case is shown to rely upon very simple conditions on the weights. We show that 'quasimonotone weights' with a simple additional condition yield a.e. convergence of weighted averages for all Dunford–Schwartz contractions of probability spaces and L_1-functions. For independent random variables, we look at weighted averages of centered random variables with bounded variances (or bounded moments of some order greater than 1), in particular the iid case, and obtain several sufficient conditions on the weights for almost sure convergence (weighted SLLN). For example, in Theorem 4.14 we show that if a weight sequence $\{w_k\}$ with divergent partial sums W_n satisfies

$$\sup_{n\geq 1}\frac{1}{W_n}\sum_{k=1}^{n} w_k(\log(w_k+1))^{\beta}<\infty \quad \text{for some } \beta>1$$

then for any iid sequence in the class $L(\log^+ L)^{1+\epsilon}$ the weighted averages converge almost surely to the expectation.

1. *Introduction*

Jamison, Orey and Pruitt [**JOP**] studied the problem of weighted strong laws of large numbers for independent identically distributed (iid) variables; for a sequence $\{w_k\}$ of positive numbers (weights), with partial sums $W_n := \sum_{k=1}^{n} w_k \to \infty$, they found necessary and sufficient conditions to ensure that for every iid sequence $\{\xi_k\}$ with finite expectation the weighted averages $(1/W_n)\sum_{k=1}^{n} w_k\xi_k$ converge almost surely (a.s.).

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Extending [**Ts**], Azuma [**Az**] obtained a sufficient condition for this convergence when $\{\xi_k\}$ are centered independent uniformly bounded.

Let τ be a measure-preserving transformation of a probability space (Ω, μ). Since the ergodic theorems of von Neumann and of Birkhoff for integrable functions are extensions of the corresponding laws of large numbers for iid random variables with finite expectation, we are interested in the convergence, almost everywhere (a.e.) or in L_p-norm, of the weighted averages

$$A_n(T)f := \frac{1}{W_n}\sum_{k=1}^{n} w_k T^k f, \tag{1.1}$$

where $Tf := f \circ \tau$ and $f \in L_p(\mu)$.

The operator T is obviously a contraction of $L_\infty(\mu)$, and an isometry of all $L_p(\mu)$, $1 \le p < \infty$. Any linear contraction T on L_1 of a σ-finite measure space, with $\|Tf\|_\infty \le \|f\|_\infty$ for $f \in L_1 \cap L_\infty$, is called a *Dunford–Schwartz contraction*, and induces a contraction on all L_p, $1 < p \le \infty$ (see [**DS-2**]; see also [**Kr-3**, p. 65] for T positive). Birkhoff's ergodic theorem has been extended to Dunford–Schwartz contractions in [**DS-1**]. We denote $E(T)f := \lim_n (1/n)\sum_{k=1}^{n} T^k f$, which exists a.e. for $f \in L_p$, $1 \le p < \infty$, and also in L_p-norm for $p > 1$ (and in L_1-norm in probability spaces). We are therefore interested in the convergence of (1.1) (particularly to $E(T)f$) for T Dunford–Schwartz and appropriate f.

Let $\{w_k\}$ be a non-null sequence of non-negative numbers. If $\sum_{k=1}^{\infty} w_k < \infty$, then for any T Dunford–Schwartz on a probability space (Ω, μ) and $f \in L_p(\mu)$, $p \ge 1$, the series $\sum_{k=1}^{\infty} w_k T^k f$ converges in L_p-norm. The inclusion $L_p \subset L_1$ yields $\sum_{k=1}^{\infty} \|w_k T^k f\|_1 < \infty$, so by Beppo Levi $\sum_{k=1}^{\infty} w_k |T^k f| < \infty$ a.e. Since $W_n \to W > 0$ finite, $A_n(T)f$ converges a.e. and in L_p-norm. Thus the convergence problem is only when $\sum_{k=1}^{\infty} w_k = \infty$.

A power-bounded operator T on a Banach space X is called *mean ergodic* if for every $f \in X$ the limit $E(T)f := \lim_{n\to\infty}(1/n)\sum_{k=1}^{n} T^k f$ exists in norm. In general, the set of f for which $E(T)f$ exists is a closed T-invariant subspace, on which the restriction of T is mean ergodic. Power-bounded operators in reflexive Banach spaces are mean ergodic.

THEOREM 1.1. *Let $\{w_k\}$ be a sequence of non-negative numbers satisfying $\sum_{k=1}^{\infty} w_k = \infty$. Then the following are equivalent:*

(i) $$\lim_{n\to\infty} \frac{1}{W_n}\left(w_n + \sum_{k=1}^{n-1} |w_k - w_{k+1}|\right) = 0.$$

(ii) *For every power-bounded mean ergodic T on a Banach space X and every $f \in X$ we have*

$$\lim_{n\to\infty} \left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k T^k f - E(T)f \right\| = 0.$$

This is an application of the general result of [**LOT**, Theorem 4.1], where it was shown that the well-known sufficient conditions are also *necessary*.

Remarks. (1) The sequence with alternating 1s and 0s does not satisfy (i), but is not a counter-example since there exists T mean ergodic with T^2 not mean ergodic [**Si**].

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(2) If $\{w_k\}$ is non-decreasing, the *ergodicity condition* (i) becomes $\lim_n (w_n/W_n) = 0$. If the sequence is non-increasing, then (i) is always satisfied (since $\sum_{k=1}^{\infty} w_k = \infty$).

The following special case of Theorem 8.2.1 of Krengel [**Kr-3**] improves the previous result when the weights are non-increasing, and applies also to pointwise convergence.

THEOREM 1.2. *Let $\{w_k\}$ be a non-increasing sequence of positive numbers satisfying $\sum_{k=1}^{\infty} w_k = \infty$. Then for any non-null sequence $\{\gamma_k\}$ of non-negative numbers and any $\{f_k\} \in L_p$, $p \geq 1$, such that $\sum_{k=1}^{n} f_k / \sum_{k=1}^{n} \gamma_k$ converges to some g a.e. (in norm), also*

$$\frac{\sum_{k=1}^{n} w_k f_k}{\sum_{k=1}^{n} w_k \gamma_k} = \frac{(1/W_n)\sum_{k=1}^{n} w_k f_k}{(1/W_n)\sum_{k=1}^{n} w_k \gamma_k} \xrightarrow[n\to\infty]{} g$$

almost everywhere (respectively, in norm).

In particular, if $\{f_k\} \in L_p$, $p \geq 1$, is such that $(1/n)\sum_{k=1}^{n} f_k$ converges to f a.e. (in norm), then

$$\frac{1}{\sum_{j=1}^{n} w_j} \sum_{k=1}^{n} w_k f_k \xrightarrow[n\to\infty]{} f.$$

Remarks. (1) The norm convergence part of the theorem applies in any Banach space.

(2) The monotonicity is used when the limit of w_k is zero. For a weight sequence $w_k \to a > 0$, even not monotonically, the second conclusion of the theorem holds when we assume also $\sup_n (1/n)\sum_{k=1}^{n} |f_k| < \infty$.

COROLLARY 1.3. *Let $\{w_k\}$ be positive numbers with $\sum w_k = \infty$. If $w_k \to a > 0$, or $w_k \to 0$ monotonically, then for every Dunford–Schwartz contraction T on a σ-finite measure space (Ω, μ), any $1 \leq p < \infty$ and $f \in L_p$, we have*

$$\frac{1}{\sum_{j=1}^{n} w_j} \sum_{k=1}^{n} w_k T^k f \xrightarrow[n\to\infty]{} E(T) f \quad \text{a.e.} \tag{1.2}$$

Remarks. (1) In general, $w_k \to 0$ does not imply even norm convergence of $A_n(T) f$.

(2) It is also possible that when $w_k \to 0$, the left-hand side of (1.2) converges a.e., for every Dunford–Schwartz T and f, but with a limit which is not necessarily $E(T) f$. For example, let $w_{2j} = 1/j$ and $w_{2j-1} = 1/2^j$; then $A_n(T) f$ converges a.e., but to $E(T^2 f)$.

(3) In [**Kr-1**], Krengel showed that for every positive contraction T on L_1 (even not Dunford–Schwartz), there is a non-increasing sequence $\{w_k\}$, *which depends on T*, such that for every $f \in L_1$ the weighted averages $A_n(T) f$ converge a.e.

The following proposition is a standard result (e.g., proof of [**LOT**, Theorem 2.7]).

PROPOSITION 1.4. *Let $\{w_k\}$ be non-negative with $W_n := \sum_{k=1}^{n} w_k$. Let T be a Dunford–Schwartz contraction on a probability space (Ω, μ), and assume that for every $f \in L_2(\mu)$ the sequence $\{A_n(T) f\}$ converges in L_2-norm; then for $1 \leq p < \infty$ and any $g \in L_p(\mu)$ the sequence $\{A_n(T) g\}$ converges in L_p-norm.*

In §2 we show that for weights $\{w_n\}$ with divergent sum (an assumption made throughout this paper), $w_n/W_n \to 0$ is necessary and sufficient for $\|A_n(T) f\|_2 \to 0$ for every contraction on a Hilbert space and f with $T^n f \to 0$ weakly. In §3 we show that

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the second part of Theorem 1.2 holds if $\{w_k/k^s\}$ is non-increasing for some $s \geq 0$. In §4 we treat weighted averages of independent random variables. We show that $w_n/W_n \to 0$ is necessary and sufficient for

$$\mathbb{E}\left(\left|\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k\right|\right) \to 0$$

for every centered iid $\{\xi_n\}$ with finite expectation. For fixed $p > 1$, we obtain sufficient conditions on the weights for

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k \to 0 \quad \text{a.s.}$$

for any centered independent $\{\xi_n\}$ with $\sup_n \mathbb{E}(|\xi_n|^p) < \infty$. We prove that *if*

$$\sup_{n\geq 1}\frac{1}{W_n}\sum_{k=1}^{n} w_k(\log(w_k+1))^{\beta} < \infty \quad \text{for some } \beta > 1$$

then for any $\{\xi_k\}$ iid with $\mathbb{E}(|\xi_1|(\log^+|\xi_1|)^{1+\epsilon}) < \infty$ we have

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k \to \mathbb{E}\xi_1 \quad \text{a.s.}$$

2. *Norm convergence of weighted averages*

In this section we treat the norm convergence of weighted averages of Hilbert space contractions. We start with a criterion for the convergence for every Hilbert space contraction. Applications to L_p-convergence for Dunford–Schwartz contractions in probability spaces follow from Proposition 1.4, and will not be stated explicitly.

THEOREM 2.1. *Let $\{w_k\}$ be a non-null sequence of non-negative numbers with partial sums W_n. Then the following are equivalent:*

(i) *For every contraction T on a Hilbert space H and any $f \in H$ the sequence $\{A_n(T)f\}$ converges in norm.*

(ii) *For every contraction T on a Hilbert space H and any $f \in H$ the sequence $\{A_n(T)f\}$ converges weakly in H.*

(iii) *For every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the limit*

$$c(\lambda) := \lim_{n\to\infty}\frac{1}{W_n}\sum_{k=1}^{n} w_k\bar{\lambda}^k$$

exists.

Proof. Apply Theorem 2.2 and the proof of Corollary 2.3 of [**BLRT**] to the matrix defined by $\alpha_{n,k} = w_k/W_n$ for $1 \leq k \leq n$, $\alpha_{n,k} = 0$ for $k > n$. □

For a contraction T on a Hilbert space put $E(T)f := \lim_n (1/n)\sum_{k=1}^{n} T^k f$. From now on the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is denoted by Γ.

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COROLLARY 2.2. *Let $\{w_k\}$ be as before. Then*

$$\left\| \frac{1}{W_n} \sum_{k=1}^{n} w_k T^k f - E(T) f \right\| \longrightarrow 0$$

for every Hilbert space contraction T and $f \in H$ if and only if

$$\lim_{n\to\infty} \frac{1}{W_n} \sum_{k=1}^{n} w_k \bar{\lambda}^k = 0 \tag{2.1}$$

for every $1 \neq \lambda \in \Gamma$.

Remarks. (1) It follows from Theorem 1.1 and Corollary 2.2 that if $\{w_k\}$ has divergent sum and satisfies the *ergodicity condition* of Theorem 1.1(i), then (2.1) holds. This can be shown directly, using Abel's summation.

(2) Condition (2.1) does not imply the ergodicity condition. Let τ be a weakly mixing probability-preserving transformation of (Ω, μ), and let $h \in L_1(\mu)$ be non-constant positive a.e. By the pointwise ergodic theorem, we have a.e.

$$\frac{1}{n} \sum_{k=1}^{n} h \circ \tau^k \to \int h \, d\mu > 0, \quad \frac{1}{n} \sum_{k=1}^{n} |h - h \circ \tau| \circ \tau^k \to \int |h - h \circ \tau| \, d\mu > 0.$$

By the Wiener–Wintner theorem, a.e. $t \in \Omega$ satisfies $\lim(1/n) \sum_{k=1}^{n} h(\tau^k t)\lambda^k = 0$ for every $1 \neq \lambda \in \Gamma$, so for a.e. t the sequence $w_k := h(\tau^k t)$ satisfies (2.1), but not the ergodicity condition.

COROLLARY 2.3. *Let $\{w_k\}$ satisfy (2.1). Then*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} w_k^2}{\left(\sum_{j=1}^{n} w_j\right)^2} = 0. \tag{2.2}$$

Proof. Integrate the square of (2.1) with respect to Lebesgue's measure on Γ, apply the dominated convergence theorem, and use orthonormality of the exponents. □

Remarks. (1) Every bounded sequence with $\sum_{k=1}^{\infty} w_k = \infty$ satisfies (2.2).

(2) Since $\|\cdot\|_{\ell_2} \leq \|\cdot\|_{\ell_1}$, condition (2.2) implies $\sum_{k=1}^{\infty} w_k = \infty$.

(3) If $\{w_k\}$ is non-decreasing, then (2.2), which implies $w_n/W_n \to 0$, is in fact equivalent to condition (i) of Theorem 1.1, and thus is also equivalent in this case to (2.1).

(4) Let $\{k_j\}$ be a strictly increasing sequence of positive integers; put $w_{k_j} = 1$, and $w_k = 0$ if $k \notin \{k_j\}$. Then the weighted averages are the averages along the subsequence $\{k_j\}$.

(5) In general (2.2) does not imply (2.1). For an example, let T be a contraction on H such that T^n does not converge weakly (to $E(T)$ is the only possibility). For $f \in H$ such that $T^n f$ does not converge weakly, there exists a subsequence $\{k_j\}$ such that the averages along this subsequence do not converge in norm [**Kr-3**, p. 253]. The weights w_k defined as in the previous remark satisfy (2.2), but by Corollary 2.2 they do not satisfy (2.1).

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THEOREM 2.4. *Let $w_k \geq 0$ with $\sum_{k=1}^{\infty} w_k = \infty$ and put $W_n = \sum_{k=1}^{n} w_k$. Then the following are equivalent:*

(i) $\dfrac{1}{W_n^2}\sum_{k=1}^{n} w_k^2 \xrightarrow[n\to\infty]{} 0.$

(ii) *For every bounded orthogonal sequence $\{f_k\} \subset H$ we have*

$$\left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k f_k \right\| \to 0.$$

(iii) *For some orthonormal sequence $\{f_k\} \subset H$ the sequence $\{(1/W_n)\sum_{k=1}^{n} w_k f_k\}$ converges in norm.*

(iv) *For every contraction T on a Hilbert space H and $f \in H$ with $T^n f \xrightarrow{\text{weakly}} 0$ we have*

$$\left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k T^k f \right\| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. (i) $\Longrightarrow$ (ii) This follows by computing

$$\left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k f_k \right\|^2 = \frac{1}{W_n^2}\sum_{k=1}^{n} w_k^2 \|f_k\|^2 \leq \sup_j \|f_j\|^2 \frac{1}{W_n^2}\sum_{k=1}^{n} w_k^2 \longrightarrow 0.$$

Obviously (ii) $\Longrightarrow$ (iii), and (iv) $\Longrightarrow$ (iii) follows by looking at the shift on the basis.

(iii) $\Longrightarrow$ (i) Let $\{f_k\}$ be orthonormal, with $f := \lim_{n\to\infty}(1/W_n)\sum_{k=1}^{n} w_k f_k$. We show that $f = 0$. The limit f is in the closed linear subspace generated by $\{f_k\}$, so we may assume that $\{f_k\}$ is a basis. Since $W_n \to \infty$, for fixed j we obtain, by orthonormality,

$$\langle f, f_j\rangle = \lim_{n\to\infty}\left\langle \frac{1}{W_n}\sum_{k=1}^{n} w_k f_k, f_j \right\rangle = \lim_{n\to\infty}\frac{w_j}{W_n} = 0.$$

Now

$$\frac{1}{W_n^2}\sum_{k=1}^{n} w_k^2 = \left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k f_k \right\|^2 \to \|f\|^2 = 0.$$

(i) $\Longrightarrow$ (iv) Eventually $W_n > 0$, and we define $\alpha_{n,k} = w_k/W_n$ for $1 \leq k \leq n$, $\alpha_{n,k} = 0$ for $k > n$. Obviously $0 \leq \alpha_{n,k} \leq 1$ and $\sum_{k=1}^{\infty}\alpha_{n,k} = 1$. Now

$$\lim_{n\to\infty}\left(\sup_{j\geq 1}\alpha_{n,j}\right) = \lim_{n\to\infty}\left(\max_{1\leq j\leq n}\frac{w_j}{W_n}\right) \leq \lim_{n\to\infty}\frac{1}{W_n}\sqrt{\sum_{k=1}^{n} w_k^2} = 0$$

by (i). Thus condition (W5) in [**Kr-3**, p. 254] is satisfied, and Theorem 8.1.3 of [**Kr-3**] yields

$$\left\| \frac{1}{W_n}\sum_{k=1}^{n} w_k T^k f \right\| = \left\| \sum_{k=1}^{\infty}\alpha_{n,k} T^k f \right\| \longrightarrow 0. \qquad \square$$

Remarks. (1) In general the assumption that $T^n f$ converges weakly cannot be dropped from (iv).

(2) In the previous theorems, the appropriate result for every T induced on L_2 by a probability-preserving transformation τ already implies it for all Hilbert space contractions.

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Theorem 2.1(iii) or, respectively, (2.1) is obtained from knowing the convergence for all rotations of the circle. Knowing the convergence of Theorem 2.4 for every T induced on L_2 by a mixing transformation τ yields (iii) of the theorem, by considering the probability-preserving shift associated with any centered iid sequence with finite variance.

(3) Condition (2.2) is in fact equivalent to $\lim_{n\to\infty}(\max_{1\le j\le n}(w_j/W_n)) = 0$, since

$$\max_{1\le j\le n}\left(\frac{w_j}{W_n}\right)^2 \le \sum_{k=1}^{n}\frac{w_k^2}{W_n^2} = \frac{1}{W_n}\sum_{k=1}^{n} w_k\frac{w_k}{W_n} \le \frac{1}{W_n}\sum_{k=1}^{n} w_k\left(\max_{1\le j\le n}\frac{w_j}{W_n}\right) = \max_{1\le j\le n}\frac{w_j}{W_n}.$$

(4) Similarly, $W_n \to \infty$ together with $w_n/W_n \to 0$ imply (so are equivalent to) (2.2).

(5) Similar computations show that (2.2) holds if and only if $(1/W_n^p)\sum_{k=1}^{n} w_k^p \to 0$ for some (every) $p > 1$.

(6) Let $\{\xi_k\}$ be iid random variables with finite expectation a. Since the associated shift is mixing, (2.2) implies that

$$\mathbb{E}\left|\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k - a\right| \to 0$$

(by Theorem 2.4 and Proposition 1.4). This is a (slight) improvement over the weak law of large numbers obtained in [**JOP**].

Recall that a bounded linear operator T on a complex Banach space X is called *weakly almost periodic* if for every $f \in X$ the 'orbit' $\{T^n f\}$ is weakly conditionally compact. In this case T is necessarily power-bounded, and $E(T)f := \lim_{n\to\infty}\sum_{k=1}^{n} T^k f$ exists for every $f \in X$, i.e., T is mean ergodic. Power-bounded operators in reflexive spaces and mean ergodic positive contractions of L_1 are weakly almost periodic.

A vector f with 0 in the weak closure of the orbit is called a *flight* vector. For T weakly almost periodic the set X_0 of flight vectors is a closed subspace, and we have the Jacobs–Deleeuw–Glicksberg decomposition [**Kr-3**, §2.4]

$$X = X_0 \oplus \operatorname{span}\{g : Tg = \lambda g \text{ for some } |\lambda| = 1\}.$$

THEOREM 2.5. *Let $\{w_k\}$ be a (non-null) sequence of non-negative numbers, with $w_N > 0$, and assume that for some $1 < p \le 2$ we have*

$$\sup_{n>N} n^{p-1}\frac{\sum_{k=1}^{n} w_k^p}{W_n^p} = C < \infty. \tag{2.3}$$

Then for every weakly almost periodic operator T on a Banach space X and every flight vector $f \in X$ we have

$$\left\|\frac{1}{W_n}\sum_{k=1}^{n} w_k T^k f\right\| \xrightarrow[n\to\infty]{} 0.$$

Proof. Let $q = p/(p-1)$ be the dual index of p. Fix T weakly almost periodic on X and f a flight vector. By the proof of [**CLO**, Theorem 4.1], we have

$$\sup_{\substack{\|\phi\|\le 1\\ \phi\in X^*}}\frac{1}{n}\sum_{k=1}^{n}|\langle\phi, T^k f\rangle|^q \xrightarrow[n\to\infty]{} 0. \tag{2.4}$$

Since $(p-1)/p = 1/q$, the result follows from Hölder's inequality, (2.3) and (2.4). □

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Remarks. (1) Condition (2.3) implies $W_n \to \infty$, since $\|\cdot\|_{\ell_p} \le \|\cdot\|_{\ell_1}$.

(2) For any $1 < p \le 2$, condition (2.3) implies (2.2), by remark 5 following Theorem 2.4.

(3) Condition (2.2) does not imply (2.3) – in remark 2 following Corollary 2.2 take $h \in L_1$ not in any L_p, $p > 1$. By Corollary 2.3, (2.2) holds, while for any $p > 1$, (2.3) does not.

(4) For $1 < p_1 < p_2$ the norms inequality yields

$$\left(\frac{n^{p_1-1}\sum_{k=1}^n w_k^{p_1}}{W_n^{p_1}}\right)^{1/p_1} = \left(\frac{1}{n}\sum_{k=1}^n\left(\frac{nw_k}{W_n}\right)^{p_1}\right)^{1/p_1} \le \left(\frac{n^{p_2-1}\sum_{k=1}^n w_k^{p_2}}{W_n^{p_2}}\right)^{1/p_2}.$$

Thus, if (2.3) holds with some $p > 2$, it also does so with $1 < p \le 2$. Condition (2.3) with $p = 2$ is the strongest of the theorem's assumptions.

(5) Let $w_k = 1/\sqrt{k}$. Then (2.3) holds for any $1 < p < 2$, but not for $p = 2$.

(6) For a strictly increasing subsequence $\{k_j\}$ and $w_k = \mathbf{1}_{\{k_j\}}(k)$, condition (2.3), with any p, is equivalent to $\sup_j k_j/j < \infty$, i.e., $\{k_j\}$ has positive lower density. For this case the theorem recovers a result of Jones and Lin (see [**Kr-3**, p. 252]).

(7) If we take $k_j = 2j$, then w_k as in the previous remark satisfies (2.3) for $1 < p \le 2$, but the convergence in (2.1) fails for $\lambda = -1$. Hence also condition (i) of Theorem 1.1 fails.

COROLLARY 2.6. *Let $\{w_k\}$ be a sequence of non-negative numbers satisfying (2.3) (for some $p > 1$) and (2.1). Then $\|A_n(T)f - E(T)f\| \to 0$ for every T weakly almost periodic and any vector f.*

Remark. If in the example in remark 2 following Corollary 2.2 we take $h \in L_2$, the sequence $\{w_k\}$ constructed there will satisfy (2.3), with $p = 2$, as well as (2.1), but still not the ergodicity condition.

COROLLARY 2.7. *Let $\{w_k\}$ be non-negative numbers with $\liminf_n(W_n/n) > 0$. If for some $p > 1$ we have*

$$\sup_{n\ge1}\frac{1}{n}\sum_{k=1}^n w_k^p < \infty \tag{2.5}$$

then $\|A_n(T)f\| \to 0$ for every T weakly almost periodic and any flight vector f.

Remarks. (1) Note that (2.5) implies $\sup_n(W_n/n) < \infty$.

(2) For complex numbers $\{a_k\}$ with $\sup_n(1/n)\sum_{k=1}^n |a_k|^p < \infty$ for some $p > 1$, the proof of [**CLO**, Theorem 4.1] yields that

$$\left\|\frac{1}{n}\sum_{k=1}^n a_k T^k f\right\| \to 0$$

for T and f as in the corollary. For $p = 2$ and T weakly mixing this is Theorem 1 of Losert and Schneider [**LoS**].

(3) The simple example of $w_k = 1/\sqrt{k}$ shows the difference between [**LoS**] (stated for T weakly mixing), which for flight vectors f yields only

$$\left\|\frac{1}{n}\sum_{k=1}^n w_k T^k f\right\| \to 0,$$

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and Theorem 2.5, which (with $1 < p < 2$) yields

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} w_k T^k f \right\| \to 0.$$

3. *Pointwise convergence of weighted averages*

In this section we look at pointwise convergence of weighted averages of Dunford–Schwartz contractions. The problem in this case is to find conditions on the weights $\{w_k\}$ such that, for every Dunford–Schwartz contraction T in a special class (e.g., mixing, weakly mixing, induced by a transformation) and every f with particular integrability conditions (e.g., in L_2, in $L \log L$, in L_p, etc.), the averages $\{A_n(T)f\}$ converge a.e. (to $E(T)f$). As noted in the introduction, we have to study only the case $\sum_{k=1}^{\infty} w_k = \infty$. In this case it was already observed in [**JOP**] that if

$$A_n(\xi_1) := \frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k$$

converges in probability for $\{\xi_k\}$ iid, then $w_n / W_n \to 0$; since, as mentioned in the introduction, we assume $W_n \to \infty$, (2.2) is a necessary condition.

As mentioned before, (2.2) is not sufficient for a.s. convergence of $\{A_n(\xi_1)\}$ for every bounded iid $\{\xi_k\}$ (see [**Ts**]). Krengel [**Kr-2**] constructed a *universally bad* increasing sequence $\{k_j\}$ of positive integers, such that for every ergodic aperiodic probability preserving τ there is a bounded f for which the averages along the $\{k_j\}$ diverge a.e.; thus $w_k := \mathbf{1}_{\{k_j\}}(k)$ satisfies (2.2), while $A_n(T)f$ diverges a.e. for a bounded function. Note that for τ mixing there is norm convergence.

We saw that (2.3) is a strengthening of (2.2). However, Conze [**Co**] constructed an increasing sequence $\{k_j\}$ of integers, with positive lower density, such that for some weakly mixing τ and bounded f the averages along the sequence diverge a.e. As noted in the previous section, $w_k := \mathbf{1}_{\{k_j\}}(k)$ satisfies (2.3).

Part (i) of the following theorem shows the necessity of (3.1) below for the conclusion of Theorem 1.2 (using $\gamma_k \equiv 1$). Parts (ii) and (iii) are in the spirit of Theorem 1.2.

THEOREM 3.1. *Let $\{w_k\}$ be a sequence of non-negative numbers with $W_n \to \infty$.*

(i) *The condition*

$$\limsup_{n\to\infty} \frac{1}{W_n} \left(n w_n + \sum_{k=1}^{n-1} k |w_k - w_{k+1}| \right) = C < \infty \tag{3.1}$$

is necessary and sufficient for the following property.

For every $\{f_k\} \in L_p$, $p \geq 1$, such that $(1/n) \sum_{k=1}^{n} f_k$ converges to f a.e. (in norm), also

$$\frac{1}{W_n} \sum_{k=1}^{n} w_k f_k \xrightarrow[n\to\infty]{} f$$

almost everywhere (respectively, in norm).

(ii) *If $\{w_k\}$ satisfies $\inf w_k > 0$, $\sup (1/n) W_n < \infty$ and (3.1), then the conclusion of Theorem 1.2 holds.*

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(iii) *If* $\{w_k\}$ *satisfies (3.1), then the conclusion of Theorem 1.2 holds for* $\{\gamma_k\}$ *with* $\inf \gamma_k > 0$ *and*

$$\sup \frac{1}{n} \sum_{k=1}^{n} \gamma_k < \infty.$$

In particular, if $W_n \to \infty$ and (3.1) holds, then for every Dunford–Schwartz contraction T on a probability space and $f \in L_p$, $1 \le p < \infty$, the weighted averages

$$A_n(T) f = \frac{1}{W_n} \sum_{k=1}^{n} w_k T^k f$$

converge to $E(T) f$ a.e. and in L_p-norm.

Proof. Since $n w_n = \sum_{k=1}^{n-1} k(w_{k+1} - w_k) + W_n$, condition (3.1) is equivalent to

$$\limsup_{n\to\infty} \frac{1}{W_n} \sum_{k=1}^{n-1} k|w_k - w_{k+1}| < \infty. \tag{3.1′}$$

(i) This part is a special case of a general result on summability methods which are stronger than the Cesàro method [**Ze**, p. 100]. If **A** is a matrix which preserves Cesàro convergence and **C** is the Cesàro matrix, then $\mathbf{AC}^{-1}$ is regular (preserves convergence), which in our case implies (3.1) [**DS-2**, p. 75]. The sufficiency of (3.1) for preserving convergence of Cesàro averages (also in norm) follows from (iii), with $\gamma_k \equiv 1$.

(ii) The proof is similar to that of Theorem 8.2.1 in [**Kr-3**], so is left to the reader. □

Remarks. (1) Condition (3.1) is a strengthening of the *ergodicity condition* given in Theorem 1.1(i). If the weights satisfy (3.1) then $(1/W_n) \sum_{k=1}^{n} w_k a_k \to a$ for every $\{a_k\}$ with averages converging to a, while the ergodicity condition yields the convergence only for *bounded* $\{a_k\}$ with $(1/n) \sum_{k=1}^{n} a_{j+k} \to a$ *uniformly in* j (since such a sequence has a unique Banach limit [**Kr-3**, p. 136] we apply [**Lor**]; we can also apply Theorem 1.1 to T the shift on these sequences).

(2) While a non-increasing $\{w_k\}$ always satisfies (3.1), in general even the ergodicity condition is not necessary for a.e. convergence of $A_n(T) f$ (though the limit may not be $E(T) f$) – see remark following Corollary 1.3.

(3) It will be shown later in this section that the ergodicity condition, and a fortiori (3.1), is not necessary for the a.e. convergence $A_n(T) f \to E(T) f$ for every T induced by a probability-preserving transformation and every $f \in L_1$.

(4) For an increasing sequence $\{w_k\}$, condition (3.1) is equivalent to $\sup_n (n w_n / W_n) < \infty$, while the ergodicity condition is equivalent to $w_n / W_n \to 0$; the example below shows that (3.1) is strictly stronger. An example in [**Ts**] shows that the ergodicity condition alone is not sufficient for a.e. convergence of $A_n(T) f$ even for f bounded and $\{T^k f\}$ iid.

(5) The ergodicity condition easily implies $A_n(T) f \to E(T) f$ a.e. for f in the dense subspace $\{g \in L_p : Tg = g\} + (I - T)L_\infty$. Condition (3.1) can be used to show the a.e. finiteness of $\sup |A_n(T) f|$ for any $f \in L_p$; however, assuming (3.1) we obtain more from Theorem 3.1.

(6) If $\sum_{k=1}^{\infty} k|w_k - w_{k+1}| < \infty$, then (3.1′) and (3.1) hold.

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COROLLARY 3.2. *Condition (3.1) is satisfied by* $\{w_k\}$ *with* $W_n \to \infty$ *(and hence all the assertions of Theorem 3.1 hold) in the following cases.*

(i) *For some* $s \geq 0$ *the sequence* $\{w_k/k^s\}$ *is non-increasing.*

(ii) *For some* $s \geq 0$ *the sequence* $\{k^s w_k\}$ *is non-decreasing and satisfies*

$$\sup_n \frac{n w_n}{\sum_{k=1}^n w_k} < \infty. \tag{3.2}$$

Proof. (i) We may of course assume $s \geq 1$. We use the given monotonicity to estimate

$$\sum_{k=1}^{n-1} k|w_k - w_{k+1}| \leq \sum_{k=1}^{n-1} k^{1+s}\left(\frac{w_k}{k^s} - \frac{w_{k+1}}{(k+1)^s}\right) + \sum_{k=1}^{n-1} \frac{(k+1)^s - k^s}{(k+1)^s}\, k\, w_{k+1}.$$

Since $s \geq 1$, the second sum is bounded by

$$\sum_{k=1}^{n-1} \frac{s(k+1)^{s-1}}{(k+1)^s} k w_{k+1} \leq s \sum_{j=2}^{n} w_j.$$

For the first sum we have the estimate

$$\begin{aligned}\sum_{k=1}^{n-1} k\left(w_k - \frac{k^s}{(k+1)^s} w_{k+1}\right) &= \sum_{k=1}^{n-1} (k w_k - (k+1) w_{k+1}) + \sum_{k=1}^{n-1} \frac{(k+1)^{s+1} - k^{s+1}}{(k+1)^s} w_{k+1} \\ &\leq w_1 - n w_n + (1+s) \sum_{k=1}^{n-1} w_{k+1}.\end{aligned}$$

We thus obtain

$$\sum_{k=1}^{n-1} k|w_k - w_{k+1}| + n w_n \leq w_1 + (1+2s) W_n,$$

which implies (3.1). Note that (i) easily implies (3.2), since

$$W_n = \sum_{k=1}^{n} k^s k^{-s} w_k \geq \sum_{k=1}^{n} k^s n^{-s} w_n \geq \frac{1}{s+1} n w_n.$$

(ii) We may now assume $s \geq 2$, and use the monotonicity to estimate

$$\begin{aligned}\sum_{k=1}^{n-1} k|w_k - w_{k+1}| &\leq \sum_{k=1}^{n-1} \frac{1}{k^{s-1}}((k+1)^s w_{k+1} - k^s w_k) + \sum_{k=1}^{n-1} \frac{(k+1)^s - k^s}{k^{s-1}} w_{k+1} \\ &= \sum_{k=1}^{n-1} ((k+1) w_{k+1} - k w_k) + \sum_{k=1}^{n-1} \left(\frac{(k+1)^s}{k^{s-1}} - (k+1)\right) w_{k+1} \\ &\quad + \sum_{k=1}^{n-1} \frac{(k+1)^s - k^s}{k^{s-1}} w_{k+1}\end{aligned}$$

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$$
\begin{aligned}
&= nw_n - w_1 + \sum_{k=1}^{n-1} \frac{(k+1)^{s-1} - k^{s-1}}{k^{s-1}}(k+1)w_{k+1} \\
&\quad + \sum_{k=1}^{n-1} \frac{(k+1)^s - k^s}{k^{s-1}} w_{k+1} \\
&\le nw_n + (s-1)\sum_{k=1}^{n-1}\left(\frac{k+1}{k}\right)^{s-1} w_{k+1} + s\sum_{k=1}^{n-1}\left(\frac{k+1}{k}\right)^{s-1} w_{k+1} \\
&\le nw_n + (2s-1)2^{s-1}\sum_{j=2}^{n} w_j,
\end{aligned}
$$

and together with (3.2) we conclude that (3.1) holds. □

Remarks. (1) When T is mean ergodic on a Banach space and $f_k = T^k f$, the norm convergence of Theorem 3.1 and Corollary 3.2 follows already from Theorem 1.1.

(2) Trigonometric series with coefficients satisfying the 'quasimonotonicity' assumptions of the corollary were considered by Lebed [**Le**].

(3) Corollary 3.2 remains true if the 'quasimonotonicity' holds only for $k > N$. If $\{w_k\}$ satisfies $W_n \to \infty$, and $\{\tilde{w}_k\}$ differs from $\{w_k\}$ only in finitely many terms, then $(1/W_n)\sum_{k=1}^n w_k f_k$ converges if and only if $(1/\tilde{W}_n)\sum_{k=1}^n \tilde{w}_k f_k$ converges (and the limits are equal).

(4) Corollary 3.2 applies also to non-monotone sequences. As an example satisfying (i), define $w_k = 2^{-js}k^s$ for $2^j \le k < 2^{j+1}$. Since $w_{2^j} = 1$ and the sequence increases in each dyadic bloc, it is not monotone. Part (ii) of the corollary applies, for example, to $w_k := 2^{js}k^{-s}$ for $2^j \le k < 2^{j+1}$; an unbounded example is $w_k := k + \frac{3}{2}\sin k$ (for which we omit the computations).

(5) Condition (3.2) is satisfied when $w_k = k^t$ for a fixed $t > 0$, or $w_k = (\log k)^t$ for a fixed $t > 0$, but not when $w_k = t^k$ for some $t > 1$ (a case in which even (2.2) does not hold).

Example. An increasing sequence $\{w_k\}$ satisfying (2.2) with $n^\gamma w_n / W_n \to \infty$ for any $\gamma > 0$. Let $w_1 = 1$, and put $w_k = e^{k/\sqrt{\log k}}$ for $k \ge 2$. Since

$$
\frac{d}{dt}(e^{at/\sqrt{\log t}}\sqrt{\log t}) \sim a e^{at/\sqrt{\log t}}
$$

for $a > 0$, we have

$$
c_1 e^{n/\sqrt{\log n}}\sqrt{\log n} \le W_n \le c_2 e^{n/\sqrt{\log n}}\sqrt{\log n}
$$

for some positive constants. Hence $w_n/W_n \le 1/(c_1\sqrt{\log n}) \to 0$, so (2.2) holds; however,

$$
\frac{n^\gamma w_n}{W_n} \ge \frac{n^\gamma}{c_2\sqrt{\log n}} \to \infty
$$

for any $\gamma > 0$. This also shows that for any $p > 1$ (2.3) is not satisfied. For $a = 2$ we obtain

$$
C_1 e^{2n/\sqrt{\log n}}\sqrt{\log n} \le \sum_{k=1}^{n} w_k^2 \le C_2 e^{2n/\sqrt{\log n}}\sqrt{\log n},
$$

which yields a rate of $1/\sqrt{\log n}$ in (2.2).

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Remark. The construction in [**JOP**] yields non-decreasing sequences with arbitrarily slow convergence to 0 of w_n/W_n; for any increasing sequence of integers $\{n_m\}$ there is $\{w_k\}$ non-decreasing satisfying (2.2) with $w_{n_m}/W_{n_m} = 1/m$.

PROPOSITION 3.3. *Let* $\{w_k\}$ *satisfy* $\sum_k w_k = \infty$. *Then for any* $\alpha > 1$ *and* $\{f_k\} \subset L_1$ *with* $\sup_k \|f_k\|_1 < \infty$ *we have*

$$\frac{\sum_{k=1}^n w_k f_k}{W_n \log^\alpha(1 + W_n)} \xrightarrow[n\to\infty]{} 0 \quad a.e.$$

Proof. For $\Phi(t) := t \log^\alpha(1+t)$ we use the well-known lemma below (e.g. [**HLP**, p. 120]), and obtain the result from Beppo Levi's theorem and Kronecker's lemma.

LEMMA. *Let* $\Phi(t) > 0$ *be non-decreasing for* $t \geq t_0 \geq 0$ *with* $\lim_{t\to\infty} \Phi(t) = \infty$. *If*

$$\sum_{k>t_0} \frac{1}{\Phi(k)} < \infty,$$

then for some k_0 *we have*

$$\sum_{k\geq k_0} \frac{w_k}{\Phi(W_k)} < \infty.$$

The lemma follows from the inequality

$$\frac{w_k}{\Phi(W_k)} \leq \int_{W_{k-1}}^{W_k} \frac{1}{\Phi(t)}\, dt,$$

valid when $W_{k-1} \geq t_0$. □

Remarks. (1) For $p > 1$ the proposition is also an immediate consequence of the remark to Corollary 1.7(c) in [**W-2**] (with $\xi_k = w_k f_k$).

(2) Taking $\Phi(t) = t \log(1 + t)[\log\log(1 + t)]^\alpha$ with $\alpha > 1$, the proof yields

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n w_k f_k}{W_n \log(1 + W_n)[\log\log(1 + W_n)]^\alpha} = 0 \quad \text{a.e.}$$

We now consider sequences of functions $\{f_k\} \subset L_2(\mu)$; without loss of generality we may assume (Ω, μ) a probability space. For a sequence of non-negative weights $\{w_k\}$ put

$$M_n := \sum_{k=1}^n w_k^2. \tag{3.3}$$

We assume the following relation between the weights and the functions $\{f_k\}$: there exists a finite constant C_0 such that

$$\left\| \sum_{k=n}^m w_k f_k \right\|_2^2 \leq C_0 \sum_{k=n}^m w_k^2 \quad \text{for all } m \geq n \geq 1. \tag{3.4}$$

Condition (3.4) obviously holds for norm-bounded orthogonal sequences, e.g. orthonormal sequences (for any weight sequence). Condition (3.4) is also realized (see [**W-1**, Lemma 7.4.3]) when

$$\sup_i \sum_j |\langle f_i, f_j \rangle| < \infty. \tag{3.5}$$

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To get (3.5) it suffices for instance that, for any integers $j \geq i \geq 1$,

$$|\langle f_i, f_j \rangle| \leq C_1 e^{-C_2|j-i|}. \tag{3.6}$$

Since the weights are non-negative, (3.4) holds for centered negatively correlated random variables with uniformly bounded variances. Another example of a sequence satisfying (3.4) is a wide-sense stationary sequence with bounded spectral density [**G**] (see also Proposition 3.4(iv) of [**CL**]).

THEOREM 3.4. *Assume that the sequence of weights* $\{w_k\}$ *with* $M_n \to \infty$ *satisfies*

$$\log \frac{1}{w_n} = \mathcal{O}(\log M_n) \tag{3.7}$$

$$\limsup_{n\to\infty} \frac{M_n \log^\gamma M_n}{W_n^2} < \infty \quad \textit{for some } \gamma > 3. \tag{3.8}$$

Then for $\{f_k\}$ *which satisfies (3.4) we have*

$$\lim_{n\to\infty} \frac{1}{W_n} \sum_{k=1}^{n} w_k f_k = 0 \quad \textit{a.s.} \tag{3.9}$$

Proof. Combine Corollary 1.5 of [**W-2**] with (3.8). □

Remarks. (1) Condition (3.8) is a rate of convergence in (2.2) which is determined by $\{w_k\}$ itself. Every bounded sequence satisfies (3.8): if $w_k \leq c$, then $M_n \leq cW_n$, and the terms in (3.8) are bounded by $c^2 \log^\gamma M_n / M_n$.

(2) When $\{w_k\}$ is a weight sequence with divergent series satisfying

$$\lim_{n\to\infty} \frac{M_n}{W_n^2} \log^\alpha W_n = 0 \quad \text{for some } \alpha > 1 \tag{3.10}$$

then, by [**SZ**, Theorem 1.2.1], (3.9) holds for any uniformly bounded orthogonal $\{f_k\}$. When $M_n \to \infty$ and $w_n^2/M_n \to 0$, the proof of [**SZ**] yields the same result if (3.10) is replaced by a weaker form of (3.8), namely

$$\lim_{n\to\infty} \frac{M_n}{W_n^2} \log^\alpha M_n = 0 \quad \text{for some } \alpha > 1. \tag{3.11}$$

The problem of a.e. convergence of weighted averages for Dunford–Schwartz contractions is closely related to the problem of the so-called *modulated ergodic theorems* (e.g., [**LOT**] and references therein). A sequence $\{a_n\}$ is called a *universally good* L_p*-modulating sequence* if for every Dunford–Schwartz contraction T on a probability space (Ω, μ) and any $f \in L_p(\mu)$ the modulated averages $(1/n)\sum_{k=1}^n a_k T^k f$ converge a.e.

If $\{w_n\}$ is a non-negative universally good L_p-modulating sequence and its mean

$$m := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} w_k$$

(which must exist) is not zero, we obtain a.e. convergence of

$$A_n(T) f = \frac{1}{W_n} \sum_{k=1}^{n} w_k T^k f$$

for every Dunford–Schwartz T and $f \in L_p$.

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An important class of universally good L_p-modulating sequences, introduced independently by Tempelman (for $p > 1$) and by Ryll-Nardzewski (for $p = 1$), is the class of q-Besicovitch sequences (with $q := p/(p-1)$ the dual index); see [**LOT**] for additional results and references. For other classes see [**BL**].

THEOREM 3.5. *Let $1 \le p \le \infty$ with dual index $q = p/(p-1)$. For $\{a_k\}$ a q-Besicovitch sequence with $\limsup_n (1/n)\sum_{k=1}^n |a_k| > 0$, define $w_k = |a_k|$. Then $A_n(T)f$ converges a.e. for every Dunford–Schwartz contraction T and $f \in L_p$.*

Proof. Recall that on any LCA group G, the space $AP(G)$ of almost periodic continuous functions is the closed linear subspace of $C(G)$ generated by the continuous characters, and is a subalgebra isometrically order isomorphic to $C(\tilde{G})$ (bounded continuous functions on the Bohr compactification). Thus the absolute value of an almost periodic function is almost periodic. When $p > 1$, this yields that also $\{w_k\}$ is q-Besicovitch. When $p = 1$ the assumption is that $\{a_n\}$ is a bounded 1-Besicovitch sequence, and we similarly obtain that so is $\{w_k\}$. Hence $\lim(W_n/n)$ exists, and by assumption is not zero, so the previous remarks yield the result. □

PROPOSITION 3.6. *Let $1 \le p < \infty$. For any $1 \le r < q$ there exists an r-Besicovitch sequence $\{w_k\}$ of positive numbers with non-zero mean, such that for any ergodic probability preserving τ there is a function $f \in L_p$ with a.e. non-convergence of*

$$\frac{1}{W_n}\sum_{k=1}^n w_k f \circ \tau^k.$$

Proof. Let $\{a_n\}$ be the sequence constructed by Demeter in Theorem 5.1 of [**De**] (see also Theorem 1.7 of [**DeJ**]): it is positive r-Besicovitch, but for some $f \in L_p$ the sequence $(1/n)\sum_{k=1}^n a_k f \circ \tau^k$ is a.e. non-convergent. This universal 'badness' is not stated in [**De**], but in the proof the sequence is constructed 'in advance'. Since $\{a_k\}$ has mean zero, we define $w_k = a_k + 1$ and obtain the divergence from $W_n/n \to 1$ and

$$\frac{1}{W_n}\sum_{k=1}^n w_k T^k f = \frac{n}{W_n}\frac{1}{n}\sum_{k=1}^n (a_k+1)T^k f = \frac{n}{W_n}\frac{1}{n}\sum_{k=1}^n a_k T^k f + \frac{n}{W_n}\frac{1}{n}\sum_{k=1}^n T^k f. \quad (3.12)$$

Because r-Besicovitch sequences are a linear space, $\{w_k\}$ is r-Besicovitch. □

Remarks. (1) If $r > 1$, the functions $\{T^k f\}$ cannot be independent (see [**BJLO**]).

(2) The weights in the proposition satisfy Theorem 2.1(iii) (with $c(\lambda) = 0$ for $\lambda \ne 1$), so norm convergence of $A_n(T)f$ holds for $f \in L_2$, hence also for $f \in L_1$ by Proposition 1.4.

Bourgain's 'return times' theorem [**Bou**] yields a.e. convergence of $A_n(T)f$ for stationary ergodic random weights: if the random weights have finite moment of order q, then the convergence holds for $f \in L_p$, where $p = q/(q-1)$ is the dual index of q (see also [**Ru**]). Assani [**As-3**] showed that, when the random weights are iid and $q > 1$, then $A_n(T)f$ converges a.e. for any $f \in L_r$, $r > 1$ (even if $r < p$).

A necessary condition for a.e. convergence of $A_n(T)f$ to $E(T)f$ for every T induced by a probability-preserving transformation and $f \in L_\infty$ is (2.1). However, the ergodicity condition of Theorem 1.1(i), which implies (2.1), is not necessary. In fact, if in the example

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following Corollary 2.2 we take $h \in L_\infty$, Bourgain's theorem yields a.e. convergence of $A_n(T)f$ for every T induced by a probability-preserving transformation and $f \in L_1$, with the limit necessarily $E(T)f$ since $\|A_n(T)f - E(T)f\|_1 \to 0$ by Corollary 2.6.

4. *Weighted averages of iid random variables*

In this section we look at the problem of almost sure convergence of the weighted averages of a sequence $\vec{\xi} := \{\xi_k\}$ of independent random variables, defined on a probability space $(\Omega, \mathbb{P})$, with $\mathbb{E}(|\xi_k|) < \infty$. For convenience we denote

$$A_n(\vec{\xi}) := \frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k.$$

It was already observed in [**JOP**] that if for some non-constant iid sequence $\{A_n(\vec{\xi})\}$ converges in probability to a constant, then $w_n/W_n \to 0$. As observed in the introduction, we have to treat only weights with $\sum_{k=1}^{\infty} w_k = \infty$, and in this case (2.2) becomes a necessary condition. The following was observed as a corollary of Theorem 2.4.

THEOREM 4.1. *If $w_n/W_n \to 0$, then $\mathbb{E}(|A_n(\vec{\xi})|) \to 0$ for every centered iid sequence $\vec{\xi}$.*

Remarks. (1) By Theorem 2.4, (2.2) implies also that $\mathbb{E}(|A_n(\vec{\xi})|^2) \to 0$ for centered independent $\{\xi_k\}$ with bounded variances.

(2) The arguments of Theorem 2.4 apply also to the general scheme of Pruitt [**P**], since his assumptions are precisely those of [**Kr-3**, Theorem 8.1.3]; thus we (slightly) strengthen his Theorem 1, replacing convergence in probability by convergence in norm.

It has been known for a long time that (2.2) is not sufficient for a.s. convergence in Theorem 4.1, even if ξ_1 is bounded. Tsuchikura [**Ts**] presents an example, essentially due to Maruyama, of $w_n \to \infty$ with

$$\frac{w_n}{W_n} = \mathcal{O}\left(\frac{1}{\log\log W_n}\right)$$

such that for $\vec{\varepsilon} := \{\varepsilon_k\}$ the Rademacher system $A_n(\vec{\varepsilon})$ is a.s. divergent (see also [**SZ**, Theorem 1.5.1]).

For iid random variables we can suppress the zero weights; in this case we may assume that $w_k > 0$ for every k. A necessary and sufficient condition for the a.s. convergence of $\{A_n(\vec{\xi})\}$ for every $\vec{\xi}$ iid with finite expectation is [**JOP**]

$$\limsup_{t\to\infty} \frac{1}{t} \#\left\{n : \frac{W_n}{w_n} \le t\right\} < \infty. \tag{4.1}$$

Remarks. (1) Note that condition (4.1) is satisfied as soon as $\sum_{n\ge1}(w_n/W_n) < \infty$, since

$$\sum_{n\ge1} \frac{w_n}{W_n} \ge \sum_{\{n:\frac{w_n}{W_n}\ge\frac{1}{t}\}} \frac{w_n}{W_n} \ge \frac{1}{t} \#\left\{n : \frac{W_n}{w_n} \le t\right\}.$$

On the other hand, taking $w_n = 1$ for every n we have $\sum_{n=1}^{\infty}(w_n/W_n) = \infty$ while (4.1) holds.

(2) Theorem 3.4 of [**BJLO**] yields that, if $\{w_n\}$ satisfies the assumptions of Corollary 2.7:

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(i) $\liminf_{n\to\infty}(1/n)\sum_{k=1}^{n} w_k > 0$;
(ii) $\sup_n(1/n)\sum_{k=1}^{n} w_k^{\alpha} < \infty$ for some $\alpha > 1$;
then $A_n(\vec{\xi}) \to \mathbb{E}(\xi_1)$ a.s. for every $\vec{\xi}$ iid with finite expectation; hence also (4.1) holds.

(3) Theorem 3.10 of [**BJLO**] together with the proof of Proposition 3.6 show that, with $\alpha = 1$ in (ii), a.s. convergence of $A_n(\vec{\xi})$ is guaranteed only for ξ_1 bounded.

(4) Strictly stationary ergodic random weights with finite moment $\alpha > 1$ satisfy (4.1) [**As-1**] (they satisfy (i) and (ii) above). Existence of only the first moment for the stationary weights is not sufficient for (4.1)—see the argument in [**As-4**] (based on [**ABM**]). The weakest assumption condition on such weights for obtaining (4.1) is given in [**DeQ**, Corollary 6] (it improves [**BJLO**, Theorem 4.1], which could have been deduced from [**As-2**, Theorem 5]).

It is shown in [**JOP**] that bounded sequences need not satisfy (4.1); it means that without (i), (ii) above does not imply (4.1). The following is Theorem 4 of [**JOP**].

THEOREM 4.2. *Let $\{w_k\}$ be bounded weights with $\sum_{k=1}^{\infty} w_k = \infty$. Then for every centered iid sequence with $\mathbb{E}(|\xi_1|\log^+|\xi_1|) < \infty$ we have $A_n(\vec{\xi}) \to 0$ a.s.*

Our purpose is to relax the assumptions on the weights, but still strengthen (2.2), to obtain a.s. convergence when more integrability conditions on ξ_1 are known.

PROPOSITION 4.3. *Let $1 \le p \le 2$ and let $w_k > 0$.*
(i) *If $\sum_{k=1}^{\infty} w_k^p < \infty$, then for any centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$, the series $\sum_{k=1}^{\infty} w_k\xi_k$ converges a.s.*
(ii) *We have a.s. convergence of the series $\sum_{k=1}^{\infty}(w_k\xi_k/W_k)$ for every centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{w_k}{W_k}\right)^p < \infty.$$

(iii) *When $W_n = \sum_{k=1}^{n} w_k \to \infty$, the following are equivalent:*
  (a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{w_k}{W_k}\right)^p < \infty$;
  (b) $\displaystyle\sum_{k=1}^{\infty}\frac{w_k\xi_k}{W_k}$ *converges a.s. for any centered independent $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$;*
  (c) $\displaystyle\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k \to 0$ *a.s. for any centered independent $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$.*

Proof. Assertion (i) follows from Marcinkiewicz–Zygmund [**MZ**].

In part (ii), if $\sum_{k=1}^{\infty}(w_k/W_k)^p < \infty$, then for $\{\xi_k\}$ as in the statement the series $\sum_{k=1}^{\infty}(w_k\xi_k/W_k)$ converges a.s. by (i). Conversely, if $\sum_{k=1}^{\infty}(w_k/W_k)^p = \infty$, then Theorem 5 of [**MZ**] yields the existence of a sequence $\{\xi_k\}$ of independent centered random variables with $\mathbb{E}(|\xi_k|^p) = 1$ for which $\sum_{k=1}^{\infty}(w_k\xi_k/W_k)$ is a.s. divergent.

In assertion (iii), (a) implies (b) by (ii), and (b) implies (c) by Kronecker's lemma. Now assume (c). An inspection of the construction of the example in Theorem 5 of [**MZ**]

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shows that, if (a) does not hold, then in fact there is $\{\xi_k\}$ centered independent with $\mathbb{E}(|\xi_k|^p) = 1$ such that $\limsup(w_k\xi_k/W_k) \geq 1$ a.s. (we define $\xi_k = W_k x_k/kw_k$, where x_k are the random variables defined in [**MZ**]). This contradicts (c), since $(1/W_n)\sum_{k=1}^n w_k\xi_k$ is then a.s. non-convergent to 0 (non-convergent at all when $w_n/W_n \to 0$), by the identity

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k - \frac{1}{W_{n-1}}\sum_{k=1}^{n-1} w_k\xi_k = \frac{w_n\xi_n}{W_n} - \frac{w_n}{W_n}\frac{1}{W_{n-1}}\sum_{k=1}^{n-1} w_k\xi_k. \qquad \square$$

Remarks. (1) Convergence of $\sum_{k=1}^{\infty}(w_k/W_k)^p$ implies

$$\limsup_{t\to\infty}\frac{1}{t^p}\#\left\{n:\ \frac{W_n}{w_n} \leq t\right\} < \infty,$$

also implied by (4.1).

(2) Any bounded weight sequence $\{w_k\}$ satisfies $\sum_{k=1}^{\infty}(w_k/W_k)^p < \infty$ for $p > 1$—see the remark to Corollary 4.11. For $p = 2$ Guy Cohen (private communication) has given a simple proof:

$$\sum_{k=2}^{n}\frac{w_k^2}{W_k^2} \leq \sum_{k=2}^{n}\frac{w_k^2}{(\sum_{j=1}^{k} w_j)(\sum_{i=1}^{k-1} w_i)} = \sum_{k=2}^{n} w_k\left(\frac{1}{W_{k-1}} - \frac{1}{W_k}\right) \leq \frac{\sup_{j>1} w_j}{w_1}.$$

He also noted that $\sum_{k=1}^{\infty}(w_k/W_k)^2 < \infty$ if (instead of boundedness) we assume convergence of $\sum_k |w_{k+1} - w_k|/W_k$ (a strengthening of condition (i) of Theorem 1.1) – apply Abel's summation to $\sum_{k=2}^{n} w_k(1/W_{k-1} - 1/W_k)$. Consequently, if $W_n \to \infty$, and $\{w_k\}$ is bounded or satisfies $\sum_k |w_{k+1} - w_k|/W_k < \infty$, then $(1/W_n)\sum_{k=1}^{n} w_k\xi_k \to 0$ a.s. for any $\{\xi_k\}$ centered independent with $\sup_k \mathbb{E}(|\xi_k|^2) < \infty$.

(3) For fixed $1 \leq p < 2$ put $w_1 = 1$ and $w_k = [(k-1)!]^{1/p}/\prod_{j=2}^{k}(j^{1/p} - 1)$. Then $W_n = n^{1/p}w_n$, so $\sum_{n=1}^{\infty}(w_n/W_n)^p = \infty$. However, for every $\{\xi_k\}$ centered iid with $\mathbb{E}(|\xi_1|^p) < \infty$ we have, by [**MZ**, Theorem 9], a.s. convergence of the series

$$\sum_{n=1}^{\infty}\frac{w_n\xi_n}{W_n} = \sum_{n=1}^{\infty}\frac{\xi_n}{n^{1/p}}.$$

Since w_n is increasing, $W_n \to \infty$, and by Kronecker's lemma $A_n(\vec{\xi}) \to 0$ a.s.

(4) For $p = 2$, the a.s. convergence of the series $\sum_{n=1}^{\infty}(w_n\xi_n/W_n)$ for *one* sequence of centered iid $\{\xi_k\}$ with $\mathbb{E}(\xi_1^2) < \infty$ already implies $\sum_{n=1}^{\infty}(w_n/W_n)^2 < \infty$ (e.g., [**MZ**, Theorem 4]).

(5) Another equivalent condition in (iii), obtained from [**Ch**, Corollary 5], is

(d) $\displaystyle\sum_{k=1}^{\infty}\frac{w_k\xi_k}{W_k}$ converges a.s. for any martingale difference $\{\xi_k\}$ with $\sup\mathbb{E}(|\xi_k|^p) < \infty$.

(6) Theorem 3.3 of [**BJLO**] shows that if $\{w_k\}$ satisfies (i) and (ii) of remark 2 after (4.1), then condition (iii)(c) of the proposition holds for *every* $p > 1$.

(7) Theorems 3.7 and 3.9 in [**BJLO**] yield methods of randomly generating weights which satisfy condition (iii)(c) for *every* $p > 1$.

THEOREM 4.4. *Let $\{w_k\}$ be positive weights such that for some $0 < \gamma \leq 1$ we have*

$$\sup_{n>0}\frac{n^\gamma w_n}{W_n} < \infty. \tag{4.2}$$

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(i) *If* $\frac{1}{2} < \gamma \leq 1$, *then* $A_n(\vec{\xi}) \to 0$ *a.s. for every centered iid with* $\mathbb{E}(|\xi_1|^{1/\gamma}) < \infty$.
(ii) *If* $\gamma \leq \frac{1}{2}$, *then* $A_n(\vec{\xi}) \to 0$ *a.s. for every centered iid with* $\mathbb{E}(|\xi_1|^{1+1/\gamma}) < \infty$.

Proof. Put $N(t) = \#\{n : W_n/w_n \leq t\}$. Since (4.2) implies $N(t) \leq (Kt)^{1/\gamma}$, for $\gamma > \frac{1}{2}$ Theorem 2 of [**JOP**] applies, which proves (i). (Assertion (i) follows also from [**ARo**]; Proposition 4.3(iii) yields the desired convergence only when $\mathbb{E}(|\xi_1|^p) < \infty$ for some $p > 1/\gamma$). Assertion (ii) follows from [**P**]. □

Remarks. (1) For $\gamma = 1$ it easy to check that (4.2) implies (4.1). On the other hand, if for some $0 < \gamma < 1$ we have $\liminf_n (n^\gamma w_n/W_n) > 0$, then (4.1) does not hold.

(2) If $\{w_k\}$ is non-decreasing and (4.2) holds with $\gamma = 1$, then Corollary 3.2 yields much more than Theorem 4.4.

(3) If $\sup_{n>0} \sqrt{n} \log n (w_n/W_n) < \infty$, then $A_n(\vec{\xi}) \to 0$ a.s. for centered iid with $\mathbb{E}(|\xi_1|^2) < \infty$, by Proposition 4.3(iii). This is an intermediate case between (i) and (ii) in the theorem. In fact, we can take $\log n$ raised to any power greater than $1/2$.

(4) Condition (2.3) implies (4.2) with $\gamma = (p-1)/p$; when $1 < p \leq 2$ we have $0 < \gamma \leq \frac{1}{2}$.

(5) Condition (4.1) does not imply $\sup(nw_n/W_n) < \infty$. Let $\{w_k\}$ be an *unbounded* sequence of positive numbers satisfying $\lim(1/n)\sum_{k=1}^n w_k = a > 0$, and $\lim(1/n)\sum_{k=1}^n w_k^\alpha = b > 0$ for some $\alpha > 1$ (e.g., a 'typical' realization of a positive ergodic unbounded strictly stationary sequence with finite moment α). Then (4.1) holds by [**BJLO**], but $\limsup(nw_n/W_n) = \limsup(w_n/(W_n/n)) = \infty$. Note that (2.3) holds for $p = \alpha$.

(6) In view of Theorem 4.2, one would like to have conditions which will be satisfied by any bounded sequence with divergent sum. As the next proposition shows, (4.2) does not satisfy this requirement.

PROPOSITION 4.5. *Let* $\{\phi_n\}$ *be an increasing sequence of positive numbers tending to infinity, which satisfies* $\limsup_n(\phi_{n+1} - \phi_n) < 1$. *Then there exists a bounded sequence* $\{w_k\}$ *of positive numbers with* $\sum_{k=1}^\infty w_k^2 = \infty$, *such that* $\limsup_{n\to\infty}(\phi_n w_n/W_n) = \infty$ *and*

$$\lim_n \frac{\phi_n}{W_n} \max_{1\leq j\leq n} w_j = \infty.$$

Proof. Put $\phi(0) = 0$, and define a continuous increasing function $\phi(t)$ linearly in each interval $[j, j+1)$. Let $\psi := \phi^{-1}$, which is also increasing to infinity, with $\liminf_n(\psi(t+1) - \psi(t)) > 1$. Put $E := \{[\psi(j^2)] : j \geq 1\} \subset \mathbb{N}$. Now define $w_k = 1$ if $k \in E$ and $w_k = 1/2^k$ if $k \notin E$. Eventually ($j > j_0$) all $[\psi(j^2)]$ are different, so $\sum_{k=1}^\infty w_k^2 = \infty$, and $W_{[\psi(n^2)]} < n+1$. Since $[\psi(n^2)] \geq \psi(n^2-1)$ and $\max_{1\leq i\leq m} w_i = 1$ for $m \geq [\psi(1)]$, we have

$$\frac{\phi_{[\psi(n^2)]} w_{[\psi(n^2)]}}{W_{[\psi(n^2)]}} = \frac{\phi_{[\psi(n^2)]} \max_{1\leq j\leq[\psi(n^2)]} w_j}{W_{[\psi(n^2)]}} \geq \frac{\phi(\psi(n^2-1))}{W_{[\psi(n^2)]}} \geq \frac{(n^2-1)}{n+1} = n-1,$$

which proves the first property. Now for $[\psi(n^2)] \leq m < [\psi((n+1)^2)]$ we have $W_m < n+2$ and $\phi_m \geq n^2 - 1$, so $\phi_m/W_m \geq n-2 \to \infty$. □

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THEOREM 4.6. *Let $\{w_k\}$ be positive weights such that*

$$\lim_{n\to\infty} \frac{\log n \max_{1\le j\le n} w_j}{W_n} = 0. \tag{4.3}$$

Then $A_n(\vec{\xi}) \to 0$ a.s. for centered iid with $\mathbb{E}(|\xi_1|^2) < \infty$ such that for some $t_0 > 0$ we have $\mathbb{E}(e^{t\xi_1}) < \infty$ whenever $|t| < t_0$.

Proof. The result follows from Theorem 1(i) of Kiesel and Stadtmüller [**KiSt**], with $a_j(n) = w_j/W_n$ for $1 \le j \le n$ and $a_j(n) = 0$ for $j > n$. Since for $p > 1$ we have

$$\sum_{j=1}^{\infty} a_j(n)^p = \frac{1}{W_n^p}\sum_{k=1}^{n} w_k^p \le \frac{1}{W_n}\sum_{k=1}^{n} w_k\left(\frac{\max_{1\le j\le n} w_j}{W_n}\right)^{p-1} = \left(\frac{\max_{1\le j\le n} w_j}{W_n}\right)^{p-1},$$

conditions (2.2) and (2.3) of [**KiSt**] are satisfied by the sequence

$$\varphi(n) := \frac{W_n}{\max_{1\le j\le n} w_j} \ge 1,$$

with $r_p = a_p = 1$ and $\delta(n) = 0$ for the 'error term' $0 \le R(p,n) \le 1$. Condition (4.3) yields $\varphi(n) \to \infty$ and the possibility of applying [**KiSt**] (when $\varphi(n)$ is not monotone, we have to rearrange appropriately the rows of the matrix $[a_j(n)]$). □

Remarks. (1) A weaker form of (4.3), namely

$$\sup_n \frac{\log n \max_{1\le j\le n} w_j}{W_n} < \infty,$$

is not sufficient for a.s. convergence of $A_n(\vec{\xi})$ even for bounded iid (Maruyama's example in [**Ts**]).

(2) Bounded weights with divergent series may fail (4.3), by Proposition 4.5.

When $\{\xi_k\}$ are centered iid with $\mathrm{Var}(\xi_1) = \mathbb{E}(\xi_1^2) = 1$, the variance of $\sum_{k=1}^n w_k\xi_k$ is $M_n := \sum_{k=1}^n w_k^2$. If $\sum_{k=1}^\infty w_k^2 < \infty$, then $\sum_{k=1}^\infty w_k\xi_k$ converges a.s., so

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k \to 0$$

a.s. when $W_n \to \infty$. If $\sum_{k=1}^\infty w_k^2 = \infty$, an estimate of the size of $|\sum_{k=1}^n w_k\xi_k|$ will allow us to obtain conditions for the almost sure convergence of the weighted averages.

THEOREM 4.7. *Let $\{w_k\}$ be weights with $M_n := \sum_{k=1}^n w_k^2 \longrightarrow \infty$ and*

$$\sup_{n>0} \frac{n w_n^2}{M_n} < \infty. \tag{4.4}$$

Then

$$\lim_n \frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k = 0$$

a.s. for every $\{\xi_k\}$ centered iid with finite variance.

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Proof. We may assume $\mathbb{E}(\xi_k^2) = 1$. When (4.4) holds, Chow and Teicher [**ChT**] proved the law of iterated logarithm (LIL)

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n w_k \xi_k}{(2M_n \log\log M_n)^{1/2}} = 1 \quad \text{a.s.} \tag{4.5}$$

A different proof was given by Fisher [**F**]. To obtain the asserted weighted strong law of large numbers (SLLN), we prove

$$\lim_{n\to\infty} \frac{(M_n \log\log M_n)^{1/2}}{W_n} = 0. \tag{4.6}$$

Given (4.5), the sufficiency of (4.6) for the weighted SLLN is obvious, while a necessary condition is

$$\liminf_{n\to\infty} \frac{(M_n \log\log M_n)^{1/2}}{W_n} = 0. \tag{4.7}$$

Note that every bounded weight sequence $\{w_k\}$ with $\lim_n M_n = \infty$ satisfies (4.6): if $\sup_k w_k = c$, then $M_n = \sum_{k=1}^n w_k^2 \le c \sum_{k=1}^n w_k = cW_n$, implying

$$\sqrt{M_n \log\log M_n}/W_n \le c\sqrt{\log\log M_n}/\sqrt{M_n} \to 0.$$

Assume now that $\{w_k\}$ is unbounded. Teicher [**T**, pp. 721–722] proved that (4.4) implies

$$\lim_n \frac{w_n^2 \log\log M_n}{M_n} = 0. \tag{4.8}$$

Denote $w_n^* = \max_{1\le j\le n} w_j$ and let j_n be the first j with $w_n^* = w_j$; then $1 \le j_n \le n$, and by unboundedness of the weights $j_n \nearrow \infty$ so $M_{j_n} \to \infty$. Since $M_n \le w_n^* W_n$, we have

$$\frac{M_n \log\log M_n}{W_n^2} \le \frac{(w_n^*)^2 \log\log M_n}{M_n} \le \frac{w_{j_n}^2 \log\log M_{j_n}}{M_{j_n}},$$

which tends to 0 by (4.8), since $j_n \to \infty$. □

Remarks. (1) Klesov [**Kl**, Corollary 3] proved the above LIL assuming $M_n \to \infty$ and

$$\sup_{n>0} \frac{n w_n^2}{M_n \log\log M_n} < \infty. \tag{4.9}$$

The conclusion of Theorem 4.7 holds if in addition $\limsup(w_n^2/M_n) < 1$, since then $M_n \le c(1+\epsilon)^n$ by [**T**], which with the help of (4.9) yields (4.8).

(2) If (4.8) holds, then $(1/W_n)\sum_{k=1}^n w_k \xi_k \to 0$ a.s. for every $\{\xi_k\}$ centered *bounded* iid, since application of Kolmogorov's LIL to $\{w_k \xi_k\}$ yields (4.5), and (4.8) implies (4.6).

(3) By Proposition 4.5, there exists $\{w_k\}$ bounded with $\sum_{k=1}^\infty w_k^2 = \infty$ such that $\limsup_n (n^{1/4} w_n / W_n) = \infty$. Since $M_n \le W_n^2$, and by boundedness $M_n \le cn$, we have

$$\limsup_{n\to\infty} \frac{n w_n^2}{M_n \log\log M_n} \ge \limsup_{n\to\infty} \frac{n w_n^2}{W_n^2 (\log n)^{1/2}} \ge \limsup_{n\to\infty} \left(\frac{n^{1/4} w_n}{W_n}\right)^2 = \infty.$$

Hence there exists a *bounded* weight sequence for which we cannot use Klesov's LIL, and Theorem 4.7 does not apply; however, Theorem 4.2 does apply.

We thus look for conditions for the weighted SLLN that express a rate in (2.2) determined only by the weights, like (4.6) or (4.8), and not by some fixed function of n.

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THEOREM 4.8. *Let $\{\zeta_k\}$ be independent $\mathcal{N}(0,1)$ distributed random variables. For $\{w_k\}$ with $\sum_{k=1}^{\infty} w_k^2 = \infty$ satisfying (4.6) we have $\lim_n (1/W_n) \sum_{k=1}^n w_k \zeta_k = 0$ a.s.*

Proof. By Hartman's LIL for normal independent random variables [**H**], there exists a number $\alpha \in [0,1]$ such that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n w_k \zeta_k}{(2M_n \log\log M_n)^{1/2}} = \sqrt{\alpha} \quad \text{a.s.} \tag{4.10}$$

The result follows from combining (4.6) and (4.10). □

Remarks. (1) Hartman [**H**] proved that when $M_n \to \infty$ and $\limsup_n (w_n^2/M_n) < 1$, we have $\alpha = 1$ in his LIL. But for any $\{w_k\}$ for which in (4.10) we have $\alpha > 0$, condition (4.7) is necessary for the weighted SLLN to hold (as remarked earlier).

(2) In the example $w_k = 2^k$ we have $w_n/W_n \to \frac{1}{2}$ and $w_n^2/M_n \to \frac{3}{4}$, which shows that $\alpha = 1$ in (4.10) may hold even when (2.2) and (4.7) fail.

(3) If

$$\lim_{n\to\infty} \frac{w_n \log\log W_n}{W_n} = 0$$

(Azuma's condition for the weighted SLLN for bounded iid) then (4.6) holds, since

$$\lim_{n\to\infty} \frac{M_n \log\log W_n}{W_n^2} = 0$$

by [**SZ**] and $M_n \le W_n^2$, and we have the weighted SLLN also for Gaussian iid.

PROPOSITION 4.9. *There exists a sequence of weights $\{w_k\}$ with $\sum_{k=1}^{\infty} w_k^2 = \infty$, satisfying assumption (2.2), such that*

$$\mathbb{P}\left\{\lim_{n\to\infty} \frac{1}{W_n} \sum_{k=1}^n w_k \zeta_k = 0\right\} = 0.$$

Proof. Let $\{w_k\}$ be the sequence of the example in §3. It is shown there that $\sum_{k=1}^{\infty} w_k^2 = \infty$, (2.2) holds, and $\sqrt{M_n}/W_n \ge C(\log n)^{-1/4}$ for large n. Moreover, we have $w_n^2/M_n \to 0$ and $\log\log M_n \sim c \log n$. Hence $\lim_n \sqrt{M_n \log\log M_n}/W_n = \infty$. If the claim of the proposition fails, then, by the $0-1$ law, $\mathbb{P}\{\lim_{n\to\infty}(1/W_n)\sum_{k=1}^n w_k\zeta_k = 0\} = 1$, which by Remark (1) above implies $\liminf_n \sqrt{M_n \log\log M_n}/W_n = 0$ – a contradiction. □

THEOREM 4.10. *Let $1 < p \le 2$, and let $\{w_k\}$ be a weight sequence with $M_n^{(p)} := \sum_{k=1}^n w_k^p \to \infty$, such that for some $\alpha > 1$ we have*

$$\limsup_{n\to\infty} \frac{[M_n^{(p)} \log M_n^{(p)} (\log\log M_n^{(p)})^{\alpha}]^{1/p}}{W_n} < \infty. \tag{4.11}$$

Then for every centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$ we have

$$\frac{1}{W_n} \sum_{k=1}^n w_k \xi_k \to 0 \quad \text{a.s.}$$

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Proof. We apply the lemma stated in the proof of Proposition 3.3 to the sequence $\{w_k^p\}$, with $\Phi(t) = t \log t (\log\log t)^\alpha$, and obtain that

$$\sum_{k=k_0}^{\infty} \frac{w_k^p}{M_k^{(p)} \log M_k^{(p)} (\log\log M_k^{(p)})^\alpha} < \infty.$$

The Marcinkiewicz–Zygmund theorem yields almost sure convergence of the series

$$\sum_{k=k_0}^{\infty} \frac{w_k \xi_k}{[M_k^{(p)} \log M_k^{(p)} (\log\log M_k^{(p)})^\alpha]^{1/p}}$$

for $\{\xi_k\}$ as in the statement. Kronecker's lemma and (4.11) finish the proof. □

For $p = 2$ we obtain, with $M_n^{(2)} = M_n$, that the condition

$$\limsup_{n\to\infty} \frac{\sqrt{M_n}[\log M_n (\log\log M_n)^\alpha]^{1/2}}{W_n} < \infty \quad \text{for some } \alpha > 1 \tag{4.12}$$

implies that $(1/W_n)\sum_{k=1}^n w_k \xi_k \to 0$ a.s. for every centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^2) < \infty$. When (4.12) is strengthened to $\sup_n (M_n/W_n) < \infty$, this follows from [**Dr**, Lemma 4.1]. Condition (4.12) clearly holds if

$$\sup_{n>0} \frac{\sqrt{M_n}(\log M_n)^\gamma}{W_n} < \infty$$

for some $\gamma > \frac{1}{2}$.

COROLLARY 4.11. *Let $\{w_k\}$ be bounded with $\sum_{k=1}^\infty w_k = \infty$. Then for every centered independent $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$ for some $p > 1$ we have*

$$\frac{1}{W_n}\sum_{k=1}^n w_k \xi_k \to 0 \quad \textit{a.s.}$$

Proof. Fix $1 < p \le 2$. If $\sum_{k=1}^\infty w_k^p < \infty$, we apply Proposition 4.3(i) and use $W_n \to \infty$. If this series diverges, we apply the previous theorem, since $M_n^{(p)} \le C W_n$ yields (4.11). □

Remarks. (1) The previous corollary is a partial generalization of Theorem 4.2.

(2) By Proposition 4.3(iii), $\sum_{k\ge 1}(w_k/W_k)^p < \infty$ for every bounded $\{w_k\}$ and $p > 1$.

THEOREM 4.12. *Let $\{w_k\}$ be a weight sequence with $M_n := \sum_{k=1}^n w_k^2 \to \infty$, such that for some $0 < \gamma \le \frac{1}{2}$ and $\alpha > 1$ we have*

$$\limsup_{n\to\infty} \frac{\sqrt{M_n}[\log M_n (\log\log M_n)^\alpha]^\gamma}{W_n} < \infty. \tag{4.13}$$

Then for any centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^{1/\gamma}) < \infty$ we have

$$\frac{1}{W_n}\sum_{k=1}^n w_k \xi_k \to 0 \quad \textit{a.s.}$$

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Proof. For $\gamma = \frac{1}{2}$ (4.13) is just (4.12) and Theorem 4.10 applies. Let $\gamma < \frac{1}{2}$ and put $p = 1/\gamma$. Since $p > 2$, for fixed integers $i < j$ we apply Rosenthal's inequality [**Ro**] to $\{w_k\xi_k\}_{k=i}^{j}$ and obtain

$$\mathbb{E}\left(\left|\sum_{k=i}^{j} w_k\xi_k\right|^p\right) \le C_p\left(\sum_{k=i}^{j}\mathbb{E}(|w_k\xi_k|^p) + \left(\sum_{k=i}^{j}\mathbb{E}(|w_k\xi_k|^2)\right)^{p/2}\right) \tag{4.14}$$

with C_p a constant which depends only on p. Since $\sup_n \mathbb{E}(|\xi_n|^2) \le (\sup_n \mathbb{E}(|\xi_n|^p))^{2/p} < \infty$ and the ℓ_p-norm does not exceed the ℓ_2-norm, (4.14) yields

$$\mathbb{E}\left(\left|\sum_{k=i}^{j} w_k\xi_k\right|^p\right) \le C'\left(\sum_{k=i}^{j}|w_k|^2\right)^{p/2} \tag{4.15}$$

with C' a constant depending on p and $\{\xi_k\}$ (but not on i and j). The following complement to the Billingsley–Stechkin theorem is given in Theorem 6 of [**Mo**] (see Theorem 2.2 of [**W-3**] for a more general result).

LEMMA. *Let $p > 2$. If a sequence of random variables $\{f_k\}$ satisfies*

$$\mathbb{E}\left(\left|\sum_{k=i}^{j} f_k\right|^p\right) \le C\left(\sum_{k=i}^{j} u_k^2\right)^{p/2}, \quad 1 \le i < j < \infty$$

for a sequence $\{u_k\}$ of reals with $M_n := \sum_{k=1}^{n} u_k^2 \to \infty$, then for every $\alpha > 1$ we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f_k}{M_n^{1/2}(\log M_n)^{1/p}(\log\log M_n)^{\alpha/p}} = 0 \quad a.s. \tag{4.16}$$

By (4.15) we can apply the above to $f_k = w_k\xi_k$, with $u_k = w_k$, and obtain

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} w_k\xi_k}{M_n^{1/2}(\log M_n)^{1/p}(\log\log M_n)^{\alpha/p}} = 0 \quad \text{a.s.} \tag{4.17}$$

Combining (4.17) with (4.13) and using $p = 1/\gamma$ we obtain the desired convergence. □

Remarks. (1) Condition (4.13) is stronger than (4.6), but is still satisfied by any bounded $\{w_k\}$. However, for bounded weights Corollary 4.11 gives a better result.

(2) Note that (4.13) does not imply (4.4), in view of remark 2 following Theorem 4.7.

COROLLARY 4.13. *Let $\{w_k\}$ be a weight sequence with $M_n := \sum_{k=1}^{n} w_k^2 \to \infty$, such that*

$$\limsup_{n\to\infty} \frac{\sqrt{M_n}\sqrt{\log M_n}}{W_n} < \infty. \tag{4.18}$$

Then for any centered independent sequence $\{\xi_k\}$ with $\sup_k \mathbb{E}(|\xi_k|^p) < \infty$ for some $p > 2$ we have

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k\xi_k \to 0 \quad a.s.$$

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Remarks. (1) Condition (4.18) is obviously implied by (3.10) or (3.11).

(2) If $M_n \to \infty$ and $\limsup_n (\sqrt{M_n}(\log M_n)^\gamma / W_n) < \infty$ for some $\gamma > 0$, then

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k \xi_k \to 0$$

a.s. for any uniformly bounded centered independent $\{\xi_k\}$.

THEOREM 4.14. *Let $\{w_k\}$ be a weight sequence with $\sum_{k=1}^{\infty} w_k = \infty$. If*

$$\sup_{n\geq 1} \frac{1}{W_n}\sum_{k=1}^{n} w_k (\log(w_k + 1))^\beta < \infty \quad \textit{for some } \beta > 1 \tag{4.19}$$

then for any iid sequence $\{\xi_k\}$ with $\mathbb{E}(|\xi_1|(\log^+ |\xi_1|)^\gamma) < \infty$ for some $\gamma > 1$ we have

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k \xi_k \to \mathbb{E}\xi_1 \quad a.s.$$

The proof of the theorem will depend on a general method for obtaining sufficient conditions, described below.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable non-decreasing function satisfying:

$$\begin{aligned}
&\text{(i)} \quad 0 \leq \varphi'(u) \leq C\varphi(u)/u,\\
&\text{(ii)} \quad u^{-1}\varphi(u) \text{ is non-decreasing for } u \geq u_0,\\
&\text{(iii)} \quad \varphi(uv) \leq C\varphi(u)\varphi(v), \qquad u \geq 1, v \geq v_0,\\
&\text{(iv)} \quad \int_t^\infty du/\varphi(u) < \infty \quad \text{for } t > 0.
\end{aligned} \tag{4.20}$$

Note that (iv) and (ii) imply that $\lim_{u\to\infty} u^{-1}\varphi(u) = \infty$. Typical examples are the functions $\varphi(u) = u^\alpha(\log(1+u))^\beta$, $\alpha \geq 1$, $\beta \in \mathbb{R}^+$, with $\beta > 1$ when $\alpha = 1$.

It can be shown that (ii) implies $\varphi(u+v) \geq \varphi(u) + \varphi(v)$. Thereby,

$$\varphi\left(\sum_{k=1}^{n} u_k\right) \geq \sum_{k=1}^{n} \varphi(u_k). \tag{4.21}$$

Let $\{w_k\}$ be a weight sequence with $W_n \to \infty$; put $T_0 = 0$ and $T_n = \sum_{k=1}^{n} \varphi(w_k)$ for $n \geq 1$. Property (4.21) implies that $T_n \leq \varphi(W_n)$ for all $n \geq 1$. In the example of $\varphi(t) = t^\alpha$ with $\alpha > 1$ this means $\sum_{k=1}^{n} w_k^\alpha \leq W_n^\alpha$, which is weaker than the necessary condition (2.2) (see the remarks following Theorem 2.4 when $\alpha \neq 2$), and thus yields no information for the weighted SLLN. We therefore introduce the following assumption:

$$\sup_n \frac{\sum_{k=1}^{n} \varphi(w_k)}{W_n} < \infty. \tag{4.22}$$

Then (4.22) implies $w_n/W_n \to 0$. Denote $\kappa = \sup_n (T_n/W_n)$. Fix $\epsilon > 0$: if $\varphi(w_n)/w_n > \kappa/\epsilon$ then

$$\frac{w_n}{W_n} = \frac{\varphi(w_n)}{W_n} \cdot \frac{w_n}{\varphi(w_n)} \leq \kappa \frac{w_n}{\varphi(w_n)} < \epsilon;$$

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if $\varphi(w_n)/w_n \le \kappa/\epsilon$ then convergence to infinity in (ii) implies $w_n \le A_\epsilon$, so $w_n/W_n \le A_\epsilon/W_n$, which is less than ϵ for large n.

Since φ is non-decreasing and $\varphi(w_n) = T_n - T_{n-1}$, for any integer $j > 0$ (iii) yields

$$\begin{aligned}\#\{n : W_n \le jw_n\} &\le j + \#\{n > j : W_n \le jw_n\} \le j + \sum_{n>j} \frac{\varphi(jw_n)}{\varphi(W_n)} \\ &\le j + C\varphi(j) \sum_{n\ge j} \frac{T_n - T_{n-1}}{\varphi(W_n)} \\ &= j + C\varphi(j)\left\{\frac{-T_{j-1}}{\varphi(W_j)} + \sum_{n=j}^{\infty} T_n\left(\frac{1}{\varphi(W_n)} - \frac{1}{\varphi(W_{n+1})}\right)\right\}.\end{aligned}$$

But by the mean value theorem and the assumptions made on φ, we have

$$\frac{1}{\varphi(W_n)} - \frac{1}{\varphi(W_{n+1})} = \frac{\varphi(W_{n+1}) - \varphi(W_n)}{\varphi(W_n)\varphi(W_{n+1})} \le C\frac{w_{n+1}}{W_{n+1}\varphi(W_n)}.$$

Reporting this and using $T_n \le T_{n+1}$ we get

$$\begin{aligned}\#\{n : W_n \le jw_n\} &\le j + C\varphi(j)\left\{\frac{-T_{j-1}}{\varphi(W_j)} + C\sum_{n=j}^{\infty} \frac{T_{n+1}}{W_{n+1}} \cdot \frac{w_{n+1}}{\varphi(W_n)}\right\} \\ &\le j + C^2\varphi(j)\left(\sup_{k\ge j} \frac{T_k}{W_k}\right) \sum_{n=j}^{\infty} \frac{w_{n+1}}{\varphi(W_n)} \le j + C^2\varphi(j)\kappa \sum_{n=j}^{\infty} \frac{w_{n+1}}{\varphi(W_n)}.\end{aligned}$$

(Recall that by (4.22) $\kappa = \sup_{k\ge1}(T_k/W_k)$ is finite.)

Now

$$\frac{w_{n+1}}{\varphi(W_n)} \le \int_{W_n}^{W_{n+1}} \frac{du}{\varphi(u)}$$

yields

$$\#\{n : W_n \le jw_n\} \le j + C^2\kappa\varphi(j) \int_{W_j}^{\infty} \frac{du}{\varphi(u)}, \tag{4.23}$$

which is finite by (4.20)(iv).

We now extend the definition of W_n to $\mathbb{R}^+$ by putting $W_0 = 0$ and $W(t) = W_{[t]}$ where $[t]$ stands for the integer part of t. For $t \in \mathbb{R}^+$ and $s \in \mathbb{R}$ define

$$G(t) = 0 \quad \text{for} \quad 0 \le t < 1; \qquad G(t) := \max\left(t, \varphi(t) \int_{W(t)}^{\infty} \frac{du}{\varphi(u)}\right) \quad \text{for} \quad t \ge 1; \tag{4.24}$$

$$H(s) = s^2 \int_{t\ge|s|} \frac{G(t)}{t^3}\, dt.$$

$G(t)$ is always finite, and (4.23) leads to

$$\sup_{j\ge1} \frac{\#\{n : W_n \le jw_n\}}{G(j)} < \infty. \tag{4.25}$$

The function H need not be finite (e.g., $\varphi(u) = u^2$ and $w_n = 1/n$). When it is, we obtain the following result.

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THEOREM 4.15. *Let φ satisfy (4.20), and let $\{w_k\}$ with divergent series satisfy (4.22). Then for any iid sequence $\vec{\xi}$ with $\mathbb{E}\,H(\xi_1) < \infty$ we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n w_k \xi_k}{\sum_{k=1}^n w_k} \overset{\text{a.s.}}{=} \mathbb{E}\xi_1.$$

Proof. The proof is built upon (4.25) and an argument that lies in the proof of Theorem 3 of [**JOP**]. For any positive real t put $N(t) := \#\{n : W_n \le t w_n\}$. For $t < 1$ we have $N(t) = 0$. For $t \ge 1$ we have $\varphi(t+1) \le \varphi(2t) \le C\varphi(2)\varphi(t)$, and $[t]+1 \le t+1$. In view of (4.25) and the definition of G, we have $\#\{n : W_n \le t w_n\} \le K'G([t]+1) \le K\,G(t)$. Thus

$$\mathbb{E}\left\{\xi^2 \int_{t\ge|\xi|} \frac{\#\{n : W_n \le t w_n\}}{t^3}\,dt\right\} \le K\mathbb{E}\left\{\xi^2 \int_{t\ge|\xi|} \frac{G(t)}{t^3}\,dt\right\} = K\mathbb{E}H(\xi) < \infty,$$

by assumption. It follows that condition (3.1) of Theorem 2 of [**JOP**] is satisfied; by this theorem, as well as the remark at the bottom of p. 41 of [**JOP**], the result obtains. □

Proof of Theorem 4.14. Since in (4.19) we can always replace β by a smaller value (still greater than 1), and also γ can be replaced by any smaller value greater than 1, we may always assume $\gamma = \beta$ and $\beta \le 2$. Put $\varphi(u) := u(\log(1+u))^\beta$. Then the assumptions on φ and $\{w_k\}$ of Theorem 4.15 are satisfied. Since $W(t) \to \infty$, we have $W(t) \ge e$ for $t \ge t_1$, and

$$\int_{W(t)}^\infty \frac{du}{\varphi(u)} \le \int_{W(t)}^\infty \frac{du}{u(\log u)^\beta} = \frac{1}{(\beta-1)(\log W(t))^{\beta-1}} \tag{4.26}$$

yields that $G(t)$ as defined in (4.24) satisfies

$$G(t) \le \frac{1}{\beta-1} t(\log(1+t))^\beta$$

for large t. For $|s|$ large enough we obtain

$$\begin{aligned}\int_{|s|}^\infty \frac{G(t)}{t^3}\,dt &\le c\int_{|s|}^\infty \frac{(\log(1+t))^\beta}{t^2}\,dt \le c'\int_{|s|}^\infty \frac{(\log t)^\beta}{t^2}\,dt \\ &\le c'\int_{|s|}^\infty 2\left(1-\frac{\beta}{\log t}\right)\frac{(\log t)^\beta}{t^2}\,dt = 2c'\frac{(\log|s|)^\beta}{|s|}.\end{aligned}$$

We can now conclude that

$$H(s) = s^2\int_{|s|}^\infty \frac{G(t)}{t^3}\,dt \le C|s|(1+(\log^+|s|)^\beta).$$

Hence $\mathbb{E}H(\xi_1) < \infty$ when $\mathbb{E}(|\xi_1|(\log^+|\xi_1|)^\beta) < \infty$. Now Theorem 4.15 yields the assertion, since $\gamma = \beta$. □

Remarks. (1) If we want to apply Theorem 4.15 to $\varphi(t) = t^\alpha$ with $\alpha > 1$, condition (4.22) becomes $\{M_n^{(\alpha)}/W_n\}$ bounded, which implies (4.19); however, without any knowledge of the size of W_n we cannot get better estimates for H with this φ, and Theorem 4.15 in this case will not improve the result of Theorem 4.14.

(2) Condition (4.19) is satisfied by bounded weights. However, Theorem 4.14 does not contain Theorem 4.2, because the latter requires a weaker integrability property of ξ_1.

We now add a condition on the weights in order to obtain the weighted SLLN under a weaker integrability condition on the iid sequence.

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THEOREM 4.16. *Let $\{w_k\}$ be a weight sequence with $\sum_{k=1}^{\infty} w_k = \infty$. If $\{w_k\}$ satisfies (4.19) and also $\inf_n (1/n) W_n > 0$, then for any iid sequence $\{\xi_k\}$ with $\mathbb{E}(|\xi_1| \log^+ |\xi_1|) < \infty$ we have*

$$\frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k \to \mathbb{E}\,\xi_1 \quad a.s.$$

Proof. Let φ be as in the proof of Theorem 4.14. By the additional assumption on W_n,

$$\inf_{t \geq 1} \frac{W(t)}{t+1} \geq \frac{1}{3} \inf_{n \geq 1} \frac{W_n}{n} > 0,$$

so by (4.26) we have

$$\int_{W(t)}^{\infty} \frac{du}{\varphi(u)} \leq C(\log(1+t))^{1-\beta}.$$

Hence for large t we have $G(t) \leq c \log(1+t)$, so for large $|s|$ we obtain

$$\int_{|s|}^{\infty} \frac{G(t)}{t^3}\, dt \leq c \int_{|s|}^{\infty} \frac{\log(1+t)}{t^2}\, dt \leq c' \int_{|s|}^{\infty} \frac{(\log t)}{t^2}\, dt = c' \frac{\log|s|+1}{|s|} \leq c'' \frac{\log|s|}{|s|}.$$

We can now conclude that

$$H(s) = s^2 \int_{|s|}^{\infty} \frac{G(t)}{t^3}\, dt \leq C|s|(1 + \log^+ |s|).$$

Hence $\mathbb{E}H(\xi_1) < \infty$ when $\mathbb{E}(|\xi_1| \log^+ |\xi_1|) < \infty$. Now Theorem 4.15 applies. □

THEOREM 4.17. *Let $\{w_k\}$ be a weight sequence with $\sum_{k=1}^{\infty} w_k = \infty$. If $\{w_k\}$ satisfies*

$$\sup_n \frac{1}{W_n} \sum_{k=1}^{n} w_k^{\alpha} < \infty \quad \text{for some } \alpha > 1 \tag{4.27}$$

and also $\inf_n (1/n) W_n > 0$, then for any iid sequence $\{\xi_k\}$ with $\mathbb{E}(|\xi_1|) < \infty$ we have

$$\frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k \to \mathbb{E}\,\xi_1 \quad a.s.$$

Proof. We take $\varphi(t) = t^{\alpha}$, which satisfies (4.20). Assumption (4.27) is condition (4.22) for our φ. It is easy to show that $G(t) \leq c t^{\alpha} W(t)^{1-\alpha}$ for large t. Since $\inf_{t \geq 1} W(t)/t \geq \frac{1}{2} \inf_n W_n/n > 0$, we obtain $G(t) \leq c't$; computations similar to the previous ones yield that $H(s) \leq C|s|$. Hence $\mathbb{E}H(\xi_1) < \infty$ when $\mathbb{E}(|\xi_1|) < \infty$, and Theorem 4.15 applies. □

Remarks. (1) The assumed linear growth of W_n allows for more precise estimates, which result in the weighted SLLN for iid with only the first moment when (4.19) is strengthened to (4.27).

(2) If $\{w_k\}$ satisfies the two conditions of remark 2 following (4.1), then Theorem 4.17 applies, and there is no need to use the modulated SLLN of [**BJLO**]. The assumptions of Theorem 4.17 are formally weaker.

(3) There are many unbounded sequences that satisfy the hypotheses of Theorems 4.16 and 4.17. For example, strictly stationary ergodic random weights with finite moment $\alpha > 1$ satisfy the hypotheses of Theorem 4.17; this also provides another proof of Corollary 3.5 of [**BJLO**]. On the other hand, if the stationary sequence is only in $L(\log^+ L)^{\beta}$ then almost

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surely the random weights satisfy (4.19), but not (4.27); nevertheless, the weighted SLLN for iid with only finite first moment still holds, by [**BJLO**, Theorem 4.1] (or [**DeQ**]).

(4) The stationary random weights above satisfy (4.19) but not (3.2), while the weights $w_k := k^t$ $(t > 0)$ satisfy (3.2) and not (4.19).

The method leading to Theorem 4.15 can be generalized. We now assume only that φ satisfies (4.20)(i)–(iii), and instead of assuming (4.20)(iv) we take another positive non-decreasing function φ_1 with

$$\int_t^\infty \frac{du}{\varphi_1(u)} < \infty \quad \text{for } t > 0. \tag{4.28}$$

For $\{w_k\}$ with divergent series we assume the following (which is (4.22) when $\varphi_1 = \varphi$):

$$\kappa := \sup_n \frac{\varphi_1(W_n)}{\varphi(W_n)} \frac{\sum_{k=1}^n \varphi(w_k)}{W_n} < \infty. \tag{4.29}$$

Adapting the two inequalities preceding (4.23), we get

$$\#\{n : W_n \le j w_n\} \le j + C^2 \kappa \varphi(j) \int_{W_j}^\infty \frac{du}{\varphi_1(u)}.$$

THEOREM 4.18. *Let φ satisfy (4.20)(i)–(iii) and φ_1 non-decreasing with (4.28). Let $\{w_k\}$ with divergent series satisfy (4.29). For $t > 0$ define*

$$G(t) = 0 \quad \text{for } 0 \le t < 1; \qquad G(t) := \max\left(t, \varphi(t) \int_{W(t)}^\infty \frac{du}{\varphi_1(u)}\right) \quad \text{for } t \ge 1.$$

If

$$H(s) := s^2 \int_{t \ge |s|} \frac{G(t)}{t^3}\, dt \quad (s \in \mathbb{R})$$

is finite, then for any iid sequence $\vec{\xi}$ with $\mathbb{E}H(\xi_1) < \infty$ we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n w_k \xi_k}{\sum_{k=1}^n w_k} \overset{\text{a.s.}}{=} \mathbb{E}\xi_1.$$

We use Theorem 4.18 for studying some *weighted modulation*. Fix $\alpha > 1$, and let $\{c_k\}$ be a sequence of positive numbers with

$$\sum_{k=1}^\infty c_k^\alpha = \infty. \tag{4.30}$$

Hence also $\sum_k c_k = \infty$. Since $C_n := \sum_{k=1}^n c_k$ and $C_n^{(\alpha)} := \sum_{k=1}^n c_k^\alpha$ are strictly increasing, there exist strictly increasing continuous functions ψ and ψ_α with $\psi(0) = 0$, $\psi_\alpha(0) = 0$, $\psi(n) = C_n$, and $\psi_\alpha(n) = C_n^{(\alpha)}$ (for example, use linear interpolation).

Let $\{b_k\}$ be a sequence of positive numbers satisfying the following:

$$\ell := \lim_{n\to\infty} \frac{\sum_{k=1}^n c_k b_k}{\sum_{k=1}^n c_k} > 0 \quad \text{exists}; \qquad \sup_{n\ge1} \frac{\sum_{k=1}^n c_k^\alpha b_k^\alpha}{\sum_{k=1}^n c_k^\alpha} < \infty. \tag{4.31}$$

As an example of such a situation, let $\{c_k\}$ with $\sum_{k=1}^\infty c_k^2 = \infty$ satisfy $\sup_n nc_n^2 / \sum_{k=1}^n c_k^2 < \infty$. Then $\sup_n \sqrt{n} c_n / \sum_{k=1}^n c_k$ is finite, and Theorem 4.4 applies to $\{c_k\}$ and to $\{c_k^2\}$. Hence for positive iid random variables $\{f_k\}$ with finite third moment, almost surely the realizations $b_k = f_k(x)$ satisfy (4.31).

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THEOREM 4.19. *Let $\alpha > 1$, and let $\{c_k\}$ be a sequence of positive numbers with $\sum_{k=1}^{\infty} c_k^{\alpha} = \infty$. Put $\varphi(t) = t^{\alpha}$ and $\varphi_1(t) := t^{1+\alpha}/\psi_\alpha \circ \psi^{-1}(t)$. Assume φ_1 is non-decreasing and satisfies (4.28), and define*

$$\tilde{G}(t) = 0 \quad \text{for } 0 \le t < 1; \qquad \tilde{G}(t) := \max\left(t, \varphi(t) \int_{C_{[t]}}^{\infty} \frac{du}{\varphi_1(u)}\right) \quad \text{for } t \ge 1.$$

If

$$\tilde{H}(s) := s^2 \int_{t \ge |s|} \frac{\tilde{G}(t)}{t^3}\, dt \quad (s \in \mathbb{R})$$

is finite, then for any positive sequence $\{b_k\}$ satisfying (4.31) and any iid sequence $\{\xi_k\}$ with $\mathbb{E}\tilde{H}(\xi_1) < \infty$ we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} c_k b_k \xi_k}{\sum_{k=1}^{n} c_k} \overset{\text{a.s.}}{=} \ell \cdot \mathbb{E}\xi_1.$$

Proof. We will apply Theorem 4.18 to the sequence $w_k = c_k b_k$. Clearly φ and φ_1 (which depend only on $\{c_k\}$) satisfy (4.20) and (4.28) respectively. We now show that $w_k = c_k b_k$ satisfies (4.29). Since $\ell > 0$, we have $\sum_{k=1}^{\infty} w_k = \infty$. Clearly we may assume $\ell > 1$. Hence for n large enough $W_n = \sum_{k=1}^{n} c_k b_k \ge \sum_{k=1}^{n} c_k = C_n$, and $\psi^{-1}(W_n) \ge \psi^{-1}(C_n) = n$. By the definitions, for large n we obtain

$$\varphi_1(W_n) := \frac{W_n^{1+\alpha}}{\psi_\alpha \circ \psi^{-1}(W_n)} \le \frac{W_n^{1+\alpha}}{\psi_\alpha(n)} = \frac{W_n^{1+\alpha}}{C_n^{(\alpha)}}.$$

Hence

$$\frac{\varphi_1(W_n)}{\varphi(W_n)} \frac{\sum_{k=1}^{n} \varphi(w_k)}{W_n} \le \frac{W_n^{1+\alpha}}{C_n^{(\alpha)} W_n^{\alpha}} \cdot \frac{\sum_{k=1}^{n} c_k^{\alpha} b_k^{\alpha}}{W_n} = \frac{\sum_{k=1}^{n} c_k^{\alpha} b_k^{\alpha}}{C_n^{(\alpha)}}$$

for large n, and (4.31) yields (4.29). Finally, for G and H defined for $\{w_k\}$ in Theorem 4.18, $W_n \ge C_n$ implies $G(t) \le \tilde{G}(t)$, and consequently $H(t) \le \tilde{H}(t)$. Hence, by Theorem 4.18 and (4.31),

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} c_k b_k \xi_k}{\sum_{k=1}^{n} c_k} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} w_k \xi_k}{\sum_{k=1}^{n} w_k} \cdot \frac{\sum_{k=1}^{n} c_k b_k}{\sum_{k=1}^{n} c_k} \overset{\text{a.s.}}{=} \ell \cdot \mathbb{E}\xi_1. \qquad \square$$

Remarks. (1) Since the functions $\tilde{G}$ and $\tilde{H}$ depend only on the weights $\{c_k\}$ and not on the 'modulators' $\{b_k\}$, the class of iid to which the result applies is the same for all positive modulators which satisfy (4.31).

(2) The theorem can be proved also for strictly monotone continuous ψ and ψ_α satisfying only $\psi(n)/C_n \to 1$ and $\psi_\alpha(n)/C_n^{(\alpha)} \to 1$. We omit the technical details.

THEOREM 4.20. *Let $\beta \ge 0$, and let $\alpha > 1$. Then for any ergodic probability-preserving transformation τ on (Ω, μ) and $0 < f \in L_\alpha(\mu)$ there exists Ω_1 with $\mu(\Omega_1) = 1$ such that, if $x \in \Omega_1$, then for any iid sequence with $\mathbb{E}|\xi_1| < \infty$ we have*

$$\lim_{n\to\infty} \frac{1}{n^{\beta+1}} \sum_{k=1}^{n} k^{\beta} f(\tau^k x) \xi_k \overset{\text{a.s.}}{=} \frac{\int f\, d\mu}{\beta + 1}\, \mathbb{E}\xi_1. \tag{4.32}$$

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Proof. Put $c_k = k^\beta$. Asymptotically

$$\sum_{k=1}^{n} c_k \sim \frac{1}{\beta+1} n^{\beta+1},$$

so we can take

$$\psi(t) = \frac{1}{\beta+1} t^{\beta+1} \quad \text{and} \quad \psi_\alpha(t) = \frac{1}{\alpha\beta+1} t^{\alpha\beta+1},$$

so $\varphi_1(u) = \gamma u^{(\alpha+\beta)/(\beta+1)}$ satisfies (4.28), and simple computations yield that for large t we have $\tilde{G}(t) \le Ct$, so $\tilde{H}(s) \le C's$. Thus $\mathbb{E}\,\tilde{H}(\xi_1)$ is finite if $\mathbb{E}(|\xi_1|) < \infty$.

Since $\{c_k\}$ and $\{c_k^\alpha\}$ both satisfy (3.2), for non-zero f Corollary 3.3 yields that for a.e. $x \in \Omega$ the sequence $b_k = f(\tau^k x)$ satisfies (4.31) with $\ell = \int f\,d\mu$. The result now follows from the previous theorem. □

Remarks. (1) The case $\beta = 0$ is Assani's result [**As-1**].

(2) If we take $f \in L_1(\mu)$, then for $x \in \Omega_1$ (4.32) holds for $\{\xi_k\}$ iid with $\mathbb{E}(|\xi_1|^p) < \infty$ for some $p > 1$: denoting $\mathbb{E}\xi_1 = a$, we obtain that for $x \in \Omega_1$ the series

$$\sum_{k=1}^{\infty} \frac{k^\beta f(\tau^k x)(\xi_k - a)}{k^{\beta+1}} = \sum_{k=1}^{\infty} \frac{f(\tau^k x)(\xi_k - a)}{k}$$

converges a.s. by [**As-4**] (see also [**BJLO**, Theorem 3.7]); (4.32) follows from Kronecker's lemma and application of Theorem 3.1 to $f_k = f \circ \tau^k$ with $w_k = k^\beta$ (using Corollary 3.2). However, Theorem 4.20 applies also when ξ_1 has no finite moments of order higher than 1 (provided f has).

(3) If in the proof of Theorem 4.20 we take $\alpha = 2$ and put $c_k = \log(k+1)$, then for $f \in L_2$, $x \in \Omega_1$ and any $\{\xi_k\}$ iid with $\mathbb{E}(|\xi_1|) < \infty$ we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log(k+1) f(\tau^k x)\xi_k}{n \log n} \overset{\text{a.s.}}{=} \int f\,d\mu\, \mathbb{E}\,\xi_1.$$

In this case $\psi(t) = t \log^+ t$ and $\psi_2(t) = t(\log^+ t)^2$. For large t we obtain

$$\int_{\psi(t)}^{\infty} \frac{du}{\varphi_1(u)} = \int_{\psi(t)}^{\infty} \frac{u^2\,du}{\psi^{-1}(u)u^3} = \int_t^{\infty} \frac{(\log s + 1)\,ds}{s^2 \log s} \le \frac{2}{t},$$

which yields $\tilde{G}(t) \le 2t$ for large t, so $\tilde{H}(s) \le Cs$. We use as before the fact that $\{c_k\}$ and $\{c_k^2\}$ both satisfy (3.2).

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