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# On the Strong Chromatic Number

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Let $G$ be a finite graph with maximum degree at most $d$. Then, for every partition of $V(G)$ into classes of size $3d-1$, there exists a proper colouring of $G$ with $3d-1$ colours in which each class receives all $3d-1$ colours.

## 1. Introduction

Let $r$ and $n$ be positive integers such that $r$ divides $n$, and let $G$ be a graph with $n$ vertices. We call $G$ *strongly $r$-colourable* if, for every partition of the vertex set $V(G)$ into parts $V_k$ of size $r$, there exists a proper vertex colouring of $G$ with $r$ colours, with the additional property that each $V_k$ receives all $r$ colours. In other words, any union of $G$ together with a collection of disjoint $r$-cliques on the vertex set $V(G)$ has chromatic number $r$. If $r$ does not divide $n$, then we say $G$ is strongly $r$-colourable if the graph obtained by adding $r\lceil n/r\rceil - n$ isolated vertices to $G$ is strongly $r$-colourable. In [5] it is shown that if $G$ is strongly $r$-colourable then it is also strongly $(r+1)$-colourable. The *strong chromatic number* $s\chi(G)$ of a graph $G$ is defined to be the minimum $r$ such that $G$ is strongly $r$-colourable. This notion was introduced independently by Alon [1] and Fellows [5].

A well-known problem concerning strong chromatic number, the so-called 'Cycle plus triangles' conjecture, was raised by Du, Hsu and Hwang (see [11] and [5]) and popularized by Erdős. It stated that $s\chi(C_{3n}) = 3$ for any cycle $C_{3n}$ of length $3n$. This conjecture was proved by Fleischner and Stiebitz [6] using a theorem of Alon and Tarsi [3]. A different proof based on elementary methods was given by Sachs [11]. For more on the study of strong chromatic number, see Problem 4.14 in Jensen and Toft [9]. The strong chromatic number for general graphs was studied by Alon in [2], in which he proved that there

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exists a constant $c$ such that $s\chi(G) \leqslant c\Delta(G)$ for every graph $G$ (here $\Delta(G)$ denotes the maximum degree of $G$). The upper bound on $c$ that follows from his argument is very large, something like $2^{10^{10}}$. The main aim of this paper is to give a better estimate for this constant.

**Theorem 1.1.** *For every graph $G$ we have $s\chi(G) \leqslant 3\Delta(G) - 1$.*

Fleischner and Stiebitz [7] gave for each $\Delta$ an example of a $\Delta$-regular graph $G$ for which $s\chi(G) \geqslant 2\Delta$, thus showing that the constant $c$ is at least 2. It would be very interesting to determine the correct value of $c$.

Strong chromatic number is closely related to the notion of *independent transversals* in graphs. For a graph $G$ whose vertex set is partitioned into classes $V_k$, an independent transversal of the classes is a subset of the vertices of $G$, one chosen from each class $V_k$, that forms an independent set in $G$. Thus a strong $r$-colouring of $G$ with given classes $V_k$, each of size $r$, is a partition of $V(G)$ into $r$ disjoint independent transversals of the classes $V_k$. In [8] it was shown that if each class $V_k$ satisfies $|V_k| \geqslant 2\Delta(G)$ then $G$ has an independent transversal of the classes $V_k$. Moreover, this statement is best possible, as for each $\Delta$ there exist $\Delta$-regular graphs $G$ and partitions of their vertex sets into classes of size $2\Delta - 1$ for which no independent transversals exist. Such examples were given by Jin [10], Yuster [13] and Szabó and Tardos [12].

## 2. Proof of Theorem 1.1

Let a positive integer $d$ be fixed, and let $G$ be a finite graph with maximum degree $\Delta(G) \leqslant d$. By adding isolated vertices if necessary, we assume that the number of vertices of $G$ is divisible by $3d - 1$. Suppose a partition of $V(G)$ into subsets $V_k$ is given, where $|V_k| = 3d - 1$ for each $k$. We show that $G$ has a proper vertex colouring that also multicolours every vertex class $V_k$. Here we assume for convenience that the sets $V_k$ are all independent sets in $G$. (If not, then any edge lying inside some $V_k$ could simply be removed.)

We begin with some definitions. Suppose on the contrary that such a vertex colouring of $G$ does not exist. By a *maximum partial colouring* of $G$, abbreviated MPC, we mean a proper vertex colouring of some induced subgraph of $G$, such that no two vertices in the same $V_k$ receive the same colour, and such that the number of vertices of the subgraph is largest among all such partial colourings. We will denote by $V(x)$ the vertex class of $G$ containing the vertex $x$, and more generally, for a vertex subset $F$ of $G$ we write $V(F)$ for the set $\{V(x) : x \in F\}$. If an MPC colours every vertex of $G$ then we have nothing to prove, so we may assume each MPC colours only a proper subset of the vertices of $G$. For a fixed MPC $\chi$, we denote by $A_\chi(V_k)$ the set of colours in $[3d-1] = \{1, \ldots, 3d-1\}$ that are not used by $\chi$ on the vertices in $V_k$ (the set of *available* colours for $V_k$ under $\chi$). Hence for every MPC there exists a $V_k$ for which $A_\chi(V_k) \neq \emptyset$. For a vertex $x$ we denote by $\Gamma_{(\chi,c)}(x)$ the set of neighbours of $x$ that get colour $c$ under $\chi$. We use the notation $\chi(y) = 0$ to indicate that $\chi$ leaves the vertex $y$ uncoloured.

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Our approach to proving Theorem 1.1 will resemble an alternating paths type argument. Starting with an MPC $\chi$ and a vertex $y_0$ not coloured by $\chi$, we will construct in $G$ a tree-like structure $F$ with certain specific properties, called a $(\chi, y_0, c)$-*alternating set*. Unlike the standard alternating paths argument, however, our method will involve a step-by-step modification of the MPC $\chi$ as well as the alternating structure $F$. We will construct a sequence of pairs $(\chi_\ell, F_\ell)$, where $F_\ell$ is a $(\chi_\ell, y_0, c)$-alternating set, and a measure $P(F_\ell)$ of the $F_\ell$ called the *profile*, such that each $F_\ell$ has a smaller profile than the previous one. Then, because an infinite descending chain of profiles cannot exist, we will reach a contradiction to the assumption that no MPC colours all vertices of $G$.

To describe what a $(\chi_\ell, y_0, c)$-alternating set $F$ is, we first give a rough outline of how such a set is constructed. We begin with an MPC $\chi$, a vertex $y_0$ not coloured by $\chi$, and a colour $c \in A_\chi(V(y_0))$. Note then that $\Gamma_{(\chi,c)}(y_0) \neq \emptyset$. This is true because otherwise we could give $y_0$ colour $c$ and obtain a valid colouring that colours more vertices than $\chi$ does, contradicting the maximality of $\chi$. The *basic* $(\chi, y_0, c)$-alternating set associated with $\chi$, $y_0$, and $c$ is just the set $\{y_0\} \cup X_0$ where $X_0 = \Gamma_{(\chi,c)}(y_0)$. To continue the construction, we next choose a vertex $y_1$ in $\bigcup\{V(x) : x \in \{y_0\} \cup X_0\}$ such that

- $y_1$ is not adjacent to any vertex of $\{y_0\} \cup X_0$,
- $y_1$ is not 'badly coloured' in the sense made precise in condition (4) below.

It can be shown (Lemma 2.1) that such a vertex exists. We then add $y_1$ and $X_1 = \Gamma_{(\chi,c)}(y_1)$ to the set $F$.

In the general step we have constructed $\{y_0, \ldots, y_j\} \cup \bigcup_{m=0}^{j} X_m$, and we are looking for a vertex $y_{j+1}$ in $\bigcup\{V(x) : x \in \{y_0\} \cup \bigcup_{m=0}^{j} X_m\}$ that satisfies

- $y_{j+1}$ is not adjacent to any vertex of $\{y_0, \ldots, y_j\} \cup \bigcup_{m=0}^{j} X_m$,
- $y_{j+1}$ is not 'badly coloured'.

Again Lemma 2.1 below will guarantee the existence of such a vertex. To continue the construction we then add $y_{j+1}$ and $X_{j+1} = \Gamma_{(\chi,c)}(y_{j+1})$ to $F$.

We now make these notions precise. Again let $\chi$ be an MPC, let $y_0$ be a vertex not coloured by $\chi$, and let $c \in A_\chi(V(y_0))$. A $(\chi, y_0, c)$-*alternating set* is a subset of distinct vertices $F = \{y_0, \ldots, y_h\} \cup \bigcup_{i=0}^{h} X_i$ in $G$ with the following properties (see Figure 1). We consider the vertices of $F$ to be labelled, but because it is not necessary to distinguish among the vertices of a fixed $X_i$, we do not assign distinct labels inside the $X_i$.

(1) For each $0 \leqslant i \leqslant h$, $X_i = \Gamma_{(\chi,c)}(y_i) \neq \emptyset$, and $X_i \cap X_j = \emptyset$ for all $i \neq j$.
(2) For each $1 \leqslant i \leqslant h$, $y_i \in U_j$ for some $j < i$, where $U_j = \bigcup\{V(x) : x \in \{y_0\} \cup \bigcup_{m=0}^{j} X_m\}$. We write $x(y_i)$ for the element $x$ of $\{y_0\} \cup \bigcup_{m=0}^{i-1} X_m$ for which $y_i \in V(x)$.
(3) For each $1 \leqslant i \leqslant h$, $y_i$ is not adjacent in $G$ to any $y_j$ or to any element of $X_j$ with $j < i$.
(4) For each $1 \leqslant i \leqslant h$,
    - if $x(y_i) \in X_j$ for some $j$ for which $|X_j| = 1$, then $\chi(y_i) \neq \chi(z)$ for any $z \in \Gamma(x(y_i)) \setminus \{y_j\}$ that is coloured by $\chi$,
    - otherwise $\chi(y_i) \neq \chi(z)$ for any $z \in \Gamma(x(y_i))$ that is coloured by $\chi$.

Before proceeding we explicitly state some simple facts about $(\chi, y_0, c)$-alternating sets $F$ that follow from the definition.

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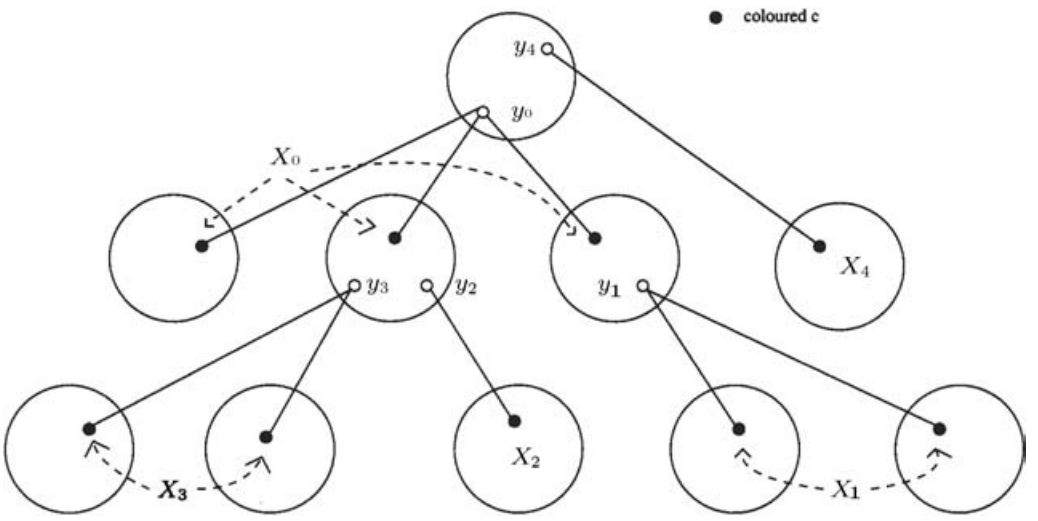



*Figure 1.* A $(\chi, y_0, c)$-alternating set

**Lemma 2.1.** *Let $F$ be a $(\chi, y_0, c)$-alternating set as defined above, and recall $V(F) = \{V(x) : x \in F\}$. Then*

(i) *all the vertices $\{y_0\} \cup \bigcup_{i=0}^{h} X_i$ lie in distinct classes $V_k$,*
(ii) $|V(F)| = 1 + \sum_{j=0}^{h} |X_j|$,
(iii) $|F| = h + 1 + \sum_{j=0}^{h} |X_j|$.

Our next claim will imply that any $(\chi, y_0, c)$-alternating set has the potential to be enlarged. We remark that this is the only step of the proof in which the assumption $|V_k| = 3d - 1$ is used. Let $\chi$ be an MPC, and let $F$ be a $(\chi, y_0, c)$-alternating set as in (1)–(4). We write $U(F) = U_h = \bigcup\{V(x) : x \in F\}$. We call a vertex $y \in U(F) \setminus F$ a *qualifying* vertex for $F$ if $y$ satisfies properties (3) and (4) above, in other words:

(3′) $y$ is not adjacent in $G$ to any vertex of $F$,
(4′) $\chi(y) \neq \chi(z)$ for any $z \in \Gamma(x(y))$ that is coloured by $\chi$ (where $x(y) \in \{y_0\} \cup \bigcup_{i=0}^{h} X_i$ is such that $V(y) = V(x(y))$), unless $X_j = \{x(y)\}$ for some $j$. In this case $\chi(y) = \chi(y_j)$ is permissible.

We write $Q(F)$ for the set of qualifying vertices for $F$. The idea here is that any qualifying vertex is a candidate for $y_{h+1}$ in a possible enlargement of $F$.

**Claim 2.2.** *Let $\chi$ be an MPC, and suppose $F$ is a $(\chi, y_0, c)$-alternating set. Then $Q(F) \neq \emptyset$.*

**Proof.** To prove the claim we simply count the vertices in $U(F) \setminus F$ that fail to satisfy (3′) or (4′).

First we look at (3′). For each $j \geqslant 0$, the number of neighbours of $\{y_j\} \cup X_j$ outside $F$ is at most $d - |X_j| + |X_j|(d-1)$. Therefore a total of at most $(h+1)d + (d-2)\sum_{j=0}^{h} |X_j|$ vertices of $U(F) \setminus F$ are disqualified by (3′).

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Now we consider (4′). Let $J_1 = \{i : 0 \leqslant i \leqslant h, |X_i| = 1\}$ and $J_0 = \{0, \ldots, h\} \setminus J_1$. First suppose $j \in J_1$, and let $X_j = \{x_j\}$. Then there are at most $d-1$ vertices in $V(x_j)$ that get the same colour under $\chi$ as a vertex in $\Gamma(x_j) \setminus \{y_j\}$, so the number of vertices in $\bigcup_{j \in J_1} V(x_j)$ that are disqualified by (4′) is at most $(d-1)|J_1|$. If $j \notin J_1$, then for each $x \in X_j$ there are at most $d$ vertices in $V(x)$ that get the same colour under $\chi$ as a vertex in $\Gamma(x)$. Hence the number of vertices in $\bigcup_{j \in J_0} \bigcup_{x \in X_j} V(x)$ that are disqualified by (4′) is at most $d \sum_{j \in J_0} |X_j|$. Finally, there are at most $d$ vertices in $V(y_0)$ that get the same colour under $\chi$ as a vertex in $\Gamma(y_0)$ (actually at most $d - |X_0|$ but we do not need this). Hence the total number of vertices in $U(F) \setminus F$ that are disqualified by (4′) is at most

$$(d-1)|J_1| + d \sum_{j \in J_0} |X_j| + d = d \sum_{j=0}^{h} |X_j| - |J_1| + d.$$

Therefore the total number of vertices in $U(F) \setminus F$ that are disqualified by either (3′) or (4′) is at most

$$(h+2)d + (2d-2) \sum_{j=0}^{h} |X_j| - |J_1|.$$

Using the assumption that $|V_k| = 3d-1$ for each $k$, and recalling from Lemma 2.1(ii) and (iii) the sizes of $F$ and $V(F)$, we conclude that the number of qualifying vertices for $F$ is at least

$$\begin{aligned}(3d-1)\left(1 + \sum_{j=0}^{h} |X_j|\right) - \left(1 + h + \sum_{j=0}^{h} |X_j|\right) - (h+2)d - (2d-2)\sum_{j=0}^{h} |X_j| + |J_1| \\ = d \sum_{j=0}^{h} |X_j| - (h+2)(d+1) + |J_1| + 3d.\end{aligned}$$

By definition of $J_1$, and property (1) of $F$, we see that $\sum_{j=0}^{h} |X_j| \geqslant |J_1| + 2|J_0| = 2h + 2 - |J_1|$. We therefore obtain the lower bound on the number of qualifying vertices of

$$2d(h+1) - (d-1)|J_1| - (h+2)(d+1) + 3d = (d-1)(h+2-|J_1|) + d.$$

But this is positive, since $|J_1| \leqslant h+1$ by definition. This completes the proof of the claim. □

The proof of Theorem 1.1 will follow quite easily from the following result.

**Theorem 2.3.** *Let $\chi$ be an MPC, let $y_0$ be a vertex not coloured by $\chi$ and let $c \in A_\chi(V(y_0))$. Let $F = \{y_0, \ldots, y_h\} \cup \bigcup_{i=0}^{h} X_i$ be a $(\chi, y_0, c)$-alternating set. Then at least one of the following holds.*

(a) *There exists $y \in Q(F)$ such that $F' = \{y_0, \ldots, y_h, y_{h+1}\} \cup \bigcup_{i=0}^{h+1} X_i$ is a $(\chi, y_0, c)$-alternating set, where $y_{h+1} = y$ and $X_{h+1} = \Gamma_{(\chi,c)}(y)$.*

(b) *There exists an MPC $\chi'$, an index $s$ where $0 \leqslant s \leqslant h$, and an element $x \in X_s$, such that $F' = \{y_0, \ldots, y_s\} \cup \bigcup_{i=0}^{s-1} X_i \cup (X_s \setminus \{x\})$ is a $(\chi', y_0, c)$-alternating set.*

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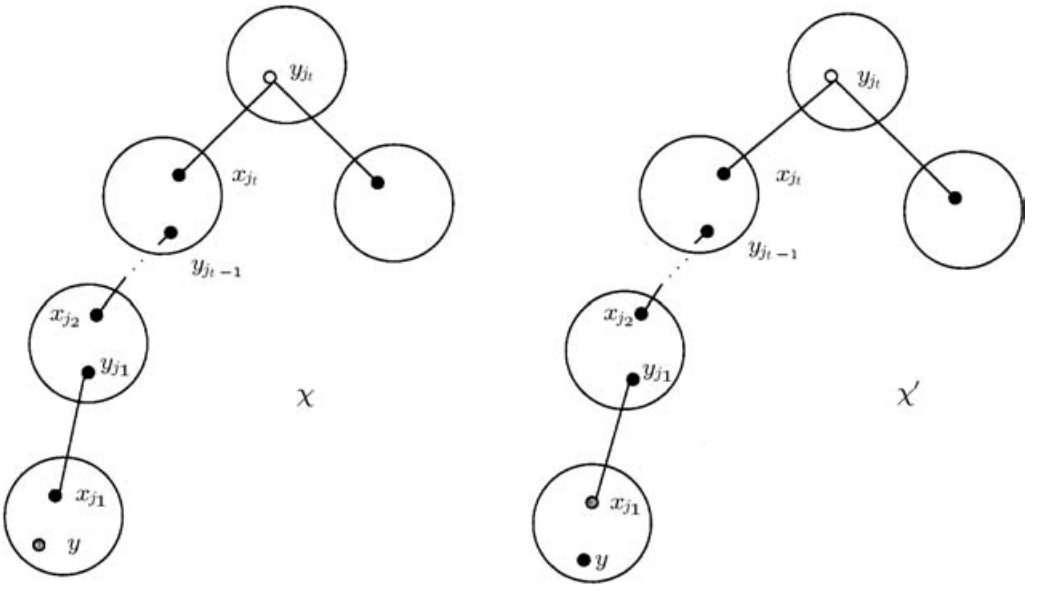



*Figure 2.* The colourings $\chi$ and $\chi'$ in Case A

**Proof.** By Claim 2.2 there exists a vertex $y \in Q(F)$. If $\Gamma_{(\chi,c)}(y) \neq \emptyset$ then the set $F'$ in (a) is a $(\chi, y_0, c)$-alternating set by definition of the fact that $y \in Q(F)$. Therefore to prove Theorem 2.3 we may assume that $\Gamma_{(\chi,c)}(y) = \emptyset$. We will now show that in this case (b) holds.

To prove (b) we will alter the colouring $\chi$ to obtain $\chi'$. First let us suppose that $V(y) = V(y_0)$. Then, by giving colour $c$ to $y$ and colour $\chi(y)$ to $y_0$, we obtain a new valid partial colouring by definition of the fact that $y \in Q(F)$. But then we have succeeded in colouring more vertices than $\chi$ did, contradicting the maximality of $\chi$. We may therefore assume that $x(y) \in X_{j_1}$ for some index $j_1$.

We now define a sequence $S = y, x_{j_1}, y_{j_1}, \ldots, x_{j_t}, y_{j_t}$ of vertices in $F \cup \{y\}$, where the index $s$ from (b) will be $s = j_t$, and the vertex $x$ from (b) will be $x_{j_t}$. To begin we let $j_1$ be the index above, and we let $x_{j_1} = x(y) \in X_{j_1}$. This defines $x_{j_1}$ and $y_{j_1}$. Assume $y, x_{j_1}, y_{j_1}, \ldots, x_{j_i}, y_{j_i}$ have been defined. Consider $y_{j_i}$. If $y_{j_i} \in V(y_0)$ (this includes the possibility that $y_{j_i} = y_0$) then set $t = i$ and $s = j_i$ and stop. If $|X_{j_i}| \geqslant 2$ then we also set $t = i$ and $s = j_i$ and stop. Otherwise $|X_{j_i}| = 1$, and by (2) there exists an index $j < j_i$ for which $y_{j_i} \in V(x_j)$ for some $x_j \in X_j$. We then set $j_{i+1} = j$, which defines $x_{j_{i+1}}$ and $y_{j_{i+1}}$. This completes the definition of the sequence $S$.

We now define the colouring $\chi'$ (see Figure 2). We consider two cases according to the reason why the sequence $S$ terminated.

**Case A:** $|X_s| = |\Gamma_{(\chi,c)}(y_{j_t})| \geqslant 2$.
In this case we set

- $\chi'(y) = \chi'(y_{j_1}) = \cdots = \chi'(y_{j_{t-1}}) = c$,

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- $\chi'(x_{j_1}) = \chi(y)$ and $\chi'(x_{j_i}) = \chi(y_{j_{i-1}})$ for $2 \leqslant i \leqslant t$,
- $\chi'(z) = \chi(z)$ for all other vertices $z$.

To see that this is a proper colouring of $G$, we note the following.

- $\{y, y_{j_1}, \ldots, y_{j_{t-1}}\}$ is an independent set by (3) and the fact that $y \in Q(F)$.
- $y$ had no neighbours coloured $c$ by $\chi$, and each $y_{j_i}$ for $1 \leqslant i \leqslant t-1$ had exactly one neighbour $x_{j_i}$ coloured $c$ by $\chi$, and this neighbour was recoloured with a different colour (or was uncoloured, *i.e.*, given colour 0).
- Consider $x_{j_i}$, which was recoloured with $\chi(y_{j_{i-1}})$. If $\chi(y_{j_{i-1}}) = 0$ then we have nothing to check, so suppose $\chi(y_{j_{i-1}}) \neq 0$. If $2 \leqslant i \leqslant t-1$, by (4) $x_{j_i}$ had no neighbour coloured $\chi(y_{j_{i-1}})$ by $\chi$ except possibly $y_{j_i}$, which was recoloured to $c$. If $i = t$, $x_{j_t}$ had no neighbour coloured $\chi(y_{j_{t-1}})$ by (4). Finally, since $y \in Q(F)$, $x_{j_1}$ had no neighbour coloured $\chi(y)$ by $\chi$ except possibly $y_{j_1}$, which was recoloured to $c$. Moreover $\{x_{j_1}, \ldots, x_{j_t}\}$ is independent since all these vertices were coloured $c$ by $\chi$.

To see that $\chi'$ is an MPC, first note that each class $V_k$ affected by $\chi'$ contains exactly two vertices of the sequence $S$, because the indices $j_i$ are strictly decreasing. Then $\chi'$ simply interchanges the colours of these two vertices. Moreover $\chi'$ colours the same number of vertices as $\chi$ does, so it is an MPC.

To complete the proof for Case A we now check that $F'$ from (b) (with $x = x_{j_t}$) is a $(\chi', y_0, c)$-alternating set, by verifying properties (1)–(4). Observe first that no element of the set $W = \{x_{j_1}, \ldots, x_{j_t}, y, y_{j_1}, \ldots, y_{j_{t-1}}\}$ of vertices that changed colour is adjacent to any $y_i$ with $0 \leqslant i \leqslant s-1$, by properties (1) and (3) of $F$, and by definition of $y \in Q(F)$. Also, the only element of $W$ that is adjacent to $y_s$ is $x = x_{j_t}$, which lost colour $c$. Therefore $\Gamma_{(\chi',c)}(y_i) = \Gamma_{(\chi,c)}(y_i) = X_i$ for $0 \leqslant i \leqslant s-1$, and $\Gamma_{(\chi',c)}(y_s) = X_s \setminus \{x\}$, verifying (1) for $F'$ (recall $|X_s \setminus \{x\}| \geqslant 1$ by the assumption of Case A). Properties (2) and (3) for $F'$ are immediate from its definition. To check property (4) for $F'$, consider a vertex $y_i$ with $0 \leqslant i \leqslant s$. Then $y_i$ did not change colour, and moreover, since no element of $W$ is adjacent to $x(y_i) \in \{y_0\} \cup \bigcup_{i=0}^{s-1} X_i$ (by properties (1) and (3) of $F$ and because $y \in Q(F)$), no neighbour of $x(y_i)$ changed colour either. This (4) still holds for $y_0, \ldots, y_s$. This verifies that $F'$ is a $(\chi', y_0, c)$-alternating set, and thus (b) holds in Case A.

**Case B:** $|X_s| = |\Gamma_{(\chi,c)}(y_{j_t})| = 1$ and $y_{j_t} \in V(y_0)$.
In this case we set

- $\chi'(y) = \chi'(y_{j_1}) = \cdots = \chi'(y_{j_t}) = c$,
- $\chi'(x_{j_1}) = \chi(y)$ and $\chi'(x_{j_i}) = \chi(y_{j_{i-1}})$ for $2 \leqslant i \leqslant t$,
- if $y_{j_t} \neq y_0$ then set $\chi'(y_0) = \chi(y_{j_t})$,
- $\chi'(z) = \chi(z)$ for all other vertices $z$.

Then $\chi'$ is a proper vertex colouring of $G$ that never gives the same colour to two vertices in the same class, for the same reasons as in Case A, together with the fact that $c \in A_\chi(V(y_0))$. However, this time $\chi'$ colours more vertices in $S$ than $\chi$ does, because $y_0$ receives colour $c$ and the colours on the other vertices are simply permuted. This contradicts the fact that $\chi$ was an MPC. Thus Case B cannot occur, and thus the theorem is proved. □

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It is now a short step to the proof of Theorem 1.1.

**Proof.** Let $G$ be a graph of maximum degree at most $d$, and suppose its vertex set is partitioned into classes $V_1 \cup \cdots \cup V_m$, where $|V_k| = 3d-1$ for each $k$. Suppose on the contrary that there is no proper colouring of $G$ with $3d-1$ colours that multicolours every class $V_k$. Then every MPC of $G$ leaves some vertices uncoloured.

Let $\chi$ be an MPC and let $F = \{y_0, \ldots, y_h\} \cup \bigcup_{i=0}^{h} X_i$ be a $(\chi, y_0, c)$-alternating set. We define the *profile* $P(F)$ of $F$ to be the sequence of positive integers $P(F) = (|X_0|, |X_1|, \ldots, |X_h|)$. Then $P(F)$ can have at most $m$ entries. Theorem 2.3 applied with $\chi$ and $F$ implies that there exists a pair $(\chi', F')$ such that either $P(F)$ is a proper initial segment of $P(F')$, or $P(F')$ is lexicographically smaller than $P(F)$.

We construct a sequence of pairs $(\chi_\ell, F_\ell)$ as follows. To begin we let $\chi_0$ be an arbitrary MPC, $y_0$ a vertex not coloured by $\chi$, and $c \in A_\chi(V(y_0))$ be a colour not used by $\chi$ on the vertices of $V(y_0)$. We let $F_0$ be the basic $(\chi, y_0, c)$-alternating set. After the sequence has been constructed up to $(\chi_\ell, F_\ell)$, we obtain the next pair by applying Theorem 2.3. For each $h$, $0 \leqslant h \leqslant m$, consider the subsequence $S_h$ consisting of those $(\chi_\ell, F_\ell)$ for which $P(F_\ell)$ has length exactly $h+1$. Then by the remark in the previous paragraph, the profiles of the $P(F_\ell)$ in $S_h$ form a strictly decreasing chain in the lexicographic order. But this is impossible, and thus our original assumption that no MPC colours all vertices of $G$ must be false. This completes the proof of Theorem 1.1. □

We note in closing that the proof of Theorem 1.1 gives an algorithm for finding a strong $(3d-1)$-colouring of $G$ with respect to the vertex classes $V_k$. Beginning with any partial colouring $\chi$, vertex $y_0$ not coloured by $\chi$, and $c \in A_\chi(V(y_0))$, we construct the sequence of pairs $(\chi_\ell, F_\ell)$ as described. At some point we must find a partial colouring $\chi'$ that colours more vertices than $\chi$ does. We then replace $\chi$ by $\chi'$ and continue, eventually reaching a colouring of the whole of $G$. Unfortunately, however, the number of steps taken to construct each colouring from the previous one could potentially be of the order $d^m$, where $m$ is the number of vertex classes $V_k$. Thus the algorithm is not efficient unless $G$ is very small.

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