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Parameter rays in the space of exponential maps

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Dedicated to Armin Leutbecher on the occasion of his 75th birthday

Abstract. We investigate the set I of parameters $\kappa \in \mathbb{C}$ for which the singular orbit $(0, e^{\kappa}, \ldots)$ of $E_\kappa(z) := \exp(z + \kappa)$ converges to ∞. These parameters are organized in curves in parameter space called *parameter rays*, together with endpoints of certain rays. Parameter rays are an important tool to understand the detailed structure of exponential parameter space. In this paper, we construct and investigate these parameter rays. Based on these results, a complete classification of the set I is given in the following paper [M. Förster, L. Rempe and D. Schleicher. Classification of escaping exponential maps. *Proc. Amer. Math. Soc.* **136** (2008), 651–663].

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1. *Introduction*

This paper is a contribution to extend the successful theory of iterated polynomials to transcendental entire functions. Douady and Hubbard [**DH**], and many others since then, have shown that the dynamical planes of polynomials can be studied in terms of *dynamic rays* and their landing points. Similarly, understanding the space of quadratic polynomials means understanding the structure of the Mandelbrot set, which itself is studied in terms of *parameter rays* and their landing points.

In recent years, it has become clear that dynamic rays also make sense for transcendental entire functions. The simplest parameter space of transcendental functions is arguably the space of exponential functions; it has been studied in [**DGH2, EL, BR, S1, S4, RS1**] and elsewhere (see also [**Fa**] and [**KK**] for studies of other transcendental parameter spaces). Specifically for exponential functions, dynamics rays were introduced in [**DK, DGH1**], and in [**SZ1**] it was shown that dynamic rays classify all *escaping points*: those points which converge to ∞ under iteration of the map. More precisely, every escaping point is either on a dynamic ray, or it is the landing point of a dynamic ray. These results were extended to larger classes of entire functions in [**DT, RoS, Ro, RRRS**].

Motivated by results about polynomial parameter spaces, we investigate *escaping parameters* of exponential maps: these are parameters $\kappa \in \mathbb{C}$ for which the singular value is an escaping point of the exponential map $E_k(z) = e^z + \kappa$. More specifically, we show that the set of all complex parameters for which the singular value is on a dynamic ray (and hence an escaping point) is the uncountable union of curves in exponential parameter space. Every such curve will be called a *parameter ray*. In the present paper, we construct and classify all parameter rays (Theorem 3.7): we show that parameter rays are indexed by infinite sequences over the integers, which we call 'external addresses' $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$. We give a precise criterion for which sequences $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ occur: so-called *exponentially bounded* external addresses. We will show that each $\underline{s}$ has a *minimal potential* $t_{\underline{s}} \in [0, \infty]$ so that the parameter ray $G_{\underline{s}}$ exists exactly for potentials $t > t_{\underline{s}}$, i.e. it is a C^1-curve $G_{\underline{s}} \colon (t_{\underline{s}}, \infty) \to \mathbb{C}$ (so that $t_{\underline{s}} = \infty$ if and only if the external address $\underline{s}$ does not occur).

Certain parameter rays were constructed in [**DGH2**]: those for which the external addresses were bounded sequences and 'regular', i.e. not containing the entry 0. All parameter rays with bounded external addresses were constructed in [**S1**, §2]; the present paper includes these (so far unpublished) results and completes the picture by allowing arbitrary external addresses. Using the methods of [**DGH2**], 'tails' of parameter rays (near infinity) for arbitrary *bounded* external addresses can be constructed relatively easily. It is surprisingly hard to extend parameter rays to their full length. It is not too hard to show that, if $G_{\underline{s}}(t_0)$ exists for some $t_0 > t_{\underline{s}}$, then there exists some $\varepsilon > 0$ so that $G_{\underline{s}}(t)$ exists for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. The problem is that there could be some $t' > t_{\underline{s}}$ for which $G_{\underline{s}}(t')$ was not defined, with $G_{\underline{s}}(t) \to \infty$ as $t \to t'$ (this problem is closely related to the old problem that hyperbolic components have connected boundaries; see below). Most of our technical work (in §4) is devoted to showing that this problem does not happen, using estimate techniques that are not known from the setting of polynomials or parameter ray tails. A completely different approach, using infinite-dimensional Teichmüller theory (or a variant thereof, infinite-dimensional spiders), has been given in the thesis [**F2**].

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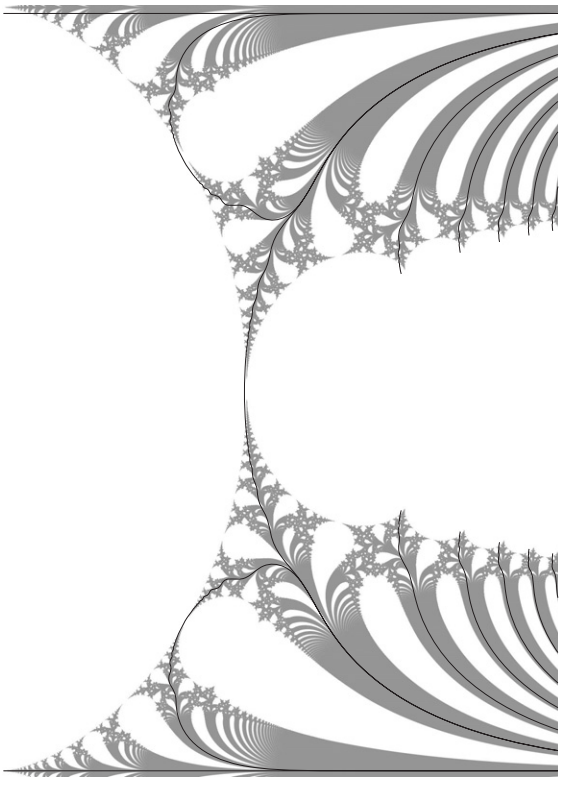


FIGURE 1. The parameter space of complex exponential maps $z \mapsto e^{z+\kappa}$, with hyperbolic components in white, the bifurcation locus in grey, and several parameter rays in black. Note that all parameter rays shown are at 'non-regular' external addresses. (Picture courtesy of Lasse Rempe.)

Many results for a detailed study of exponential parameter space use the structure given by parameter rays in an essential way (see Figure 1), in the same way that parameter rays are used in most deep studies of the Mandelbrot set, including the famous puzzle techniques of Yoccoz [**HY**] (since hyperbolic components in exponential parameter space are unbounded, they can sometimes be used instead of hyperbolic components). We mention a few examples that are already known.

(i) In contrast to the polynomial case, not all escaping parameters are on parameter rays. Every such parameter is either on a unique parameter ray, or it is the landing point of a unique parameter ray with precisely described combinatorics [**FRS**].

(ii) In exponential dynamics, all periodic dynamic rays land at periodic points, and every repelling periodic point is the landing point of a periodic dynamic ray, possibly with one or two exceptional cycles. This is much harder to prove than for polynomials. In fact, the proof by Rempe [**R3**] uses landing properties of parameter rays (at non-regular addresses): in an unorthodox reversal of Douady's strategy, he plows in parameter space and then harvests in the dynamical plane.

(iii) Every hyperbolic component has connected boundary, which had been conjectured in the early 1980s by Baker and Rippon, and by Eremenko and Lyubich. This

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conjecture is now established [**S1, S4, RS1, RS2**], and at least its original proof [**S1, S4**] heavily uses parameter rays at bounded and unbounded external addresses because parameter rays give a lot of structure to the exponential bifurcation locus. Parameter rays landing at a common endpoint partition the parameter plane (which requires parameter rays at non-regular addresses); moreover, families of parameter rays allow us to 'squeeze off' certain directions in which hyperbolic components could possibly be unbounded (this often requires parameter rays at unbounded addresses; see also the next item).

(iv) One of the main conjectures about exponential dynamics is that hyperbolic dynamics is dense in parameter space, i.e. there are no 'queer' components (non-hyperbolic components in the complement of the bifurcation). It is current work in progress to prove that any queer component must be bounded, by 'squeezing it off' near infinity using a family of parameter rays.

(v) One of the fundamental results about quadratic polynomials is the famous theorem by Douady and Hubbard that the bifurcation locus of quadratic polynomials (the boundary of the Mandelbrot set) is connected. The analogous result about exponential parameter space is that the exponential bifurcation locus is connected. This has been shown in [**RS2**] as an extension to the aforementioned proof that hyperbolic components have connected boundaries.

(vi) Any attempt to introduce puzzle or fiber techniques into the study of exponential parameter space will presumably use parameter rays landing at common points (hence at non-regular external addresses); compare [**RS3**]. The exponential bifurcation locus is not locally connected. However, a more appropriate conjecture (which would imply that hyperbolic dynamics was dense) could be that every parameter in the bifurcation locus is either on a unique parameter ray, or it is the landing point of finitely many parameter rays. Parameter rays allow us to define 'fibers' in exponential parameter space (see [**RS3**]) in analogy to the polynomial case [**S2**], and the conjecture could then be phrased as 'fibers are points'. This approach is especially tempting now that Lyubich *et al.* [**AKLS**] have extended the famous results of Yoccoz [**HY**] to unicritical polynomials of arbitrary degrees, so one could hope that there is a chance for analogous results in the limiting exponential case.

It turns out that the set of escaping parameters yields a nice dimension paradox: the union of all parameter rays has Hausdorff dimension 1, while the set of only those endpoints of parameter rays that are themselves escaping parameters has dimension 2 (see [**BBS, FRS, Q**]). This is the parameter space analog to a well-known situation in the dynamical planes of exponential maps [**K, SZ1, S5, S6**].

2. *Dynamic rays*

2.1. *Notation and definitions.*

We investigate the family

$$\{E_\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \exp(z + \kappa) \mid \kappa \in \mathbb{C}\}.$$

Translating κ by an integer multiple of $2\pi i$ yields the same mapping, but with different combinatorics. Therefore, we consider the complex plane as the parameter space, even

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though the 'true parameter space' is the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$, parametrized by $\lambda = \exp(\kappa)$. The asymptotic value 0 is the only singular value of E_κ (i.e. there are no other asymptotic or critical values), and we call $(E_\kappa^{\circ n}(0))_{n\in\mathbb{N}} = (0, e^\kappa, \exp(e^\kappa + \kappa), \dots)$ the *singular orbit*.

Every $E_\kappa : z \mapsto \exp(z + \kappa)$ is conjugate by translation to $z \mapsto e^z + \kappa$, which is used in a number of publications; the difference is that $\exp(z + \kappa)$ has constant singular value 0, while $\exp(z) + \kappa$ has better asymptotics near ∞.

The iterates of a function f are denoted by

$$f^{\circ(n+1)}(z) := f \circ f^{\circ n}(z) = f \circ \cdots \circ f(z),$$

with $f^{\circ 0} := \mathrm{id}$. If f is bijective, we also define $f^{\circ(-n)} := (f^{-1})^{\circ n}$. Let

$$\mathbb{N} := \{0, 1, 2, 3, \dots\}, \quad \mathbb{N}^* := \mathbb{N} \setminus \{0\},$$
$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}, \quad \mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$$

(and similarly $\mathbb{R}^-$, $\mathbb{R}_0^-$), as well as $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_0^-$. We shall write $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$.

Let $\sigma : \mathcal{S} \to \mathcal{S}$ denote the shift map on the space $\mathcal{S} := \mathbb{Z}^{\mathbb{N}^*}$ of sequences over the integers. We are going to use

$$F : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \quad F(t) := e^t - 1$$

as a bijective model function for exponential growth.

The following discussion will take place in the dynamical plane of a fixed map E_κ. We define

$$I(E_\kappa) := \{z \in \mathbb{C} : |(E_\kappa)^{\circ n}(z)| \to \infty \text{ as } n \to \infty\},$$
$$I := \{\kappa \in \mathbb{C} : 0 \in I(E_\kappa)\}.$$

LEMMA 2.1. (Characterization of escaping points) *For all* $\kappa \in \mathbb{C}$,

$$z \in I(E_\kappa) \iff \mathrm{Re}(E_\kappa^{\circ n}(z)) \to +\infty \quad \text{as } n \to \infty.$$

Proof. This follows from $|E_\kappa^{\circ(n+1)}(z)| = \exp(\mathrm{Re}(E_\kappa^{\circ n}(z) + \kappa))$. □

To start with, we would like to endow the plane with dynamical structure so as to obtain symbolic dynamics. On the slit plane $\mathbb{C}'$, there is a biholomorphic branch $\mathrm{Log} : \mathbb{C}' \to \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ of the logarithm, which we will refer to as the principal branch. Thus the branches of E_κ^{-1} on $\mathbb{C}'$ are given by

$$L_{\kappa,j} : \mathbb{C}' \to \mathbb{C}, \quad L_{\kappa,j}(z) = \mathrm{Log}\, z - \kappa + 2\pi i j \quad (j \in \mathbb{Z}).$$

Define the partition

$$R_j := \{z \in \mathbb{C} : -\mathrm{Im}\,\kappa - \pi + 2\pi j < \mathrm{Im}\, z < -\mathrm{Im}\,\kappa + \pi + 2\pi j\} \quad (j \in \mathbb{Z})$$

(see Figure 2): these are the components of $\mathbb{C} \setminus E_\kappa^{-1}(\mathbb{R}^-)$. Clearly, every $L_{\kappa,j}$ maps $\mathbb{C}'$ biholomorphically onto R_j. The following definition gives rise to symbolic dynamics and is the key idea for understanding the set $I(E_\kappa)$.

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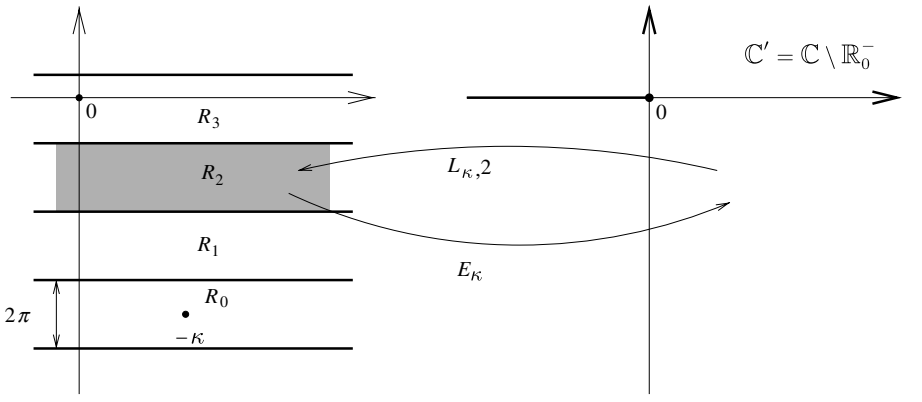


FIGURE 2. The (static) partition and $L_{\kappa,j}$.

Definition 2.2. (External addresses) Consider an exponential map E_κ and let $z = z_1 \in \mathbb{C}$ be such that $z_{n+1} := E_\kappa^{\circ n}(z) \notin \mathbb{R}^-$ for all $n \in \mathbb{N}$. Then the *external address* $\underline{s}(z) = (s_1, s_2, \dots) \in \mathcal{S}$ of z is defined to be the sequence of labels such that $z_n \in R_{s_n}$ for all $n \geq 1$.

Remark. The difference between a parameter $\kappa \in \mathbb{C}$ and its translate $\kappa' = \kappa + 2\pi i k$ (with $k \in \mathbb{Z}$) is a different labeling: a point z has external address $\underline{s}$ for the map $E_{\kappa'}$ if and only if it has external address $(s_1 + k, s_2 + k, s_3 + k, \dots)$ for the map E_κ.

Definition 2.3. (Exponential boundedness) A sequence $\underline{s} \in \mathcal{S}$ is said to be *exponentially bounded* if there is an $x > 0$, called a *growth parameter for* $\underline{s}$, such that $|s_{n+1}| \leq F^{\circ n}(x)$ for all $n \in \mathbb{N}$. The set of exponentially bounded sequences is denoted by $\mathcal{S}_0$.

The following lemma is taken from [**SZ1**, Lemma 2.4].

LEMMA 2.4. (External addresses are exponentially bounded) *For every κ and every $z \in \mathbb{C}$ there is an $x > 0$ such that, for all $n \geq 0$, we have $|E_\kappa^{\circ n}(z)| \leq F^{\circ n}(x)$. Thus every sequence that is realized as an external address is exponentially bounded.* □

2.2. *Dynamic rays.* This section summarizes necessary results from [**SZ1**] on dynamic rays.

Definition 2.5. (Minimal potential) For every $\underline{s} = (s_1, s_2, \dots) \in \mathcal{S}$ define

$$t_{\underline{s}} := \limsup_{n \geq 1} F^{\circ(-(n-1))}(|s_n|) \in [0, \infty] \tag{1}$$

(the *minimal potential* of $\underline{s}$) and

$$t_{\underline{s}}^* := \sup_{n \geq 1} F^{\circ(-(n-1))}(|s_n|).$$

The significance of the minimal potential is that, if an external address is exponentially bounded but grows very fast, then only points $z \in \mathbb{C}$ with sufficiently large real parts can have this external address. This is measured by the minimal potential $t_{\underline{s}}$: the dynamic ray

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$g_{\underline{s}}$ and the parameter ray $G_{\underline{s}}$ exist only for potentials $t > t_{\underline{s}}$. Rays are easier to control for potentials $t > t^*_{\underline{s}} \geq t_{\underline{s}}$; we will need this quantity in some of the proofs.

LEMMA 2.6. (Minimal potential) *The minimal potential $t_{\underline{s}}$ and the quantity $t^*_{\underline{s}}$ have the following properties:*

(i) $t_{\underline{s}} = \inf\{x > 0 : \limsup_{n\to\infty}(|s_n|/F^{\circ(n-1)}(x)) = 0\}$ *(i.e. the definition in (1) coincides with the one from* [**SZ1**]*);*

(ii) *if $\underline{s} \in \mathcal{S}_0$, then $0 \leq t_{\underline{s}} \leq t^*_{\underline{s}} < \infty$, otherwise $t_{\underline{s}} = t^*_{\underline{s}} = \infty$;*

(iii) $|s_{n+1}| \leq F^{\circ n}(t^*_{\underline{s}})$ *for all $n \in \mathbb{N}$;*

(iv) $t_{\sigma(s)} = F(t_{\underline{s}})$ *and* $t^*_{\sigma(s)} \leq F(t^*_{\underline{s}})$*; and*

(v) *for all $t > t_{\underline{s}}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$*

$$F^{\circ n}(t) > t^*_{\sigma^n \underline{s}} \quad \text{and} \quad F^{\circ n}(t) \geq |s_{n+1}|.$$

Proof. Let $c(x) := \limsup_{n\geq 1}(|s_n|/F^{\circ(n-1)}(x))$. If $x > t_{\underline{s}}$, then $c(x) < \infty$. Similarly $c(x') < \infty$ for $x' \in (t_{\underline{s}}, x)$ and thus $c(x) = 0$. Conversely, if $x < t_{\underline{s}}$, then $c(x) > 0$. Together this implies the first assertion.

The second assertion is clear (any sequence has finite sup if and only it has finite lim sup), and the third assertion follows directly from the definition. The subsequent assertion is not hard to check.

Finally, $t > t_{\underline{s}} = \limsup F^{\circ(-(n-1))}(|s_n|)$ means that $t > F^{\circ(-(n-1))}(|s_n|)$ and thus $F^{\circ(n-1)}(t) > |s_n|$ for all but finitely many n. But $t > F^{\circ(-(n-1))}(|s_n|)$ for all $n \geq m \geq N$ means that $F^{\circ m}(t) > F^{\circ(-(k-1))}(|s_{m+k}|)$ for all $k \geq 0$ (setting $n = m + k$) and hence $F^{\circ m}(t) > t^*_{\sigma^m(\underline{s})}$. □

According to [**SZ1**, Definition 6.7], we divide $\mathcal{S}_0$ into so-called *slow* and *fast* sequences: a sequence $\underline{s} \in \mathcal{S}_0$ is called slow if there is a common growth parameter x for infinitely many shifts $\sigma^k \underline{s}$ of $\underline{s}$: i.e. if there is an $x > 0$ so that infinitely many $k > 0$ satisfy $|s_{n+k}| \leq F^{\circ n}(x)$ for all $n \in \mathbb{N}$. Otherwise $\underline{s}$ is called fast. Consider the set $X \subset \mathcal{S}_0 \times \mathbb{R}^+_0$ defined by

$$X := \{(\underline{s}, t) \in \mathcal{S}_0 \times \mathbb{R} \mid t > t_{\underline{s}}\} \cup \bigcup_{\underline{s}\in\mathcal{S}_0 \text{ fast}} \{(\underline{s}, t_{\underline{s}})\}.$$

Let us endow X with the product topology induced by the discrete topology on $\mathcal{S}_0$ and the standard topology on $\mathbb{R}$. (See [**R2**] for a deeper discussion of the topology of the sets $I(E_\kappa)$.)

THEOREM 2.7. (Dynamic rays)

(1) *For every $\kappa \notin I$ there is a continuous bijection*

$$g^\kappa : X \to I(E_\kappa).$$

If $\kappa \in I$ then we have to restrict the map g^κ: there is a preferred pair $(\underline{s}^\kappa, t^\kappa)$ and a set

$$X^\kappa := X \setminus \{(\underline{s}, t) : \exists\, n \geq 1 : \sigma^n \underline{s} = \underline{s}^\kappa \text{ and } F^{\circ n}(t) \leq t^\kappa\} \tag{2}$$

with a continuous injection $g^\kappa : X^\kappa \to I(E_\kappa)$ with $g^\kappa(\underline{s}^\kappa, t^\kappa) = 0$. For every $z \in I(E_\kappa) \setminus g^\kappa(X^\kappa)$ there is a unique $n \geq 1$ and a $t < t^\kappa$ such that $E^{\circ n}_\kappa(z) = g^\kappa(\underline{s}^\kappa, t)$.

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(2) *For every* $\underline{s} \in \mathcal{S}_0$ *and* $\kappa \in \mathbb{C}$, *we define the* dynamic ray *at external address* $\underline{s}$ *by*

$$g^\kappa_{\underline{s}}(t) := g^\kappa(\underline{s}, t) \quad (t > t_{\underline{s}})$$

wherever this is defined. For fixed $t > t_{\underline{s}}$, $g^\kappa_{\underline{s}}(t)$ *depends analytically on* κ *(on* κ*-disks where* $g^\kappa_{\underline{s}}(t)$ *is defined). For fixed* κ *the rays have the following properties.*

(a) *For all* t, $\underline{s}$ *where* $g^\kappa_{\underline{s}}(t)$ *is defined also* $g^\kappa_{\sigma\underline{s}}(F(t))$ *is defined and we have*

$$E_\kappa \circ g^\kappa_{\underline{s}}(t) = g^\kappa_{\sigma\underline{s}} \circ F(t).$$

(b) *Suppose that* $K \geq |\kappa|$. *Then* $g^\kappa_{\underline{s}}(t)$ *is defined for all* $t \geq t^K_{\underline{s}} := t^*_{\underline{s}} + 2\log(K+3)$, *and*

$$g^\kappa_{\underline{s}}(t) = t - \kappa + 2\pi i s_1 + r_{\kappa,\underline{s}}(t)$$
$$\text{with } |r_{\kappa,\underline{s}}(t)| \leq 2e^{-t}(K + 2\pi|s_2| + 12) < 5. \tag{3}$$

(c) *The orbit* $z_n := E^{\circ(n-1)}_\kappa(z_1)$ *of any* $z_1 = g^\kappa_{\underline{s}}(t)$ *satisfies*

$$z_n = F^{\circ(n-1)}(t) - \kappa + 2\pi i s_n + o(1) \quad \text{as } n \to \infty. \tag{4}$$

Remark. We call the curve $g^\kappa_{\underline{s}}(t)$ $(t > t_{\underline{s}})$ the *dynamic ray* at external address $\underline{s}$. For a fast sequence $\underline{s}$, the point $g^\kappa(\underline{s}, t_{\underline{s}})$ (if defined) is called the *endpoint* of the dynamic ray $g^\kappa_{\underline{s}}$. The variable t is referred to as the *potential* (in analogy to the terminology for polynomial external rays).

Some remarks on the meaning of this result might be helpful. The set X provides a topological model of the set $I(E_\kappa)$ of escaping points. However, if the singular value escapes, say $0 = g_{\underline{s}}(t)$, then the dynamic ray $g_{\underline{s}}$ does not have complete preimages, and we need the exceptions as stated in the theorem. (This is exactly in analogy to the well-known case of quadratic polynomials. If the only finite critical point does not escape, then the set I of escaping points is conformally isomorphic to $\{z \in \mathbb{C}\colon |z| > 1\}$, and dynamic rays are defined for all external angles. However, if the critical value does escape, then the ray containing the critical value does not have complete preimages. Only countably many rays are affected: those on the backwards orbit of the ray containing the critical value; all others exist at full length.)

Proof of Theorem 2.7. All results are taken from [**SZ1**]: the first assertion is [**SZ1**, Corollary 6.9]. Analytic dependence of $g^\kappa_{\underline{s}}(t)$ is shown in [**SZ1**, Proposition 3.4] for sufficiently large $t > t_{\underline{s}}$, and every point on a dynamic ray maps onto such a point after finitely many iterations (see [**SZ1**, proof of Theorem 4.2]: the essential estimate is Lemma 2.6(v) above).

Assertion 2(a) is [**SZ1**, Theorem 4.2] (the claim that $g^\kappa_{\underline{s}}(t)$ is defined only if $g^\kappa_{\sigma\underline{s}}(F(t))$ is defined follows from the fact that rays are defined via pullbacks). Assertion 2(b) is shown in [**SZ1**, Proposition 3.4]: the estimate there is $|r_{\kappa,\underline{s}}(t)| \leq 2e^{-t}(K + 2 + 2\pi|s_2| + 2\pi AC)$, where $C < 1.5$ is a universal constant and $A > 1/2\pi$ is such that $|s_{n+1}| < AF^{\circ(n+1)}(t)$ for all n. We have $t > t^*_{\underline{s}} + 2\log(K+3) > t^*_{\underline{s}} + 2$ and $|s_{n+1}| < F^{\circ(n+1)}(t^*_{\underline{s}})$ (Lemma 2.6(ii)), so we can use $A = 1$ and $|s_2| < F(t^*_{\underline{s}}) < e^{t^*_{\underline{s}}}$ and we get

$$|r_{\kappa,\underline{s}}(t)| \leq 2e^{-t}(K + 2\pi|s_2| + 12) \leq \frac{2K + 4\pi|s_2| + 24}{e^{t^*_{\underline{s}}}(K+3)^2} < \frac{2K + 4\pi + 24}{(6K+9)} < 5$$

as claimed in equation (3). Finally, assertion 2(c) is [**SZ1**, Proposition 4.5]. □

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Remark. Viana [**Vi**] showed that dynamic rays are C^∞-smooth. (For C^2, see §4.) Smoothness of rays at escaping endpoints (i.e. for fast addresses) is discussed in [**R2**].

Remark. By Lemma 2.6, for every $z = g_{\underline{s}}^\kappa(t)$ with $t > t_{\underline{s}}$ there is an N such that $F^{\circ N}(t) \geq t_{\sigma^N \underline{s}}^{|\kappa|}$, i.e. for all $n \geq N$, $E_\kappa^{\circ n}(z) = g_{\sigma^n \underline{s}}^\kappa(F^{\circ n}(t))$ satisfies the condition required for equation (3). This is crucial for several arguments given in this paper (see Definition 4.1 and Lemma 4.2).

3. *Construction of parameter rays*

Now we turn our attention to parameter space. We are interested in the set I of *escaping parameters*, which are those parameters for which the singular value 0 is an escaping point. Again, this investigation will lead to curves, called *parameter rays*, which (together with some of their endpoints) classify the set of escaping parameters by the external address $\underline{s}$ and the potential t (i.e. speed of escape) of the singular orbit under E_κ.

Based on dynamic rays, we start the construction of *parameter ray tails*: these are restrictions of parameter rays to large potentials where it is comparably easy to find an escaping parameter κ with given combinatorial data $(\underline{s}, t)$. Afterwards we will extend the parameter ray tails to the full domain $(t_{\underline{s}}, \infty)$ of potentials.

Definition 3.1. (Pointwise definition of parameter rays) For an external address $\underline{s} \in \mathcal{S}_0$ and a potential $t > t_{\underline{s}}$, let

$$\mathcal{D}_{\underline{s}}(t) := \mathbb{C} \setminus \{\kappa \in I \mid \text{there are } n \geq 1 \text{ and } t^\kappa \geq F^{\circ n}(t) \text{ with } g_{\sigma^n \underline{s}}^\kappa(t^\kappa) = 0\}$$

be the set of parameters for which $g_{\underline{s}}^\kappa(t)$ is defined in the sense of Theorem 2.7, and

$$\mathcal{G}_{\underline{s}}(t) := \{\kappa \in \mathcal{D}_{\underline{s}}(t) \mid g_{\underline{s}}^\kappa(t) = 0\}. \tag{5}$$

Remark. As our main result (Theorem 3.7), we will show that, for every exponentially bounded external address $\underline{s} \in \mathcal{S}_0$ and for every potential $t > t_{\underline{s}}$, we have $|\mathcal{G}_{\underline{s}}(t)| = 1$, and that the unique map $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to I$, $t \mapsto \kappa \in \mathcal{G}_{\underline{s}}(t)$, is a curve. This curve will be called the *parameter ray* at external address $\underline{s}$.

3.1. *Parameter ray tails.*

PROPOSITION 3.2. (Existence of parameter ray tails) *For every $\underline{s} \in \mathcal{S}_0$ there is a constant $T_{\underline{s}} > t_{\underline{s}}$ and a unique map $G_{\underline{s}} : [T_{\underline{s}}, \infty) \to \mathbb{C}$, called a* parameter ray tail*, such that, for all $t \geq T_{\underline{s}}$,*

$$G_{\underline{s}}(t) \in \mathcal{G}_{\underline{s}}(t) \quad \textit{and} \quad |G_{\underline{s}}(t)| < 2\pi t.$$

The parameter $G_{\underline{s}}(t)$ is a simple root of the map $\kappa \mapsto g_{\underline{s}}^\kappa(t)$, and the parameter ray tails carry the asymptotics

$$G_{\underline{s}}(t) = t + 2\pi i s_1 + R_{\underline{s}}(t),$$

with $|R_{\underline{s}}(t)| < 2e^{-t}(2\pi t + 2\pi |s_2| + 12) < 5$.

Proof. Define $T_{\underline{s}} := 20 + 2t_{\underline{s}}^*$. Consider an arbitrary fixed potential $t \geq T_{\underline{s}}$ and define $K := 2\pi t$. Consider the auxiliary function $h\colon B_K(0) \to \mathbb{C}$ with $h(\kappa) := g_{\underline{s}}^\kappa(t)$; we will show that h is well defined on the open disk $B_K(0)$ and has a unique root κ_0 (see Figure 3).

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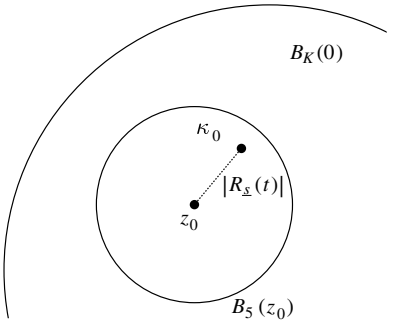


FIGURE 3. The setting in the proof of Proposition 3.2.

Indeed, by Theorem 2.7 part (2b), this map is well defined because $t \geq T_{\underline{s}} \geq 20$ implies that $t/2 > 2\log(2\pi t + 3)$ and thus

$$t = t/2 + t/2 > 2\log(2\pi t + 3) + t^*_{\underline{s}} = t^K_{\underline{s}}.$$

In fact, this implies by equation (3) that, for all $\kappa \in B_K(0)$,

$$g^\kappa_{\underline{s}}(t) = t - \kappa + 2\pi i s_1 + r_{\kappa,\underline{s}}(t) \quad \text{with } |r_{\kappa,\underline{s}}(t)| < 5. \tag{6}$$

Now for given κ, define $z_0 := t + 2\pi i s_1$, so that $g^\kappa_{\underline{s}}(t) = z_0 - \kappa + r_{\kappa,\underline{s}}(t)$. Since $|r_{\kappa,\underline{s}}(t)| < 5$, we have $g^\kappa_{\underline{s}}(t) \neq 0$ for $|z_0 - \kappa| \geq 5$. Within the disk $B_K(0)$, the only parameters κ with $g^\kappa_{\underline{s}}(t) = 0$ are thus contained in $B_5(z_0)$.

Note that $B_7(z_0) \subset B_K(0)$ because every $\kappa \in B_7(z_0)$ satisfies

$$|\kappa| \leq t + 2\pi|s_1| + 7 \leq t + 2\pi t^*_{\underline{s}} + 7 \leq t + \pi T_{\underline{s}} + 7 < 2\pi t = K.$$

By (6), $h(\partial B_6(z_0))$ winds exactly once around 0. Analyticity of h and Rouché's theorem imply therefore that there is exactly one $\kappa_0 =: G_{\underline{s}}(t)$ (counting multiplicities) with $|\kappa_0| < K$ for which $h(\kappa_0) = 0$. This is a simple root of $\kappa \mapsto g^\kappa_{\underline{s}}(t)$.

Since $g^{\kappa_0}_{\underline{s}}(t) = t - \kappa_0 + 2\pi i s_1 + r_{\kappa_0,\underline{s}}(t) = 0$, Theorem 2.7 part (2b) yields

$$|G_{\underline{s}}(t) - t - 2\pi i s_1| \leq |r_{2\pi t,\underline{s}}(t)|. \qquad \square$$

Notice that we have not yet shown that $G_{\underline{s}}$ is a curve (we could do it here, but we wait until we have constructed parameter rays at their full length). The following proposition implies uniqueness of the parameter ray tails (without the restriction on $|\kappa| < 2\pi t$) and is the main argument for extending the ray tails onto the full domain of definition $(t_{\underline{s}}, \infty)$. We defer the proof, which is the technical heart of this paper, to §4.

PROPOSITION 3.3. (A bound on the growth of parameter rays) *For every $\underline{s} \in \mathcal{S}_0$ there is a continuous function $\xi_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{R}^+$ such that, for every $t > t_{\underline{s}}$,*

$$\mathcal{G}_{\underline{s}}(t) \subset B_{\xi_{\underline{s}}(t)}(0).$$

Moreover, for sufficiently large t we can choose $\xi_{\underline{s}}(t) = 2t$.

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COROLLARY 3.4. (Parameter ray tails are unique) *For every $\underline{s} \in \mathcal{S}_0$ and every sufficiently large t, $\mathcal{G}_{\underline{s}}(t)$ contains exactly one element κ_0. The multiplicity of κ_0 as a root of $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$ is* 1.

Proof. By Proposition 3.3 we have $\mathcal{G}_{\underline{s}}(t) \subset B_{2t}(0) \subset B_{2\pi t}(0)$, and the claim follows from Proposition 3.2. □

3.2. *Parameter rays at their full length.*

LEMMA 3.5. (The domain of definition of $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$) *Let $\underline{s} \in \mathcal{S}_0$ be an external address.*

(1) *For every bounded set Λ of parameters and every compact interval $J \subset (t_{\underline{s}}, \infty)$ of potentials there is an $N \in \mathbb{N}$ such that*

$$\text{for all } n \geq N, \text{ for all } \kappa \in \Lambda, \text{ for all } t \in J: \quad \kappa \in \mathcal{D}_{\sigma^n \underline{s}}(F^{\circ n}(t)).$$

(2) *Let $t > t_{\underline{s}}$. For every $\kappa \in \mathcal{D}_{\underline{s}}(t)$ there are neighborhoods $J \subset (t_{\underline{s}}, \infty)$ and $\Lambda \subset \mathbb{C}$ of t and κ respectively such that*

$$\text{for all } t' \in J, \text{ for all } \kappa' \in \Lambda: \quad \kappa' \in \mathcal{D}_{\underline{s}}(t').$$

In particular, $\mathcal{D}_{\underline{s}}(t)$ is open for every $t > t_{\underline{s}}$.

(3) *If $t_k \to t \in (t_{\underline{s}}, \infty)$ and $\kappa_k \to \kappa \in \mathbb{C}$ are sequences such that $\kappa_k \in \mathcal{G}_{\underline{s}}(t_k)$ for all $k \geq 1$, then $\kappa \in \mathcal{G}_{\underline{s}}(t)$. In particular, $\mathcal{G}_{\underline{s}}(t)$ is closed in $\mathbb{C}$.*

Proof. Recall from Theorem 2.7 part (1) that, if $t > t_{\underline{s}}$, then $g^{\kappa}_{\underline{s}}$ is defined unless there is an $n \geq 1$ such that $g^{\kappa}_{\sigma^n \underline{s}}(t_0) = 0$ with $t_0 \geq F^{\circ n}(t)$.

For the first claim, let $K := \sup_{\kappa \in \Lambda} |\kappa|$. It follows from Lemma 2.6(v) that there is an $N \in \mathbb{N}$ such that $F^{\circ N}(t) > \max\{t^K_{\sigma^N \underline{s}}, K + 5\}$ for all $t \in J$. By Theorem 2.7 part (2b), this implies that, for all $\kappa \in \Lambda$ and all $t \in J$, the dynamic ray $g^{\kappa}_{\sigma^N \underline{s}}$ is defined at the potential $F^{\circ N}(t)$ with

$$\operatorname{Re}(g^{\kappa}_{\sigma^N(\underline{s})}(F^{\circ N}(t))) > F^{\circ N}(t) - K - 5 > 0. \tag{7}$$

This shows the first assertion.

We continue with assertion (3). Let $t_k \to t$ and $\kappa_k \to \kappa$ be sequences such that $\kappa_k \in \mathcal{G}_{\underline{s}}(t_k)$ for all $k \geq 1$. We have to show that $\kappa \in \mathcal{G}_{\underline{s}}(t)$. We have just proved that there is an $N \in \mathbb{N}$ such that, for all $m \geq N$, we have $\{\kappa, \kappa_1, \kappa_2, \ldots\} \subset \mathcal{D}_{\sigma^m \underline{s}}(F^{\circ m}(t_k))$. For every k we have $g^{\kappa_k}_{\sigma^N \underline{s}}(F^{\circ N}(t_k)) = E^{\circ N}_{\kappa_k}(0)$ and thus by continuity $g^{\kappa}_{\sigma^N \underline{s}}(F^{\circ N}(t)) = E^{\circ N}_{\kappa}(0)$. If $\kappa \notin \mathcal{D}_{\underline{s}}(t)$, then there is an $m \geq 1$ and a $t_0 \geq F^{\circ m}(t) > t$ such that $g^{\kappa}_{\sigma^m \underline{s}}(t_0) = 0$ and thus

$$E^{\circ N}_{\kappa}(0) = g^{\kappa}_{\sigma^{m+N} \underline{s}}(F^{\circ N}(t_0)).$$

We get two different potentials for $E^{\circ N}_{\kappa}(0)$, which contradicts the injectivity of g^{κ} in Theorem 2.7. Therefore $\kappa_0 \in \mathcal{D}_{\underline{s}}(t)$, and by continuity $\kappa_0 \in \mathcal{G}_{\underline{s}}(t)$.

For assertion (2), let (t, κ) be a pair consisting of a potential $t > t_{\underline{s}}$ and a parameter $\kappa \in \mathbb{C}$ such that $\kappa \in \mathcal{D}_{\underline{s}}(t)$. Suppose by way of contradiction that there are sequences $(t_k)_{k \geq 1}$ and $(\kappa_k)_{k \geq 1}$ with $t_k \to t$ and $\kappa_k \to \kappa$, such that

$$\text{for all } k \geq 1, \text{ there exists } n_k \geq 1, \text{ there exists } t'_k \geq F^{\circ n_k}(t_k): \quad g^{\kappa_k}_{\sigma^{n_k}(\underline{s})}(t'_k) = 0.$$

By assertion (1), we may pass to a subsequence so that all n_k are equal to some n.

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Let $t_- := \inf t_k > t_{\underline{s}}$; then $t'_- := \inf t'_k \geq F^{\circ n}(t_-) > t_{\sigma^n \underline{s}}$. Similarly, (7) implies that $t'_+ := \sup t'_k < \infty$. So by passing to a subsequence once more we may assume that the t'_k converge to some $t' \in [t'_-, t'_+]$. Assertion (3) implies that $g^{\kappa}_{\sigma^n \underline{s}}(t') = 0$.

Since $t'_k \geq F^{\circ n}(t_k)$, we have $t' \geq F^{\circ n}(t)$. By Theorem 2.7 part (1), $\kappa \notin \mathcal{D}_{\underline{s}}(t)$, in contradiction to our hypothesis. □

PROPOSITION 3.6. (Discreteness and local continuous extension of $G_{\underline{s}}$) *Consider a sequence $\underline{s} \in \mathcal{S}_0$ and a potential $t > t_{\underline{s}}$.*

(1) *The set $\mathcal{G}_{\underline{s}}(t)$ is discrete in $\mathbb{C}$.*

(2) *For every $\kappa_0 \in \mathcal{G}_{\underline{s}}(t)$ there are neighborhoods $\Lambda \subset \mathbb{C}$ and $J \subset (t_{\underline{s}}, \infty)$ containing κ_0 and t respectively, such that, for every $t' \in J$, the number of elements of $\mathcal{G}_{\underline{s}}(t') \cap \Lambda$ (counting multiplicities) equals the finite multiplicity of κ_0 as a root of the map $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$. In particular, for every sequence $t_k \to t$ there is an $N \in \mathbb{N}$ and a sequence $(\kappa_k)_{k \geq N} \to \kappa_0$ such that $\kappa_k \in \mathcal{G}_{\underline{s}}(t_k)$ for all $k \geq N$.*

Proof. Consider a parameter $\kappa_0 \in \mathcal{G}_{\underline{s}}(t)$ and suppose that κ_0 is not isolated in $\mathcal{G}_{\underline{s}}(t)$. Let U be the connected component of $\mathcal{D}_{\underline{s}}(t)$ that contains κ_0. Since $\mathcal{D}_{\underline{s}}(t)$ is open, U is open in $\mathbb{C}$. Analyticity of $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$ and the identity principle imply that $U \subset \mathcal{G}_{\underline{s}}(t) \subset \mathcal{D}_{\underline{s}}(t)$. But $\mathcal{G}_{\underline{s}}(t)$ is closed, hence $\overline{U} \subset \mathcal{D}_{\underline{s}}(t)$, so U is open and closed in $\mathbb{C}$. This implies that $U = \mathbb{C}$, a contradiction.

For the last assertion, consider neighborhoods J_0, Λ_0 of t, κ_0 respectively as provided by Lemma 3.5 part (2). Since $\mathcal{G}_{\underline{s}}(t)$ is discrete, there is an $\varepsilon > 0$ such that κ_0 is the only root of the map $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$ within $\Lambda := D_{\varepsilon}(\kappa_0)$ and such that $\overline{\Lambda} \subset \Lambda_0$. Define a circular loop $\gamma : [0, 1] \to \Lambda_0$ by $\gamma(s) := \kappa_0 + \varepsilon e^{is}$. By Rouché's theorem, the multiplicity of κ_0 as a zero equals the winding number $\eta(g^{\gamma(s)}_{\underline{s}}(t), 0)$ of $g^{\gamma(s)}_{\underline{s}}(t)$ around 0. On $\Lambda_0 \times J_0$, $g : (\kappa, t) \to g_{\kappa}(t)$ is continuous, so we have $\eta(g^{\gamma}_{\underline{s}}(t'), 0) = \eta(g^{\gamma}_{\underline{s}}(t), 0)$ for potentials t' sufficiently close to t, and Λ contains a parameter κ' with $g^{\kappa'}_{\underline{s}}(t') = 0$; in fact, the total multiplicity of such parameters within Λ equals the multiplicity of κ_0. Since this is true for every $\varepsilon > 0$, the claim follows. □

We are now ready to state and to prove the main result.

THEOREM 3.7. (Parameter rays at their full length) *For every external address $\underline{s} \in \mathcal{S}_0$ there is a unique curve $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to I \subset \mathbb{C}$, called a* parameter ray, *such that, for all $t > t_{\underline{s}}$, $\kappa_0 = G_{\underline{s}}(t)$ satisfies $g^{\kappa_0}_{\underline{s}}(t) = 0$ and is a simple root of the map $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$. Conversely, if $g^{\kappa}_{\underline{s}}(t) = 0$ with $t > t_{\underline{s}}$, then $\kappa = G_{\underline{s}}(t)$. The parameter rays are injective and pairwise disjoint, and they carry the asymptotics*

$$G_{\underline{s}}(t) = t + 2\pi i s_1 + R_{\underline{s}}(t) \quad \text{with } |R_{\underline{s}}(t)| = O(te^{-t}) \text{ as } t \to \infty.$$

Proof. By Propositions 3.3 and 3.6 part (1), $\mathcal{G}_{\underline{s}}(t)$ is bounded and discrete and thus finite. Set $n(t) := |\mathcal{G}_{\underline{s}}(t)|$ (counting multiplicities) and let

$$J_{\underline{s}} := \{T > t_{\underline{s}} \mid n(t) \geq 1 \ \forall\, t \geq T\}.$$

By Corollary 3.4 we have $n(t) = 1$ for sufficiently large t, so $J_{\underline{s}}$ is non-empty. It follows from Proposition 3.6 part (1) that $J_{\underline{s}}$ is open. We show that $J_{\underline{s}}$ is also closed in $(t_{\underline{s}}, \infty)$.

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Let $t_* := \inf J_{\underline{s}}$ and suppose that $t_* > t_{\underline{s}}$. A function $t \mapsto G_{\underline{s}}(t) \in \mathcal{G}_{\underline{s}}(t)$ can be defined on $J_{\underline{s}}$, possibly involving a choice. By Proposition 3.3, the set $\{G_{\underline{s}}(t) \mid t \in (t_*, t_* + 1)\}$ is contained in a compact set. Thus the set L of all limits $\lim_{t \searrow t_*} G_{\underline{s}}(t)$ is a non-empty compact subset of $\mathbb{C}$. By Lemma 3.5 part (3), it follows that $L \subset \mathcal{G}_{\underline{s}}(t_*)$. Hence $J_{\underline{s}}$ is closed in $(t_{\underline{s}}, \infty)$ and $J_{\underline{s}} = (t_{\underline{s}}, \infty)$.

Similarly we show that the set $J'_{\underline{s}} := \{t > t_{\underline{s}} \mid n(t) \geq 2\}$ is empty: by using Proposition 3.6 as above, this set is open and closed relative to $(t_{\underline{s}}, \infty)$. However, the complement is non-empty because it contains an interval of the form (T, ∞) (on which Corollary 3.4 holds). Therefore $J'_{\underline{s}} = \emptyset$, and for every $t > t_{\underline{s}}$ we have $n(t) = 1$. This shows that the choice for $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to I$ above was unique.

Now this means that $G_{\underline{s}}$ is continuous because of Proposition 3.6 part (2). Injectivity of $G_{\underline{s}}$ and disjointness of the parameter rays follow from the injectivity of g^κ in Theorem 2.7, since every parameter κ has at most one external address and one potential. The asymptotic behavior follows from Proposition 3.2. □

Remark. Note that, unlike dynamic rays, parameter rays are always defined on the entire interval $(t_{\underline{s}}, \infty)$.

In [**FRS**], the above result is extended to endpoints for fast sequences $\underline{s}$. This yields a complete classification of escaping parameters: there is a continuous bijection $G : X \to I$, where X is defined as in §2, and the path components of I are exactly the parameter rays, including the endpoints at fast addresses.

Remark. Our proof strategy explained above depends crucially on Proposition 3.3, which will be proved in §4. An alternative strategy could proceed as follows. Proposition 3.2 shows the (pointwise) existence of any given parameter ray $G_{\underline{s}}$ for sufficiently large potentials; Proposition 3.6 proves continuity and shows that the maximal domain of definition of $G_{\underline{s}}$ as a curve is open in $(t_{\underline{s}}, \infty)$. The only possible obstacle to the existence of parameter rays at their full length is if there is a $T^*_{\underline{s}} > t_{\underline{s}}$ so that $G_{\underline{s}} : (T^*_{\underline{s}}, \infty) \to \mathbb{C}$ is a well-defined curve with $G_{\underline{s}}(t) \to \infty$ as $t \searrow T^*_{\underline{s}}$. This problem is excluded by the squeezing lemma [**RS1**]. (However, the proof of the squeezing lemma (in [**S1, S4**] or [**RS1**]) requires deep *a priori* knowledge of exponential parameter space; its original proof needs parameter ray tails, but it does not need the existence of parameter rays at their full length.)

3.3. *Vertical order of parameter rays.* We show that parameter rays have a natural vertical order, which coincides with the lexicographic order of their external addresses.

Definition 3.8. (Vertical order) Let Γ be a family of disjoint injective curves $\gamma : \mathbb{R}^+ \to \mathbb{C}$ with $\operatorname{Re} \gamma(t) \to \infty$ as $t \to \infty$, and consider $\gamma_1, \gamma_2 \in \Gamma$. Let H_R be a right half plane so that γ_2 disconnects H_R into two unbounded components (one upper and one lower component, plus possibly some bounded components). Denote by $H_R^+(\gamma_2)$ and $H_R^-(\gamma_2)$, respectively, the upper and lower components of $H_R \setminus \gamma_2$. Then

$$\gamma_1 \succ \gamma_2 : \quad \Longleftrightarrow \quad \gamma_1 \cap H_R^+(\gamma_2) \text{ is unbounded}$$

defines a linear order on Γ that is independent of the choice of R. We say that γ_1 is *above* γ_2. □

Equip $\mathcal{S} = \mathbb{Z}^{\mathbb{N}^*}$ with the lexicographic order '$>$'.

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LEMMA 3.9. (Vertical order of dynamic rays) *For all $\kappa \in \mathbb{C}$ and all exponentially bounded addresses $\underline{s}$, $\underline{s}' \in \mathcal{S}_0$,*

$$g_{\underline{s}}^{\kappa} \succ g_{\underline{s}'}^{\kappa} \quad \Longleftrightarrow \quad \underline{s} > \underline{s}'.$$

Proof. Without loss of generality, say $\underline{s} > \underline{s}'$. If the initial entries satisfy $s_1 > s_1'$, then $g_{\underline{s}}^{\kappa} \succ g_{\underline{s}'}^{\kappa}$ follows directly from the asymptotic estimate of equation (3) in Theorem 2.7. Otherwise, let $k+1 > 1$ be the first position in which $\underline{s}$ and $\underline{s}'$ differ. Then by the same argument, $g_{\sigma^k \underline{s}}^{\kappa} \succ g_{\sigma^k \underline{s}'}^{\kappa}$. Since

$$g_{\underline{s}}^{\kappa}(t) = L_{\kappa,s_1} \circ \cdots \circ L_{\kappa,s_k} \circ g_{\kappa,\sigma^k \underline{s}}^{\kappa}(F^{\circ k}(t))$$

and the translated logarithms $L_{\kappa,s} = \mathrm{Log} - \kappa + 2\pi i s$ preserve the vertical order, the claim follows. □

PROPOSITION 3.10. (Vertical order of parameter rays) *For all exponentially bounded addresses $\underline{s}$, $\underline{s}' \in \mathcal{S}_0$,*

$$G_{\underline{s}} \succ G_{\underline{s}'} \quad \Longleftrightarrow \quad \underline{s} > \underline{s}'.$$

Proof. The claim follows from the asymptotic estimate in Theorem 3.7 if the first entries in $\underline{s}$ and $\underline{s}'$ are different, so we assume that they are equal and $\underline{s} > \underline{s}'$. Write $\underline{s} = s_1 s_2 s_3 \ldots$. There is a $\xi > 0$ depending on $\underline{s}$ so that all $x \geq 0$ satisfy

$$\xi + x > t_{\underline{s}}^* + 16 + 2\log(\xi + x + 2\pi(|s_1| + 2) + 3),$$

where $t_{\underline{s}}^*$ is as in Lemma 2.6. Define the strip

$$S := \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq \xi,\ 2\pi(s_1 - 2) \leq \mathrm{Im}\, z \leq 2\pi(s_1 + 2)\}.$$

We claim that every $\kappa \in S$ has a potential $t_\kappa > t_{\underline{s}}$ so that the dynamic ray $g_{\underline{s}}^{\kappa}(t)$ is defined for $t \geq t_\kappa$ and satisfies $\mathrm{Re}\, g_{\underline{s}}^{\kappa}(t_\kappa) < -11$.

Indeed, if $\kappa \in S$ has $\mathrm{Re}\,\kappa = \xi + x$, then $|\kappa| < K := \xi + x + 2\pi(|s_1| + 2)$; set $t_\kappa := t_{\underline{s}}^* + 2\log(K+3)$. We then have

$$\xi + x > t_{\underline{s}}^* + 16 + 2\log(K+3) = t_\kappa + 16.$$

By Theorem 2.7 part (2b), $g_{\underline{s}}^{\kappa}(t)$ is defined for $t \geq t_\kappa$ and satisfies

$$\mathrm{Re}\, g_{\underline{s}}^{\kappa}(t_\kappa) \leq t_\kappa - \xi - x + 5 < -11.$$

In the dynamical plane for $\kappa \in S$, consider the right half plane

$$H_\kappa := \{z \in \mathbb{C} \mid \mathrm{Re}\, z > \mathrm{Re}\, g_{\underline{s}}^{\kappa}(t_\kappa)\}.$$

The ray tail $g_{\underline{s}}^{\kappa}((t_\kappa, \infty))$ cuts H_κ into two unbounded components H_κ^+ and H_κ^- (plus possibly some bounded components), so that in H_κ^+, imaginary parts are unbounded above, while in H_κ^-, they are unbounded below. The asymptotics in equation (3) from Theorem 2.7 together with the condition $\mathrm{Re}(g_{\underline{s}}^{\kappa}(t_\kappa)) < -11$ imply that any bounded component in $H_\kappa \setminus g_{\underline{s}}^{\kappa}((t_\kappa, \infty))$ has real parts less than -1. (If not, there would have to be two potentials $t_2 > t_1 > t_\kappa$ with $\mathrm{Re}\, g_{\underline{s}}^{\kappa}(t_2) \leq -11$ and $\mathrm{Re}\, g_{\underline{s}}^{\kappa}(t_1) \geq -1$, while

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equation (3) implies that $\operatorname{Re} g_{\underline{s}}^{\kappa}(t_2) - \operatorname{Re} g_{\underline{s}}^{\kappa}(t_1) > t_2 - t_1 - 10 > -10$.) Therefore, every $\kappa \in S$ has the property that $0 \in H_{\kappa}^{+} \dot\cup H_{\kappa}^{-} \dot\cup g_{\underline{s}}^{\kappa}((t_{\kappa}, \infty))$ (where $\dot\cup$ denotes disjoint unions). Define $S_0^{+} := \{\kappa \in S\colon 0 \in H_{\kappa}^{+}\}$ and analogously S_0^{-}. By construction, $S \subset S_0^{+} \dot\cup S_0^{-} \dot\cup G_{\underline{s}}((t_{\underline{s}}, \infty))$.

The asymptotics of the parameter ray $G_{\underline{s}}$ from Theorem 3.7 imply that $S \setminus G_{\underline{s}}((t_s, \infty))$ contains two unbounded components (plus possibly some bounded ones); let S^{+} and S^{-}, respectively, be the unbounded component above and below $G_{\underline{s}}((t_s, \infty))$ in the obvious sense.

We claim that $S^{+} \subset S_0^{+}$ and $S^{-} \subset S_0^{-}$. Indeed, the set S^{+} contains a tail of the parameter ray $G_{\underline{s}''}$ with $\underline{s}'' = (s_1 + 1)s_2s_3s_3 \ldots$. For parameters κ on this tail, the vertical order of dynamic rays implies that $0 \in H_{\kappa}^{+}$, hence $\kappa \in S_0^{+}$. Since S^{+} is connected and $g_{\underline{s}}((t_{\kappa}, \infty))$ depends continuously on κ, it follows that $S^{+} \subset S_0^{+}$, and analogously $S^{-} \subset S_0^{-}$.

The parameter ray $G_{\underline{s}'}$ also has a tail in S, and, for parameters κ on this tail, the vertical order of dynamic rays (Lemma 3.9) implies that $\kappa \in S_0^{-}$. If $\operatorname{Re} \kappa$ is sufficiently large, then $\kappa \in S^{+} \cup S^{-}$. Finally, $\kappa \in S^{+}$ would imply that $\kappa \in S_0^{+}$, a contradiction, so $\kappa \in S^{-}$. □

Now we show that the asymptotic bounds of parameter rays from Theorem 3.7 extend to vertical bounds on the entire rays, and show that this has non-trivial dynamical consequences. As usual, we say that a dynamic ray $g_{\underline{s}}$ *lands* at a point z_0 if $z_0 = \lim_{t \searrow t_{\underline{s}}} g_{\underline{s}}(t)$, and analogously for a parameter ray $G_{\underline{s}}$.

PROPOSITION 3.11. (Vertical bounds)

(1) *Every parameter ray $G_{\underline{s}}$ at external address $\underline{s}$ is entirely contained in the strip $(2s_1 - 1)\pi < \operatorname{Im} \kappa < (2s_1 + 1)\pi$.*

(2) *In any dynamical plane, if the singular value escapes so that $0 = g_{\underline{s}}^{\kappa}(t)$ and $t > t_{\underline{s}}$, then s_1 is the unique closest integer to $\operatorname{Im} \kappa / 2\pi$.*

(3) *If the singular value is the landing point of the dynamic ray at preperiodic address $\underline{s}$, then again s_1 is the closest integer to $\operatorname{Im} \kappa / 2\pi$; moreover, the corresponding parameter κ_0 is the landing point of the parameter ray $G_{\underline{s}}$.*

Proof. The first claim follows for sufficiently large potentials t from the asymptotics in Theorem 3.7, and for arbitrary t it follows because the lines $\{\kappa \in \mathbb{C} \mid \operatorname{Im} \kappa \in (2\mathbb{Z} + 1)\pi\}$ only contain exponential maps with attracting or parabolic orbits of period 1 or 2 and are thus disjoint from parameter rays. (If $\operatorname{Im} \kappa \in (2\mathbb{Z} + 1)\pi$, then $E_{\kappa}(z) = -c \exp(z)$ for $c \in \mathbb{R}^{+}$, and such maps are well understood.) The second claim is an equivalent reformulation.

The third assertion follows by perturbation. The landing point of a preperiodic dynamic ray must be preperiodic. If it equals the singular value, then the preperiodic orbit is repelling, and there is a neighborhood Λ_0 of κ_0 in parameter space so that, for all $\kappa \in \Lambda$, the perturbed preperiodic point is still the landing point of the perturbed preperiodic ray. It follows easily that Λ contains parameters κ with $0 = g_{\underline{s}}(t)$ for arbitrarily small positive t (if κ makes a small loop around κ_0, there must be a point where g_s intersects 0). This implies that κ_0 is a limit point of $G_{\underline{s}}$. Conversely, any limit point κ of $G_{\underline{s}}$ within Λ must have the property that $g_{\underline{s}}^{\kappa}$ lands at 0, so 0 is preperiodic under E_{κ}. Such parameters are discrete; since the limit set of $G_{\underline{s}}$ is connected, it follows indeed that $G_{\underline{s}}$ lands at κ_0. □

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The second assertion in Proposition 3.11 has a purely dynamical meaning, even though the choice of the branch of $\kappa = \log \lambda$ is dynamically insignificant. The branch determines the choice of labels in the partition $\{R_j\}$ used for external addresses. A dynamical formulation that is free of all choices, and independent of all parametrizations, is the following: if the singular value v escapes on a dynamic ray $g_{\underline{s}}$, then $\lim_{t\to\infty} \operatorname{Im}(g_{\underline{s}}(t) - v) \in (-\pi, \pi)$. This result seems to be far from obvious.

3.4. *Differentiability of parameter rays.*

THEOREM 3.12. (Differentiability of parameter rays) *All parameter rays* $G_{\underline{s}}\colon (t_{\underline{s}}, \infty) \to \mathbb{C}$ *are continuously differentiable with*

$$G'_{\underline{s}}(t) = \frac{d\kappa}{dt} = -\frac{\partial g^{\kappa}_{\underline{s}}(t)/\partial t}{\partial g^{\kappa}_{\underline{s}}(t)/\partial \kappa} \neq 0.$$

Proof. By definition, $\kappa = G_{\underline{s}}(t)$ if and only if $g^{\kappa}_{\underline{s}}(t) = 0$. By [**Vi**] (or Proposition 4.6), the map $g^{\kappa}_{\underline{s}}(t)$ is continuously differentiable in t. By Lemma 3.5 part (2), for fixed t, the domain of definition of $\kappa \mapsto g^{\kappa}_{\underline{s}}(t)$ is open, and by Theorem 2.7 part (2), this map is holomorphic in κ. Finally, by Theorem 3.7, $(\partial/\partial\kappa) g^{\kappa}_{\underline{s}}(t) \neq 0$.

Implicit differentiation of $g^{\kappa}_{\underline{s}}(t) = 0$ yields

$$\frac{\partial}{\partial \kappa} g^{\kappa}_{\underline{s}}(t) \frac{d\kappa}{dt} + \frac{\partial}{\partial t} g^{\kappa}_{\underline{s}}(t) = 0,$$

and this gives the claimed formula. Finally, we show in Proposition 4.6 that $\partial g^{\kappa}_{\underline{s}}(t)/\partial t \neq 0$.

□

Remark. We believe that parameter rays are in fact C^{∞}-curves; for this, it would be enough to show that $\partial g^{\kappa}_{\underline{s}}(t)/\partial\kappa$ is C^{∞}. It seems feasible to adapt Viana's [**Vi**] proof to this question. As far as we know, it is still an open question whether dynamic or parameter rays are analytic in general.

4. *The proof of the bound on parameter rays*

4.1. *First derivative of dynamic rays.* In order to prove Proposition 3.3, we will need estimates on the derivative of dynamic rays, which lead to estimates on the winding numbers of dynamic rays. These in turn will help us control the rays at small potentials in order to obtain a bound on the absolute value of all $\kappa \in \mathcal{G}_{\underline{s}}(t)$ with prescribed combinatorics $(\underline{s}, t)$.

We will often be concerned with obtaining estimates on some 'tail pieces' of dynamic rays. In order to simplify the discussion without having to keep track of exact constants, we make the following definition.

Definition 4.1. (Properties on ray tails) We say that a property $P(\kappa, \underline{s}, t)$ holds *on ray tails* if there are $A \geq 1$ and $B, C \geq 0$ such that, for all $\underline{s} \in \mathcal{S}_0$ and all $K \geq 1$, the property $P(\kappa, \underline{s}, t)$ holds whenever $|\kappa| \leq K$ and

$$t \geq A t^*_{\underline{s}} + B \log K + C,$$

where $t^*_{\underline{s}}$ is the constant from Definition 2.5.

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Remark. The problem is that it is much easier to control tails of dynamic rays than to control entire rays. This control is non-uniform in κ: if $|\kappa|$ is large, then we have good control only for large potentials t. The following result often allows us to transfer results from ray tails to whole rays.

LEMMA 4.2. (From ray tails to entire rays) *Suppose a property $P(\kappa, \underline{s}, t)$ holds on ray tails and is backward invariant, i.e. it holds for $g_{\kappa,\underline{s}}(t)$ whenever it holds for $g_{\kappa,\sigma(\underline{s})}(F(t))$. Then it holds on the whole of all dynamic rays.*

Proof. The property holds on $g_{\kappa,\underline{s}}(t)$ as soon as there is an $n \in \mathbb{N}$ such that it holds on $g_{\kappa,\sigma^n(\underline{s})}(F^{\circ n}(t))$. This is true as soon as there is an N such that

$$F^{\circ N}(t) \geq A t^*_{\sigma^N(\underline{s})} + B \log K + C.$$

By Lemma 2.6, every $t > t_{\underline{s}}$ has an n so that $F^{\circ n}(t) > t^*_{\sigma^n \underline{s}}$; writing $F^{\circ n}(t) = t^*_{\sigma^n \underline{s}} + x_n$, we have

$$F^{\circ(n+1)}(t) = F(t^*_{\sigma^n \underline{s}} + x_n) > e^{x_n} F(t^*_{\sigma^n \underline{s}}) \geq e^{x_n} t^*_{\sigma^{n+1} \underline{s}}$$

and thus also $F^{\circ(n+1)}(t) = t^*_{\sigma^{n+1} \underline{s}} + x_{n+1}$ with $x_{n+1} \geq x_n F(t^*_{\sigma^n \underline{s}})$. Finite iteration of this last calculation proves the claim. □

The quantifier 'on ray tails' commutes with finite (but not infinite) conjunctions: if $P_1, \ldots, P_n$ are finitely many properties that each hold on ray tails, then $P_1 \wedge \cdots \wedge P_n$ also holds on ray tails (the constants A, B, C may depend on P_i).

Using this notation, we can now say that the asymptotic bound in equation (3) of Theorem 2.7 holds on ray tails: it holds for all

$$t \geq t^*_{\underline{s}} + 2 \log K + 2 \log 3 \geq t^*_{\underline{s}} + 2 \log(K + 3) = t^K_{\underline{s}}.$$

This gives us good control on the orbit of points on dynamic rays, except for at most finitely many steps. The following lemma helps to estimate after how many iteration steps good control takes over.

LEMMA 4.3. (Bound on initial iteration steps) *Fix $\underline{s} \in \mathcal{S}_0$, $t > t_{\underline{s}}$, $A \geq 1$ and $B, C \geq 0$. Then, for every $K \geq 1$, there is an $N \in \mathbb{N}$ such that, for all $n \geq N$, we have*

$$F^{\circ n}(t) \geq A t^*_{\sigma^n(\underline{s})} + B \log K + C. \tag{8}$$

The value of N has the following properties for fixed $\underline{s}$:

(1) *for fixed K, it is (weakly) monotonically decreasing in t;*
(2) *if $t \geq A t^*_{\underline{s}} + B \log K + C$, then $N = 0$; and*
(3) *if, for fixed t, N_0 is such that*

$$F^{\circ N_0}(t) \geq A t^*_{\sigma^{N_0}(\underline{s})} + C + 1,$$

then (8) holds for $N = N_0 + N_1$ as soon as $F^{\circ N_1}(1) \geq B \log K$.

Finally, $F(x + y) \geq F(x) + F(y)$ and $F(Ax) \geq A F(x)$ for all $x, y \geq 0$ and $A \geq 1$.

Proof. Note first that $0 \leq (e^x - 1)(e^y - 1) = F(x + y) - F(x) - F(y)$ for all $x, y \geq 0$, so $F(x + y) \geq F(x) + F(y)$. We have $F(x) = e^x - 1 \geq x$ for all x and even

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$F(Ax) \geq AF(x)$: to see this, it is enough to check that $F(A \cdot 0) = AF(0)$ and $(d/dx)(F(Ax) - AF(x)) \geq 0$. Moreover, by definition, $F(t^*_{\underline{s}}) \geq t^*_{\sigma(\underline{s})}$.

This implies that, if (8) holds for n, then it also holds for $n+1$. The second claim follows, and the first is trivial. The third claim is verified as follows:

$$\begin{aligned} F^{\circ N}(t) = F^{\circ(N_0+N_1)}(t) &\geq F^{\circ N_1}(At^*_{\sigma^{N_0}(\underline{s})} + C + 1) \\ &\geq F^{\circ N_1}(At^*_{\sigma^{N_0}(\underline{s})}) + F^{\circ N_1}(C) + F^{\circ N_1}(1) \\ &\geq At^*_{\sigma^N(\underline{s})} + C + B \log K. \end{aligned}$$ □

Using the terminology of 'properties on ray tails', the following statements follow easily from [**SZ1**, Lemma 3.3 and Proposition 3.4].

LEMMA 4.4. (Further properties of dynamic rays) *Let $\underline{s} \in \mathcal{S}_0$, $K > 0$ and κ be a parameter with $|\kappa| \leq K$. On the interval $(t^K_{\underline{s}}, \infty)$, the curve $g^\kappa_{\underline{s}}$ is the uniform limit of the functions $g^n_{\kappa,\underline{s}}$ defined by $g^0_{\kappa,\underline{s}} := \mathrm{id}$ and $g^{n+1}_{\kappa,\underline{s}}(t) := L_{\kappa,s_1} \circ g^n_{\kappa,\sigma\underline{s}}(F(t))$, i.e.*

$$g^n_{\kappa,\underline{s}}(t) := L_{\kappa,s_1} \circ \cdots \circ L_{\kappa,s_n}(F^{\circ n}(t)). \tag{9}$$

On ray tails, they satisfy, for all $n, k \geq 1$,

$$|g^n_{\kappa,\underline{s}}(F^{\circ k}(t))| \geq 2^k, \tag{10}$$

$$|g^k_{\kappa,\underline{s}}(t) - g^\kappa_{\underline{s}}(t)| \leq 2^{-k}. \tag{11}$$

Proof. Uniform convergence of $g^n_{\kappa,\underline{s}}(t)$ to $g^\kappa_{\underline{s}}$ and (9) is shown verbatim in [**SZ1**]. By [**SZ1**, Lemma 3.3], we have, for all n,

$$\operatorname{Re} g^n_{\kappa,\underline{s}}(F^{\circ k}(t)) \geq F^{\circ k}(t) - (K+2)$$

and this exceeds 2^k for all k if t satisfies a lower bound as specified in a condition 'on ray tails'.

If $t \geq t^*_{\underline{s}}$, then $|s_{n+1}| \leq F^{\circ n}(t)$ for all n, and (5) in the proof of [**SZ1**, Proposition 3.4] gives, for $k \geq 2$,

$$|g^k_{\kappa,\underline{s}}(t) - g^\kappa_{\underline{s}}(t)| < 4\pi e^{-t} \sum_{j=k}^{\infty} \prod_{m=1}^{j-2} e^{-m} < 4\pi e^{-t} \sum_{j=k}^{\infty} e^{-(j-2)} < 4\pi e^{-t} \frac{e^{-(k-2)}}{1-e^{-1}}$$

and this is less than 2^{-k} on ray tails. A similar bound for $|g^2_{\kappa,\underline{s}}(t) - g^1_{\kappa,\underline{s}}(t)|$ follows on ray tails from (6) in the proof of [**SZ1**, Proposition 3.4]. □

We omit the straightforward but technical proof of the following lemma.

LEMMA 4.5. (Some properties of F) *If $x \geq 0$, then we have, for sufficiently large $t > 0$,*

$$\sum_{k=1}^{\infty} \frac{1}{F^{\circ k}(t)+1} < 3e^{-t} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{F^{\circ k}(x)}{F^{\circ k}(t)+1} < (e^x+1)e^{-t},$$

and both sums converge to their limits uniformly in t. Moreover,

$$\frac{d}{dt} F^{\circ n}(t) = \prod_{k=1}^{n} (F^{\circ k}(t) + 1) \tag{12}$$

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and

$$\text{there exists } T > 0 : \text{ for all } t \geq T,\ n \geq 1 : \quad \frac{(F^{\circ n})'(t)}{(F^{\circ n}(t))^2} \leq \frac{1}{F^{\circ n}(t-1)+1}. \tag{13}$$

□

Differentiability of dynamic rays has already been proven in 1988 by Viana da Silva [**Vi**]. We will prove it again in order to obtain explicit estimates on the first and second derivatives.

PROPOSITION 4.6. (The derivative of dynamic rays) *For every sequence $\underline{s} \in \mathcal{S}_0$ and every parameter κ, the dynamic ray $g^\kappa_{\underline{s}}(t)$ is continuously differentiable with respect to the potential t and satisfies*

$$(g^\kappa_{\underline{s}})'(t) = \prod_{k=1}^{\infty} \frac{F^{\circ k}(t)+1}{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t))} \neq 0. \tag{14}$$

Moreover, on ray tails,

$$|(g^\kappa_{\underline{s}})'(t) - 1| < C_{\kappa,\underline{s}} e^{-t}, \tag{15}$$

with $C_{\kappa,\underline{s}} = 12(|\kappa|+6) + 8\pi(e^{t^*_{\underline{s}}}+1)$.

Proof. Recall the functions $g^n_{\kappa,\underline{s}}(t)$ from Lemma 4.4, which converge uniformly to $g^\kappa_{\underline{s}}(t)$. The chain rule gives, for $\kappa \in \mathbb{C}$, $\underline{s} \in \mathcal{S}_0$, $n \geq 1$ and t (where defined),

$$\begin{aligned}\frac{d}{dt} g^n_{\kappa,\underline{s}}(t) &= \frac{d}{dt} L_{\kappa,s_1} \circ \cdots \circ L_{\kappa,s_n}(F^{\circ n}(t)) = \frac{dF^{\circ n}(t)}{dt} \prod_{k=1}^{n} (g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)))^{-1} \\ &= \left(\prod_{k=1}^{n} \frac{F^{\circ k}(t)+1}{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t))}\right) \prod_{k=1}^{n} \underbrace{\frac{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t))}{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}}_{=:P^n_k(t)},\end{aligned} \tag{16}$$

where we used (12) in Lemma 4.5 in the last step.

Our first claim is that (on ray tails) $\prod_{k=1}^n P^n_k(t)$ converges uniformly to 1 as $n \to \infty$. Indeed, on ray tails,

$$|P^n_k(t) - 1| = \left|\frac{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t)) - g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right| \leq \frac{1}{2^{n-k}}\,\frac{1}{2^k} = \frac{1}{2^n}$$

by Lemma 4.4. Thus $|\prod_{k=1}^n P^n_k(t) - 1| \leq (1+2^{-n})^n - 1$, and this proves the claim.

Now we show that the left product of (16) converges uniformly on ray tails and satisfies the uniform bound (15) there. Note that

$$\operatorname{Log} \prod_{k=1}^{n} \frac{F^{\circ k}(t)+1}{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t))} = -\sum_{k=1}^{n} \operatorname{Log}\left(1 + \underbrace{\frac{g^\kappa_{\sigma^k\underline{s}}(F^{\circ k}(t)) - F^{\circ k}(t) - 1}{F^{\circ k}(t)+1}}_{=:x_k(t)}\right).$$

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By Theorem 2.7 part (2b), we have, on ray tails,

$$|x_k(t)| \le \frac{|\kappa| + 2\pi|s_{k+1}| + 6}{F^{\circ k}(t) + 1} \le \frac{|\kappa| + 6}{F^{\circ k}(t) + 1} + 2\pi \frac{F^{\circ k}(t_{\underline{s}}^*)}{F^{\circ k}(t) + 1}.$$

Thus, on ray tails, by the first two inequalities from Lemma 4.5,

$$\sum_{k=1}^{\infty} |x_k(t)| < (3(|\kappa| + 6) + 2\pi(e^{t_{\underline{s}}^*} + 1))e^{-t}$$

and this sum converges uniformly in t. Therefore, (16) converges uniformly in t. Continuous differentiability of $g_{\underline{s}}^{\kappa}$ and (14) follow from Weierstrass's theorem, and $\sum |x_k| < \infty$ implies that $(g_{\underline{s}}^{\kappa})'(t) \neq 0$; all this is on ray tails.

Since $|\mathrm{Log}(1 + x)| < 2|x|$ for $|x| \le 1/2$, we have, on ray tails,

$$|\mathrm{Log}(g_{\underline{s}}^{\kappa})'(t)| = \left|\mathrm{Log} \prod_{k=1}^{\infty} \frac{F^{\circ k}(t) + 1}{g_{\sigma^k \underline{s}}^{\kappa}(F^{\circ k}(t))}\right| \le \sum_{k=1}^{\infty} |\mathrm{Log}(1 + x_k(t))| \le \sum_{k=1}^{\infty} 2|x_k(t)|.$$

Since $|z - 1| \le 2|\log z|$ for $|\log z| \le 1/2$, (15) follows on ray tails.

We conclude the proof by showing that continuous differentiability and (14) hold on entire rays, not just on ray tails. Using Lemma 4.2, it is sufficient to prove that these conditions are preserved under $E_\kappa : g_{\underline{s}}^{\kappa}(t) \mapsto g_{\sigma\underline{s}}^{\kappa}(F(t))$. Differentiation yields

$$(g_{\underline{s}}^{\kappa})'(t) = \frac{(g_{\sigma\underline{s}}^{\kappa} \circ F)'(t)}{g_{\sigma\underline{s}}^{\kappa}(F(t))} = \left(\prod_{k=1}^{\infty} \frac{F^{\circ(k+1)}(t) + 1}{g_{\sigma^{k+1}}^{\kappa}(F^{\circ(k+1)}(t))}\right) \frac{F(t) + 1}{g_{\sigma\underline{s}}^{\kappa}(F(t))} = \prod_{k=1}^{\infty} \frac{F^{\circ k}(t) + 1}{g_{\sigma^k \underline{s}}^{\kappa}(F^{\circ k}(t))}$$

and this proves the claims. □

4.2. *Second derivative of dynamic rays.*

PROPOSITION 4.7. (The second derivative of dynamic rays) *Every dynamic ray $g_{\underline{s}}^{\kappa} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ is twice continuously differentiable. On ray tails,*

$$|(g_{\underline{s}}^{\kappa})''(t)| < C_{\underline{s}} e^{-t} \tag{17}$$

with $C_{\underline{s}} = 16\pi e(e^{t_{\underline{s}}^*} + 1)$, and

$$\left|\frac{(g_{\underline{s}}^{\kappa})''(t)}{(g_{\underline{s}}^{\kappa})'(t)}\right| < e^{-t/2}. \tag{18}$$

Proof. Define $f_{\underline{s}}(t) := (t + 1)/g_{\underline{s}}^{\kappa}(t)$ and, for $k \ge 1$,

$$P_k(t) := \frac{F^{\circ k}(t) + 1}{g_{\sigma^k \underline{s}}^{\kappa}(F^{\circ k}(t))} = f_{\sigma^k \underline{s}}(F^{\circ k}(t)).$$

We showed in Proposition 4.6 that the partial products $h_N(t) := \prod_{k=1}^{N} P_k(t)$ converge uniformly on ray tails to $(g_{\underline{s}}^{\kappa})'$. We thus need to show that the derivatives h_N' converge uniformly and their limit satisfies (17). By the product rule,

$$h_N'(t) = h_N(t) \sum_{k=1}^{N} \frac{P_k'(t)}{P_k(t)}. \tag{19}$$

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We start by showing that $\sum P_k'(t)$ converges on ray tails. To do this, we use

$$g^{\kappa}_{\sigma^k \underline{s}}(F^{\circ k}(t)) = F^{\circ k}(t) - \kappa + 2\pi i s_{k+1} + O(e^{F^{\circ k}(t^*_{\underline{s}}) - F^{\circ k}(t)})$$

(from Theorem 2.7 part (2b)) and

$$(g^{\kappa}_{\sigma^k \underline{s}})'(F^{\circ k}(t)) = 1 + O(e^{F^{\circ k}(t^*_{\underline{s}}) - F^{\circ k}(t)})$$

(from Proposition 4.6) and estimate

$$\begin{aligned} |f'_{\sigma^k \underline{s}}(F^{\circ k}(t))| &= \left| \frac{g^{\kappa}_{\sigma^k \underline{s}}(F^{\circ k}(t)) - (F^{\circ k}(t) + 1)(g^{\kappa}_{\sigma^k \underline{s}})'(F^{\circ k}(t))}{(g^{\kappa}_{\sigma^k \underline{s}}(F^{\circ k}(t)))^2} \right| \\ &= \left| \frac{-\kappa + 2\pi i s_{k+1} - 1 + F^{\circ k}(t) O(e^{F^{\circ k}(t^*_{\underline{s}}) - F^{\circ k}(t)})}{(F^{\circ k}(t) - \kappa + 2\pi i s_{k+1} + O(e^{F^{\circ k}(t^*_{\underline{s}}) - F^{\circ k}(t)}))^2} \right| \\ &\le \frac{2\pi F^{\circ k}(t^*_{\underline{s}})(1 + O(F^{\circ k}(t) e^{-F^{\circ k}(t)})}{((F^{\circ k}(t) + O(1))^2} < 4\pi \frac{F^{\circ k}(t^*_{\underline{s}})}{((F^{\circ k}(t))^2} \end{aligned}$$

if t is sufficiently large (uniformly for all k). Therefore (13) in Lemma 4.5 implies that

$$|P_k'(t)| = |f'_{\sigma^k \underline{s}}(F^{\circ k}(t))|\, (F^{\circ k})'(t) \le 4\pi \frac{F^{\circ k}(t^*_{\underline{s}})}{F^{\circ k}(t-1) + 1}.$$

Using Lemma 4.5 once more,

$$\sum_{k=1}^{\infty} |P_k'(t)| < 4\pi (e^{t^*_{\underline{s}}} + 1) e^{-t+1}, \tag{20}$$

with uniform convergence on ray tails.

Since $|s_{k+1}| \le F^{\circ k}(t^*_{\underline{s}})$ and $g^{\kappa}_{\sigma^k \underline{s}}(F^{\circ k}(t)) = F^{\circ k}(t) - \kappa + 2\pi i s_{k+1} + o(1)$, we get, on ray tails,

$$|P_k(t)| \ge \frac{F^{\circ k}(t) + 1}{F^{\circ k}(t) + |\kappa| + 2\pi |s_{k+1}| + 1} \ge 1/2 \quad \text{for every } k \ge 1.$$

It follows similarly that $|h_N(t)| \le 2$ on ray tails. This shows that the h'_N converge uniformly, and by Weierstrass's theorem, $(g^{\kappa}_{\underline{s}})'(t) = \lim h_N(t)$. The estimate (17) follows using (19) and (20).

If $t > 2t^*_{\underline{s}} + c$ for an appropriate $c > 0$, then $|(g^{\kappa}_{\underline{s}})''(t)| < C_{\underline{s}} e^{-t} < e^{-t/2}/3$, and this is a condition on ray tails. Similarly, on ray tails, $|(g^{\kappa}_{\underline{s}})'(t)| > 1 - e^{-t/2}/3$, and together this implies (18) on ray tails. □

4.3. *Variation numbers of dynamic rays.* Several key ideas in this section are due to Niklas Beisert.

Recall that, for a C^1-curve $\gamma : [t_0, t_1] \to \mathbb{C}$ and $a \in \mathbb{C} \setminus \gamma([t_0, t_1])$, the *winding number of γ around a* is defined by

$$\eta(\gamma, a) := \frac{1}{2\pi} \int_{t_0}^{\infty} \operatorname{Im} \frac{\gamma'(t)}{\gamma(t) - a}\, dt = \frac{1}{2\pi} \int_{t_0}^{t_1} d \arg(\gamma(t) - a). \tag{21}$$

This is most often defined if γ is closed, i.e. $\gamma(t_1) = \gamma(t_0)$. In this case, $\eta(\gamma) \in \mathbb{Z}$. We will consider curves that are not closed, and we need to keep track of the variation of the integrand in (21). We thus make the following definitions.

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Definition 4.8. (Variation number) Consider a C^1-curve $\gamma : (t_0, \infty) \to \mathbb{C}$ with $t_0 \geq -\infty$ and $a \notin \gamma((t_0, \infty))$. Define the *variation number of γ around a* by

$$\alpha(\gamma, a) := \frac{1}{2\pi} \int_{t_0}^{\infty} \left| \operatorname{Im} \frac{\gamma'(t)}{\gamma(t) - a} \right| dt = \frac{1}{2\pi} \int_{t_0}^{\infty} |d \arg(\gamma(t) - a)| \quad \in [0, \infty].$$

Unlike the winding number, the variation number also measures local oscillations of the curve. It is preserved under all reparametrizations of γ, orientation preserving or reversing.

Definition 4.9. (Admissible curves) An *admissible curve* is an injective C^2-curve $\gamma : (t_0, \infty) \to \mathbb{C}$ with non-vanishing derivative γ' such that, for $t \to \infty$,

$$|\gamma(t)| = t + O(1), \quad |\gamma'(t)| = 1 + O(1/t) \quad \text{and} \quad |\gamma''(t)| = O(1/t^2).$$

Remark. The definitions are made so that dynamic rays $g_{\underline{s}}^{\kappa} : ((t_{\underline{s}}, \infty)) \to \mathbb{C}$ are admissible curves (by Theorem 2.7 part (2b) and Propositions 4.6 and 4.7).

LEMMA 4.10. (Admissible curves have variation numbers) *Let $\gamma : (t_0, \infty) \to \mathbb{C}$ be an admissible curve.*

(1) *If $a \in \mathbb{C} \setminus \overline{\gamma((t_0, \infty))}$ and $|\gamma'(t)|$ is bounded as $t \searrow t_0$, then $\alpha(\gamma, a)$ is finite.*
(2) *For every $t_1 > t_0$, $\alpha(\gamma|_{(t_1,\infty)}, \gamma(t_1))$ is finite.*
(3) *For every $t_1 > t_0$, $\alpha(\gamma'|_{(t_1,\infty)}, 0)$ is finite.*

Proof. In all statements the integrands are locally Riemann integrable. Therefore we only have to show that the integrals $\int_{t_0}^{\infty} |\operatorname{Im}(\gamma'/\gamma - a)|$ are finite near the boundaries of integration.

Let us first discuss the lower boundary of integration. In case (1), we can bound $|\gamma(t) - a|$ below and $|\gamma'(t)|$ above. Fix $\varepsilon > 0$. In case (3) the continuous function $|\gamma''(t)/\gamma'(t)|$ is bounded on $[t_1, t_1 + \varepsilon]$. In case (2), however, the denominator tends to 0. By the Taylor theorem applied to $\gamma \in C^2$, for every $t \in (t_1, t_1 + \varepsilon)$ there is a $\xi \in [t_1, t]$ such that

$$\begin{aligned} \left| \operatorname{Im} \frac{\gamma'(t)}{\gamma(t) - \gamma(t_1)} \right| &= \left| \operatorname{Im} \frac{\gamma'(t)}{\gamma'(t)(t - t_1) + \gamma''(\xi)(t - t_1)^2/2} \right| \\ &= \left| \operatorname{Im} \frac{1}{t - t_1 + (\gamma''(\xi)/2\gamma'(t))\,(t - t_1)^2} \right|. \end{aligned}$$

For $\xi, t \in [t_1, t_1 + \varepsilon]$, $(\gamma''(\xi)/2\gamma'(t)) =: c(t)$ is uniformly bounded and thus of class $O(1)$. Now for $t - t_1 =: \delta \searrow 0$ we observe

$$\operatorname{Im} \frac{1}{\delta + c\delta^2} = \frac{\delta^2 \operatorname{Im} \bar{c}}{|\delta + c\delta^2|^2} = \frac{-\operatorname{Re} c}{|1 + c\delta|^2} = O(1).$$

It is left to show that the limits $\lim_{x \to \infty} \int^x$ are finite. For the first two cases we have (with $\gamma(t_1) =: a$ in case (2))

$$\begin{aligned} \operatorname{Im} \frac{\gamma'(t)}{\gamma(t) - a} &= \operatorname{Im} \frac{1 + O(t^{-1})}{t + O(1)} = \operatorname{Im} \frac{(1 + O(t^{-1}))\overline{(t + O(1))}}{|t + O(1)|^2} \\ &= \frac{\operatorname{Im}(t + O(1))}{|t + O(1)|^2} = O(t^{-2}), \end{aligned}$$

and for the last case we estimate $|\operatorname{Im}(\gamma''(t)/\gamma'(t))| \leq |\gamma''(t)/\gamma'(t)| = O(1/t^2)$. □

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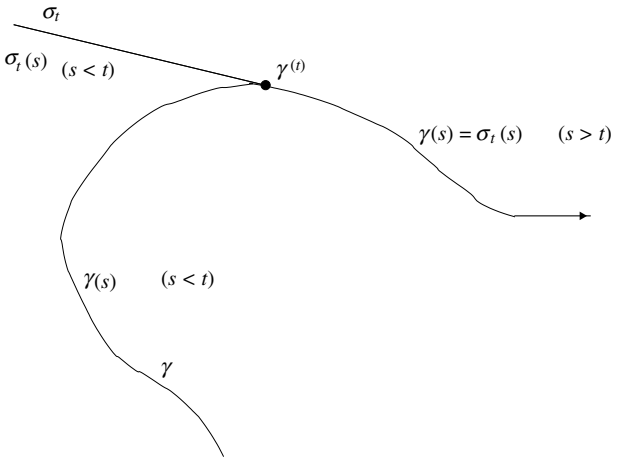


FIGURE 4. The linear continuation of γ as defined in the proof of Lemma 4.12.

LEMMA 4.11. (The variation number of a half line) *For the curve* $\ell : \mathbb{R}^+ \to \mathbb{C}$, $\ell(s) = \lambda s$ *(*$\lambda \in \mathbb{C}^*$*), we have, for every* $a \notin \lambda\mathbb{R}_0^+$,

$$\alpha(\ell, a) = \frac{|\arg(a/\lambda) - \pi|}{2\pi},$$

if $\arg(a/\lambda)$ *is chosen in the interval* $[0, 2\pi]$.

Proof. The quantity $\alpha(\ell, a)$ measures the change of argument with respect to a as a point traverses the line $\lambda\mathbb{R}_0^+$. This is invariant under multiplication by $1/\lambda$ and translation by $-a/\lambda$: then $\alpha(\ell, a)$ measures the change of argument along the half line $\ell' := -a/\lambda + \mathbb{R}^+$ with respect to the origin. If $\operatorname{Im} a/\lambda < 0$, i.e. $\arg(a/\lambda) > \pi$, then the two endpoints of ℓ' are at angles 0 and $\arg(-a/\lambda) = \arg(a/\lambda) - \pi$; if $\operatorname{Im} a/\lambda > 0$, i.e. $\arg(a/\lambda) < \pi$, then the two endpoints are at angles 2π and $\arg(-a/\lambda) = \arg(a/\lambda) + \pi$; and the boundary case is $a/\lambda < 0$ with $\arg(a/\lambda) = \pi$ and $\alpha(\ell, a) = 0$. □

The following lemma gives a very useful connection between the variation number of a curve and of its derivative.

LEMMA 4.12. (Variation numbers of a curve and its derivative) *Let* $\gamma_0 : (t_0, \infty) \to \mathbb{C}$ *be an admissible curve, and* $\gamma := \gamma_0|_{(t_1,\infty)}$ *its restriction for some* $t_1 > t_0$. *Then, for every* $a \notin \gamma_0([t_1, \infty))$,

$$\alpha(\gamma, a) \le \alpha(\gamma', 0) + 1/2.$$

Proof. For every $t \ge t_1$ define a C^1-curve $\sigma_t : \mathbb{R} \to \mathbb{C}$ to be the linear continuation of γ at t, i.e. $\sigma_t(s) := \gamma(s)$ if $s \ge t$ and $\sigma_t(s) := \gamma(t) + (s - t)\gamma'(t)$ if $s \le t$ (see Figure 4).

As $t \to +\infty$, the curves σ_t approximate a horizontal straight line with $\lim_{t\to\infty} \alpha(\sigma_t, a) = 1/2$. For $t = t_1$, the curve σ_t is the union of γ extended by a straight half line, and $\alpha(\sigma_{t_1}, a) \ge \alpha(\gamma, a)$. The key relation is

$$\frac{d}{dt}\alpha(\sigma_t, a) = -\frac{1}{2\pi}|d \arg \gamma'| = -\frac{d}{dt}\alpha(\gamma', 0) \tag{22}$$

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provided $a \notin \sigma_t$. Assuming this key relation, we obtain

$$\alpha(\gamma, a) \le \alpha(\sigma_{t_1}, a) = \lim_{\tau\to\infty} \left(- \int_{t_1}^{\tau} \frac{d\alpha(\sigma_t, a)}{dt}\, dt + \alpha(\sigma_\tau, a) \right) = \alpha(\gamma', 0) + 1/2.$$

Equation (22) is justified as follows. The change from σ_t to $\sigma_{t+\delta}$ has the effect that the straight part of σ_t approaches ∞ in a different direction (as long as the straight parts in σ_t do not contain a, the sign of $d \arg(\gamma(t) - a)$ in the definition of $\alpha(\gamma, a)$ is unchanged). The change of direction is measured by $\arg \gamma'(t+\delta) - \arg \gamma'(t)$. The minus sign in (22) comes from the fact that larger values of t mean that σ_t follows γ along a shorter subcurve before turning into a straight line.

Extra care must be taken when t is so that $a \in \sigma_t(\mathbb{R})$ (i.e. a is on the straight part of σ_t). In the neighborhood of such values t, there are branch cuts of arg, and a different branch needs to be used. Details are not enlightening and thus omitted. □

PROPOSITION 4.13. (Variation numbers and pullbacks) *Let $\gamma_0 : (t_0, \infty) \to \mathbb{C}$ be an admissible curve such that $\alpha((\gamma_0)', 0) \le 1/2$. For every $n \in \mathbb{N}$ choose $a_n \in \mathbb{C} \setminus \overline{\gamma_n(t_0, \infty)}$ and $\gamma_{n+1} : (t_0, \infty) \to \mathbb{C}$ such that $\exp(\gamma_{n+1}(t)) = \gamma_n(t) - a_n$. Then*

$$\begin{aligned} &\text{for all } n \in \mathbb{N}, \quad \text{for all } t_1 > t_0, \text{ for all } b \notin \overline{\gamma_n(t_1, \infty)} : \\ &\alpha(\gamma_n|_{(t_1,\infty)}, b) \le 2^n \quad \text{and} \quad \alpha((\gamma_n)'|_{(t_1,\infty)}, 0) \le 2^n - 1/2. \end{aligned}$$

Proof. It follows from the calculations below that all the variation numbers are indeed defined. The case $n = 0$ follows from Lemma 4.12. Let a, γ, $\tilde{\gamma}$ denote a_n, γ_n, $\gamma_{n+1} = \log(\gamma_n - a)$ respectively. We estimate (the branches of the log are inessential for the derivative)

$$\begin{aligned} \alpha(\tilde{\gamma}', 0) &= \alpha\left(\frac{\gamma'}{\gamma - a}, 0\right) = \frac{1}{2\pi} \int \left| d\left(\operatorname{Im} \log \left(\frac{\gamma'}{\gamma - a} \right) \right) \right| \\ &= \frac{1}{2\pi} \int |d(\operatorname{Im} \log \gamma' - \operatorname{Im} \log(\gamma - a))| \\ &= \frac{1}{2\pi} \int \left| \operatorname{Im}\left(\frac{\gamma''}{\gamma'}\right) - \operatorname{Im}\left(\frac{\gamma'}{\gamma - a}\right) dt \right| \\ &\le \frac{1}{2\pi} \int \left| \operatorname{Im}\left(\frac{\gamma''}{\gamma'}\right) \right| dt + \frac{1}{2\pi} \int \left| \operatorname{Im}\left(\frac{\gamma'}{\gamma - a}\right) \right| dt \\ &= \alpha(\gamma', 0) + \alpha(\gamma, a) \overset{\text{Lemma 4.12}}{\le} 2\alpha(\gamma', 0) + 1/2. \end{aligned}$$

Induction yields $\alpha(\gamma'_{n+1}, 0) \le 2(2^n - 1/2) + 1/2 = 2^{n+1} - 1/2$. Lemma 4.12 thus gives us $\alpha(\gamma_n, a) \le \alpha((\gamma_n)', 0) + 1/2 \le 2^n$ for all $n \in \mathbb{N}$. □

4.4. *The bound on parameter rays.* In this final subsection, we will complete the proof of Proposition 3.3. It may be helpful to outline the general line of argument before going into details. We want to construct parameter rays $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$; we know these exist as curves for sufficiently large potentials (Proposition 3.2). The danger is that there is a $\tilde{t} > t_{\underline{s}}$ such that, as $t \searrow \tilde{t}$, $G_{\underline{s}}(t) \to \infty$.

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We will of course use our estimates on dynamic rays (Theorem 2.7). The problem is that these estimates depend on κ and become worse as $\kappa \to \infty$. Rescue comes from ∞ in a different way. For a given parameter $\kappa = G_{\underline{s}}(t)$ with $t > t_{\underline{s}}$, one needs to iterate the dynamic ray tail $g_{\underline{s}}(t, \infty)$ only a finite number of times until the iterated image ray is almost horizontal. If N is this number of iterations, then the winding number of $g_{\underline{s}}(t, \infty)$ will be bounded by 2^N (Proposition 4.14). This induces a partition of the dynamical plane with vertical uncertainty of approximately 2^N (Lemma 4.15). But if now $\operatorname{Re}\kappa$ is too large, then the imaginary bounds imply that the singular orbit must escape very fast (Proposition 4.16), which means it must have large potential t. For bounded potential t, this yields an upper bound for $\operatorname{Re}\kappa$.

The success of this argument depends on the fact that, as $|\kappa|$ increases, the number N of necessary iterations, and thus the upper bound for $\operatorname{Re}\kappa$, grows extremely slowly in $|\kappa|$, much more slowly than the errors arising from the growth of $|\kappa|$ itself.

PROPOSITION 4.14. (Variation number of dynamic rays) *Consider an arbitrary parameter κ and an external address $\underline{s} \in \mathcal{S}_0$. Let $z = g^{\kappa}_{\underline{s}}(t_0)$ be any point on the dynamic ray of address $\underline{s}$ with potential $t_0 > t_{\underline{s}}$. Suppose that $N \in \mathbb{N}$ is such that, for all $t \geq F^{\circ N}(t_0)$,*

$$\left| \frac{g''_{\kappa,\sigma^N\underline{s}}(t)}{g'_{\kappa,\sigma^N\underline{s}}(t)} \right| < e^{-t/2}.$$

Then

$$\alpha(g^{\kappa}_{\underline{s}}|_{(t_0,\infty)}, z) \leq 2^N.$$

Proof. On ray tails, we have bounds on the rays (Theorem 2.7 part (2b)), their first derivatives (Proposition 4.6) and their second derivatives (Proposition 4.7), so ray tails (and hence entire dynamic rays) are admissible curves.

Specifically, the curve $\gamma := g^{\kappa}_{\sigma^N\underline{s}} : (F^{\circ N}(t_0), \infty) \to \mathbb{C}$ satisfies

$$\begin{aligned} \alpha(\gamma', 0) &= \frac{1}{2\pi} \int_{F^{\circ N}(t_0)}^{\infty} \left| \operatorname{Im}\left(\frac{\gamma''(t)}{\gamma'(t)} \right) \right| dt \leq \frac{1}{2\pi} \int_{F^{\circ N}(t_0)}^{\infty} \left| \frac{g''_{\kappa,\sigma^N\underline{s}}(t)}{g'_{\kappa,\sigma^N\underline{s}}(t)} \right| dt \\ &< \frac{1}{2\pi} \int_{F^{\circ N}(t_0)}^{\infty} e^{-t/2}\, dt \leq \pi^{-1} e^{-F^{\circ N}(t_0)/2} < 1/2. \end{aligned}$$

If we define $\gamma_0 := \gamma$ and $\gamma_{k+1} := L_{\kappa,s_{N-k}}(\gamma_k)$, then

$$\gamma_N = g^{\kappa}_{\underline{s}}|_{t>t_0},$$

and applying Proposition 4.13 settles the claim. □

LEMMA 4.15. (Bounding the imaginary parts) *Suppose that $s \in \mathcal{S}_0$ and that κ is a parameter such that $g^{\kappa}_{\underline{s}}(t_0) = 0$ for some potential $t_0 > t_{\underline{s}}$. Let N be as in Proposition 4.14. Then the singular orbit $(z_k)_{k\geq 1} = (E^{\circ(k-1)}_{\kappa}(0))_{k\geq 1}$ satisfies, for all $k \geq 1$,*

$$|\operatorname{Im} z_k| \leq 2\pi(2^N + 1 + |s_k|) + |\operatorname{Im}\kappa|. \tag{23}$$

Furthermore,

$$|\operatorname{Im}\kappa| \leq 2\pi(2^N + 1 + |s_1|).$$

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Proof. The inverse images of the ray $g_{\underline{s}}^{\kappa}|_{(t_0,\infty)}$ provide a *dynamic partition* of the plane, as opposed to the *static partition* introduced in §2. Note that dynamic rays cannot intersect the boundary of the dynamic partition, so that each ray $g_{\underline{s}'}^{\kappa}$ has to be contained in one of the two components that are asymptotic to the line $t - \kappa + 2\pi i s_1'$ for large t. By Proposition 4.14, the vertical variation of any boundary component of the dynamic partition is bounded by $2\pi \cdot 2^N$; since the boundaries are a vertical distance 2π apart, (23) follows.

The additional inequality follows similarly. A single strip in the dynamic partition contains the singular value 0 and the dynamic ray $g_{\underline{s}}^{\kappa}$ with asymptotic imaginary parts $-\operatorname{Im}\kappa + 2\pi s_1$, so we have $|-\operatorname{Im}\kappa + 2\pi s_1| \le 2\pi(2^N + 1)$. The triangle inequality gives the desired estimate. □

PROPOSITION 4.16. (The behavior of the singular orbit) *Let κ be an escaping parameter such that* $\operatorname{Re}\kappa \ge 4$ *and* $|\operatorname{Im}\kappa| \le \operatorname{Re}\kappa - 2$. *Let* $K := |\kappa|$ *and* $(z_k)_{k\ge1} := (E_\kappa^{\circ(k-1)}(0))_{k\ge1}$ *be the singular orbit, and suppose that, for every* $k \ge 2$,

$$|\operatorname{Im} z_k| < F^{\circ(k-1)}(\operatorname{Re}\kappa - 2) - K.$$

Then, for all $k \ge 1$,

$$\operatorname{Re} z_k \ge F^{\circ(k-1)}(\operatorname{Re}\kappa - 1) - \operatorname{Re}\kappa.$$

Proof. The calculations of the proof are tedious, but the idea is simple. If $\operatorname{Re} z_k$ is large and positive, then $|z_{k+1}| = |E_\kappa(z_k)|$ is exponentially large. Since the imaginary parts of the orbit are bounded, then $|\operatorname{Re} z_{k+1}|$ must be almost as large as $|z_{k+1}|$. If $\operatorname{Re} z_{k+1} < 0$, then z_{k+2} is extremely close to the origin, and there is an attracting orbit of period at most $k + 2$; thus $\operatorname{Re} z_{k+1} > 0$.

The calculations go as follows. For $k = 1$, we clearly have

$$\operatorname{Re} z_1 = 0 > -1 = F^{\circ 0}(\operatorname{Re}\kappa - 1) - \operatorname{Re}\kappa.$$

By induction, assume that $\operatorname{Re} z_k \ge F^{\circ(k-1)}(\operatorname{Re}\kappa - 1) - \operatorname{Re}\kappa$. We get

$$|z_{k+1}| = |E_\kappa(z_k)| \ge \exp(F^{\circ(k-1)}(\operatorname{Re}\kappa - 1)) > F^{\circ k}(\operatorname{Re}\kappa - 1)$$

and

$$\begin{aligned}
|\operatorname{Re} z_{k+1}| &= \sqrt{|z_{k+1}|^2 - |\operatorname{Im} z_{k+1}|^2} \\
&> \sqrt{(F^{\circ k}(\operatorname{Re}\kappa - 1))^2 - (F^{\circ k}(\operatorname{Re}\kappa - 2) - K)^2} \\
&= (F^{\circ k}(\operatorname{Re}\kappa - 1))\sqrt{1 - \left(\frac{F^{\circ k}(\operatorname{Re}\kappa - 2) - K}{F^{\circ k}(\operatorname{Re}\kappa - 1)}\right)^2} \\
&> F^{\circ k}(\operatorname{Re}\kappa - 1) - \frac{(F^{\circ k}(\operatorname{Re}\kappa - 2) - K)^2}{2F^{\circ k}(\operatorname{Re}\kappa - 1)}. \qquad (24)
\end{aligned}$$

We claim that the last fraction is less than $\operatorname{Re}\kappa$. All $x \ge 4$ and all $k \ge 1$ satisfy

$$(F^{\circ k}(x - 2))^2 \le 2x F^{\circ k}(x - 1)$$

and

$$x \le F^{\circ k}(x - 2)$$

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(we omit the details). Together, this implies that

$$(F^{\circ k}(x-2))^2 + 2xK - 2KF^{\circ k}(x-2) \le 2xF^{\circ k}(x-1).$$

Setting $x = \operatorname{Re}\kappa$, then $K \le |\operatorname{Re}\kappa| + |\operatorname{Im}\kappa| < 2|\operatorname{Re}\kappa|$ yields

$$(F^{\circ k}(\operatorname{Re}\kappa - 2))^2 + K^2 - 2KF^{\circ k}(\operatorname{Re}\kappa - 2) < 2\operatorname{Re}\kappa F^{\circ k}(\operatorname{Re}\kappa - 1).$$

Together with (24), we obtain

$$|\operatorname{Re} z_{k+1}| > F^{\circ k}(\operatorname{Re}\kappa - 1) - \operatorname{Re}\kappa$$

as claimed.

It remains to show that $\operatorname{Re} z_{k+1} > 0$. Suppose that $\operatorname{Re} z_{k+1} < 0$. We claim that E_κ has an attracting orbit, in contradiction to the assumption that $\kappa \in I$. Let D_{k+1} be the disk of radius 1 around z_{k+1}. Then D_{k+1} can be pulled back along the orbit of z_1 so that there is a neighborhood D_1 of z_1 and a conformal isomorphism $E_\kappa^{\circ k}\colon D_1 \to D_{k+1}$. We will show that $E_\kappa(D_{k+1}) \subset D_1$; this would imply that there was an attracting periodic point within D_1.

First observe that $E_\kappa(D_{k+1})$ is contained in the disk around 0 with radius given by $\exp(\operatorname{Re} z_{k+1} + 1 + \operatorname{Re}\kappa)$; call this disk B_1. We will show that $B_1 \subset D_1$.

The next step is to observe that, for $m = 1, 2, \ldots, k$,

$$|z_{m+1}| = \exp(\operatorname{Re}(z_m + \kappa)) > 4(\operatorname{Re} z_m)^2 > (\operatorname{Re} z_m + \operatorname{Im} z_m)^2 \ge |z_m|^2.$$

An easy induction proves that

$$|(E_\kappa^{\circ k})'(z_1)| = \prod_{m=1}^{k} |E_\kappa'(z_m)| = \prod_{m=1}^{k} |E_\kappa(z_m)| = \prod_{m=1}^{k} |z_{m+1}| < |z_{k+1}|^2.$$

By the Koebe 1/4 theorem, it follows that D_1 contains the disk around $z_1 = 0$ of radius $|z_{k+1}|^{-2}/4$. We thus have an attracting orbit as soon as we know that

$$|z_{k+1}|^{-2}/4 \ge \exp(\operatorname{Re} z_{k+1} + 1 + \operatorname{Re}\kappa).$$

But $|z_{k+1}| \ge |z_2| = \exp(\operatorname{Re}\kappa) \ge e^4$, hence $|\operatorname{Re} z_{k+1}| > \exp(\operatorname{Re}\kappa)/2 > 2\operatorname{Re}\kappa + 2$. This implies that

$$\exp(-(\operatorname{Re} z_{k+1} + 1 + \operatorname{Re}\kappa)) > \exp(-\operatorname{Re} z_{k+1}/2) > 16(\operatorname{Re} z_{k+1})^2 > 4|z_{k+1}|^2$$

and this proves the claim. □

Proof of Proposition 3.3. Fix an external address $\underline{s}$ and a potential $t > t_{\underline{s}}$. We need to show that there is a $\xi_{\underline{s}}(t) > 0$ so that $g_{\underline{s}}^\kappa(t) = 0$ implies $|\kappa| < \xi_{\underline{s}}(t)$, and that $\xi(\underline{s}) \le 2t$ for sufficiently large t. It suffices to consider the case $|\kappa| > e$. Set $K := |\kappa|$.

By Proposition 4.7, the inequality

$$\left| \frac{(g_{\sigma^n(\underline{s})}^\kappa)''(t)}{(g_{\sigma^n(\underline{s})}^\kappa)'(t)} \right| < e^{-t/2} < 1$$

holds on ray tails; recall from Definition 4.1 that this means that there are constants $A \ge 1$, B, $C \ge 0$ so that the inequality holds for any $n \ge 0$ provided $t \ge At^*_{\sigma^n(\underline{s})} + B\log K + C$.

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By Lemma 4.3, there is an $N_0 \in \mathbb{N}$ so that $F^{\circ N_0}(t) \geq A t^*_{\sigma^{N_0}(\underline{s})} + C + 1$. Let $N_1 \in \mathbb{N}$ be minimal with $F^{\circ N_1}(1) \geq B \log K > 1$. Then $N_1 \geq 1$ and $F^{\circ(N_1-1)}(1) < B \log K$.

By Lemma 4.3 part (3), we have, for all $n \geq N_0 + N_1 =: N$,

$$F^{\circ n}(t) \geq A t^*_{\sigma^n(\underline{s})} + B \log K + C.$$

There is a constant $c_1 > 0$ such that all $n \in \mathbb{N}$ satisfy $2^n \leq c_1 F^{\circ(n-1)}(1)$.

By Lemma 4.15, we have the estimates

$$\begin{aligned} |\operatorname{Im} \kappa| &\leq 2\pi(2^N + 1 + |s_1|) \leq 2\pi(2^{N_0} c_1 F^{\circ(N_1-1)}(1) + 1 + |s_1|) \\ &\leq 2\pi(2^{N_0} c_1 B \log K + 1 + |s_1|) = c_2 + c_3 2^{N_0} \log K \end{aligned} \tag{25}$$

with constants $c_2, c_3 > 0$ depending only on $|s_1|$, and also

$$|\operatorname{Im} z_k| \leq 2\pi(2^N + 1 + |s_k|) + |\operatorname{Im} \kappa|. \tag{26}$$

There is an $M \geq 1$ so that every $K \geq M$ satisfies the following two implicit equations:

$$K \geq 2c_2 + 2c_3 2^{N_0} \log K + 2, \tag{27}$$

$$K \geq 2\pi t^*_{\underline{s}} + (2\pi 2^{N_0} c_1 B + c_3 2^{N_0} + 1) \log K + 2\pi + 4 + c_2. \tag{28}$$

Now suppose that $K \geq M$. By (25) and (27) this implies that

$$2|\operatorname{Im} \kappa| \leq 2c_2 + 2c_3 2^{N_0} \log K \leq K - 2 \leq |\operatorname{Re} \kappa| + |\operatorname{Im} \kappa| - 2$$

and hence

$$|\operatorname{Im} \kappa| \leq |\operatorname{Re} \kappa| - 2. \tag{29}$$

Similarly, using (28), (25) and $F(\log K + 2) \geq 2K$ we have

$$\begin{aligned} |\operatorname{Re} \kappa| - 2 &\geq K - |\operatorname{Im} \kappa| - 2 \\ &\geq 2\pi t^*_{\underline{s}} + 2\pi 2^{N_0} c_1 B \log K + \log K + 2\pi + 2 \\ &\geq 2\pi t^*_{\underline{s}} + 2\pi 2^{N_0} c_1 F^{\circ(N_1-1)}(1) + 2\pi + F^{-1}(2K) \\ &\geq 2\pi t^*_{\underline{s}} + 2\pi 2^{N_0} 2^{N_1} + 2\pi + F^{-1}(2K). \end{aligned} \tag{30}$$

Parameters with $\operatorname{Re} \kappa < -1$ are known (and easily seen) to be attracting. Since our parameter κ is escaping, (30) holds for $\operatorname{Re} \kappa$ instead of $|\operatorname{Re} \kappa|$; in particular we have $\operatorname{Re} \kappa > 2 + 2\pi$. We obtain for $k \geq 2$ (using convexity of F and $F^{\circ k}(2\pi x) \geq 2\pi F(x)$ from Lemma 4.3)

$$\begin{aligned} F^{\circ(k-1)}(\operatorname{Re} \kappa - 2) &\geq F^{\circ(k-1)}(2\pi t^*_{\underline{s}} + 2\pi 2^N + 2\pi + F^{-1}(2K)) \\ &> 2\pi F^{\circ(k-1)}(t^*_{\underline{s}}) + 2\pi 2^N + 2\pi + 2K \\ &\geq 2\pi t^*_{\sigma^{k-1}(\underline{s})} + 2\pi 2^N + 2\pi + 2K \\ &\geq 2\pi(|s_k| + 2^N + 1) + 2K \geq |\operatorname{Im} z_k| + K. \end{aligned}$$

We can thus invoke Proposition 4.16 and obtain, for all $k \geq 2$,

$$F^{\circ(k-1)}(\operatorname{Re} \kappa - 1) - \operatorname{Re} \kappa \leq \operatorname{Re} z_k \overset{\text{(4) in Theorem 2.7}}{\leq} F^{\circ(k-1)}(t) - \operatorname{Re} \kappa + o(1).$$

Since $\operatorname{Re} \kappa > 2 + 2\pi$, both sides of this inequality tend to ∞ as $k \to \infty$. Comparing the growth of the left- and the right-hand sides as $k \to \infty$, we conclude that $\operatorname{Re} \kappa \leq t + 1$. The triangle inequality thus yields $K \leq |\operatorname{Re} \kappa| + |\operatorname{Im} \kappa| < 2|\operatorname{Re} \kappa| - 2 \leq 2t$.

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Throughout this argument, we had $\underline{s}$ fixed. Let us review the dependence on $t > t_{\underline{s}}$. Depending on t we had chosen N_0 and thus M; the values of c_1, c_2, c_3 do not depend on t. We then showed that $K \geq M$ implies that $K \leq 2t$; this amounts to

$$K \leq \max\{2t, M\}.$$

Increasing t does not increase N_0 and thus M; we can thus choose M so that it depends continuously and monotonically decreasing on t. We thus have a valid choice $\xi_{\underline{s}}(t) := \max\{2t, M\}$; monotonicity of M shows that $\xi_s(t) = 2t$ for large t. This proves the claim. □

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