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# The join of several subnormal subgroups

By J. P. WILLIAMS
*Trinity College, Cambridge*

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## 1. *Introduction*

In (2) Lennox and Stonehewer proved:

Let $G$ be a group with subnormal subgroups $H$ and $K$, and let $J = \langle H, K \rangle$. Suppose that $J/J'$ has finite rank. Then

(i) given positive integers $a$ and $b$, there exists $c$ such that $\gamma_c(J) \leqslant \gamma_a(H).\gamma_b(K)$ and

(ii) $J$ is subnormal in $G$.

Here $\gamma_n(X)$ denotes the $n'$th term of the lower central series of a group $X$, and 'rank' means Prüfer rank, i.e. a group $X$ is said to have rank $r$ if each finitely generated subgroup of $X$ can be generated by at most $r$ elements.

Lennox and Stonehewer also conjectured that these results could be extended to the join of an arbitrary finite number of subnormal subgroups. In this paper we shall establish the following generalization of that result:

THEOREM A. *Let $H_1, H_2, \ldots, H_r$ be subnormal subgroups of $G$ and let $J = \langle H_1, H_2, \ldots, H_r \rangle$. Suppose that $\gamma_2(J)/\gamma_3(J)$ has finite rank. Then*

(i) *given positive integers $a_1, a_2, \ldots, a_r$ there exists $c$ such that*

$$\gamma_c(J) \leqslant \gamma_{a_1}(H_1).\gamma_{a_2}(H_2) \ldots \gamma_{a_r}(H_r)$$

*and*

(ii) *$J$ is subnormal in $G$.*

Of course if $J/J'$ has finite rank it follows immediately that $\gamma_2(J)/\gamma_3(J)$ has the same property.

Suppose that $J$ is the join of subnormal subgroups $H_1, H_2, \ldots, H_r$. Let us say that the *augmentation condition* holds if, whenever $N$ is a normal subgroup of $J$ such that $X = J/N$ is nilpotent, and $b_1, b_2, \ldots, b_r$ are positive integers, there exists an integer $m$ such that

$$\mathbf{x}^m \leqslant \mathbb{Z}X(\mathbf{y}_1^{b_1}) + \mathbb{Z}X(\mathbf{y}_2^{b_2}) + \ldots + \mathbb{Z}X(\mathbf{y}_r^{b_r}),$$

where $Y_i = H_i N/N$ and $\mathbf{x}, \mathbf{y}_i$ denote the augmentation ideals of $\mathbb{Z}X$ and $\mathbb{Z}Y_i$ respectively.

Theorem A is an immediate consequence of the following three results:

THEOREM B. *Suppose that $J$ is the join of subnormal subgroups $H_1, H_2, \ldots, H_r$. If $\gamma_2(J)/\gamma_3(J)$ has finite rank, then the augmentation condition holds for $J$.*

THEOREM C. *Let $H_1, H_2, \ldots, H_r$ be subnormal subgroups of $G$ and let $J = \langle H_1, H_2, \ldots, H_r \rangle$. Suppose that the augmentation condition holds for $J$. Then*

(i) *given positive integers $a_1, a_2, \ldots, a_r$ there exists $c$ such that*

$$\gamma_c(J) \leqslant \gamma_{a_1}(H_1).\gamma_{a_2}(H_2) \ldots \gamma_{a_r}(H_r)$$

*and*

(ii) *there exists $h$ such that $\gamma_h(J)$ is subnormal in $G$.*

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THEOREM D. *Let $H_1, H_2, \ldots, H_r$ be subnormal subgroups of $G$ and let*

$$J = \langle H_1, H_2, \ldots, H_r \rangle.$$

*Suppose that $\gamma_2(J)/\gamma_3(J)$ has finite rank. If $\gamma_h(J)$ is subnormal in $G$ for some integer $h$, then $J$ is subnormal in $G$.*

We prove Theorem B in Section 2 and Part (i) of Theorem C in Section 3. Our proof here proceeds in four steps, which closely parallel the corresponding arguments in (1) and (2):

(i) the case $J$ abelian-by-nilpotent,
(ii) the case $J$ nilpotent-by-nilpotent,
(iii) the case $J$ soluble,
(iv) the general case.

Part (ii) of Theorem C follows in Section 4 by a more complicated argument along the same lines. Finally Section 5 gives a brief proof of Theorem D.

## 2. *Proof of Theorem B*

In this section, a *simple commutator* of weight $c$ will mean an element of the form $[x_1, x_2, \ldots, x_c]$, where as usual we make the inductive definitions

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 \quad \text{and} \quad [x_1, x_2, \ldots, x_c] = [[x_1, \ldots, x_{c-1}], x_c].$$

The proof of the following lemma is an extension of an argument in the proof of the main theorem of (3).

LEMMA 2·1. *Let $X$ be a nilpotent group of class 2, and suppose $\gamma_2(X)$ has finite rank $d$. Let $E$ be a finitely generated subgroup of $\gamma_2(X)$. Then there exists a subgroup $E_1$ such that $E \leqslant E_1 \leqslant \gamma_2(X)$ and $E_1$ is generated by at most $2d$ simple commutators of weight 2.*

*Proof.* As $E$ is finitely generated, there exists a finitely generated subgroup $X_1$ of $X$ such that $E \leqslant E_1 = \gamma_2(X_1)$.

So it suffices to prove the result with $X_1$ replacing $X$ and $E_1$ replacing $E$, i.e. we may assume that $X$ is finitely generated and that $E = \gamma_2(X)$. Suppose firstly that $X$ is finite. $X$ is the direct product of finite $p_i$-groups $X_i$ for various distinct primes $p_1, p_2, \ldots, p_s$. It suffices to prove that each $\gamma_2(X_i)$ is generated by at most $d$ simple commutators. For then if generators of $\gamma_2(X_i)$ are $[a_{i1}, b_{i1}], [a_{i2}, b_{i2}], \ldots, [a_{id}, b_{id}]$, a set of $d$ generators of $\gamma_2(X)$ is $\{[a_{1j}a_{2j} \ldots a_{sj}, b_{1j}b_{2j} \ldots b_{sj}], 1 \leqslant j \leqslant d\}$. So we may assume $X$ is a finite $p$-group.

If the exponent of $\gamma_2(X)$ is $p^{e_1}$, there must be a commutator $l_1$ of order precisely $p^{e_1}$, for if a smaller power of $p$ annihilates all the commutators, it annihilates $\gamma_2(X)$.

If the exponent of $\gamma_2(X)/\langle l_1 \rangle$ is $p^{e_2}$ there must be another commutator $l_2$ of order precisely $p^{e_2}$ modulo $\langle l_1 \rangle$. Similarly we can find $l_3$ whose order modulo $\langle l_1, l_2 \rangle$ is $p^{e_3}$, where $p^{e_3}$ is the exponent of $\gamma_2(X)/\langle l_1, l_2 \rangle$, and so on.

The numbers $p^{e_1}, p^{e_2}, \ldots$ are the invariants of $\gamma_2(X)$, and there are at most $m$ of these greater than 1. The corresponding elements $l_1, l_2, \ldots, l_m$ generate $\gamma_2(X)$.

Next suppose that $X$ is infinite, but $\gamma_2(X)$ is finite.

There is a finite number of elements generating $X$, and as $\gamma_2(X)$ is finite each of these generators has finitely many conjugates in $X$, so the centralizer of each generator has finite index in $X$. The intersection of all these centralizers will also have finite index in $X$, and this intersection is the centre of $X$.

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Let $N$ be a maximal torsion-free subgroup of the centre of $X$. From the above remark we see that $X/N$ is finite. Also $N \cap \gamma_2(X) = 1$ as $\gamma_2(X)$ is finite. By the first case applied to $X/N$, there exist at most $d$ commutators which generate $\gamma_2(X)$ modulo $N$. As $\gamma_2(X) \cap N = 1$, these commutators in fact generate $\gamma_2(X)$.

Finally we deal with the general case, in which $\gamma_2(X)$ is infinite. As the torsion-free rank of $\gamma_2(X)$ is at most $d$, there are at most $d$ commutators which generate a subgroup $M$ of finite index in $\gamma_2(X)$. For example, we could choose a basis of $\gamma_2(X) \otimes \mathbb{Q}$ from amongst the elements $[a, b] \otimes 1\,(a, b \in X)$. The corresponding commutators $[a, b]$ would be the required generators of $M$.

We then have $\gamma_2(X/M) = \gamma_2(X).M/M$ finite, so the previous case applied to $X/M$ gives a further $d$ commutators, at most, generating $\gamma_2(X)$ modulo $M$.

LEMMA 2·2. *Let $X$ be a nilpotent group, of class $c$, and suppose $\gamma_2(X)/\gamma_3(X)$ has finite rank $d$.*

*Then there is a number $m$, depending only on $c$ and $d$, such that if $E$ is a finitely generated subgroup of $\gamma_c(X)$ then there exists a subgroup $E_1$, such that $E \leqslant E_1 \leqslant \gamma_c(X)$ and $E_1$ is generated by at most $m$ simple commutators of weight $c$.*

*Proof.* The case $c = 2$ is just Lemma 2·1, so we may proceed by induction on $c$. As $E$ is finitely generated, there exist finitely generated subgroups A of $\gamma_{c-1}(X)$ and $B$ of $X$ such that $E \leqslant [A, B]$. By the induction hypothesis, we may assume $A$ is generated by $m_1 = m(c-1, d)$ simple commutators of weight $c-1$.

Each generator of $A$ is of the shape $[c, d]$, where $c$ is a simple commutator of weight $c-2$ and $d$ is an element of $X$. Let $C$ be the subgroup of $\gamma_{c-2}(X)$ generated by all such $c$, and let $D$ be the subgroup of $X$ generated by all such $d$. $C$ and $D$ are both $m_1$-generator groups, and $A \leqslant [C, D]$, so $E \leqslant [C, B, D]$.

Each of $[D, B, C]$ and $[B, C, D]$ is a subgroup of $\gamma_c(X)$ so is central in $X$. Hence, by the three-subgroup lemma, we have $E \leqslant [D, B, C][B, C, D]$. As $[D, B]$ is a finitely generated subgroup of $\gamma_2(X)$, Lemma 2·1 gives us $2d$ simple commutators of weight 2 generating $[D, B]$ modulo $\gamma_3(X)$. Thus there are $2d$-generator subgroups $F$ and $G$ of $X$ such that $[D, B] \leqslant [F, G].\gamma_3(X)$.

As $[\gamma_3(X), C] \leqslant \gamma_{c+1}(X) = 1$ we have $[D, B, C] \leqslant [F, G, C]$.

Another application of the three-subgroup lemma gives $[F, G, C] \leqslant [G, C, F][C, F, G]$.

As $C$ is generated by $m_1$ simple commutators of weight $c-2$, $[G, C]$ is generated by $2m_1 d$ simple commutators of weight $c-1$, so $[G, C, F]$ is generated by $4m_1 d^2$ commutators of weight $c$. The same applies to $[C, F, G]$. As $[B, C]$ is a finitely generated subgroup of $\gamma_{c-1}(X)$, the induction hypothesis gives $m_1$ simple commutators of weight $c-1$ generating a subgroup $H$ of $\gamma_{c-1}(X)$ such that $[B, C] \leqslant H.\gamma_c(X)$.

As $[\gamma_c(X), D] \leqslant \gamma_{c+1}(X) = 1$ we have $[B, C, D] \leqslant [H, D]$.

$D$ is generated by $m_1$ elements of $X$, so $[H, D]$ is generated by $m_1^2$ simple commutators of weight $c$.

Finally

$$E \leqslant [D, B, C][B, C, D] \leqslant [F, G, C][H, D] \leqslant [G, C, F][C, F, G][H, D]$$

and

$$E_1 = [G, C, F][C, F, G][H, D]$$

is generated by $(4m_1 d^2).(4m_1 d^2).(m_1^2) = 16m_1^4 d^4$ simple commutators of weight $c$.

The above two lemmas are used to prove the key result needed in this section:

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PROPOSITION 2·3. *Let $X = \langle Y_1, Y_2, \ldots, Y_r \rangle$ be nilpotent of class $c$ and suppose $\gamma_2(X)/\gamma_3(X)$ has finite rank $d$. Then there exists an integer $n$, depending only on $c$ and $d$, such that if $F$ is a finitely generated subgroup of $X'$ there exist $n$-generator subgroups $A_i$ of $Y_i$ with $F \leqslant \langle A_1, A_2, \ldots, A_r \rangle$.*

*Proof.* We proceed by induction on $c$, the assertion being vacuously true for $c = 1$.

Thus we may assume that there are $n_1$-generator subgroups $C_i$ of $Y_i$, where $n_1 = n(c-1, d)$, such that $F \leqslant \langle C_1, C_2, \ldots, C_r \rangle . \gamma_c(X)$.

For each of the finitely many generators $f$ of $F$ there exist $c \in \langle C_1, C_2, \ldots, C_r \rangle$ and $e \in \gamma_c(X)$ such that $f = ce$. Let $E$ be the subgroup of $\gamma_c(X)$ generated by all such $e$. $E$ is a finitely generated subgroup of $\gamma_c(X)$ and $F \leqslant \langle C_1, C_2, \ldots, C_r, E \rangle$.

By Lemma 2·2 there exist $m = m(c, d)$ and $E_1$ such that $E \leqslant E_1 \leqslant \gamma_c(X)$ and $E_1$ is generated by (at most) $m$ simple commutators of weight $c$.

Consider such a commutator, say $[x_1, x_2, \ldots, x_c]$.

As $X = \langle Y_1, Y_2, \ldots, Y_r \rangle$ there exist $y_{ij} \in Y_i$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant c)$ and $w_j \in X'$ such that

$$x_j = y_{1j} y_{2j} \ldots y_{rj} . w_j.$$

Let $B_i$ be the subgroup of $Y_i$ generated by all the $y_{ij}$ appearing in some such expansions of all the $x_j$ in the $m$ generators of $E_1$. Each $B_i$ is generated by $mc$ elements of $Y_i$.

Standard commutator identities give

$$[x_1, x_2, \ldots, x_c] = \Pi [y_{i_1, 1}, y_{i_2, 2}, \ldots, y_{i_c, c}],$$

where the product is over all $1 \leqslant i_1 \leqslant r$, $1 \leqslant i_2 \leqslant r, \ldots, 1 \leqslant i_c \leqslant r$.

So each generator of $E_1$, and hence the whole of $E_1$, is contained in $\langle B_1, B_2, \ldots, B_r \rangle$.

Thus $F \leqslant \langle C_1, C_2, \ldots, C_r, B_1, B_2, \ldots, B_r \rangle$ so the result follows on setting $A_i = \langle C_i, B_i \rangle$ which is a $(n_1 + mc)$-generator subgroup of $Y_i$.

We now prove Theorem B.

Let $X = J/N$ be a nilpotent image of $J$, of class $c$, and put $Y_i = H_i N/N$. Let $F$ be any finitely generated subgroup of $X'$, and let $C = \langle A_1, A_2, \ldots, A_r \rangle$ where the $A_i$ are as given by the proposition. Repeated application of theorem B of (1) gives an integer $s$ such that

$$\mathbf{f}^s \leqslant \mathbf{e}^s \leqslant \mathbb{Z}C(\mathbf{a}_1^{b_1}) + \ldots + \mathbb{Z}C(\mathbf{a}_r^{b_r}) \leqslant \mathbb{Z}X(\mathbf{y}_1^{b_1}) + \ldots + \mathbb{Z}X(\mathbf{y}_r^{b_r}),$$

and as in the proof of theorem C of (2) we see that $s$ depends only on $c$, $d$, $r$ and the choice of $b_1, b_2, \ldots, b_r$, and so is independent of the choice of $F$.

But $X'$ is the union of all its finitely generated subgroups, so plainly

$$\mathbf{w}^c \leqslant \mathbb{Z}X(\mathbf{y}_1^{b_1}) + \mathbb{Z}X(\mathbf{y}_2^{b_2}) + \ldots + \mathbb{Z}X(\mathbf{y}_r^{b_r}),$$

where $\mathbf{w}$ is the augmentation ideal of $X'$.

Finally $X = Y_1 Y_2 \ldots Y_r . X'$ so repeated application of proposition $C$ of (1) gives an integer $m$ such that

$$\mathbf{x} \leqslant \mathbb{Z}X(\mathbf{y}_1^{b_1}) + \ldots + \mathbb{Z}X(\mathbf{y}_r^{b_r}) + \mathbb{Z}X(\mathbf{w}^s),$$

and this completes the proof.

## 3. *Proof of Theorem C* (i)

We recall the following results which we shall need repeatedly in the course of the proof:

LEMMA 3·1 (lemma 2 of (6)). *If $J = HK$ with $H$ and $K$ subnormal in $J$, then given $a, b$ there exists $c$ such that $\gamma_c(J) \leqslant \gamma_a(H) . \gamma_b(K)$.*

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LEMMA 3·2 (corollary B 1 of (5)). *If $J = \langle H, K\rangle$ with $H$ and $K$ subnormal in $J$, then there exists $\lambda$ such that $H^{(\lambda)} \leqslant P_H(K)$.*

Here $P_H(K)$ denotes the permutizer of $H$ in $K$, i.e. the largest subgroup of $K$ which permutes with $H$.

LEMMA 3·3 (corollary to lemma 2·4 of (4)). *Let $H$, $K$ be subnormal subgroups of $G$ and $J = \langle H, K\rangle$. If $J = HK$ then $J$ is subnormal in $G$.*

*Step* (i). Firstly let us assume that $J$ is abelian-by-nilpotent, i.e. there exists $A \lhd J$ such that $A$ is abelian and $J/A$ is nilpotent.

By Lemma 3·1, we may choose $b_i\,(1 \leqslant i \leqslant r)$ such that $\gamma_{b_i}(H_iA) \leqslant \gamma_{a_i}(H_i)$. Put $X = J/A$ and $Y_i = H_iA/A$. As $X$ is nilpotent, the hypothesis that the augmentation condition applies to $J$ gives us an integer $m_1$ such that

$$\mathbf{x}^{m_1} \leqslant \mathbb{Z}X(\mathbf{y}_1^{b_1}) + \mathbb{Z}X(\mathbf{y}_2^{b_2}) + \ldots + \mathbb{Z}X(\mathbf{y}_r^{b_r}),$$

where $\mathbf{x}$, $\mathbf{y}_i$ denote the augmentation ideals of $\mathbb{Z}X$ and $\mathbb{Z}Y_i$ respectively. Thus

$$A\mathbf{x}^{m_1} \leqslant A\mathbf{y}_1^{b_1} + \ldots + A\mathbf{y}_r^{b_r},$$

i.e.

$$[\underbrace{A, J, J, \ldots, J}_{\longleftarrow m_1 \longrightarrow}] \leqslant [\underbrace{A, H_1, \ldots, H_1}_{\longleftarrow b_1 \longrightarrow}][\underbrace{A, H_2, \ldots, H_2}_{\longleftarrow b_2 \longrightarrow}] \ldots [\underbrace{A, H_r, \ldots, H_r}_{\longleftarrow b_r \longrightarrow}].$$

Now $J/A$ is nilpotent, of class $m_2$ say, so $\gamma_{m_2}(J) \leqslant A$. Put $m_3 = m_1 + m_2$. Then

$$\begin{aligned}\gamma_{m_3}(J) &\leqslant \gamma_{b_1+1}(H_1A).\gamma_{b_2+1}(H_2A)\ldots\gamma_{b_r+1}(H_rA)\\ &\leqslant \gamma_{a_1}(H_1).\gamma_{a_2}(H_2)\ldots\gamma_{a_r}(H_r).\end{aligned}$$

*Step* (ii). Now let us assume that $J$ is nilpotent-by-nilpotent, i.e. that there exists $N \lhd J$ such that $N$ and $J/N$ are nilpotent.

For sufficiently large $c_i$ we have $\gamma_{c_i}(H_iN) \leqslant \gamma_{a_i}(H_1) \cap N$.

Then $\gamma_{c_i}(H_iN)$ is normalized by $H_iN$ and so a fortiori by $N$, and so is a normal subgroup of $N$. Thus all the $\gamma_{c_i}(H_iN)$ normalize each other. By induction on the class of $N$, the class 1 case being step (i), we may assume that there exists $m_4$ such that

$$\gamma_{m_4}(JZ) \leqslant \gamma_{c_1}(H_1Z)\ldots\gamma_{c_r}(H_rZ).Z,$$

where $Z$ is the centre of $N$. Thus

$$\gamma_{m_4}(J) \leqslant \gamma_{c_1}(H_1N)\ldots\gamma_{c_r}(H_rN).Z.$$

Since the $\gamma_{c_i}(H_iN)$ normalize each other,

$$(\gamma_{m_4}(J))' \leqslant \gamma_{c_1}(H_1N)\ldots\gamma_{c_r}(H_rN).$$

The abelian-by-nilpotent case applied to $J/A'$, where $A = \gamma_{m_4}(J)$, gives $m_5$ such that

$$\gamma_{m_5}(J) \leqslant \gamma_{c_1}(H_1A')\ldots(\gamma_{c_r}(H_rA').A'.$$

Therefore

$$\begin{aligned}\gamma_{m_5}(J) &\leqslant \gamma_{c_1}(H_1N)\ldots\gamma_{c_r}(H_rN).\gamma_{c_1}(H_1N)\ldots\gamma_{c_r}(H_rN)\\ &\leqslant \gamma_{a_1}(H_1)\ldots\gamma_{a_r}(H_r).\end{aligned}$$

Moreover, writing $C_i = \gamma_{c_i}(H_iN)$ we see $C_i \lhd H_i$ so $C_i\,\mathrm{s_n}\,G$, and the $C_i$ permute, whence their product $C_1.C_2\ldots C_r$ is subnormal in $G$. But $\gamma_{m_5}(J) \lhd J$ and $\gamma_{m_5}(J) \leqslant C_1.C_2\ldots C_r$ so $\gamma_{m_5}(J) \lhd C_1.C_2\ldots C_r$, whence $\gamma_{m_5}(J)\,\mathrm{s_n}\,G$.

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*Step* (iii). We now assume that $J$ is polynilpotent, i.e. that $J$ has a series

$$1 \leqslant N_p \leqslant N_{p-1} \leqslant \ldots \leqslant N_1 \leqslant N_0 = J$$

with $\gamma_{q_j}(N_j) = N_{j+1}$ for some integers $q_0, q_1, \ldots, q_p$.

Of course all soluble groups have such a series, with each $q_j = 2$. We proceed by induction on $p$, the case $p = 1$ having been done in step (ii). So we assume the result is true for $p-1$.

By Lemma 3·1, there exist integers $s_{j,i} (1 \leqslant j \leqslant p, 1 \leqslant i \leqslant r)$ such that

$$\gamma_{s_{1,i}}(H_i N_1) \leqslant \gamma_{a_i}(H_i) . N_2,$$

$$\gamma_{s_{2,i}}\gamma_{s_{1,i}}(H_i N_1) \leqslant \gamma_{a_i}(H_i) . N_3,$$

and so on, until we get

$$D_i = \gamma_{s_{p,i}}\gamma_{s_{p-1,i}} \ldots \gamma_{s_{1,i}}(H_i N_1) \leqslant \gamma_{a_i}(H_i).$$

Moreover, by increasing each $s_{j,i}$ if necessary, we can ensure $s_{j,i} \geqslant q_j$. Note that $D_i \mathrm{s_n} H_i N_1$ and $H_i \mathrm{s_n} J$, $N_1 \mathrm{s_n} J$ so by Lemma 3·3 $H_i N_1 \mathrm{s_n} J$, whence $D_i \mathrm{s_n} J$.

The idea of the proof is to find $A \lhd J$ such that $A \leqslant D_1 . D_2 \ldots D_r$ and $J/A$ is polynilpotent with a series of length $p-1$, so that the induction hypothesis can be applied to $J/A$.

By the induction hypothesis applied to $J/N_p$ there exists $r_1$ such that

$$\begin{aligned}\gamma_{r_1}(J) &\leqslant \gamma_{s_{1,1}}(H_1 N_p) . \gamma_{s_{1,2}}(H_2 N_p) \ldots \gamma_{s_{1,r}}(H_r N_p) . N_p \\ &\leqslant \gamma_{s_{1,1}}(H_1 N_1) \ldots \gamma_{s_{1,r}}(H_r N_1) . N_p.\end{aligned}$$

As $s_{1,i} \geqslant q_1$, we have $\gamma_{s_{1,i}}(H_i N_1) \leqslant N_1$, so each $\gamma_{s_{1,i}}(H_i N_1)$ is a normal subgroup of $N_1$. So too is $N_p$.

Thus $G_1 = \gamma_{s_{1,1}}(H_1 N_1) . \gamma_{s_{1,2}}(H_2 N_1) \ldots \gamma_{s_{1,r}}(H_r N_1) . N_p$ is a group, as it is a product of normal subgroups of $N_1$.

Applying Lemma 3·1 repeatedly to $G_1$ gives an integer $r_2$ such that

$$\gamma_{r_2}\gamma_{r_1}(J) \leqslant \gamma_{r_2}(G_1) \leqslant \gamma_{s_{2,1}}\gamma_{s_{1,1}}(H_1 N_1) \ldots \gamma_{s_{2,r}}\gamma_{s_{1,r}}(H_r N_1).$$

Each $\gamma_{s_{2,i}}\gamma_{s_{1,i}}(H_i N_1)$ is contained in $N_2$ and so is a normal subgroup of $N_2$. So their product $G_2$ is a group.

Applying Lemma 3·1 to $G_2$ gives an integer $r_3$ such that

$$\begin{aligned}\gamma_{r_3}\gamma_{r_2}\gamma_{r_1}(J) &\leqslant \gamma_{r_3}(G_2) \\ &\leqslant \gamma_{s_{3,1}}\gamma_{s_{2,1}}\gamma_{s_{1,1}}(H_1 N_1) \ldots \gamma_{s_{3,r}}\gamma_{s_{2,r}}\gamma_{s_{1,r}}(H_r N_1).\end{aligned}$$

Continuing thus, we eventually get

$$\begin{aligned}\gamma_{r_p}\gamma_{r_{p-1}} \ldots \gamma_{r_1}(J) &\leqslant \gamma_{s_{p,1}}\gamma_{s_{p-1,1}} \ldots \gamma_{s_{1,1}}(H_1 N_1) \ldots \gamma_{s_{p,r}}\gamma_{s_{p-1,r}} \ldots \gamma_{s_{1,r}}(H_r N_1) \\ &= D_1 . D_2 \ldots D_r.\end{aligned}$$

Choose, by Lemma 3·1, $e_i$ such that $\gamma_{e_i}(H_i N_p) \leqslant \gamma_{a_i}(H_i)$.

By increasing $e_i$ if necessary, we may assume $e_i \geqslant q_1$.

Writing $A = \gamma_{r_p}\gamma_{r_{p-1}} \ldots \gamma_{r_1}(J)$ we see that $J/A$ is polynilpotent with $p-1$ factors, so the induction hypothesis applies to $J/A$ and gives $c$ such that

$$\begin{aligned}\gamma_c(J) &\leqslant \gamma_{e_1}(H_1 A) . \gamma_{e_2}(H_2 A) \ldots \gamma_{e_r}(H_r A) . A \\ &\leqslant \gamma_{e_1}(H_1 N_p) \ldots \gamma_{e_r}(H_r N_p) . A.\end{aligned}$$

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Now $\gamma_{e_i}(H_i N_p) \leqslant N_1$ (as we have chosen $e_i \geqslant q_1$), so $E_i = \gamma_{e_i}(H_i N_p)$ normalizes each $D_j$ as $D_j$ is a normal subgroup of $N_1$.

Thus we have

$$\begin{aligned}\gamma_c(J) &\leqslant E_1 E_2 \dots E_r . D_1 D_2 \dots D_r \leqslant (E_1 D_1)(E_2 D_2) \dots (E_r D_r)\\ &\leqslant \gamma_{a_1}(H_1).\gamma_{a_2}(H_2) \dots \gamma_{a_r}(H_r).\end{aligned}$$

*Step* (iv). Finally we deal with the general case.

Put $A_i = \gamma_{a_i}(H_i)$.

By Lemma 3·2 we can choose $\lambda_{12}$ such that $A_1^{(\lambda_{12})} \leqslant P_{A_1}(A_2)$. Then we can choose $\lambda_{13}$ such that $(A_1^{(\lambda_{12})})^{(\lambda_{13})} \leqslant P_{A_1^{(\lambda_{12})}}(A_3)$ and so on, until we get $\lambda_{12}, \lambda_{13}, \dots, \lambda_{1r}$ such that for each $j$, $2 \leqslant j \leqslant r$, we have

$$A_1^{(\lambda_{12}+\lambda_{13}+\dots+\lambda_{1j})} \leqslant P_{A_1^{(\lambda_{12}+\lambda_{13}+\dots+\lambda_{1,j-1})}}(A_j).$$

Similarly we may find $\lambda_{i,j}$ $(1 \leqslant i < j \leqslant r)$ such that

$$A_i^{(\lambda_{i,i+1}+\lambda_{i,i+2}+\dots+\lambda_{i,j})} \leqslant P_{A_i^{(\lambda_{i,i+1}+\lambda_{i,i+2}+\dots+\lambda_{i,j-1})}}(A_j).$$

Write $C_{i,j} = A_i^{(\lambda_{i,i+1}+\lambda_{i,i+2}+\dots+\lambda_{i,j})}$, $B_i = C_{i,r}$ and $\lambda_i = \lambda_{i,i+1} + \lambda_{i,i+2} + \dots + \lambda_{i,r}$.

By theorem A of (5) there exists $n$ such that

$$J^{(n)} \leqslant H_1^{(a_1+\lambda_1)}.H_2^{(a_2+\lambda_2)} \dots H_r^{(a_r+\lambda_r)} \leqslant B_1.B_2 \dots B_r.$$

By Lemma 3·1 we can find $d_1, d_2, \dots, d_r$ such that

$$\gamma_{d_i}(H_i J^{(n)}) \leqslant \gamma_{a_i}(H_i).J^{(n)}.$$

Step (iii) applied to $J/J^{(n)}$ gives an integer $c$ such that

$$\begin{aligned}\gamma_c(J) &\leqslant \gamma_{d_1}(H_1 J^{(n)}).\gamma_{d_2}(H_2 J^{(n)}) \dots \gamma_{d_r}(H_r J^{(n)})\\ &\leqslant A_1, A_2 \dots A_r . J^{(n)}.\end{aligned}$$

So $\gamma_c(J) \leqslant A_1 A_2 \dots A_r . B_1 B_2 \dots B_r$.

Now $B_1 = C_{1,r} \leqslant P_{C_{1,r-1}}(A_r)$ so

$$A_r B_1 \leqslant A_r . P_{C_{1,r-1}}(A_r) = P_{C_{1,r-1}}(A_r).A_r \leqslant C_{1,r-1}.A_r.$$

Also

$$C_{1,r-1} \leqslant P_{C_{1,r-2}}(A_{r-2}), \quad \text{so} \quad A_{r-1}.C_{1,r-1} \leqslant C_{1,r-2}.A_{r-1},$$

whence

$$A_{r-1}.A_r.C_{1,r} \leqslant C_{1,r-2}.A_{r-1}.A_r.$$

Continuing thus we get

$$A_1 A_2 \dots A_r . B_1 = A_1 . C_{1,1} . A_2 \dots A_r \leqslant A_1 A_2 \dots A_r.$$

A similar argument passes $B_2$ across $A_r, A_{r-1}, \dots, A_3$, and so on.

Finally we get $\gamma_c(J) \leqslant A_1 A_2 \dots A_r$ as required.

## 4. *Proof of Theorem C* (ii)

*Step* (i). Firstly let us assume that $J$ is polynilpotent, i.e. $J$ has a series

$$1 \leqslant N_p \leqslant N_{p-1} \leqslant \dots \leqslant N_1 \leqslant N_0 = J$$

with $\gamma_{q_j}(N_j) = N_{j+1}$ for some integers $q_j$ and that $J = \langle H_1, \dots, H_r \rangle$ with each $H_i$ sn $J$.

The case $p = 1$ was dealt with at the end of step (ii) of the proof of part (i) of Theorem C, and our aim is to extend the argument given there, by induction on $p$.

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We claim that, given integers $a_1, \ldots, a_r$, there is an integer $h$ and subgroups $C_{p,j,i}$ and $L_{p,j}$ of $J$ $(1 \leqslant j \leqslant p, 1 \leqslant i \leqslant r)$ such that:

(*a*) $1 = L_{p,p} \leqslant L_{p,p-1} \leqslant \ldots \leqslant L_{p,1}$ and each $L_{p,j}$ is normal in $J$,

(*b*) each $C_{p,j,i}$ is subnormal in $J$,

(*c*) $C_{p,j,i_1}$ and $C_{p,j,i_2}$ permute, for all $j, i_1, i_2$,

(*d*) $L_{p,j} \leqslant C_{p,j+1,1} C_{p,j+1,2} \ldots C_{p,j+1,r}$ for all $j < p$,

(*e*) $C_{p,j,i} \leqslant \gamma_{a_i}(H_i) . L_{p,j}$ for all $j, i$,

(*f*) $\gamma_h(J) \leqslant C_{p,1,1} C_{p,1,2} \ldots C_{p,1,r}$.

We establish this claim by induction on $p$.

If $p = 1$, $J$ is nilpotent-by-nilpotent. We put $L_{1,1} = 1$, $C_{1,1,i} = C_i$ and $h = m_5$ where $C_i$ and $m_5$ are as defined in step (ii) of the proof of Theorem C (i).

Conditions (*a*)–(*f*) plainly hold in this case.

Now suppose that $p > 1$ and that the claim is true for all smaller $p$. Put $C_{p,p,i} = D_i$ as defined in step (iii) of the proof of Theorem C (i). Write $A = \gamma_{r_p} \gamma_{r_{p-1}} \ldots \gamma_{r_1}(J)$ where the $r_j$ are as in step (iii). Then $A \leqslant C_{p,p,1} C_{p,p,2} \ldots C_{p,p,r}$. Choose $e_i$ such that $\gamma_{e_i}(H_i A) \leqslant \gamma_{a_i}(H_i)$. The inductive hypothesis applied to $J/A$ with the integers $e_i$ replacing the $a_i$ gives $h$ and subgroups $C_{p-1,j,i}$ and $L_{p-1,j}$ of $J/A$ for $1 \leqslant j \leqslant p-1$. Define $C_{p,j,i}$ for $1 \leqslant j \leqslant p-1$ to be the full preimage in $J$ of $C_{p-1,j,i}$. Define $L_{p,j}$ for $1 \leqslant j \leqslant p-1$ to be the full preimage in $J$ of $L_{p-1,j}$ (so in particular $L_{p,p-1} = A$) and define $L_{p,p} = 1$.

It is readily seen that conditions (*a*)–(*f*) hold for this choice of $h$, $C_{p,j,i}$ and $L_{p,j}$.

Having established our claim, our next step is to prove that each $C_{p,j,i}$ is subnormal in $G$. This is certainly true for $j = p$, as $C_{p,p,i} \mathrm{s_n} J$ and $C_{p,p,i} \leqslant \gamma_{a_i}(H_i)$ together give $C_{p,p,i} \mathrm{s_n} \gamma_{a_i}(H_i)$, and of course $\gamma_{a_i}(H_i) \lhd H_i \mathrm{s_n} G$.

So we may proceed by reverse induction on $j$ and assume that each $C_{p,j+1,i}$ is subnormal in $G$.

As they permute, their product is also subnormal in $G$. But by (*a*) and (*d*) we see $L_{p,j}$ is a normal subgroup of this product, so $L_{p,j}$ is also subnormal in $G$.

As $L_{p,j}$ is normal in $J$ it permutes with $\gamma_{a_i}(H_i)$ which is subnormal in $G$ so by Lemma 3·3 we see $\gamma_{a_i}(H_i) . L_{p,j} \mathrm{s_n} G$. But combining (*b*) and (*e*) we see $C_{p,j,i} \mathrm{s_n} \gamma_{a_i}(H_i) . L_{p,j}$ so we have $C_{p,j,i} \mathrm{s_n} G$ as claimed.

In particular each $C_{p,1,i}$ is subnormal in $G$, so repeated application of Lemma 3·3 shows that their product is subnormal in $G$. But as $\gamma_h(J) \lhd J$ it follows from (*f*) that $\gamma_h(J)$ is a normal subgroup of this product. Hence $\gamma_h(J)$ is subnormal in $G$.

*Step* (ii). We now deal with the general case.

By theorem A of (5), there is an integer $n$ such that $J^{(n)}$ is subnormal in $G$.

Now $J/J^{(n)}$ is soluble, and hence polynilpotent. So the above argument with $p = n-1$ and each $a_i = 1$ gives an integer $h$ and subgroups $C_{p,j,i}$ and $L_{p,j}$ of $J/J^{(n)}$ satisfying the conditions (*a*)–(*f*) stated above. For $1 \leqslant j \leqslant p = n-1$ and $1 \leqslant i \leqslant r$ define $F_{j,i}$ to be the full preimage in $J$ of $C_{p,j,i}$ and $K_j$ to be the full preimage in $J$ of $L_{p,j}$ (so in particular $K_p = J^{(n)}$).

The $F_{j,i}$, $K_j$ and $N = \gamma_h(J)$ satisfy:

(*a'*) $1 \leqslant K_p \leqslant \ldots K_1$ and each $K_j$ is normal in $J$,

(*b'*) each $F_{j,i}$ is subnormal in $J$,

(*c'*) $F_{j,i_1}$ and $F_{j,i_2}$ permute, for all $j, i_1, i_2$,

(*d'*) $K_j \leqslant F_{j+1,1} F_{j+1,2} \ldots F_{j+1,r}$ for all $j < p$,

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($e'$) $F_{j,i} \leqslant H_i K_j$ for all $j$, $i$,

($f'$) $N = \gamma_h(J) \leqslant F_{1,1} F_{1,2} \dots F_{1,r}$.

Our argument is now exactly as in step (i).

We claim that each $F_{j,i}$ is subnormal in $G$.

From ($b'$) and ($c'$) we have $F_{p,i} \mathrm{s_n} H_i . J^{(n)}$, but $H_i \mathrm{s_n} G$ and $J^{(n)} \mathrm{s_n} G$ so Lemma 3·3 gives $H_i . J^{(n)} \mathrm{s_n} G$, whence $F_{p,i} \mathrm{s_n} G$.

By induction on $j$ decreasing, we may assume each $F_{j+1,i}$ is subnormal in $G$.

Hence, by repeated application of Lemma 3·3, so is their product $F_{j+1,1} F_{j+1,2} \dots F_{j+1,r}$.

Now $K_j \lhd J$ and $K_j \leqslant F_{j+1,1} \dots F_{j+1,r} \leqslant J$ so $K_j \lhd F_{j+1,1} \dots F_{j+1,r}$ so $K_j \mathrm{s_n} G$. $K_j$ is normal in $J$ and $H_i \leqslant J$ so $H_i$, $K_j$ permute, so by Lemma 3·3 $H_i K_j \mathrm{s_n} G$. But $F_{j,i} \mathrm{s_n} J$ and $F_{j,i} \leqslant H_i K_j \leqslant J$ so $F_{j,i} \mathrm{s_n} H_i K_j$, whence $F_{j,i} \mathrm{s_n} G$ as required.

This shows that each $F_{1,i}$ is subnormal in $G$. Hence so too is their product.

But $N \lhd J$ and $\mathrm{N} \leqslant F_{1,1} F_{1,2} \dots F_{1,r} \leqslant J$ so $N \lhd F_{1,1} \dots F_{1,r}$, whence $N \mathrm{s_n} G$.

This proves the theorem.

## 5. *Proof of Theorem D*

Write $N = \gamma_h(J)$.

Let $F$ be any subgroup of $J'$ such that $F \geqslant N$ and $F/N$ is finitely generated. By Proposition 2·3 we can find subgroups $A_i$ of $H_i N$ such that $F \leqslant \langle A_1, A_2, \dots, A_r \rangle$ and each $A_i/N$ is generated by $n$ elements, where $n$ is independent of the choice of $F$.

As $H_i$ and $N$ are subnormal in $G$, so too is $H_i N$, and $H_i N/N$ is nilpotent of class $h$, so each $A_i$ is subnormal in $G$. Moreover, the defect of $A_i$ in $G$ is bounded in terms of $h$ and the defects of $H_i$ and of $N$, and is thus independent of the choice of $F$.

So repeated application of lemma 5 of (2) proves that $A = \langle A_1, A_2, \dots, A_r \rangle$ is subnormal in $G$, with defect independent of the choice of $F$.

Now $N \leqslant F \leqslant A$ and $A/N$ is nilpotent of class $h$, so $F$ is subnormal in $G$, with defect bounded independently of the choice of $F$.

But $J'$ is the union of all its finitely generated subgroups, so theorem 1 in section 10 of (7) gives $J'$ subnormal in $G$.

Finally $J = H_1 H_2 \dots H_r . J'$ so repeated application of Lemma 3·3 gives $J \mathrm{s_n} G$ as required.

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