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A refined lower bound theorem for d-polytopes with at most $2d$ vertices

Guillermo Pineda-Villavicencio, Jie Wang, and David Yost

Abstract. In 1967, Grünbaum conjectured that the function

$$\phi_k(d+s,d) := \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}, \text{ for } 2 \leqslant s \leqslant d$$

provides the minimum number of k-faces for a d-dimensional polytope (abbreviated as a d-polytope) with $d+s$ vertices. In 2021, Xue proved this conjecture for each $k \in [1, d-2]$ and characterized the unique minimizers, each having $d+2$ facets. In this article, we refine Xue's theorem by considering d-polytopes with $d+s$ vertices ($2 \leqslant s \leqslant d$) and at least $d+3$ facets. If $s=2$, then there is precisely one minimizer for many values of k. For other values of s, the number of k-faces is at least $\phi_k(d+s,d) + \binom{d-1}{k} - \binom{d+1-s}{k}$, which is met by precisely two polytopes in many cases, and up to five polytopes for certain values of s and k. We also characterize the minimizing polytopes.

1 Introduction

We denote by f_k the number of k-faces in a polytope; the *f-vector* of a d-polytope is the sequence $(f_0, \ldots, f_{d-1})$ of the number of faces in the polytope. Xue's theorem settling Grünbaum's 1967 conjecture [5, Section 10.2] reads as follows.

Theorem 1.1 (d-polytopes with at most $2d$ vertices, Xue 2021) *Let $d \geqslant 2$ and $1 \leqslant s \leqslant d$. If P is a d-polytope with $d+s$ vertices, then*

$$f_k(P) \geqslant \phi_k(d+s,d), \text{ for all } k \in [1, d-1].$$

Furthermore, if $k \in [1, d-2]$ and $f_k(P) = \phi_k(d+s,d)$, then P is (combinatorially) unique and has exactly $d+2$ facets.

Naturally, one wants to extend this result to d-polytopes with more than $2d$ vertices: $(d-1)$-polytopes with at most $2(d-1)$ vertices have featured prominently as potential facets in the proofs of such results. Lower bound theorems for d-polytopes with $2d+1$ vertices [13, 15] (see also Theorem 2.4) and $2d+2$ vertices [8], reveal an interesting dichotomy: minimizers may have either $d+2$ or $d+3$ facets. Pineda-Villavicencio [7, Problem 8.7.11] conjectured that this dichotomy extends to

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minimizers of the number of k-faces among d-polytopes with at most $3d-1$ vertices. (This conjecture has recently been proved; see [12]. The main result in the present article serves as an important ingredient.)

In light of these results, we find it natural to investigate this dichotomy for d-polytopes with $2d$ or fewer vertices. That is, we determine the minimizers of f_k amongst all d-polytopes with at most $2d$ vertices and at least $d+3$ facets. Our main theorem thus furnishes a refinement of Theorem 1.1. It will be stated in more detail and proved in subsequent sections.

Since this expression will appear frequently, it is convenient to define

$$\zeta_k(d+s,d) := \phi_k(d+s,d) + \binom{d-1}{k} - \binom{d+1-s}{k}.$$

Theorem 1.2 (Refined theorem for d-polytopes with at most $2d$ vertices (short version)) *Given parameters $d \geqslant 3$ and $2 \leqslant s \leqslant d$, and a d-polytope P with $d+s$ vertices, the following statements hold:*

(i) *If $s = 2$ and P has at least $d+\ell$ facets where $\ell \in [3,d]$, then $f_k(P) \geqslant \phi_k(d+2,d) + \binom{d-1}{k-1} - \binom{d-\ell+1}{k-\ell+1}$ for all $k \in [1,d-2]$. Moreover, for each $k \geqslant \ell - 1$, there is a unique polytope whose number of k-faces equals this lower bound.*

(ii) *If $s \in [3,d]$ and P has at least $d+3$ facets, then $f_k(P) \geqslant \zeta_k(d+s,d)$. There are two fundamental examples for which the number of k-faces equals $\zeta_k(d+s,d)$ for each $k \in [1,d-2]$. Additionally, if $s < d$ and $k \geqslant d-s+2$, then a third minimizer arises, and when $s = 4$, two further examples appear.*

A full version of Theorem 1.2, describing the minimizing polytopes in detail, is provided at the beginning of Section 3.

2 Minimizers

This section presents the minimizers in Theorems 1.1 and 1.2.

The m-dimensional simplex will be denoted by $T(m)$. A prism based on $T(d-1)$ is called a simplicial d-prism; we will often refer to them simply as prisms. When we need to discuss other prisms, the base will be clearly specified. When $k \in [1,d-2]$, each minimizer in Theorem 1.1 is a $(d-s)$-fold pyramid over a simplicial s-prism for $s \in [1,d]$; this polytope is called the $(s,d-s)$-*triplex* and denoted by $M(s,d-s)$. Triplices were introduced by Pineda-Villavicencio et al. [10, Section 3]. Each triplex has $d+2$ facets. The structure of d-polytopes with $d+2$ facets is well understood (see, for instance, [5, Section 6.1]). We will use the formulation provided by McMullen and Shephard [6, Section 3].

Lemma 2.1 (McMullen and Shephard 1970) *Let P be a d-dimensional polytope with $d+2$ facets, where $d \geqslant 2$. Then, there exist integers $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$ such that P is a $(d-a)$-fold pyramid over $T(m) \times T(a-m)$. The number of k-faces of P is*

$$\binom{d+2}{k+2} - \binom{d-a+m+1}{k+2} - \binom{d+1-m}{k+2} + \binom{d-a+1}{k+2}. \tag{2.1}$$

In particular, $f_0(P) = d+1+m(a-m)$.

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Recall that the *Cartesian product* of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ is the Cartesian product of the sets P and P':

$$P \times P' = \{(p, p')^t \in \mathbb{R}^{d+d'} \mid p \in P,\, p' \in P'\}.$$

Dually to Lemma 2.1, we have a lemma for d-polytopes with $d+2$ vertices, which is from Grünbaum [5, Section 6.1]. The *direct sum* $P \oplus P'$ of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ with the origin in their relative interiors is the $(d+d')$-polytope:

$$P \oplus P' = \operatorname{conv}\left(\left\{\begin{pmatrix} \boldsymbol{p} \\ \mathbf{0}_{d'} \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, \boldsymbol{p} \in P\right\} \bigcup \left\{\begin{pmatrix} \mathbf{0}_d \\ \boldsymbol{p'} \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, \boldsymbol{p'} \in P'\right\}\right).$$

Lemma 2.2 *Let P be a d-dimensional polytope with $d+2$ vertices, where $d \geqslant 2$. Then, there exist integers $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$ such that P is a $(d-a)$-fold pyramid over $T(m) \oplus T(a-m)$. The number of k-dimensional faces of P is*

$$\binom{d+2}{d-k+1} - \binom{d-a+m+1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.$$

In particular, $f_{d-1}(P) = d+1+m(a-m)$.

As Grünbaum [5, Section 6.1], we denote the $(d-a)$-fold pyramid over $T(m) \oplus T(a-m)$ by $T_m^{d,d-a}$.

Grünbaum [5, p. 101] established inequalities for the number of k-faces in d-polytopes with $d+2$ vertices.

Lemma 2.3 *For $k \in [0, d-1]$, the following hold:*

(i) *If $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor - 1$, then $f_k(T_m^{d,d-a}) \leqslant f_k(T_{m+1}^{d,d-a})$, with strict inequality if and only if $m \leqslant k$.*

(ii) *If $2 \leqslant a \leqslant d-1$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$, then $f_k(T_m^{d,d-a}) \leqslant f_k(T_m^{d,d-(a+1)})$, with strict inequality if and only if $a - m \leqslant k$.*

A vertex v in a d-polytope P is *simple* if it is contained in exactly d edges. Otherwise, it is *nonsimple*. A nonsimple vertex in P may be simple in a proper face of P, and we often need to make this distinction. If K is a closed halfspace in $\mathbb{R}^d$ whose boundary does not contain any vertices of P and such that v is the only vertex of P not in K, we say that a polytope P' is obtained by *truncating* v if $P' = P \cap K$.

If we truncate a simple vertex from the $(2, d-2)$-triplex ($d \geqslant 2$), we obtain a d-polytope $Pm(d)$ whose number of faces is

$$f_k(Pm(d)) = \begin{cases} 2d+1, & \text{if } k = 0; \\ \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k}, & \text{if } k \in [1, d-1]. \end{cases} \tag{2.2}$$

Pineda-Villavicencio and Yost [13], also [9], called the d-polytope $Pm(d)$ the *d-pentasm*.

Theorem 2.4 [13, Theorem 25] and [15, Theorem 5.1] *Let $d \geqslant 4$, P a d-polytope with **at least** $2d+1$ vertices, and $k \in [1, d-2]$.*

(i) *If P has at least $d+3$ facets, then $f_k(P) \geqslant f_k(Pm(d))$, with equality for some $k \in [1, d-2]$ only if $P = Pm(d)$.*

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(ii) *If P has $d+2$ facets, then $f_k(P) \geqslant f_k((T_2^{d,d-(\lfloor d/2\rfloor+2)})^*)$. If d is even, then $(T_2^{d,d-(\lfloor d/2\rfloor+2)})^*$ has $2d+1$ vertices and this bound can be attained.*

Since it appears frequently, it will be convenient to denote the a-fold pyramid over a b-pentasm, by $Pm(b,a)$. We will denote as usual the pyramid over an arbitrary polytope P by $\mathrm{pyr}(P)$ and the t-fold pyramid over P by $\mathrm{pyr}_t(P)$.

The number of k-faces in the polytope $Pm(s-1,d+1-s)$ is obtained by counting the k-faces among the $d+1-s$ apices of the pyramid and the k-faces obtained from an i-face in an $(s-1)$-pentasm and $k-i$ vertices from the $d+1-s$ apices of the pyramid, $0 \leqslant i \leqslant k$. That is, for $k \in [0, d-1]$, we have that

$$f_k(Pm(s-1,d+1-s)) = \binom{d+1-s}{k+1} + (2s-1)\binom{d+1-s}{k} + \sum_{i=1}^{k}\binom{d+1-s}{k-i}\left[\binom{s}{i+1} + \binom{s-1}{i+1} + \binom{s-2}{i}\right].$$

Let us compare this expression with the function $\zeta_k(d+s,d)$. The following three identities are routine to establish:

$$\binom{d+1}{k+1} = \binom{d+1-s}{k+1} + s\binom{d+1-s}{k} + \sum_{i=1}^{k}\binom{d+1-s}{k-i}\binom{s}{i+1},$$

$$\binom{d}{k+1} - \binom{d+1-s}{k+1} = (s-1)\binom{d+1-s}{k} + \sum_{i=1}^{k}\binom{d+1-s}{k-i}\binom{s-1}{i+1},$$

$$\binom{d-1}{k} - \binom{d+1-s}{k} = \sum_{i=1}^{k}\binom{d+1-s}{k-i}\binom{s-2}{i}.$$

Comparing these expressions with the definition of ζ_k, we arrive at the following relation:

$$f_k(Pm(s-1,d+1-s)) = \zeta_k(d+s,d). \tag{2.3}$$

The wedge construction will be very useful to us. Let P be a d-polytope embedded in the hyperplane $x_{d+1}=0$ of $\mathbb{R}^{d+1}$. Let F be a proper face of P, and let C be the halfcylinder $P\times[0,\infty) \subset \mathbb{R}^{d+1}$. We cut the halfcylinder with a hyperplane H' through $F\times\{0\}$ so that C is partitioned into two parts, one bounded and one unbounded. The *wedge* of P at F is the bounded part; it is denoted by $\mathrm{W}_F(P)$. The sets P and $H'\cap C$, the *bases* of $\mathrm{W}_F(P)$, define facets of $\mathrm{W}_F(P)$ that are combinatorially isomorphic to P and intersect at the face $F\times\{0\}$. The wedge W over a d-polytope $P\times\{0\}\subseteq\mathbb{R}^{d+1}$ at a face $F\times\{0\}$ of $P\times\{0\}$ is combinatorially isomorphic to a prism Q over $P\times\{0\}$ where the face prism $(F\times\{0\})$ of Q has collapsed into $F\times\{0\}$. For further information, refer to [7, Section 2.6]. The following basic facts about wedges will be used several times. Their proofs are routine.

Lemma 2.5 *Let $\mathrm{W}_F(P)$ be a wedge of a d-polytope P at a face F of P.*

(i) *A k-face of $\mathrm{W}_F(P)$ is either a k-face of one of the bases of $\mathrm{W}_F(P)$, or the wedge of a $(k-1)$-face F' of P at the proper face $F\cap F'$, or a prism over a $(k-1)$-face of P disjoint from F.*

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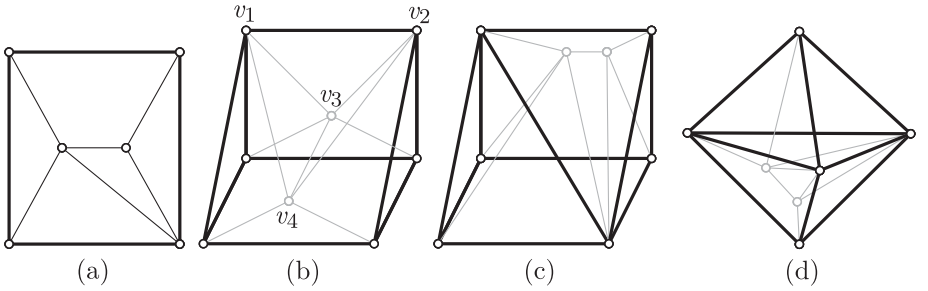


Figure 1: Schlegel diagrams of polytopes. (a) The tetragonal antiwedge. (b) The 4-polytope WP. (c) The 4-polytope $TA(4)$. (d) The 4-polytope $Z(4)$.

(ii) *If F is a facet, then for each value of $k \in [0, d+1]$, we have*

$$f_k(\mathrm{W}_F(P)) = 2f_k(P) + f_{k-1}(P) - f_k(F) - f_{k-1}(F).$$

(iii) *If W_1, P_1, F_1 denote pyramids over $\mathrm{W}_F(P), P, F$, respectively, then W_1 is combinatorially equivalent to $\mathrm{W}_{F_1}(P_1)$.*

We let $TA(2)$ be a quadrilateral and let $TA(3)$ be the *tetragonal antiwedge*, the unique nonpyramidal 3-polytope with six vertices and six 2-faces (four triangles and two quadrilaterals) (see Figure 1(a)). For $d \geqslant 4$, we iteratively define the d-polytope $TA(d)$ as the wedge of $TA(d-1)$ at a facet $TA(d-2)$ of $TA(d-1)$. The polytope $TA(d)$ has $TA(d-1)$ as a facet, and thus this construction is sound (see Figure 1(c)).

Remark 2.6 (Structure of $TA(d)$) The polytope $TA(d)$ has $2d$ vertices. It has one vertex v with maximum degree $2d-2$, $d-2$ vertices with degree $d+1$, and every other vertex is simple. Its $d+3$ facets are as follows:

(i) $d-2$ copies of $TA(d-1)$;
(ii) three simplices;
(iii) one simplicial $(d-1)$-prism;
(iv) one $M(d-2,1)$.

The facets not containing v are the prism and one of the simplices.

The polytope $TA(d)$ and a pyramid over a $(d-1)$-pentasm have the same f-vector. This follows from noticing that the pyramid over the pentagon (the 2-pentasm) and the tetragonal antiwedge $TA(3)$ both have f-vector $(6,10,6)$ and that a pyramid over a $(d-1)$-pentasm is a wedge of a pyramid over a $(d-2)$-pentasm at a facet that is itself a pyramid over $(d-3)$-pentasm. In particular, the f-vectors of the four-dimensional $TA(4)$ and a pyramid over $Pm(3)$ are both $(8,18,17,7)$, which serves as a base for induction arguments (see also Proposition 3.4). As a consequence, a $(d-s)$-fold pyramid over the s-polytope $TA(s)$, denoted by $TA(s, d-s)$, and the d-polytope $Pm(s-1, d+1-s)$ both have the same f-vector:

$$f_k(TA(s, d-s)) = \zeta_k(d+s, d). \tag{2.4}$$

We will see that, for fixed $s \in [3, d]$, the polytopes $TA(s, d-s)$ and $Pm(s-1, d+1-s)$ minimize the number of k-faces for every $k \in [1, d-2]$. For many values of

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s and k, these are the only two minimizers. But for some values of s and k, there may be up to three more.

Consider $s = d = 4$. The polytopes $TA(4)$ and $Pm(3,1)$ both have f-vector $(8,18,17,7)$. The wedge of a square pyramid at one of its edges also has f-vector $(8,18,17,7)$; the combinatorial type depends on whether or not the edge contains the apex of the pyramid. If the edge is in the base of the pyramid, we obtain the example called Polytope 2 in [11, Lemma 18(ii)]; here, we denote this 4-polytope by WP. If the edge contains the apex, then Lemma 2.5 tells us the polytope is a pyramid over the wedge of a square at one of the vertices; we use $\Sigma(3)$ to denote the wedge of a square at one of the vertices.[1] Therefore, for $s = 4$, $\mathrm{pyr}_{d-4}(WP)$ (the $(d-4)$-fold pyramid over WP) and $\mathrm{pyr}_{d-3}(\Sigma(3))$ (the $(d-3)$-fold pyramid over $\Sigma(3)$) also minimize the number of k-faces for every $k \in [1, d-2]$.

Let $Z(3)$ be a triangular bipyramid and let $Z(2)$ be a triangular face of $Z(3)$. For $d \geqslant 4$, we define $Z(d)$ inductively by $Z(d+1) := \mathrm{W}_{Z(d-1)}(Z(d))$.

Remark 2.7 (Structure of $Z(d)$) The polytope $Z(d)$ is d-dimensional with $2d-1$ vertices and $d+3$ facets. It has two vertices, say v and w, with maximum degree $2d-2$, $d-2$ vertices with degree $d+1$, and $d-1$ simple vertices. The $d+3$ facets are

(i) $d-2$ copies of $Z(d-1)$;
(ii) three simplices;
(iii) two copies of $M(d-2,1)$.

The facets not containing v are one of the $M(d-2,1)$ copies and one of the simplices. The facets not containing w are the other $M(d-2,1)$ copy and another one of the simplices (see Figure 1(d)).

The next result says that the polytope $Z(d)$ minimizes f_k among d-polytopes with $2d-1$ vertices and at least $d+3$ facets only for $k \geqslant 3$.

Lemma 2.8 *For $d \geqslant 2$, the f-vector of the polytope $Z(d)$ is given by*

$$f_k = \begin{cases} 2d-1, & \text{if } k = 0; \\ d^2 + d - 3 = \zeta_1(2d-1, d) + 1, & \text{if } k = 1; \\ \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k}, & \text{if } k \geqslant 2. \end{cases}$$

In particular, $f_2(Z(d)) = \zeta_2(2d-1, d) + 1$, and $f_k(Z(d)) = \zeta_k(2d-1, d)$ for $k \geqslant 3$.

Proof Since $Z(3)$ has exactly two more vertices than $Z(2)$, the expression for f_0 is clear. An induction argument on $d \geqslant 3$ establishes that $f_1(Z(d)) = (d-2)^2 + 5(d-2) + 3$. Similarly, the expression $f_2(Z(d)) = \frac{1}{3}(d^3 - 4d + 3)$ can be proved by induction on d, starting with the base cases $d = 2, 3$; the induction step uses the expression for f_1 and Lemma 2.5.

For computing f_k for $k \geqslant 3$, we proceed again by induction on $d \geqslant 3$ for all k, with $d = 4, 3$ as the base cases. Suppose the conclusion is valid up to dimension d. Repeated

[1] There are two natural generalizations of $\Sigma(3)$ to higher dimensions, denoted $\Sigma(d)$ and $C(d)$ in [10]. They both have $3d-2$ vertices and are not relevant in this article.

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applications of Pascal's identity yield

$$\begin{aligned}
f_k(Z(d+1)) &= 2f_k(Z(d)) - f_k(Z(d-1)) + f_{k-1}(Z(d)) - f_{k-1}(Z(d-1)) \\
&= 2\binom{d+1}{k+1} + 2\binom{d}{k+1} + 2\binom{d-1}{k} - \binom{d}{k+1} - \binom{d-1}{k+1} - \binom{d-2}{k} \\
&\quad + \binom{d+1}{k} + \binom{d}{k} + \binom{d-1}{k-1} - \binom{d}{k} - \binom{d-1}{k} - \binom{d-2}{k-1} \\
&= 2\binom{d+1}{k+1} + \binom{d+1}{k} + \binom{d}{k+1} - \binom{d-1}{k+1} + \\
&\quad + \binom{d-1}{k} - \binom{d-2}{k} + \binom{d-1}{k-1} - \binom{d-2}{k-1} \\
&= 2\binom{d+1}{k+1} + \binom{d+1}{k} + \binom{d-1}{k} + \binom{d-2}{k-1} + \binom{d-2}{k-2} \\
&= \binom{d+1}{k+1} + \binom{d+2}{k+1} + \binom{d-1}{k} + \binom{d-1}{k-1} \\
&= \binom{d+2}{k+1} + \binom{d+1}{k+1} + \binom{d}{k}.
\end{aligned}$$

Hence, the conclusion is also valid in dimension $d+1$. ■

We are also interested in the $(d-s-1)$-fold pyramid over $Z(s+1)$, denoted by $Z(s+1, d-s-1)$, which has $d+s$ vertices and $d+3$ facets. If P is a pyramid over a facet F, then $f_k(P) = f_k(F) + f_{k-1}(F)$ for each $k \geqslant 0$. Iteration of this leads to the following conclusion for the $(d-s-1)$-fold pyramid over $Z(s+1)$:

(i) If $s < d-1$, then the expression for f_k only involves terms of the form f_j with $3 \leqslant j \leqslant k$, leading to $f_k(Z(s+1, d-s-1)) = \zeta_k(d+s, d)$ for $k \geqslant d-s+2$.
(ii) However, for $k \leqslant d-s+1$, we have $f_k(Z(s+1, d-s-1)) = \zeta_k(d+s, d) + 1$.

Thus, $Z(s+1, d-s-1)$ minimizes the number of k-faces among d-polytopes with at least $d+3$ facets and exactly $d+s$ ($3 \leqslant s \leqslant d$) vertices only for $k \in [d-s+2, d-1]$.

3 Proof of the main theorem

We list the full version of the main theorem.

Theorem 3.1 (Refined theorem for d-polytopes with at most $2d$ vertices) *Given parameters $d \geqslant 3$ and $2 \leqslant s \leqslant d$, and a d-polytope P with $d+s$ vertices, the following statements hold:*

(i) *If $s = 2$, then P has the form $T_m^{d,d-a}$ for some $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$. If P has at least $d+\ell$ facets, where $\ell \in [3, d]$, then for all $k \in [1, d-2]$,*

$$f_k(P) \geqslant \phi_k(d+2, d) + \binom{d-1}{k-1} - \binom{d-\ell+1}{k-\ell+1}.$$

Moreover, if $m = 1$ and $P \neq T_1^{d,d-\ell}$, then $f_k(P) > f_k(T_1^{d,d-\ell})$, for each $k \geqslant \ell - 1$. If $m \geqslant 2$, then $f_k(P) > f_k(T_1^{d,d-\ell})$, for each $k \in [1, d-2]$.

(ii) *If $s \in [3, d]$ and P has at least $d+3$ facets, then $f_k(P) \geqslant \zeta_k(d+s, d)$. Additionally, if $k \in [1, d-2]$, and P is a d-polytope whose number of k-faces equals $\zeta_k(d+s, d)$, then either*

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(a) *P is a $(d-s+1)$-fold pyramid over an $(s-1)$-pentasm, or*
(b) *P is a $(d-s)$-fold pyramid over $TA(s)$, or*
(c) *$s<d$, $k\geqslant d-s+2$, and P is a $(d-s-1)$-fold pyramid over $Z(s+1)$, or*
(d) *$s=4$, and P is a $(d-3)$-fold pyramid over $\Sigma(3)$, or*
(e) *$s=4$, and P is a $(d-4)$-fold pyramid over WP.*

We start from the beginning, at d-polytopes with $d+2$ vertices. This theorem is a variation of the dual statement in Theorem 1.1.

Theorem 3.2 *($d+2$ vertices)* *Let $d\geqslant 3$, $2\leqslant \ell, a\leqslant d$, and $1\leqslant m\leqslant \lfloor a/2\rfloor$. Let $P:=T_m^{d,d-a}$ be a d-polytope with **at least** $d+\ell$ facets, other than $T_1^{d,d-\ell}$. The following hold:*

(i) *If $m=1$, then $f_k(P)>f_k(T_1^{d,d-\ell})$, for each $k\geqslant \ell-1$.*
(ii) *If $m\geqslant 2$, then $f_k(P)>f_k(T_1^{d,d-\ell})$, for each $k\in[1,d-2]$.*

Proof If $m=1$, Lemma 2.2 gives $f_{d-1}(P)=d+1+m(a-m)=d+a$. Since $P\neq T_1^{d,d-\ell}$, we have that $a>\ell$, in which case Lemma 2.3(ii) gives that $f_k(P)>f_k(T_1^{d,d-\ell})$, for each $k\geqslant \ell-1$. Henceforth, we assume that $m\geqslant 2$. We prove (ii) by an induction argument on $d\geqslant 3$ for all $a,\ell\in[2,d]$ and $2\leqslant m\leqslant\lfloor a/2\rfloor$. The case $d=3$ is simple. We then assume that $d\geqslant 4$, $f_{d-1}(P)\geqslant d+\ell$, and that (ii) holds for all $(d-1)$-polytopes $T_{m'}^{d-1,d-a'-1}$ with at least $d-1+\ell'$ facets for all $a',\ell'\in[2,d-1]$ and $2\leqslant m'\leqslant\lfloor a'/2\rfloor$.

Lemma 2.3(i) yields that $f_k(T_1^{d,d-\ell})<f_k(T_2^{d,d-\ell})$, for each $k\geqslant 1$. Additionally, when $\ell\leqslant a$, Lemma 2.3 gives that

$$f_k(T_2^{d,d-\ell})\leqslant f_k(T_2^{d,d-a})\leqslant f_k(T_m^{d,d-a}).$$

Hence, for $k\in[1,d-2]$ and $\ell\leqslant a$, we get that $f_k(T_1^{d,d-\ell})<f_k(T_m^{d,d-a})$.

We now assume that $d\geqslant \ell>a$. Hence, P is a pyramid with base $F:=T_m^{d-1,d-1-a}$, where $a\in[2,d-1]$ and $m\geqslant 2$. The induction hypothesis holds for F when $\ell\in[2,d-1]$, since F has $d-1+2$ vertices and at least $d-1+\ell(d-2)$-faces. As a consequence, for each $k\in[1,d-2]$, we find that

$$\begin{aligned} f_k(P)&=f_k(F)+f_{k-1}(F)\\ &>f_k(T_1^{d-1,d-1-\ell})+f_{k-1}(T_1^{d-1,d-1-\ell})\\ &=f_k(T_1^{d,d-\ell}).\end{aligned}$$

For the case $f_{d-2}(P)$, we remark that $f_{d-2}(F)\geqslant d-1+\ell=f_{d-2}(T_1^{d-1,d-1-\ell})$ by assumption and that $f_{d-3}(F)>f_{d-3}(T_1^{d-1,d-1-\ell})$ by induction.

Suppose $\ell=d$. Then $f_{d-2}(F)\geqslant d-1+\ell\geqslant 2(d-1)$, and thus, for each $k\in[1,d-2]$, the induction hypothesis on $d-1$ for $\ell'=d-1$ gives that

$$\begin{aligned} f_k(P)&=f_k(F)+f_{k-1}(F)\\ &\geqslant f_k(T_1^{d-1,0})+1+f_{k-1}(T_1^{d-1,0})+1\\ &=\left[\binom{d+1}{d-k}-\binom{2}{d-k}-\binom{d-1}{d-k}+\binom{1}{d-k}+1\right]\\ &\quad+\left[\binom{d+1}{d-k+1}-\binom{2}{d-k+1}-\binom{d-1}{d-k+1}+\binom{1}{d-k+1}+1\right]\end{aligned}$$

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$$= \binom{d+2}{d-k+1} - \binom{3}{d-k+1} - \binom{d}{d-k+1} + \binom{2}{d-k+1} + 2$$
$$> \binom{d+2}{d-k+1} - \binom{1}{d-k} - \binom{d}{d-k+1} = f_k(T_1^{d,0}).$$

This completes the proof of the theorem. ∎

Proposition 3.3 [14, Proposition 3.1] *Let $d \geqslant 2$ and let P be a d-polytope. In addition, suppose that $r \leqslant d+1$ is given and that $S := (v_1, v_2, \ldots, v_r)$ is a sequence of distinct vertices in P. Then the following hold:*

(i) *There is a sequence $F_1, F_2, \ldots, F_r$ of faces of P such that each F_i has dimension $d-i+1$ and contains v_i, but does not contain any v_j with $j < i$.*

(ii) *For each $k \geqslant 1$, the number of k-faces of P that contain at least one of the vertices in S is bounded from below by*

$$\sum_{i=1}^{r} f_{k-1}(F_i/v_i) \geqslant \sum_{i=1}^{r} \binom{d+1-i}{k}.$$

(iii) *In case of equality in (ii) for some $k \in [1, d-2]$, then for any ordering $v_{\ell 1}, \ldots, v_{\ell r}$ of the vertices of S, we must have that the number of k-faces containing $v_{\ell i}$ and not containing any vertex $v_{\ell j}$ with $j < i$ is precisely $\binom{d-i+1}{k}$. In particular, every vertex in S is simple.*

Proof Statements (i) and (ii) are from [14, Proposition 3.1]. Part (iii) is proven in [7, Corollary 8.2.6]; it follows from noting that the minimum of the right-hand side is attained when each vertex figure $F_i/v_{\ell i}$ is a $(d-i)$-simplex. ∎

We now deal with dimensions 3 and 4 in Theorem 3.1.

Proposition 3.4 *For $d = 3$ or 4 and $3 \leqslant s \leqslant d$, the d-polytopes with precisely $d+s$ vertices, at least $d+3$ facets, and at most $\zeta_k(d+s, d)$ k-faces for each $k \in [1, d-2]$, are as follows:*

(i) *For $d = 3$, the minimizers are $TA(3)$ and the pyramid over the pentagon, both with f-vector $(6, 10, 6)$.*

(ii) *For $d = 4$, the minimizers with seven vertices are the pyramid over $TA(3)$ and the 2-fold pyramid over a pentagon, both with f-vector $(7, 16, 16, 7)$.*

(iii) *The minimizers with eight vertices, all with f-vector $(8, 18, 17, 7)$, are the polytopes $TA(4)$, $Pm(3,1)$, WP, and $\mathrm{pyr}(\Sigma(3))$.*

Proof The catalogs of 3-polytopes with six vertices (see, for instance, [2, Figure 3]) reveal that the two possible minimizers are $TA(3)$ and the pyramid over the pentagon; both have f-vector $(6, 10, 6)$. This can be deduced easily from Steinitz's theorem.

The 4-polytopes with exactly seven vertices were completely characterized by Grünbaum, in terms of their Gale diagrams: a complete list, together with the corresponding f-vectors, can be found in [4, Figure 5]. The minimizers with seven vertices are the pyramid over $TA(3)$ and the 2-fold pyramid over a pentagon, with f-vector (7,16,16,7). Every other 4-polytope with seven vertices and at least seven facets has $f_1 \geqslant 17$ and $f_2 \geqslant 17$.

A 4-polytope with 8 vertices and 16 edges must be a prism, which has $d+2$ facets. There is no 4-polytope with 8 vertices and 17 edges [5, Section 10.4]. There

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are exactly four 4-polytopes with 8 vertices and 18 edges, namely, $TA(4)$, $Pm(3,1)$, WP, and a pyramid over $\Sigma(3)$, all with f-vector $(8,18,17,7)$ (see [11, Lemma 18]). Any 4-polytope with eight vertices and at least seven facets must satisfy $f_2 = f_1 + f_3 - f_0 \geqslant 18 + 7 - 8 = 17$, with strict inequality unless $f_1 = 18$ and $f_3 = 7$. Thus, these four polytopes are also the only minimizers of f_2 for 4-polytopes with eight vertices and at least seven facets. Any other 4-polytope with eight vertices and at least seven facets has more edges and 2-faces than these examples. ■

It is worth noting that the f-vectors of all 4-polytopes with up to nine vertices, together with the corresponding number of combinatorial types, can be found in [3, Tables 6 and 7].

We recall the smallest simple d-polytopes (see [9, Lemma 2.19]).

Remark 3.5 The smallest vertex counts of simple d-polytopes are $d+1$ (the d-simplex), $2d$ (the simplicial d-prism), and $3d-3$ (the polytope $T(2) \times T(d-2)$). These all have $d+2$ facets.

Theorem 3.6 *Let $d \geqslant 3$ and $3 \leqslant s \leqslant d$, and let P be a d-polytope with $d+s$ vertices and at least $d+3$ facets. Then $f_k(P) \geqslant \zeta_k(d+s,d)$ for each $k \in [1, d-2]$.*

Proof Proceed by induction on $d \geqslant 3$ for all $3 \leqslant s \leqslant d$. The cases $d = 3, 4$ are covered in Proposition 3.4, so we assume that $d \geqslant 5$. Remark 3.5 gives that P is not simple. Let v be a nonsimple vertex of P with maximum degree and let F be a facet of P that does not contain v and has the maximum number $d+m$ of vertices where $0 \leqslant m \leqslant d-1$. ■

Case 1 $f_0(F) = f_0(P) - 1$; that is, $m = s - 1$.

The induction hypothesis on $d-1$ for $3 \leqslant s \leqslant d-1$ yields that

$$\begin{aligned} f_k(P) &= f_k(F) + f_{k-1}(F) \\ &\geqslant f_k(Pm(s-1, d-s)) + f_{k-1}(Pm(s-1, d-s)) \\ &= f_k(Pm(s-1, d+1-s) = \zeta_k(d+s, d). \end{aligned} \tag{3.1}$$

If $s = d$, then F has $2d-1 = 2(d-1)+1$ vertices and at least $(d-1)+3$ facets. In this case, Theorem 2.4 gives that $f_k(F) \geqslant f_k(Pm(d-1))$. This combined with the relation $f_k(P) = f_k(F) + f_{k-1}(F)$ yields the result.

Case 2 $f_0(F) = d + m = d - 1 + m + 1$, where $m \in [2, s-2]$.

Subcase 2.1 *The facet F has at least* d+2=(d-1)+3 facets.

Since $\deg_P(v) \geqslant d+1$, by Proposition 3.3, the number of k-faces containing at least one of the vertices outside F is at least $\phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k}$. The induction

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hypothesis on $d-1$ yields that

$$
\begin{aligned}
f_k(P) &\geqslant f_k(F) + \phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&\geqslant \zeta_k(d+m, d-1) + \phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \phi_k(d+m, d-1) + \binom{d-2}{k} - \binom{d-m-1}{k} + \phi_{k-1}(d+1, d-1) \\
&\quad + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-m-1}{k+1} + \binom{d-2}{k} - \binom{d-m-1}{k} \\
&\quad + \binom{d}{k} + \binom{d-1}{k} - \binom{d-2}{k} + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d-m}{k+1} + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \qquad (3.2) \\
&= \zeta_k(d+s, d) + \binom{d+2-s}{k+1} - \binom{d-m}{k+1} - \binom{d-1}{k} + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \binom{d+2-s}{k+1} - \binom{d-m}{k+1} + \sum_{i=3}^{s-m} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=0}^{d+1-s} \binom{i}{k} - \sum_{i=0}^{d-m-1} \binom{i}{k} + \sum_{i=d-s+m+1}^{d-2} \binom{i}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=0}^{d+1-s} \binom{i}{k} - \left[\sum_{i=0}^{d+1-s} \binom{i}{k} + \sum_{i=d+2-s}^{d-m-1} \binom{i}{k} \right] + \sum_{i=d-s+m+1}^{d-2} \binom{i}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=d+2-s}^{d-m-1} \left[\binom{m-1+i}{k} - \binom{i}{k} \right] \\
&\geqslant \zeta_k(d+s, d).
\end{aligned}
$$

The last inequality holds with equality if and only if $m = s-2$.

Subcase 2.2 *The facet F has $d+1 = (d-1)+2$ facets and is not a simple polytope.*

If F is not a simple $(d-1)$-polytope, which occurs when either $f_0(F) \leqslant 2d-3$ or $f_0(F) = 2d-2$ and F is not a simplicial $(d-1)$-prism (Remark 3.5), then, by Lemma 2.1, F must be a pyramid whose apex has degree at least $f_0(F) = d+m$ in P. Thus, $\deg_P(v) \geqslant d+m = d-1+m+1$.

By Proposition 3.3, the number of k-faces containing at least one of the vertices outside F is at least $\phi_{k-1}(d+m, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k}$. Furthermore,

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$f_k(F) \geqslant \phi_k(d+m, d-1)$ by Theorem 1.1. Hence, we have that

$$\begin{aligned}
f_k(P) &\geqslant f_k(F) + \phi_{k-1}(d+m, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&\geqslant \phi_k(d+m, d-1) + \phi_{k-1}(d+m, d-1) + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-m-1}{k+1} + \binom{d}{k} + \binom{d-1}{k} \\
&\quad - \binom{d-m-1}{k} + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \binom{d+2-s}{k+1} - \binom{d-m}{k+1} - \binom{d-1}{k} + \sum_{i=2}^{s-m} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=0}^{d+1-s} \binom{i}{k} - \sum_{i=0}^{d-m-1} \binom{i}{k} - \binom{d-1}{k} + \sum_{i=d-s+m+1}^{d-1} \binom{i}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=d+2-s}^{d-m-1} \left[\binom{m-1+i}{k} - \binom{i}{k} \right] \\
&\geqslant \zeta_k(d+s, d).
\end{aligned} \tag{3.3}$$

As in the previous scenario, the last inequality holds with equality if and only if $m = s-2$.

Remark 3.7 (Number of facets containing the vertex v) In case of equality, the vertex v can have degree either $d+s-2$ or $d+s-1$ in P. Additionally, the number of $k-1$ faces in the vertex figure of v must equal $\phi_{k-1}(d+s-2, d-1)$.

(i) If its degree is $d+s-2$, then its vertex figure must be an $(s-1, d-s)$-triplex, implying that v is contained in exactly $d+1$ facets of P.

(ii) If the degree of v is $d+s-1$ and $s < d$, since

$$\phi_{k-1}(d+s-2, d-1) \leqslant \phi_{k-1}(d+s-1, d-1) \leqslant f_{k-1}(P/v),$$

then Theorem 1.1 gives that P/v is an $(s, d-1-s)$-triplex, implying that v is contained in exactly $d+1$ facets of P. From $\phi_{k-1}(d+s-1, d-1) = \phi_{k-1}(d+s-2, d-1)$, which amounts to $\binom{d-s}{k} = \binom{d-s+1}{k}$, it follows that $\binom{d-s}{k-1} = 0$. Hence, the equality in this case happens provided that $k \geqslant d-s+2$.

(iii) Suppose that the degree of v is $d+s-1$ and $s = d$. If $f_{d-2}(P/v) \geqslant d+2$, then Theorem 2.4 gives that $f_{k-1}(P/v) \geqslant \binom{d}{k} + 2\binom{d-1}{k} - \binom{d-2}{k} > \phi_{k-1}(2d-2, d-1)$. If $f_{d-2}(P/v_1) = d+1$, then Theorem 2.4 gives that $f_{k-1}(P/v) \geqslant f_{k-1}((T_2^{d-1, d-1-(\lfloor (d-1)/2 \rfloor + 2)})^*) > \phi_{k-1}(2d-2, d-1)$. Thus, equality does not arise in this case.

Subcase 2.3 *The facet F has $d+1 = (d-1)+2$ facets and is a simple polytope.*

This implies that F has $2d-2$ vertices and is a simplicial $(d-1)$-prism. Label the vertices in a simplex $(d-2)$-face R of F as $w_1, \ldots, w_{d-1}$ and the vertices in the other simplex $(d-2)$-face R' of F as $w'_1, \ldots, w'_{d-1}$. Then $R \cap R' = \varnothing$. Additionally, label the two vertices outside F as v, v'. Since v has degree at least $d+1$ in P, there are at least

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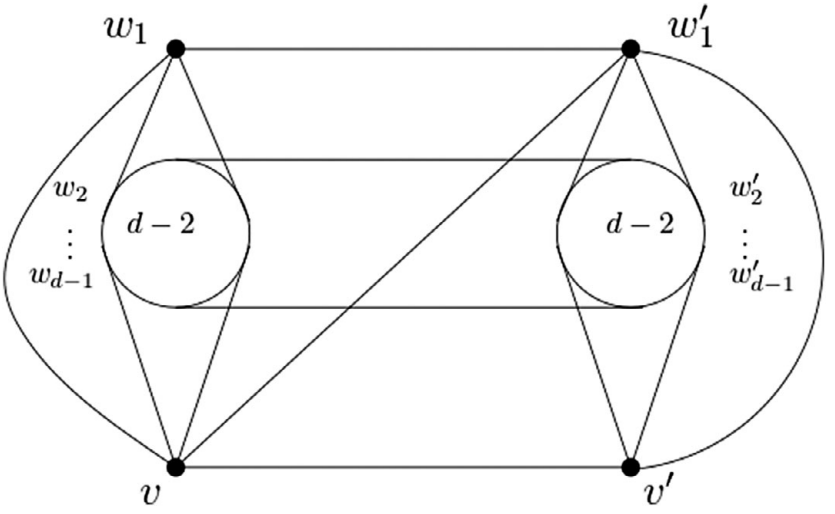


Figure 2: F is a prism.

$2d-1$ edges between $\{v, v'\}$ and F. Thus, there is a vertex in F, say w_1', that is adjacent to both v and v' (see Figure 2).

Consider the other facet F_1 that intersects F at the simplicial $(d-2)$-prism $K_1 = \operatorname{conv}\{w_i, w_i' : i = 2, \ldots, d-1\}$. It follows that $f_0(F_1) = 2d-3$ or $2d-2$.

First suppose that $f_0(F_1) = 2d-3$. Then F_1 is a pyramid over K_1 and thus, has $d+1(d-2)$-faces. The apex of F_1, namely, v or v', has degree at least $2d-2$ in P as it is adjacent to w_1'.

The computation in (3.3) also applies here:

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + \phi_{k-1}(2d-2, d-1) + \binom{d-1}{k} \\ &\geqslant \left[\binom{d}{k+1} + \binom{d-1}{k+1} - \binom{1}{k+1}\right] + \left[\binom{d}{k} + \binom{d-1}{k} - \binom{1}{k}\right] \\ &\quad + \binom{d-1}{k} \\ &= \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{2}{k+1} + \binom{d-1}{k} \\ &= \zeta_k(2d, d). \end{aligned} \tag{3.4}$$

In case of equality in (3.4), we find that v' cannot be the apex of F_1, as v' is present in exactly $\binom{d-1}{k}$ k-faces not containing v. (We can actually apply Equation (3.3), then in case of equality, it implies that v' has to be simple in every facet not containing v, so v' cannot be the apex of F_1.) Furthermore, by reasoning as in Remark 3.7(iii), since $f_{k-1}(P/v) = \phi_{k-1}(2d-2, d-1)$, we find that P/v must have $d+1$ $(d-2)$-faces and be a simplicial $(d-1)$-polytope. In particular, the vertex v has degree exactly $2d-2$.

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Suppose that F_1 has $2d-2$ vertices. Then F_1 is not a pyramid. If F_1 has at least $d+2$ facets, then, since w_1' is a nonsimple vertex outside F_1, we settle this by the calculation of Equation (3.2). We can then assume that F_1 has $d+1$ facets, in which case F_1 is a simplicial $(d-1)$-prism (see Lemma 2.1). Without loss of generality, we may assume that, within F_1, the vertex v is adjacent to $w_2, \ldots, w_{d-1}$ and v' is adjacent to $w_2', \ldots, w_{d-1}'$. It follows that $w_2, \ldots, w_{d-1}$, $w_2', \ldots, w_{d-1}'$ are simple vertices in P.

As F_1 is a prism, the lines containing the edges $[w_i, w_i']$, $i = 2, \ldots, d-1$ and the line containing the edge $[v, v']$ have the property that every two of them are coplanar. Likewise, since F is a prism, every two of the lines containing the edges $[w_i, w_i']$, $i = 1, \ldots, d-1$, are coplanar. It follows that $[w_1, w_1']$ is coplanar with $[v, v']$. Since the vertex w_1' is adjacent to both v and v', while the vertex w_1 is adjacent to either v or v', we have a violation of convexity. Therefore this case does not arise.

Case 3 The facet F has $d+1$ vertices.

There are $s-1$ vertices outside F, say $\{u_1, \ldots, u_{s-1}\}$, where $u_1 = v$. By Proposition 3.3, there is a sequence $F_1, \ldots, F_{s-1}$ of faces of P such that each F_i has dimension $d-i+1$ and contains u_i, but does not contain any u_j with $j < i$. The number of k-faces in P is bounded from below as follows:

$$
\begin{aligned}
f_k(P) &\geqslant f_k(F) + \phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&\geqslant \phi_k(d+1, d-1) + \phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&= \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-2}{k+1} + \binom{d}{k} + \binom{d-1}{k} - \binom{d-2}{k} \\
&\quad + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \binom{d+2-s}{k+1} - \binom{d}{k+1} + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=0}^{d+1-s} \binom{i}{k} - \sum_{i=0}^{d-1} \binom{i}{k} + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) - \sum_{i=d+2-s}^{d-1} \binom{i}{k} + \sum_{i=2}^{s-1} \binom{d-i+1}{k} \\
&= \zeta_k(d+s, d).
\end{aligned}
\tag{3.5}
$$

This settles this case.

Case 4 The facet F (and any other facet not containing v) is a simplex.

Let v_1 be a vertex not in F other than v. By assumption, there must be a simplex facet F' containing v_1 but not v. Let $v_1, \ldots, v_a$ represent the vertices in $F' \backslash F$ and $v_{a+1}, \ldots, v_s$

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the vertices outside $F \cup F'$. We consider two scenarios according to the number $s - a$ of vertices outside $F \cup F'$.

Subcase 4.1 *The inequality* $s - a \geqslant 2$ holds.

By Proposition 3.3, there is a sequence $J_1, \ldots, J_a$ of faces in F' such that each J_i has dimension $d-1-i+1$ and contains v_i but does not contain v_j with $j < i$. Additionally, there is a sequence $F_{a+1}, \ldots, F_s$ of faces in P such that each F_ℓ has dimension $d - \ell + (a+1)$. Consequently, we have the following:

$$\begin{aligned}
f_k(P) &\geqslant f_k(F) + \#\text{ of } k\text{-faces in } F'\backslash F \text{ containing a vertex in } \{v_1, \ldots, v_a\} \\
&\quad + \#\text{ of } k\text{-faces outside } F \cup F' \text{ containing a vertex in } \{v_{a+1}, \ldots, v_s\} \\
&\geqslant \binom{d}{k+1} + \left[\sum_{i=1}^{a}\binom{d-i}{k}\right] + \left[\phi_{k-1}(d+1, d-1) + \sum_{i=2}^{s-a}\binom{d-i+1}{k}\right] \\
&= \binom{d}{k+1} + \sum_{i=1}^{a}\binom{d-i}{k} + \binom{d}{k} + \binom{d-1}{k} - \binom{d-2}{k} + \sum_{i=2}^{s-a}\binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \binom{d+2-s}{k+1} - \binom{d}{k+1} - \binom{d-2}{k} + \sum_{i=1}^{a}\binom{d-i}{k} + \sum_{i=2}^{s-a}\binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) + \sum_{i=0}^{d+1-s}\binom{i}{k} - \sum_{i=0}^{d-1}\binom{i}{k} - \binom{d-2}{k} + \sum_{i=1}^{a}\binom{d-i}{k} + \sum_{i=2}^{s-a}\binom{d-i+1}{k} \\
&= \zeta_k(d+s, d) - \sum_{i=d+2-s}^{d-1}\binom{i}{k} - \binom{d-2}{k} + \sum_{i=d-a}^{d-1}\binom{i}{k} + \sum_{i=d+a+1-s}^{d-1}\binom{i}{k} \\
&= \zeta_k(d+s, d) - \sum_{i=d+2-s}^{d+a-s}\binom{i}{k} - \binom{d-2}{k} + \sum_{i=d-a}^{d-1}\binom{i}{k} \\
&= \zeta_k(d+s, d) + \left[\binom{d-1}{k} - \binom{d-2}{k}\right] + \sum_{i=d+2-s}^{d+a-s}\underbrace{\binom{i+s-a-2}{k} - \binom{i}{k}}_{\geqslant 0 \text{ since } s-a-2 \geqslant 0} \\
&> \zeta_k(d+s, d).
\end{aligned}$$

This subcase is now settled.

Subcase 4.2 *The equality* $s - a = 1$ holds; that is, v is the only vertex that is outside $F \cup F'$.

In this case, $\#(F \cap F') \geqslant 1$, and each vertex in $F \cap F'$ is adjacent to every other vertex in $F \cup F'$, thereby implying that such a vertex has degree at least $d + s - 2 = \#(F \cup F')$ in P. By maximality, $\deg(v) \geqslant d + s - 2$. Again, by Proposition 3.3, the number of k-faces containing a vertex outside F is at least $\sum_{i=2}^{s}\binom{d-i+1}{k} + \phi_{k-1}(d+s-2, d-1)$. Hence, we

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have the following:

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + \sum_{i=2}^{s} \binom{d-i+1}{k} + \phi_{k-1}(d+s-2, d-1) \\ &\geqslant \binom{d}{k+1} + \sum_{i=2}^{s} \binom{d-i+1}{k} + \phi_{k-1}(d+s-2, d-1) \\ &= \binom{d}{k+1} + \binom{d}{k} + \binom{d-1}{k} - \binom{d+1-s}{k} + \sum_{i=2}^{s} \binom{d-i+1}{k} \\ &= \zeta_k(d+s, d) - \binom{d+1}{k+1} + \binom{d+1-s}{k+1} + \binom{d}{k} + \sum_{i=2}^{s} \binom{d-i+1}{k} \\ &= \zeta_k(d+s, d) - \sum_{i=1}^{s} \binom{d-i+1}{k} + \binom{d}{k} + \sum_{i=2}^{s} \binom{d-i+1}{k} \\ &= \zeta_k(d+s, d). \end{aligned} \tag{3.6}$$

This subcase is also settled.

We delve deeper into the degree of v and the facets containing it.

Remark 3.8 In case of equality in (3.6), we have the following:

(i) If $s \leqslant d-1$, then $\#(F \cap F') \geqslant 2$. Since v has degree at least $d+s-2$, it must be adjacent to a vertex $w \in F \cap F'$, as there are not $d+s-2$ vertices outside the intersection. Consequently, the vertex w has degree $d+s-1$ in P, which implies that v has degree $d+s-1$ in P.

(ii) Suppose that $s = d$. If $\deg(v) = 2d-1$, then the vertex figure P/v has $2d-1$ vertices. Reasoning as in Remark 3.7(iii), we see that this case does not arise.

Assume that $\deg(v) = 2d-2$. There is a facet that contains v and has $2d-2$ vertices in total. Indeed, let w be the unique vertex in $F \cap F'$. Then we get that $\deg(w) = 2d-2$ and that w is not adjacent to v (otherwise $\deg(w) = 2d-1$). Furthermore, since $f_k(P/v) = \phi_k(2d-2, d-1)$, we find that P/v is a simplicial $(d-1)$-prism. Consider a simplicial $(d-2)$-prism R in P/v. The facet J of P containing v (and R) has at least $2(d-2)+1$ vertices. It follows that J must also contain w; otherwise, J is a nonsimplex facet missing a vertex of largest degree. That is, $f_0(J) = 2d-2$.

In every scenario of the remark, v is contained in exactly $d+1$ facets of P.

The proof of the theorem is complete.

3.1 Equality analysis

Theorems 3.2 and 3.6 deal with the inequality part of our main theorem (Theorem 3.1). We now characterize the equality cases.

Remark 3.9 For the parameters (d, s, m), where $d \geqslant 5$, $3 \leqslant s \leqslant d$ and $0 \leqslant m \leqslant s-1$, let P be a d-polytope with $d+s$ vertices and at least $d+3$ facets, let v be a nonsimple vertex with the largest degree in P, let F be a facet not containing v and with the largest possible number $d+m$ of vertices (subject to not containing v), and let X be the set of vertices outside F. The proof of Theorem 3.6 establishes strict inequality if $2 \leqslant m \leqslant s-3$. The possible equality cases can occur only in the following settings:

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(i) $\mathbf{m = s - 1}$ **(Theorem 3.6, Case 1)**: The facet F satisfies $f_k(F) = f_k(Pm(s-1, d-s)) = \zeta_k(d+s-1, d-1)$ for each $k \in [1, d-2]$ (see (3.1)). If $s = d$, then $f_k(F) = f_k(Pm(d-1))$ for each $k \in [1, d-2]$.

(ii) $\mathbf{m = s - 2}$ **(Theorem 3.6, Subcase 2.1)**: The facet F has at least $d + 2(d-2)$-faces and satisfies $f_k(F) = f_k(Pm(s-2, d-s+1)) = \zeta_k(d+s-2, d-1)$ for $k \in [1, d-2]$. Additionally, the vertex v has degree $d+1$ and its vertex figure is a $(2, d-3)$-triplex. The number of k-faces containing the other vertex in $X\backslash\{v\}$ but not v is $\sum_{i=2}^{2}\binom{d-i+1}{k} = \binom{d-1}{k}$ (see (3.2)).

(iii) $\mathbf{m = s - 2}$ **(Theorem 3.6, Subcase 2.2)**: The facet F has $d+1$ $(d-2)$-faces and is an $(s-1, d-s)$-triplex. The vertex v has degree $d+s-2$ and its vertex figure is an $(s-1, d-s)$-triplex. Additionally, the number of k-faces containing the other vertex in $X\backslash\{v\}$ but not v is $\sum_{i=2}^{2}\binom{d-i+1}{k} = \binom{d-1}{k}$ (see (3.3)).

(iv) $\mathbf{m = s - 2}$ **(Theorem 3.6, Subcase 2.2)**: The facet F has $d+1$ $(d-2)$-faces and is an $(s-1, d-s)$-triplex. The vertex v has degree $d+s-1$ and its vertex figure is an $(s, d-1-s)$-triplex with $s \leqslant d-1$. Furthermore, $\phi_{k-1}(d+s-1, d-1) = \phi_{k-1}(d+s-2, d-1)$, which implies that $\binom{d-s}{k} = \binom{d-s+1}{k}$ or, equivalently, that $k \geqslant d-s+2$ (see Remark 3.7(ii)). Additionally, the number of k-faces containing the other vertex in $X\backslash\{v\}$ but not v is $\sum_{i=2}^{2}\binom{d-i+1}{k} = \binom{d-1}{k}$ (see (3.3)).

(v) $\mathbf{m = d - 2}$ **(Theorem 3.6, Subcase 2.3)**: The facet F is a simplicial $(d-1)$-prism, there is a facet F' which is a pyramid over a simplicial $(d-2)$-prism from F and whose apex is v. The degree of v in P is exactly $2d-2$ and its vertex figure is a simplicial $(d-1)$-prism. Additionally, the number of k-faces containing the vertex in $X\backslash\{v\}$ but not v is $\binom{d-1}{k}$ (see (3.4)).

(vi) $\mathbf{m = 1}$ **(Theorem 3.6, Case 3)**: The facet F is a $(2, d-3)$-triplex. The vertex v has degree $d+1$ and its vertex figure is a $(2, d-3)$-triplex, and the number of k-faces containing some vertex in $X\backslash\{v\}$ but not v is $\sum_{i=2}^{s-1}\binom{d-i+1}{k}$ (see (3.5)).

(vii) $\mathbf{m = 0}$ **(Theorem 3.6, Subcase 4.2)**: Every facet not containing a nonsimple vertex with the largest degree is a simplex. Moreover, there is a simplex facet F' such that $F \cup F'$ contains $d+s-1$ vertices. If $s \leqslant d-1$, then the vertex v has degree $d+s-1$ and its vertex figure is an $(s, d-1-s)$-triplex. If $s = d$, then the vertex v has degree $2d-2$ and its vertex figure is a simplicial $(d-1)$-prism. Additionally, the number of k-faces containing some vertex in $X\backslash\{v\}$ but not v is $\sum_{i=2}^{s}\binom{d-i+1}{k}$ (see (3.6) and Remark 3.8).

Lemma 3.10 *Let $d \geqslant 5$ and $3 \leqslant s \leqslant d$ be given parameters, and let P be a nonpyramidal d-polytope with $d+s$ vertices and at least $d+3$ facets. If $f_k(P) = \zeta_k(d+s, d)$ for some $k \in [1, d-2]$, then P has exactly $d+3$ facets.*

Proof We are inspired by the reasoning in [14, Section 4]. The scenarios where we get equality in the proof of Theorem 3.6 are listed in Remark 3.9. In every case, the following hold:

(i) There is a nonsimple vertex v_1 (with the maximum degree) contained in exactly $d+1$ facets.

(ii) There is a facet J with $d+m$ vertices not containing v_1 where $0 \leqslant m \leqslant s-2$.

(iii) Let $Y := \{v_1, \ldots, v_{s-m}\}$ be the set of vertices outside J. The vertices outside J other than v_1 are contained in exactly $\binom{d-1}{k}$ k-faces of P.

In Remark 3.9, the vertex v_1 is the vertex v, $Y = X$, and $J = F$.

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Claim 1 The number of k-faces in P containing the vertex v_i and not v_j with $i \geqslant 2$ and $j < i$ is precisely $\binom{d-i+1}{k}$. ■

Proof of claim The minimum number of k-faces within a $(d-i+1)$-face that contains the vertex v_i is attained when the vertex figure is a simplex, namely, a $(d-i)$-simplex. Since the number of k-faces in P containing some vertex in $Y\backslash\{v_1\}$ is precisely $\sum_{i=2}^{s-m}\binom{d-i+1}{k}$, each vertex figure must be a minimizer. ■

The facets of P can be partitioned into the facet J and the sets X_i of facets containing the vertex v_i but no vertex v_j with $j < i$. From the first paragraph, we get that $\#X_1 = d+1$. We prove that $\#X_2 = 1$ and $\#X_\ell = 0$ for $\ell \in [3, s-m]$.

Suppose that there are two facets J_2 and J_2' in X_2. Then there must exist a k-face R of J_2' that is not a face of J_2. Since there are already $\binom{d-1}{k}$ k-faces in J_2 containing v_2 (and not v_1), the number of k-faces in P containing v_2 but not v_1 is greater than $\binom{d-1}{k}$, contradicting Claim 1. Thus, $\#X_2 = 1$.

The same idea proves that $\#X_3 = 0$. If there were a facet J_3 containing v_3 but not v_1 or v_2, then the number of k-faces in P containing v_3 but not v_1 or v_2 would be at least the number of k-faces in J_3 containing v_3, namely, $\binom{d-1}{k}$, which again contradicts Claim 1. Since we have the freedom to choose a different ordering of the vertices in $Y\backslash\{v_1\}$, any vertex in $Y\backslash\{v_1, v_2\}$ can be picked as v_3. Thus, $\#X_\ell = 0$ for $\ell \in [3, r]$, yielding that P has $d+3$ facets.

We write down a corollary of Lemma 3.10 to aid our future discussion.

Corollary 3.11 *Let $d \geqslant 5$ and $3 \leqslant s \leqslant d$, let P be a d-polytope with $d+s$ vertices and at least $d+3$ facets, let v be a nonsimple vertex with the maximum degree, and let F be a facet not containing v and with a largest possible number of vertices. In the equality condition of Subcases 2.2 and 2.3 of Theorem 3.6, the facet F has $d+s-2$ vertices and $d+1$ $(d-2)$-faces. Moreover, for the vertex $v' \neq v$ outside F, the following hold:*

(i) *There is a unique facet containing both v and v', which intersects F at a $(d-3)$-face.*
(ii) *There is a unique facet containing v' but not v, which intersects F at a $(d-2)$-face.*
(iii) *There are either $d-2$ or $d-1$ facets containing both v and v' that intersect with F at a $(d-2)$-face.*

Proof In the notation of Lemma 3.10, we assume that the facet F here is the facet J there, the vertex v is the vertex v_1 there, and the vertex v' is the vertex v_2 there.

(i) Since F contains exactly $d+1$ $(d-2)$-faces and, by Lemma 3.10, P contains exactly $d+3$ $(d-1)$-faces, there must exist a unique facet in P consisting of the convex hull of a $(d-3)$-face of F and the vertices v and v'.

(ii) The proof of Claim 1 in Lemma 3.10 shows that the set X_2 of facets containing v' but not v has cardinality one.

(iii) Parts (i) and (ii) count two facets in P containing v'. The remaining facets containing v' must also contain v and a $(d-2)$-face of F. If v' is simple in P, there are $d-2$ such facets; otherwise, there are $d-1$. ■

Theorem 3.12 (Equality) *Let $d \geqslant s \geqslant 3$ be parameters and let P be a d-polytope with $d+s$ vertices and at least $d+3$ facets. If, for some $k \in [1, d-2]$,*

$$f_k(P) = \zeta_k(d+s, d),$$

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then the following statements hold:

(i) *If $s = 3$, then P is either the $(d-2)$-fold pyramid over a pentagon or the $(d-3)$-fold pyramid over $TA(3)$.*
(ii) *If $5 \leqslant s \leqslant d-1$ and $k \leqslant d-s+1$, then P is either the $(d-s+1)$-fold pyramid over an $(s-1)$-pentasm or the $(d-s)$-fold pyramid over $TA(s)$.*
(iii) *If $5 \leqslant s \leqslant d-1$ and $d-s+2 \leqslant k \leqslant d-2$, then P is either the $(d-s+1)$-fold pyramid over an $(s-1)$-pentasm, the $(d-s)$-fold pyramid over $TA(s)$, or the $(d-s-1)$-fold pyramid over $Z(s+1)$.*
(iv) *If $s = 4$ and either $k \leqslant d-3$ or $d = 4$ and $k = 2$, then P is either the $(d-3)$-fold pyramid over a 3-pentasm, the $(d-3)$-fold pyramid over $\Sigma(3)$, the $(d-4)$-fold pyramid over $TA(4)$, or the $(d-4)$-fold pyramid over WP.*
(v) *If $s = 4$, $k = d-2$, and $d \geqslant 5$, then P is either the $(d-3)$-fold pyramid over a 3-pentasm, the $(d-3)$-fold pyramid over $\Sigma(3)$, the $(d-4)$-fold pyramid over $TA(4)$, the $(d-4)$-fold pyramid over WP, or the $(d-5)$-fold pyramid over $Z(5)$.*
(vi) *If $s = d$ and $d \geqslant 5$, then P is either $TA(d)$ or the pyramid over a $(d-1)$-pentasm.*

Proof We proceed by induction on the dimension. The induction base is $d = 3$ and 4, which is covered by Proposition 3.4.

Let P be a d-polytope as in the hypothesis. Suppose now the theorem holds for dimensions $3, 4, \ldots, d-1$. As before, let v be a nonsimple vertex of P with maximum degree and let F be a facet of P that does not contain v and has a maximum number $d+m$ of vertices (among such facets). We consider each of the scenarios listed in Remark 3.9. ∎

Remark 3.9(i) The facet F has $d+s-1$ vertices. If $s \leqslant d-1$, by induction on $d-1$, the facet is one of the minimizers. Thus, P is a pyramid over F and has the desired form. If $s = d$, by Theorem 3.6, Case 1 (and [13]), the facet F is a $(d-1)$-pentasm. Hence, P is a pyramid over the pentasm; that is, a $Pm(d-1,1)$.

We now consider the scenarios in Remark 3.9(ii)–(v). Here, the facet F has $d+s-2$ vertices with $s \geqslant 4$. (If $s = 3$, then F has $d+1$ vertices, a scenario considered in Remark 3.9(vi)). Denote by v' the vertex other than v and outside F. In Remark 3.9(ii)–(v), note that the vertex v' is simple in the unique facet that contains it but does not contain v.

Remark 3.9(ii) The facet F has at least $d+2$ $(d-2)$-faces and satisfies $f_k(F) = \zeta_k(d+s-2, d-1)$. By the induction hypothesis, F is either $Pm(s-2, d-s+1)$, $TA(s-1, d-s)$, $Z(s, d-s-1)$, $\mathrm{pyr}_{d-5}(WP)$, or $\mathrm{pyr}_{d-4}(\Sigma(3))$.

First suppose that F is not a pyramid. For $d \geqslant 5$, this means that the facet F could be only $TA(d-1)$, $Z(d-1)$, or WP. Remarks 2.6 and 2.7 assert that $TA(d-1)$ and $Z(d-1)$ both have a vertex with degree $2d-4$, which results in a degree at least $2d-3$ in P. Since the vertex v has degree $d+1$, so does every nonsimple vertex in P. Thus, we find that $d \leqslant 4$, which has been already covered.

It remains to consider the scenario where F is WP. We denote by v_1, v_2, v_3, v_4 the four nonsimple vertices of F (see Figure 1(b)). Since v' is simple in the unique facet containing it but not v, this facet must be the simplex with vertices v', v_1, v_2, v_3, v_4. Additionally, because the degree of v is six, every nonsimple vertex in P must have degree six. Thus, the vertices v_1, v_2, v_3, v_4 each have exactly one neighbor outside F,

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which is v'. As a consequence, the neighbors of v in P other than v' must be the four vertices in F other than v_1, v_2, v_3, v_4, ensuring that the degree of v in P is five, a contradiction.

Assume that F is a pyramid. Since $f_0(F) \geqslant d+2$, its apex has at least $d+1$ neighbors in F and, therefore has degree at least $d+2$ in P, contradicting the degree of v.

Remark 3.9(iii) The facet F has $d+1(d-2)$-faces and is an $(s-1, d-s)$-triplex. Then $d-s \geqslant 1$. Moreover, the vertex v has degree $d+s-2$.

Suppose that $d-s \geqslant 2$. There are at least two apices x_1 and x_2 in F. For $i = 1, 2$, consider the other facet F_i in P intersecting F at a $(d-2)$-face that does not contain x_i. Since P is not a pyramid, each F_i must be pyramid with apex either v or v' (but not both). Since $s \geqslant 4$, each facet F_i is not a simplex. Therefore, as v' is simple in each of the facets containing it, for each $i = 1, 2$, $v' \notin F_i$. Thus, $v \in F_i$ for each $i = 1, 2$. It follows that v is adjacent to every other vertex in the polytope; that is, the degree of v is $d+s-1$, contradicting our assumption.

Assume that $d-s = 1$. Then the facet F is a pyramid over a simplicial $(d-2)$-prism R. Then R is the convex hull of two $(d-3)$-simplices K and K'. Denote by x the apex of F.

Consider a facet F' in P containing R but not x. The facet F' is a pyramid with apex either v or v'; otherwise, $f_0(F') = d+s-1$ and P would be a pyramid whose apex x has degree $2d-2$, which is larger than the degree of v, contradicting our assumption. It follows that $v \in F'$, as v' is simple in each facet that contains it but does not contain v. Because $\deg_P(v) = 2d-3$, v cannot be adjacent to x.

It is true that $\deg_P(v') < \deg_P(v)$; otherwise, we could pick $v' \notin F$ as our nonsimple vertex with the maximum degree. Additionally, v would need to be simple in the facet that contains it but does not contain v'. However, v is not simple in F'.

Consider a facet J' in P containing v' but not v. Then J' must be a pyramid with apex v' and intersecting F at a ridge. Since v' is simple in J', the facet J' must be a simplex. Without loss of generality, we may assume that $J' = \operatorname{conv}(\{v', x\} \cup K')$. Since $\deg(v') < \deg(v) = 2d-3$, we find that v' is not adjacent to at least two vertices in K. Finally, we consider another facet J in P containing the $(d-2)$-simplex $\operatorname{conv}\{x\} \cup K$. Since v' is not adjacent to two vertices in K, the degree of v' in J would be at most $d-2$, a contradiction. Thus, $v' \notin J$ and, therefore $v \in J$. On the other hand, because v is not adjacent to x, the vertex $v \notin J$, a contradiction.

Remark 3.9(iv) The facet F has $d+1$ $(d-2)$-faces and is an $(s-1, d-s)$-triplex. Denote by $x_1, \ldots, x_{d-s}$ the apices in F, where $d-s \geqslant 1$. Moreover, the vertex v has degree $d+s-1$.

For $i = 1, \ldots, d-s$, consider the facet F_i in P intersecting F at the $(d-2)$-face of F not containing x_i. If F_i contained both v and v', then the polytope would be a pyramid, a case already covered. Thus, F_i is a pyramid with apex either v or v'. Since $f_0(F_i) \geqslant d+2$, and v' is simple in any facet that contains it but does not contain v, the latter is impossible. So the apex of F_i is v. It follows that, for each $i = 1, \ldots, d-s$, F_i contains v but not v'. Additionally, consider a facet J' in P containing v' but not v. Then J' must be a pyramid with apex v' and intersecting F at a ridge. Since v' is simple in J', any such facet J' must be a simplex.

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Suppose that $d - s \geqslant 3$. Since F does not contain v', there are at least four facets in P not containing v'. However, by Lemma 3.10, the polytope has exactly $d + 3$ facets, leaving at most $d - 1$ facets containing v', which is not possible.

Suppose $d - s = 2$. Then F is a two-fold pyramid over a simplicial $(d-3)$-prism R. Then R is the convex hull of two $(d-4)$-simplices K and K'. Without loss of generality, we assume that $J' = \mathrm{conv}(\{v', x_1, x_2\} \cup K)$. By Lemma 3.10, the polytope has exactly $d + 3$ facets. The vertex v' is not present in precisely three of those facets, namely, F_1, F_2, and F. Thus, v' is simple in P, and so is not adjacent to any of the vertices in K'. For $i = 1, 2$, there is a facet S_i in P containing v' but not x_i. Since v' is simple in P, each S_i must intersect F at a $(d-3)$-face that contains K and the other apex, and so must contain v. This violates Corollary 3.11(i).

Assume that $d - s = 1$. Then F is a pyramid over a simplicial $(d-2)$-prism R, and so $f_0(F) = 2d - 3$. The face R is the convex hull of two $(d-3)$-simplices $K := \mathrm{conv}\{u_1, \ldots, u_{d-2}\}$ and $K' := \mathrm{conv}\{u'_1, \ldots, u'_{d-2}\}$ so that the vertices u_i and u'_i are adjacent. Without loss of generality, we assume that $J' = \mathrm{conv}(\{v', x_1\} \cup K)$. The facet F_1 is a pyramid with apex v over R. This means that the vertices in K have degree exactly $d + 1$ and the vertices in K' have degree at most $d + 1$.

For $i = 1, \ldots, d - 2$, consider the other facet J_i in P containing the $(d-2)$-face I_i of F missing the edge $[u_i, u'_i]$. By Lemma 3.10, the polytope has exactly $d + 3$ facets. The vertex v' is not present in two of those facets, namely, F_1 and F. As a consequence, v' needs to be present in at least $d - 3$ of such facets J_i, say in those for $i \in [1, d-3]$. Since v' is simple in the facet that contains it but does not contain v, the vertex v must be present in all the facets J_i. Hence, $f_0(J_i) = f_0(F) = 2d - 3$, for $i \in [1, d-3]$.

For $i \in [1, d-3]$, we claim that each facet J_i is neither a pyramid nor a simple polytope, and so, by Lemma 2.1, has at least $d + 2$ $(d-2)$-faces. Indeed, if J_i were a pyramid, then either v or v' would be the apex. In case of v being the apex, $I_i \cup \{v'\}$ would be the base but I_i is already a $(d-2)$-face; a similar argument establishes that v' cannot be the apex of J_i. It is also straightforward that J_i is not a simple polytope.

For each $i \in [1, d-3]$, since the vertex u_i (which is outside J_i) has degree $d + 1$, J_i is a largest facet with at least $d + 2$ $(d-2)$-faces, and $f_k(P) = \zeta_k(d + s, d)$ for some $k \in [1, d-2]$, the computation (3.2) in Subcase 2.1 applies and gives that $f_k(J_i) = \zeta_k(d + s - 2, d - 1)$. In this scenario, the induction hypothesis on $d - 1$ ensures that J_i is one of the polytopes in Parts (ii)–(v) of the theorem. Since $d \geqslant 5$, J_i is not a pyramid, and v (and x_1) is adjacent to every other vertex in J_i, the facet J_i can be only $Z(d-1)$. According to the structure of $Z(d-1)$ (Remark 2.7), for $i \in [1, d-3]$, the $d - 3$ vertices in K and J_i have degree d in $J_i = Z(d-1)$, so v' and the $d - 3$ vertices in K' and J_i must be simple in $Z(d-1)$. Since $d - 3 \geqslant 2$, this implies that v' is not adjacent to any vertex in K'. This gives the full graph of P, which consists of the union of the graphs of F, F_1, and J'. In particular,

- $f_0(P) = 2d - 1$;
- the $d - 1$ vertices in $K' \cup \{v'\}$ are simple in P;
- the $d - 2$ vertices in K have degree $d + 1$ in P;
- the vertices v and x_1 have degree $2d - 2$ in P.

See Figure 1(d) and Remark 2.7.

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We now establish the vertex–facet incidence of the polytope. For each $i \in [1, d-2]$, let us focus on one J_i and the other facets that intersect J_i at a ridge (Remark 2.7). Since J_i contains exactly $d+2$ ridges of P, every other facet of P intersects J_i at a $(d-2)$-face of J_i. Because the degree of u_i in P is $d+1$ and that of u_i' is d, the other facet intersecting J_i at $Z(d-2)$ must contain both u_i and u_i' and thus must be one of the facets J_ℓ. The $d-3$ copies of $Z(d-2)$ in J_i plus J_i result in $d-2$ facets $Z(d-1)$ in P and establish the vertex–facet incidence among them. Similarly, a facet that intersects J_i at a one of the two copies of $M(d-3,1)$ in J_i must contain both u_i and u_i'. This gives two facets $M(d-2,1)$ of P and the vertex–facet incidence among them. There are three simplex $(d-2)$-faces in J_i (Remark 2.7):

$$\begin{aligned}
&\operatorname{conv}(\{v', v\} \cup \{u_j \in K : j \neq i\}),\\
&\operatorname{conv}(\{v', x_1\} \cup \{u_j \in K : j \neq i\}),\\
&\operatorname{conv}(\{v, x_1\} \cup \{u_j' \in K' : j \neq i\}).
\end{aligned}$$

Since v' is simple in P, one can see that the other facet containing one of such $(d-2)$-faces will be, respectively,

$$\begin{aligned}
&\operatorname{conv}(\{v', v\} \cup K),\\
&\operatorname{conv}(\{v', x_1\} \cup K),\\
&\operatorname{conv}(\{v, x_1\} \cup K').
\end{aligned}$$

The complete vertex–facet incidence of P is established and so, the polytope is $Z(d)$.

Remark 3.9(v) The facet F is a simplicial $(d-1)$-prism and the polytope P has $2d$ vertices. Moreover, the vertex v has degree exactly $2d-2$.

Following the proof of Subcase 2.3 of Theorem 3.6, we label the vertices of one $(d-2)$-simplex R of F by $w_1, \ldots, w_{d-1}$ and the vertices of the other $(d-2)$-simplex R' of F by $w_1', \ldots, w_{d-1}'$, with $R \cap R' = \varnothing$. Additionally, let v' be the vertex outside F and distinct from v. Let S be the facet of P containing v' but not v. Then S must be a pyramid (with apex v') over a $(d-2)$-face contained in F. Since v' has to be simple in S, it follows that S is a simplex and either $S = \operatorname{conv}(R' \cup \{v'\})$ or $S = \operatorname{conv}(R \cup \{v'\})$.

Only one vertex of P is not adjacent to v, so we can assume without loss of generality that w_1' is adjacent to both v and v'.

Now consider the other facet F_1 that intersects F at the simplicial $(d-2)$-prism $K_1 = \operatorname{conv}\{w_i, w_i' : i = 2, \ldots, d-1\}$. It follows that $f_0(F_1) = 2d-3$ or $2d-2$.

First suppose that $f_0(F_1) = 2d-3$. Then the facet F_1 is a pyramid with apex v over the simplicial $(d-2)$-prism $\operatorname{conv}((R\backslash\{w_1\}) \cup (R'\backslash\{w_1'\}))$ of F. It follows that v is adjacent to every vertex in P except for w_1. Thus, w_1 is adjacent to v'.

It follows that $\deg(v') < 2d-2$. Otherwise, v' would also be a vertex with maximum degree, so the role of v and v' can be swapped. As F_1 is a facet containing v but not v' where v is not simple, we would find a contradiction to the assumption that the unique facet containing v' and not v must be simple.

First suppose that $S = \operatorname{conv}(R' \cup \{v'\})$. Since $\deg_P(v') < 2d-2$, the vertex v' is not adjacent to at least two vertices from $w_2, \ldots, w_{d-1}$. Consider the other facet J of P that intersects F at R. Since v' is not adjacent to at least two vertices in R, the vertex v' would have at most $d-2$ neighbors in J, so $v' \notin J$. Then, since v is not adjacent to w_1,

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the vertex v would have at most $d-2$ neighbors in J, so $v \notin J$. This is absurd and this case does not arise.

Now suppose that $S = \operatorname{conv}(R \cup \{v'\})$. Consider a facet G that intersects F at the simplicial $(d-2)$-prism $\operatorname{conv}\{w_i, w_i' : i = 1, \ldots, d-2\}$. The facet G cannot be a pyramid, as v is not adjacent to $w_1 \in G$ and $\deg_P(v') < 2d-2$. Thus, $v, v' \in G$. Furthermore, G cannot be a simple polytope, since the vertex w_j (with $j \neq 1, d-1$) is not simple in G as it is adjacent to both v and v'. From Lemma 2.1, it now follows that G has $d + 2(d-2)$-faces. Since w_{d-1} is not a simple vertex and since $f_k(P) = \zeta_k(d+s, d)$ for some $k \in [1, d-2]$, the computation (3.2) in Subcase 2.1 applies and gives that $f_k(G) = \zeta_k(d+s-2, d-1)$ for some $k \in [1, d-2]$. By induction, the facet G must be one of the nonpyramidal minimizers with $2d-2$ vertices, namely, $TA(d-1)$ or WP. Since the $d-1$ vertices $w_1', v', w_2, \ldots, w_{d-2}$ have degree d in G, the facet G cannot be $TA(d-1)$ (Remark 2.6). When $d = 5$, the degree of v in G is 6, so G cannot be WP (Figure 1). Hence, this case does not arise.

It remains to consider the case $f_0(F_1) = 2d-2$. The proof of Subcase 2.1 gives that F_1 has at least $d+2$ facets, as it is not a pyramid or a simple polytope. As w_1' is a nonsimple vertex outside F_1, the facet F_1 is a largest facet with at least $d+2$ $(d-2)$-faces, and $f_k(P) = \zeta_k(d+s, d)$ for some $k \in [1, d-2]$, the computation (3.2) in Subcase 2.1 applies and gives that $f_k(F_1) = \zeta_k(d+s-2, d-1)$. In this scenario, the induction hypothesis on $d-1$ ensures that F_1 is one of the nonpyramidal minimizers with $2d-2$ vertices, namely, $TA(d-1)$ or WP. Since $\deg_P(v) = 2d-2$, the degree of v in F_1 is at least $2d-4$. Thus, when $d = 5$, the facet F_1 cannot be WP (Figure 1). Consequently, F_1 can be only $TA(d-1)$. By the structure of $TA(d-1)$ (Remark 2.6), the degree of v in F_1 is $2d-4$, so v must be adjacent to w_1, and v' must be simple in F_1. Moreover, without loss of generality, we may assume that w_{d-1} and w_{d-1}' are simple in F_1, and therefore simple in P. It follows that v is not adjacent to either w_{d-1} or w_{d-1}'.

By the structure of $TA(d-1)$ (Remark 2.6), either w_i or w_i' is not simple in F_1 (for each $i \neq 1, d-1$) and therefore not simple in P, respectively (see Figure 1).

For $i = 2, \ldots, d-2$, consider the other facet F_i in P containing the $(d-2)$-face of F missing the edge $[w_i, w_i']$. None of these facets can be pyramids, as v cannot be an apex; recall that the vertices w_{d-1} and w_{d-1}' are in each such F_i and v is not adjacent to one of them. None of these facets are simple polytopes, as either w_j or w_j' (for $j \neq i, d-1$) is nonsimple in F_i. Thus, each F_i has $d+2$ facets and $2d-2$ vertices. The same reasoning used with F_1 ensures that the facets F_i (for $i = 2, \ldots, d-2$) are all $TA(d-1)$. Hence, this gives $d-2$ copies of $TA(d-1)$ as facets of P.

Since w_1' is adjacent to both v' and v, and the vertex v' is simple in each of the facets F_i, the graph of P is now clear: the vertex v' is adjacent to the vertices $v, w_1', \ldots, w_{d-1}'$; the vertex v is adjacent to the vertices $v', w_1', \ldots, w_{d-2}'$, and $w_1, \ldots, w_{d-1}$.

With the graph of P in place and using Corollary 3.11, we find that the unique facet containing v' but not v must be the simplex $\operatorname{conv}(R' \cup \{v'\})$. Since w_{d-1} is simple in P, there is a further simplex facet $\operatorname{conv}(R \cup \{v\})$. Furthermore, the unique facet containing both v and v' and intersecting F at a $(d-3)$-face must be the simplex $\operatorname{conv}(\{v, v'\} \cup \{w_i' : i = 1, \ldots, d-2\})$. Hence, this gives three simplex facets of P.

Since v' is simple in P and there are already d facets containing v', the other facet F_{d-1} in P containing the $(d-2)$-face of F missing the edge $[w_{d-1}, w_{d-1}']$, does not contain v'. Thus, F_{d-1} must contain v, implying that F_{d-1} is an $M(d-2, 1)$. We have

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now produced all the $d+3$ facets of P. These facets now give the vertex–facet incidence of the polytope, implying that P is $TA(d)$.

Remark 3.9(vi) The facet F has $d+1$ vertices and is a $(2, d-3)$-triplex. Moreover, the vertex v has degree exactly $d+1$. Every vertex outside F other than v is simple in the facet of P containing it but not v.

As in Case 3 of Theorem 3.6, we denote the $s-1$ vertices outside F by $u_1, \ldots, u_{s-1}$ where $u_1 = v$. Denote the apices of F by $x_1, \ldots, x_{d-3}$. For $i \in [1, d-3]$, consider the facet F_i intersecting F at the $(d-2)$-face $F\backslash\{x_i\}$. It follows that F_i has exactly $d+1$ vertices, and so F_i is a pyramid. No vertex in $\{u_2, \ldots, u_{s-1}\}$ can be the apex of F_i, since they need to be simple in the facet containing them but not v. Thus, v is the apex of F_i for each i. It follows that v is adjacent to all the vertices of F, and since the vertices outside a facet form a connected subgraph [7, Theorem 4.1.6], the degree of v in P would be at least $d+2$, contradicting our assumption.

Remark 3.9(vii) The facet F is a simplex, and so is every facet not containing a vertex of the largest degree. Every vertex outside F other than v is simple in every facet of P containing it but not v. There is another simplex facet F' such that $F \cup F'$ contains $d+s-1$ vertices.

Suppose that $s \leqslant d-1$. By Remark 3.8(i), $\deg(v) = d+s-1$, and the same holds for every vertex in $F \cap F'$. The vertex figure P/v of v is $M(s, d-s-1)$. If $d-s-1 \geqslant 1$, P/v is a pyramid. Then the polytope is a pyramid: consider the facet containing v but not containing the vertex in P corresponding to the apex of P/v; this facet necessarily contains $d+s-1$ vertices. The case of P being a pyramid has already been covered. In case $d-s-1=0$, P/v is a simplicial $(d-1)$-prism and $\#(F \cap F') = 2$. A vertex w in $F \cap F'$ has the same degree $2d-2$ as v. Denote by w' a vertex in P/v corresponding to w. In P/v, there is a face that is a simplicial $(d-2)$-prism R and does not contain w'. In P, there is a corresponding facet J that contains v and R. The facet J contains v but not w and has $2d-3$ vertices. The facet J has more vertices than F and misses a vertex of largest degree, contrary to our assumption on F.

Assume that $s = d$. By Remark 3.8(ii), we find that $\deg(v) = 2d-2$ and that its vertex figure is a simplicial $(d-1)$-prism R. The prism R is the convex hull of two $(d-2)$-simplices $\operatorname{conv}\{w_1, \ldots, w_{d-1}\}$ and $\operatorname{conv}\{w'_1, \ldots, w'_{d-1}\}$ so that the vertices w_i and w'_i are adjacent. For $i = 1, \ldots, d-1$, let u_i and u'_i represent the vertices in $F \cup F'$ corresponding to the vertices $w_i, w'_i \in P/v$, respectively. Moreover, the unique vertex $w \in F \cap F'$ has degree $2d-2$ and is not adjacent to v.

For $i = 1, \ldots, d-1$, consider the other facet J_i in P containing v and arising from the $(d-2)$-face of R missing the edge $[u_i, u'_i]$. As in Remark 3.8(ii), each facet J_i has $2d-2$ vertices and contains the vertex w. If a vertex u_j or u'_j has degree $2d-3$ in a facet J_i $(i \neq j)$, then J_i would be missing a vertex of largest degree in P, contradicting our assumption. Thus, no vertex u_j or u'_j can be an apex of a facet J_i. Each facet J_i is not a pyramid: none of the vertices u_j or u'_j $(i \neq j)$ can serve as the apex, nor can v or w due to their degrees in J_i. Additionally, each facet J_i is not a simple polytope, as both v and w have degree $2d-4$ in J_i. By Lemma 2.1, each facet J_i has at least $d+2$ $(d-2)$-faces.

Suppose that, for some $i \in [1, d-1]$, the vertex u_i or u'_i is not simple in P. Then, J_i is a facet with $2d-2$ vertices and at least $d+2$ $(d-2)$-faces, and since $f_k(P) = \zeta_k(2d, d)$

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for some $k \in [1, d-2]$, the computation (3.2) in Subcase 2.1 applies and gives that $f_k(J_i) = \zeta_k(2d-2, d-1)$. In this scenario, the induction hypothesis on $d-1$ ensures that J_i is one of the nonpyramidal polytopes in Parts (ii)–(v) of the theorem with $2(d-1)$ vertices, namely, $TA(d-1)$, or WP (Figure 1). In $TA(d-1)$, all the nonsimple vertices are pairwise adjacent, while in J_i, the vertex v is not adjacent to w. If J_i were WP, then $d = 5$ and the degree of v in J_i would be $2d - 4 = 6$, but no vertex in WP has degree six. Hence, this scenario does not arise.

We can now assume that all the vertices u_i, u_i' are simple in P. Thus, there are exactly two nonsimple vertices in P, namely, v and w. The graph of P is determined: there are two disjoint $(d-2)$-simplices $\operatorname{conv}\{u_1, \ldots, u_{d-1}\}$ and $\operatorname{conv}\{u_1', \ldots, u_{d-1}'\}$ whose vertices are all adjacent to both v and w. However, this is not the graph of a d-polytope with $d \geqslant 5$, as the removal of v and w would disconnect the graph, contradicting the vertex d-connectivity of the graph by Balinski's theorem [1].

References

[1] M. L. Balinski, *On the graph structure of convex polyhedra in n-space*. Pac. J. Math. **11**(1961), 431–434.
[2] D. Britton and J. D. Dunitz, *A complete catalogue of polyhedra with eight or fewer vertices*. Acta Cryst. Ser. A **29**(1973), 362–371.
[3] M. Firsching, *The complete enumeration of 4-polytopes and 3-spheres with nine vertices*. Isr. J. Math. **240**(2020), 417–441.
[4] B. Grünbaum, *Polytopes, graphs, and complexes*. Bull. Amer. Math. Soc. **76**(1970), 1131–1201.
[5] B. Grünbaum, *Convex polytopes*. 2nd ed., Graduate Texts in Mathematics, 221, Springer-Verlag, New York, NY, 2003. Prepared and with a preface by V. Kaibel, V. Klee and G. M. Ziegler.
[6] P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, London Mathematical Society Lecture Note Series, 3, Cambridge University Press, London, 1970.
[7] G. Pineda-Villavicencio, *Polytopes and graphs*, Cambridge Studies in Advanced Mathematics, 211, Cambridge University Press, Cambridge, 2024.
[8] G. Pineda-Villavicencio, A. Tritama, J. Wang, and D. Yost, *A lower bound theorem for d-polytopes with* $2d + 2$ *vertices*. Electron. J. Combin. **33**(2026), no. 2, #P2.26.
[9] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *The excess degree of a polytope*. SIAM J. Discrete Math. **32**(2018), 2011–2046.
[10] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Lower bound theorems for general polytopes*. Eur. J. Comb. **79**(2019), 27–45.
[11] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Minimum number of edges of d-polytopes with* $2d + 2$ *vertices*. Electron. J. Combin. **29**(2022), no. 3, P3.18.
[12] G. Pineda-Villavicencio and J. Wang, *A lower bound theorem for d-polytopes with at most* $3d - 1$ *vertices*. Preprint, 2025, https://arxiv.org/abs/2512.07456
[13] G. Pineda-Villavicencio and D. Yost, *A lower bound theorem for d-polytopes with* $2d + 1$ *vertices*. SIAM J. Discrete Math. **36**(2022), 2920–2941.
[14] L. Xue, *A proof of Grünbaum's lower bound conjecture for polytopes*. Isr. J. Math. **245**(2021), 991–1000.
[15] L. Xue, *A lower bound theorem for strongly regular CW spheres with up to* $2d + 1$ *vertices*. Discrete Comput. Geom. **72**(2024), 1348–1368.

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