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PROOF SYSTEMS FOR VARIOUS FDE-BASED MODAL LOGICS

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Abstract. We present novel proof systems for various FDE-based modal logics. Among the systems considered are a number of Belnapian modal logics introduced in Odintsov & Wansing (2010) and Odintsov & Wansing (2017), as well as the modal logic KN4 with strong implication introduced in Goble (2006). In particular, we provide a Hilbert-style axiom system for the logic $\mathsf{BK}^{\Box-}$ and characterize the logic BK as an axiomatic extension of the system $\mathsf{BK}^{\mathsf{FS}}$. For KN4 we provide both an FDE-style axiom system and a decidable sequent calculus for which a contraction elimination and a cut elimination result are shown.

§1. Introduction. In this article we present novel proof systems for various FDE-based modal logics, i.e., modal logics extending first-degree entailment logic, FDE, see, for example, Omori & Wansing (2017). In Odintsov & Wansing (2017) the relationship between the FDE-based paraconsistent modal logics BK, KN4 (Goble, 2006), and MBL (modal bilattice logic; Jung & Rivieccio, 2013; Rivieccio, 2011; Rivieccio, Jung, & Jansana, 2017) has been considered and clarified. The system BK can be regarded as the Belnapian version of the basic modal logic K in the language of FDE extended by a falsity constant, $\bot$, classical implication, $\rightarrow$, and the modal operators $\Box$ (necessity) and $\Diamond$ (possibility). A Hilbert-style axiomatization of BK was given in Odintsov & Wansing (2010), and an axiomatization of $\mathsf{BK}^{\Box}$, the $\Diamond$-free fragment of BK, can be found in Odintsov & Wansing (2017).

Some preliminary information will be given in §2, where, in particular, all the discussed logics will be introduced.

In §3 we present an axiom system for the logic $\mathsf{BK}^{\Box-}$, the $\bot$-free fragment of $\mathsf{BK}^{\Box}$, and show it to be strongly sound and complete with respect to the relational semantics for $\mathsf{BK}^{\Box-}$. This axiomatization makes use of monotonicity rules for $\Box$ and $\sim\Box\sim$, and we also present an alternative axiom system that uses necessitation for $\Box$ instead of the monotonicity rules.

§4 is concerned with BK and the Fischer Servi-style (see Fischer Servi, 1984) modal logic $\mathsf{BK}^{\mathsf{FS}}$. The system $\mathsf{BK}^{\mathsf{FS}}$ is defined as the set of all modal formulas valid under a modified standard translation from the language of BK into first-order FDE. In Odintsov & Wansing (2017), $\mathsf{BK}^{\mathsf{FS}}$ is shown to be characterized by the class of all models for $\mathsf{BK}^{\Box}$ ×

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$\mathsf{BK}^{\Box}$, the fusion of $\mathsf{BK}^{\Box}$ with itself (see, e.g., Kracht, 1999). Moreover, in the same article an axiom system for $\mathsf{BK}^{\mathsf{FS}}$ is presented and shown to be strongly sound and complete with respect to the class of models for $\mathsf{BK}^{\Box} \times \mathsf{BK}^{\Box}$. In §4 we show that BK is an axiomatic extension of $\mathsf{BK}^{\mathsf{FS}}$.

§5–§7 are dedicated to proof systems for Goble's (Goble, 2006) paraconsistent modal logic $\mathsf{KN4}$. The language of $\mathsf{KN4}$ is obtained from the language of $\mathsf{BK}^{\Box-}$ by replacing the 'weak' implication $\rightarrow$ by the contraposable 'strong' implication, $\Rightarrow$. This replacement, however, creates problems for an axiomatic presentation. The axiomatization in Goble (2006) is quite complicated and makes use of an infinite set of rules. In §5 we present an FDE-style axiomatization $\mathsf{KN4}_{fde}$ of $\mathsf{KN4}$ with a finite set of rules and show it to be strongly sound and complete with respect to the relational semantics for $\mathsf{KN4}$. (We also point out that along the same lines one can obtain an FDE-style axiomatization $\mathsf{BK}^{\Box-}_{fde}$ of $\mathsf{BK}^{\Box-}$.) Eventually, in §6 and §7, we develop a Gentzen-style sequent system, $\mathsf{G3KN4}$, for $\mathsf{KN4}$. The completeness proof for $\mathsf{G3KN4}$ is based on the soundness of $\mathsf{BK}^{\Box-}_{fde}$. Since the structural rules of contraction and cut are admissible, root-first proof search gives a decision procedure for $\mathsf{KN4}$. We conclude with a brief summary in §8.

§2. Preliminaries. In this section we will introduce some of the systems and notions we will be using throughout the article. Among the systems introduced are BK and its fragments $\mathsf{BK}^{\Box}$ and $\mathsf{BK}^{\Box-}$, $\mathsf{BK}^{\mathsf{FS}}$ (Odintsov & Wansing, 2017) and Goble's $\mathsf{KN4}$ (Goble, 2006).

We will be dealing only with propositional logics in languages containing the connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\Rightarrow$ (strong implication), $\sim$ (negation), $\Box$ (necessity), $\Diamond$ (possibility) and $\bot$ (falsity constant). We distinguish the following languages, where $\mathcal{L}^{\mathsf{BK}}$ is the language of both BK and $\mathsf{BK}^{\mathsf{FS}}$, $\mathcal{L}^{\Box}$ and $\mathcal{L}^{\Box-}$ are the languages of $\mathsf{BK}^{\Box}$ and $\mathsf{BK}^{\Box-}$, respectively, and $\mathcal{L}^{\mathsf{KN4}}$ is the language of $\mathsf{KN4}$:

$$\mathcal{L}^{\Box-} := \{\wedge, \vee, \rightarrow, \sim, \Box\}; \qquad \mathcal{L}^{\Box} := \mathcal{L}^{\Box-} \cup \{\Diamond\};$$
$$\mathcal{L}^{\mathsf{KN4}} := \{\wedge, \vee, \Rightarrow, \sim, \Box\}; \qquad \mathcal{L}^{\mathsf{BK}} := \mathcal{L}^{\Box} \cup \{\bot\}.$$

For any language $\mathcal{L}$ we construct *$\mathcal{L}$-formulas* in the usual way using propositional variables from a fixed denumerable set Prop and connectives in $\mathcal{L}$. The set of all $\mathcal{L}$-formulas is denoted $\mathsf{For}\,\mathcal{L}$. Here are some definable connectives we will use when applicable:

$$\begin{aligned}
\neg\varphi &:= (\varphi \rightarrow \bot);\\
(\varphi \leftrightarrow \psi) &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi);\\
(\varphi \Rightarrow \psi) &:= (\varphi \rightarrow \psi) \wedge (\sim\psi \rightarrow \sim\varphi);\\
(\varphi \Leftrightarrow \psi) &:= (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi).
\end{aligned}$$

Observe that strong implication will be employed as a primitive connective in $\mathsf{KN4}$ and as a definable connective in all other systems.

For the most part (with one exception) we will identify logics with sets of formulas, thus by a *logic* we will understand any set of formulas closed under substitution and *modus ponens*:

$$\frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)}\ \mathit{Sub} \qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ \mathit{MP}.$$

With any logic L we will associate its Scott consequence relation $\vdash_L$ (see Scott, 1982): for a set of formulas $\Gamma \cup \Delta$ put $\Gamma \vdash_L \Delta$ if $\varphi_1 \vee \cdots \vee \varphi_n$ can be derived from $L \cup \Gamma'$ using *modus*

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ponens for some $\varphi_1, \ldots, \varphi_n \in \Delta$ and finite $\Gamma' \subseteq \Gamma$, where, moreover, $\vdash_L$ is required to satisfy the following properties:

- *Reflexivity.* $\Gamma \vdash_L \Delta$, whenever $\Gamma \cap \Delta \neq \varnothing$.
- *Monotonicity.* If $\Gamma \vdash_L \Delta$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, then $\Gamma' \vdash_L \Delta'$.
- *Cut.* If $\Gamma, \varphi \vdash_L \Delta$ and $\Gamma \vdash_L \varphi, \Delta$, then $\Gamma \vdash_L \Delta$.

By definition, $\vdash_L$ also satisfies

- *Compactness.* If $\Gamma \vdash_L \Delta$ then $\Gamma' \vdash_L \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.

Observe that the definition of $\vdash_L$ is such that $\Gamma \vdash_L \Delta$ can only hold if $\Delta \neq \varnothing$. In cases when L contains at least one *counter-theorem* φ (i.e., a formula such that $\varphi \vdash_L \psi$ holds for any ψ), it is a common practice to put $\bigvee \varnothing := \varphi$ (similarly, as we will do later on, one often puts $\bigwedge \varnothing := \varphi$, where φ is some theorem). Among the systems discussed in the article the only ones that do have a counter-theorem are BK and $\mathsf{BK}^{\mathsf{FS}}$, so, to keep things uniform, throughout the article we will only allow inferences $\Gamma \vdash_L \Delta$ with $\Delta \neq \varnothing$.

We employ a number of usual notational conventions: in particular, we write $\vdash_L \Delta$ instead of $\varnothing \vdash_L \Delta$, replace union signs with commas and omit curly brackets in the notation for finite sets of formulas. As usual, $\vdash_L \varphi$ is equivalent to $\varphi \in L$.

Next, let us introduce all of the abovementioned logics semantically. Let us begin with BK and its fragments $\mathsf{BK}^{\Box}$ and $\mathsf{BK}^{\Box-}$. A BK-*frame* is a pair $\mathcal{W} = \langle W, R\rangle$, where W is a nonempty set and $R \subseteq W^2$ is an *accessibility relation*. A BK-*model* is a tuple $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ consisting of a $\mathsf{BK}^{\Box-}$-frame $\mathcal{W}$ and a pair of valuations $v^+ : \mathsf{Prop} \to 2^W$ and $v^- : \mathsf{Prop} \to 2^W$. We define inductively two *satisfaction relations* between formulas and worlds of a BK-model:

$$
\begin{array}{lll}
\mu, x \vDash^+ p & \iff & x \in v^+(p);\\
\mu, x \vDash^- p & \iff & x \in v^-(p);\\
\mu, x \nvDash^+ \bot; \quad \mu, x \vDash^- \bot; & & \\
\mu, x \vDash^+ \varphi \wedge \psi & \iff & \mu, x \vDash^+ \varphi \text{ and } \mu, x \vDash^+ \psi;\\
\mu, x \vDash^- \varphi \wedge \psi & \iff & \mu, x \vDash^- \varphi \text{ or } \mu, x \vDash^- \psi;\\
\mu, x \vDash^+ \varphi \vee \psi & \iff & \mu, x \vDash^+ \varphi \text{ or } \mu, x \vDash^+ \psi;\\
\mu, x \vDash^- \varphi \vee \psi & \iff & \mu, x \vDash^- \varphi \text{ and } \mu, x \vDash^- \psi;\\
\mu, x \vDash^+ \varphi \to \psi & \iff & \mu, x \vDash^+ \varphi \text{ implies } \mu, x \vDash^+ \psi;\\
\mu, x \vDash^- \varphi \to \psi & \iff & \mu, x \vDash^+ \varphi \text{ and } \mu, x \vDash^- \psi;\\
\mu, x \vDash^+ {\sim}\varphi & \iff & \mu, x \vDash^- \varphi;\\
\mu, x \vDash^- {\sim}\varphi & \iff & \mu, x \vDash^+ \varphi;\\
\mu, x \vDash^+ \Box\varphi & \iff & \forall y\,(xRy \text{ implies } \mu, y \vDash^+ \varphi);\\
\mu, x \vDash^- \Box\varphi & \iff & \exists y\,(xRy \text{ and } \mu, y \vDash^- \varphi);\\
\mu, x \vDash^+ \Diamond\varphi & \iff & \exists y\,(xRy \text{ and } \mu, y \vDash^+ \varphi);\\
\mu, x \vDash^- \Diamond\varphi & \iff & \forall y\,(xRy \text{ implies } \mu, y \vDash^- \varphi).
\end{array}
$$

Using this semantics as an example, let us demonstrate how some familiar notions can be defined in a standard way. First, for a BK-model $\mu = \langle W, R, v^+, v^-\rangle$ write $\mu \vDash^+ \varphi$ if $\mu, x \vDash^+ \varphi$ holds for all $x \in W$; for a BK-frame $\mathcal{W} = \langle W, R\rangle$ write $\mathcal{W} \vDash \varphi$ if $\mu \vDash \varphi$ holds for

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every BK-model $\mu = \langle \mathcal{W}, v^+, v^- \rangle$ over $\mathcal{W}$. For a set of $\mathcal{L}^{\mathsf{BK}}$-formulas Γ we write $\mu, x \vDash \Gamma$ if $\mu, x \vDash^+ \varphi$ for all $\varphi \in \Gamma$ and similarly for $\mu \vDash^+ \Gamma$ and $\mathcal{W} \vDash^+ \Gamma$. We define the semantic consequence relation for BK in the following way: for two sets of $\mathcal{L}^{\mathsf{BK}}$-formulas Γ and Δ with $\Delta \neq \varnothing$ put $\Gamma \vDash_{\mathsf{BK}} \Delta$ if

$$\mu, x \vDash^+ \Gamma \implies \mu, x \vDash^+ \psi \text{ for some } \psi \in \Delta$$

for every BK-model $\mu = \langle W, R, v^+, v^- \rangle$ and $x \in W$.

Semantics for $\mathsf{BK}^{\Box}$ and $\mathsf{BK}^{\Box-}$ can be obtained from the one above by restricting the language to $\mathcal{L}^{\Box}$ and $\mathcal{L}^{\Box-}$, respectively. This way we can define the notions of a $\mathsf{BK}^{\Box}$-($\mathsf{BK}^{\Box-}$-)*frame/model* and semantic consequence relations $\vDash_{\mathsf{BK}^{\Box}}$ and $\vDash_{\mathsf{BK}^{\Box-}}$.

The definition of strong implication above gives us the following satisfaction clauses for it:

$$\begin{aligned} \mu, x \vDash^+ \varphi \Rightarrow \psi &\iff (\mu, x \vDash^+ \varphi \text{ implies } \mu, x \vDash^+ \psi) \text{ and} \\ &\quad\ (\mu, x \vDash^- \psi \text{ implies } \mu, x \vDash^- \varphi); \\ \mu, x \vDash^- \varphi \Rightarrow \psi &\iff \mu, x \vDash^+ \varphi \text{ and } \mu, x \vDash^- \psi. \end{aligned}$$

Adding these to the semantics for BK and restricting the language to $\mathcal{L}^{\mathsf{KN4}}$ give us the semantics for $\mathsf{KN4}$ in a similar fashion. Observe that semantically the difference between $\mathsf{KN4}$ and $\mathsf{BK}^{\Box-}$ boils down to whether strong or weak implication is primitive in the language.

The situation is slightly different with $\mathsf{BK}^{\mathsf{FS}}$. By a $\mathsf{BK}^{\mathsf{FS}}$-*frame* we mean a triple $\mathcal{W} = \langle W, R, R' \rangle$ consisting of a nonempty set and two accessibility relations as above; a $\mathsf{BK}^{\mathsf{FS}}$-*model* is a $\mathsf{BK}^{\mathsf{FS}}$-frame together with two satisfaction relations as before. We use the same satisfaction clauses as for BK, except for the case of the possibility operator for which we use:

$$\begin{aligned} \mu, x \vDash^+ \Diamond\varphi &\iff \exists y\, (xRy \text{ and } \mu, y \vDash^+ \varphi); \\ \mu, x \vDash^- \Diamond\varphi &\iff \forall y\, (xR'y \text{ implies } \mu, y \vDash^- \varphi). \end{aligned}$$

The semantic consequence relation $\vDash_{\mathsf{BK}^{\mathsf{FS}}}$ is defined as before.

We proceed by providing known axiom systems for the listed logics.

Logic BK was introduced in Odintsov & Wansing (2010) to fill the role of the least normal modal logic K for Belnapian modal logics. The Hilbert-style axiom system HBK consists of the following axioms:

1. axioms of classical propositional logic in the language $\{\wedge, \vee, \to, \bot\}$;
2. axioms of strong negation:

$$\begin{aligned} &\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q); \qquad &&\sim(p \to q) \leftrightarrow (\sim p \wedge \sim q); \\ &\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q); \qquad &&\sim\sim p \leftrightarrow p; \qquad \sim\bot; \end{aligned}$$

3. modal axioms

$$\begin{aligned} &\Box(p \to p); \qquad &&\Box p \wedge \Box q \to \Box(p \wedge q); \\ &\neg\Box p \leftrightarrow \Diamond\neg p; \qquad &&\Box p \Leftrightarrow \sim\Diamond\sim p; \\ &\neg\Diamond p \leftrightarrow \Box\neg p; \qquad &&\Diamond p \Leftrightarrow \sim\Box\sim p; \end{aligned}$$

and the rules of inference *substitution*, *modus ponens* as well as two modal rules:

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ M\Box \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ M\Diamond$$

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In Odintsov & Wansing (2017) BK was compared to two other modal logics over FDE: system $\mathsf{KN4}$ (Goble, 2006) and the modal bilattice logic MBL (Rivieccio, 2011; Jung & Rivieccio, 2013; Rivieccio *et al.*, 2017). In particular, it was shown that $\mathsf{KN4}$ is definitionally equivalent to $\mathsf{BK}^{\Box-}$ — the $\bot$-free, $\Diamond$-free fragment of BK, and that MBL can be faithfully embedded into the fusion $\mathsf{BK}^{\Box} \times \mathsf{BK}^{\Box}$, where $\mathsf{BK}^{\Box}$, as already mentioned, is the $\Diamond$-free fragment of BK. Observe that the axiomatization of BK above does not allow one to extract an axiomatization of $\mathsf{BK}^{\Box}$ in a straightforward way, so in the same article $\mathsf{BK}^{\Box}$ was axiomatized in the form of a Hilbert-style system $\mathsf{HBK}^{\Box}$, consisting of all nonmodal axioms and rules of HBK plus the following

$$\Box(p \to q) \to (\Box p \to \Box q); \qquad \frac{\varphi}{\Box\varphi}\ N\Box.$$
$$\neg\sim\Box p \leftrightarrow \Box\neg\sim p;$$

Another system introduced in Odintsov & Wansing (2017) is $\mathsf{BK}^{\mathsf{FS}}$, which is meant to serve as a connecting point between MBL and BK. In particular, it was shown to be weakly definitionally equivalent (a notion introduced in the same article) to MBL. Axiom system $\mathsf{HBK}^{\mathsf{FS}}$ for $\mathsf{BK}^{\mathsf{FS}}$ was formulated by extending $\mathsf{HBK}^{\Box}$ with the following axioms and rule

$$\neg\Diamond p \leftrightarrow \Box\neg p\,; \qquad \frac{\sim\varphi \to \sim\psi}{\sim\Diamond\varphi \to \sim\Diamond\psi}\ M\sim\Diamond.$$
$$\sim\Diamond(p \vee q) \leftrightarrow (\sim\Diamond p \wedge \sim\Diamond q);$$

Finally, a Hilbert-style axiom system for $\mathsf{KN4}$ with an infinite set of rules was introduced in Goble (2006). Both Goble (2006) and Odintsov & Wansing (2017) draw attention to the problem of providing a more natural axiomatization for $\mathsf{KN4}$.

Observe that different styles of axiom systems for all of the systems outlined here make comparing them not straightforward even though the semantics suggest that they are clearly very closely related. As one specific consequence we have to introduce a notion of extension in different ways for all of these systems. Thus, by an *extension* of one of the above logics let us understand any logic closed under the corresponding modal rules: $M\Box$ and $M\Diamond$ for extensions of BK; $N\Box$ and $M\sim\Diamond$ for extensions of $\mathsf{BK}^{\mathsf{FS}}$, etc. With any extension we associate a Scott consequence relation as for logics in general.

§3. Axiomatizing $\mathsf{BK}^{\Box-}$. The following section is dedicated to axiomatizing $\mathsf{BK}^{\Box-}$. Without further ado we introduce a Hilbert-style system $\mathsf{HBK}^{\Box-}$ with the following axioms:

1. axioms of the positive fragment of classical propositional logic in the language $\{\wedge, \vee, \to\}$;
2. axioms of strong negation $\sim$ (in the language without $\bot$);
3. modal axioms:

$\Box 1.\ \Box(p \to p)$; $\quad \Box 2.\ (\Box p \wedge \Box q) \Rightarrow \Box(p \wedge q)$; $\quad \Box 3.\ \Box(p \vee \sim q) \Rightarrow (\Box p \vee \sim\Box q)$.

Rules of inferences of $\mathsf{HBK}^{\Box-}$ are substitution, *modus ponens*, $M\Box$ and $M\sim\Box$:

$$\frac{\varphi \to \psi}{\sim\Box\varphi \to \sim\Box\psi}\ M\sim\Box.$$

Since derivations are generated by using *modus ponens* only, we have a kind of deduction theorem in the following form:

THEOREM 3.1 (Deduction). *Suppose L is an extension of $\mathsf{BK}^{\Box-}$. For any sets of $\mathcal{L}^{\Box-}$-formulas Γ and $\Delta \neq \varnothing$, we have $\Gamma \vdash_L \Delta$ iff there are $\mathcal{L}^{\Box-}$-formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ and*

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$\psi_1, \ldots, \psi_k \in \Delta$ *such that*[1]

$$\vdash_L \bigwedge_{i \le n} \varphi_i \to \bigvee_{j \le k} \psi_j.$$

It is also easy to show that L satisfies the following weak version of the replacement property:

THEOREM 3.2 (Weak replacement). *Any extension L of $\mathsf{BK}^{\Box-}$ is closed under the following* weak replacement rule*:*

$$\frac{\varphi \Leftrightarrow \psi}{\chi[\varphi] \Leftrightarrow \chi[\psi],}$$

where $\chi[\varphi]$ and $\chi[\psi]$ are the results of replacing some occurrence of a propositional variable in χ by φ and ψ, respectively.

The result is proved in the usual way (for the modal case use the modal inference rules). We will use weak replacement in conjunction with the following equivalences to simplify some proofs:

$$\sim\sim\varphi \Leftrightarrow \varphi; \qquad \sim(\varphi \wedge \psi) \Leftrightarrow (\sim\varphi \vee \sim\psi);$$
$$\Box(\varphi \wedge \psi) \Leftrightarrow (\Box\varphi \wedge \Box\psi); \qquad \sim(\varphi \vee \psi) \Leftrightarrow (\sim\varphi \wedge \sim\psi).$$

For the remainder of the section let us fix some extension L of $\mathsf{BK}^{\Box-}$.

We say that a set of $\mathcal{L}^{\Box-}$-formulas Γ is a *prime L-theory* if it is i) not *trivial* (i.e., does not coincide with the set of all $\mathcal{L}^{\Box-}$formulas); ii) contains L; iii) closed under *modus ponens*; iv) satisfies the *disjunction property*:

$$(\varphi \vee \psi) \in \Gamma \text{ implies } \varphi \in \Gamma \text{ or } \psi \in \Gamma.$$

Notice that any prime L-theory is also closed under the strong implication version of *modus ponens*: from $\varphi \in \Gamma$ and $\varphi \Rightarrow \psi \in \Gamma$ we can infer that $\psi \in \Gamma$.

As usual we have

LEMMA 3.3 (Extension). *Suppose $\Gamma \nvdash_L \Delta$ for some sets of $\mathcal{L}^{\Box-}$-formulas Γ and $\Delta \neq \varnothing$. Then there is a prime L-theory Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_L \Delta$.*

The *canonical L-model* is $\mu_L = \langle W_L, R_L, v_L^+, v_L^- \rangle$, where

- W_L is the set of all prime L-theories;
- $\Gamma R_L \Delta$ holds for some prime L-theories Γ and Δ if

  $(r1)\ \forall\varphi\ (\Box\varphi \in \Gamma \text{ implies } \varphi \in \Delta)$ and $(r2)\ \forall\varphi\ (\sim\varphi \in \Delta \text{ implies } \sim\Box\varphi \in \Gamma)$.

- for any propositional variable $p \in \mathsf{Prop}$

$$v_L^+(p) := \{\Gamma \in W_L \mid p \in \Gamma\}; \qquad v_L^-(p) := \{\Gamma \in W_L \mid \sim p \in \Gamma\}.$$

LEMMA 3.4 (Canonical). *Let $\mu_L = \langle W_L, R_L, v_L^+, v_L^- \rangle$ be the canonical L-model. Then for any prime L-theory Γ and $\mathcal{L}^{\Box-}$-formula φ:*

$$\mu_L, \Gamma \vDash^+ \varphi \iff \varphi \in \Gamma; \qquad \mu_L, \Gamma \vDash^- \varphi \iff \sim\varphi \in \Gamma.$$

[1] Here, we assume that $\bigwedge \varnothing := p \to p$; as before, there is no dual definition of $\bigvee \varnothing$ as there are no counter-theorems in $\mathsf{BK}^{\Box-}$.

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Proof. The proof is by a standard induction on the complexity of φ, we only consider the modal case.

Given the induction hypothesis it is enough to show that

$$\text{(i)} \quad \Box\varphi \in \Gamma \iff \forall \Delta \in W_L \ (\Gamma R_L \Delta \text{ implies } \varphi \in \Delta);$$
$$\text{(ii)} \ {\sim}\Box\varphi \in \Gamma \iff \exists \Delta \in W_L \ (\Gamma R_L \Delta \text{ and } {\sim}\varphi \in \Delta).$$

(i $\Longrightarrow$) and (ii $\Longleftarrow$) are both proved simply using the definition of the accessibility relation.

(i $\Longleftarrow$). Suppose $\Box\varphi \notin \Gamma$. Put

$$\Gamma_\Box := \{\psi \mid \Box\psi \in \Gamma\}, \qquad \Gamma^{\sim\Box} := \{{\sim}\chi \mid {\sim}\Box\chi \notin \Gamma\}.$$

Notice that $\Gamma_\Box \neq \varnothing$ due to axiom $\Box 1$. Now suppose that $\Gamma_\Box \vdash_L \{\varphi\} \cup \Gamma^{\sim\Box}$. By the deduction Theorem 3.1 it follows without loss of generality that

$$\vdash_L \bigwedge_{i\leq n} \psi_i \to (\varphi \vee \bigvee_{j\leq k} {\sim}\chi_j)$$

for some $\varphi_1, \ldots, \varphi_n \in \Gamma_\Box$ and ${\sim}\psi_1, \ldots, {\sim}\psi_k \in \Gamma^{\sim\Box}$. Then by De Morgan laws and weak replacement we have

$$\vdash_L \bigwedge_{i\leq n} \psi_i \to (\varphi \vee {\sim} \bigwedge_{j\leq k} \chi_j)$$

and by the monotonicity rule $M\Box$ we obtain

$$\vdash_L \Box \bigwedge_{i\leq n} \psi_i \to \Box(\varphi \vee {\sim} \bigwedge_{j\leq k} \chi_j).$$

Now, since $\Box\psi_i \in \Gamma$ for $i \leq n$ and by axiom $\Box 2$ we obtain $\Box \bigwedge_{i\leq n} \psi_i \in \Gamma$ and thus, since Γ is closed under modus ponens, we have $\Box(\varphi \vee {\sim} \bigwedge_{j\leq k} \chi_j) \in \Gamma$. Using axiom $\Box 3$ we also know that $\Box\varphi \vee {\sim}\Box \bigwedge_{j\leq k} \chi_j \in \Gamma$. From this by the disjunction property either $\Box\varphi \in \Gamma$ or ${\sim}\Box \bigwedge_{j\leq k} \chi_j \in \Gamma$. The former cannot hold by the assumption so the latter has to be true. Using De Morgan laws, axiom $\Box 2$ and weak replacement, we derive that $\bigvee_{j\leq k} {\sim}\Box\chi_j \in \Gamma$ and by the disjunction property again ${\sim}\Box\chi_j \in \Gamma$ for some $j \leq k$, which contradicts the choice of χ_j.

Consequently, $\Gamma_\Box \nvdash_L \{\varphi\} \cup \Gamma^{\sim\Box}$. By the extension lemma there is a prime L-theory Δ such that $\Gamma_\Box \subseteq \Delta$ and $\Delta \nvdash_L \{\varphi\} \cup \Gamma^{\sim\Box}$, so that $\varphi \notin \Delta$. It remains to show that $\Gamma R_L \Delta$. Suppose $\Box\psi \in \Gamma$ for some ψ. Then $\varphi \in \Gamma_\Box$, hence $\varphi \in \Delta$, which shows $(r1)$. For $(r2)$ suppose by contraposition that ${\sim}\Box\psi \notin \Gamma$, then ${\sim}\psi \in \Gamma^{\sim\Box}$, hence $\Delta \nvdash_L {\sim}\psi$ and ${\sim}\psi \notin \Delta$.

(ii $\Longrightarrow$) is proved analogously, so we will be even less generous with details. Suppose ${\sim}\Box\varphi \in \Gamma$ and assume that $\Gamma_\Box \cup \{{\sim}\varphi\} \vdash_L \Gamma^{\sim\Box}$, where $\Gamma_\Box$ and $\Gamma^{\sim\Box}$ are defined as above. Then, as before, there are $\varphi_1, \ldots, \varphi_n \in \Gamma_\Box$ and ${\sim}\psi_1, \ldots, {\sim}\psi_k \in \Gamma^{\sim\Box}$ such that

$$\vdash_L ({\sim}\varphi \wedge \bigwedge_{i\leq n} \psi_i) \to \bigvee_{j\leq k} {\sim}\chi_j.$$

Applying strong negation axioms, weak replacement and $M{\sim}\Box$ we derive

$$\vdash_L {\sim}\Box(\varphi \vee {\sim} \bigwedge_{i\leq n} \psi_i) \to {\sim}\Box \bigwedge_{j\leq k} \chi_j.$$

By axiom $(\Box 3)$ we also know that

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$$\vdash_L \sim(\Box\varphi \vee \sim\Box \bigwedge_{i\leq n} \psi_i) \to \sim\Box(\varphi \vee \sim \bigwedge_{i\leq n} \psi_i),$$

and from the last two facts using the transitivity of $\to$ and De Morgan laws we get

$$\vdash_L (\sim\Box\varphi \wedge \Box \bigwedge_{i\leq n} \psi_i) \to \sim\Box \bigwedge_{j\leq k} \chi_j.$$

From this, using our assumption, we derive a contradiction as before. □

THEOREM 3.5 (Strong completeness). *For any sets of $\mathcal{L}^{\Box-}$-formulas Γ, Δ with $\Delta \neq \varnothing$ the following equivalence holds*:

$$\Gamma \vdash_{\mathsf{BK}^{\Box-}} \Delta \iff \Gamma \models_{\mathsf{BK}^{\Box-}} \Delta.$$

Proof. First we show that the equivalence holds when Δ is a singleton.

Soundness is proved routinely as usual.

Suppose $\Gamma \nvdash_{\mathsf{BK}^{\Box-}} \varphi$, then there is a prime $\mathsf{BK}^{\Box-}$-theory Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_{\mathsf{BK}^{\Box-}} \varphi$. Hence, by the canonical lemma, we have $\mu_{\mathsf{BK}^{\Box-}}, \Gamma' \models^+ \psi$ for all $\psi \in \Gamma$ and $\mu_{\mathsf{BK}^{\Box-}}, \Gamma' \not\models^+ \varphi$, where $\mu_{\mathsf{BK}^{\Box-}}$ is the canonical $\mathsf{BK}^{\Box-}$-model. Hence, by definition, $\Gamma \nvdash_{\mathsf{BK}^{\Box-}} \varphi$.

For the case when Δ is arbitrary, it is enough to observe that $\Gamma \models_{\mathsf{BK}^{\Box-}} \Delta$ iff $\Gamma \models_{\mathsf{BK}^{\Box-}} \psi_1 \vee \cdots \vee \psi_n$ for some $\psi_1, \ldots, \psi_n \in \Delta$. □

Let us outline how this axiomatization has been obtained. Observing the semantic satisfaction clauses, one can notice that $\Box$ and $\sim\Box\sim$ are essentially necessity and possibility operators that share the same accessibility relation. In Dunn (1995) Michael Dunn studied such operators in a very bare-bones framework: the only other connectives he had were conjunction and disjunction, and instead of adding a conditional, he used FDE-style formula–formula sequents to define an FDE-style calculus. It turns out that keeping in mind our definition of a possibility operator as $\sim\Box\sim$ and translating his system to include implication is enough to obtain an axiomatization of $\mathsf{BK}^{\Box-}$. The finishing touches include using properties of strong negation to combine pairs of axioms into axioms in strong implication form as we did here.

There is another typical way of axiomatizing modal operators: by having a necessitation rule instead of monotonicity rules for necessity and possibility. Thus we can define an alternative Hilbert-style calculus $\mathsf{H'BK}^{\Box-}$ by replacing both modal rules of $\mathsf{HBK}^{\Box-}$ with the $N\Box$ (see §2) and modal axioms $\Box$1–$\Box$3 with the following:

$$\Box 1'.\ \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi);$$
$$\Box 2'.\ \Box(\sim\varphi \to \sim\psi) \to (\sim\Box\varphi \to \sim\Box\psi);$$
$$\Box 3'.\ \sim\Box(\varphi \wedge \psi) \to (\sim\Box\varphi \vee \sim\Box\psi);$$
$$\Box 4'.\ \Box(\varphi \vee \sim\psi) \Rightarrow (\Box\varphi \vee \sim\Box\psi).$$

It is an easy exercise to establish that $\mathsf{HBK}^{\Box-}$ and $\mathsf{H'BK}^{\Box-}$ are equivalent.

§4. BK is an extension of $\mathsf{BK}^{\mathsf{FS}}$. In this short section we are going to show that BK can be characterized as the smallest extension of $\mathsf{BK}^{\mathsf{FS}}$ with axiom $\Box\sim p \leftrightarrow \sim\Diamond p$. For brevity let us denote this extension by BK'.

In Odintsov & Wansing (2017) the completeness result for $\mathsf{BK}^{\mathsf{FS}}$ was proved via the canonical model method. Let us outline some key notions. For an extension L of $\mathsf{BK}^{\mathsf{FS}}$

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a *prime L-theory* is defined exactly as in the previous section; the *canonical L-model* is $\mu_L = \langle W_L, R_L, R'_L, v^+_L, v^-_L\rangle$, where W_L, v^+_L and v^-_L are defined as before, $\Gamma R_L \Delta$ is defined by condition $(r1)$ from the previous section, and

$$\Gamma R'_L \Delta :\Longleftrightarrow \forall\varphi\,(\sim\Diamond\varphi \in \Gamma \Longrightarrow \sim\varphi \in \Delta),$$

where Γ and Δ are prime L-theories. Then we have

THEOREM 4.1. *For any sets of $\mathcal{L}^{\mathsf{BK}}$-formulas Γ and $\Delta \neq \varnothing$:*

$$\Gamma \vdash_{\mathsf{BK^{FS}}} \Delta \iff \Gamma \models_{\mathsf{BK^{FS}}} \Delta.$$

PROPOSITION 4.2. *Suppose $\mathcal{W} = \langle W, R, R'\rangle$ is a $\mathsf{BK^{FS}}$-frame. Then*

$$\mathcal{W} \models^+ \Box\sim p \leftrightarrow \sim\Diamond p \iff R = R'.$$

Proof. ($\Longleftarrow$). Suppose $R = R'$ and $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ is an arbitrary $\mathsf{BK^{FS}}$-model over $\mathcal{W}$. Then we have the following equivalences for all $x \in W$:

$$\mu, x \models^+ \Box\sim p \iff \forall y\,(xRy \text{ implies } \mu, y \models^+ \sim p) \iff \forall y\,(xRy \text{ implies } \mu, y \models^- p) \iff$$

$$\forall y\,(xR'y \text{ implies } \mu, y \models^- p) \iff \mu, x \models^- \Diamond p \iff \mu, x \models^+ \sim\Diamond p,$$

from which the result easily follows.

($\Longrightarrow$). Suppose $R \neq R'$, then there are $x, y \in W$ such that (1) xRy and not $xR'y$ or (2) $xR'y$ and not xRy. Consider (1) and define a $\mathsf{BK^{FS}}$-model $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ by putting $v^-(p) = \{z \in W \mid xR'z\}$ (it does not matter how $v^+(p)$ is defined, one might put, for instance, $v^+(p) = W$). Then it is easy to see that $\mu, x \models^+ \sim\Diamond p$ and $\mu, x \not\models^+ \Box\sim p$ as xRy and $\mu, y \not\models^- p$. Hence, $\mu, x \not\models^+ \Box\sim p \leftrightarrow \sim\Diamond p$ and $\mathcal{W} \not\models^+ \Box\sim p \leftrightarrow \sim\Diamond p$. The second case is considered in a similar way. □

LEMMA 4.3. *Suppose L is an extension of BK' and $\mu_L = \langle W_L, R_L, R'_L, v^+_L, v^-_L\rangle$ is its canonical model. Then $R_L = R'_L$.*

Proof. Suppose $\Gamma R_L \Delta$ for some $\Gamma, \Delta \in W_L$. Take also an arbitrary $\sim\Diamond\varphi \in \Gamma$. Then $\Box\sim\varphi \in \Gamma$ by the new axiom, hence $\sim\varphi \in \Delta$ by the definition of R_L, from which we infer by the definition of R'_L that $\Gamma R'_L \Delta$.

Suppose, on the other hand, that $\Gamma R'_L \Delta$ and take an arbitrary $\Box\varphi \in \Gamma$. Since $\Box\varphi \to \Box\sim\sim\varphi$ is a valid $\mathsf{BK^{FS}}$-formula, we can infer that $\Box\sim\sim\varphi \in \Gamma$, thus by the new axiom $\sim\Diamond\sim\varphi \in \Gamma$ and $\sim\sim\varphi \in \Delta$. Hence $\varphi \in \Delta$ and, since φ was arbitrary, $\Gamma R \Delta$. □

THEOREM 4.4. *For any set $\Gamma \cup \Delta$ of $\mathcal{L}^{\mathsf{BK}}$-formulas we have*

$$\Gamma \vdash_{\mathsf{BK}} \Delta \iff \Gamma \vdash_{\mathsf{BK}'} \Delta.$$

In particular, BK *coincides with* BK'.

Proof. Simply use the completeness results for both systems. □

§5. FDE-style axiomatization of KN4. In Goble (2006), the logic $\mathsf{KN4}$ was defined both semantically and in the form of a Hilbert-style calculus, and the corresponding soundness and completeness result was proved. Even though the relational semantics of $\mathsf{KN4}$ is very natural, the Hilbert-style calculus from Goble (2006) includes an infinite number of inference rule schemas. The question whether a simpler axiomatization exists was raised both in Goble (2006) and in Odintsov & Wansing (2017). We now turn to this problem.

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Naturally, all of the above suggest that KN4 is the $\mathcal{L}^{\mathsf{KN4}}$-fragment of $\mathsf{BK}^{\Box-}$, which seems to give us hope of somehow transferring $\mathsf{HBK}^{\Box-}$ into a Hilbert-style calculus of KN4, since the modal part of $\mathsf{HBK}^{\Box-}$is already formulated using strong implication instead of weak implication in most places. Unfortunately, things are not that simple.

In Arieli & Avron (1996) it was shown that not only strong implication can be defined by means of weak implication but also the other way around via the following definition:

$$(\varphi \rightarrow \psi) := \psi \vee (\varphi \Rightarrow (\varphi \Rightarrow \psi)).$$

This suggests that the following deduction theorem holds for KN4:

$$\Gamma, \varphi \vDash_{\mathsf{KN4}} \psi \iff \Gamma \vDash_{\mathsf{KN4}} \psi \vee (\varphi \Rightarrow (\varphi \Rightarrow \psi)).$$

Naturally, this means that reducing all inferences of KN4 to ones in the form $\vdash_{\mathsf{KN4}} \varphi$ (which is what Hilbert-style axiom systems are about) will not go as smoothly as for logics with the usual deduction theorem. Things turn out to go not too badly in the nonmodal case Brady (1982), yet it is naturally to be expected that adding new components might significantly complicate the picture. For instance, having in mind the modal rules of $\mathsf{HBK}^{\Box-}$ we know that the set of theorems of KN4 is closed under both

$$\frac{\psi \vee (\varphi \Rightarrow (\varphi \Rightarrow \psi))}{\Box\psi \vee (\Box\varphi \Rightarrow (\Box\varphi \Rightarrow \Box\psi))} \qquad \frac{\sim\psi \vee (\sim\varphi \Rightarrow (\sim\varphi \Rightarrow \sim\psi))}{\sim\Box\psi \vee (\sim\Box\varphi \Rightarrow (\sim\Box\varphi \Rightarrow \sim\Box\psi))}$$

which suggests that having a modal rule in the form

$$\frac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi}$$

will not be enough. The question is then: if reducing all inferences to ones in the form $\vdash \varphi$ does not go too well, why not reduce them to some other inferences, which are not too hard to handle, like, for instance, inferences of the form $\varphi \vdash \psi$? Thus, here we propose to axiomatize KN4 in the form of an FDE-style calculus $\mathsf{KN4}_{fde}$ over formula–formula type sequents, cf. Anderson & Belnap (1975). There are two advantages in doing so: firstly, we make a closer connection of KN4 to the FDE-based modal logics and secondly, we get to transfer the modal part of $\mathsf{HBK}^{\Box-}$ onto $\mathsf{KN4}_{fde}$ almost directly. For the remainder of this section, by a *sequent* we mean an expression $\varphi \vdash \psi$, where φ and ψ are $\mathcal{L}^{\mathsf{KN4}}$-formulas.

The axioms of $\mathsf{KN4}_{fde}$ are:

1. axioms of FDE:

$$a1_{fde}.\ \varphi \wedge \psi \vdash \varphi; \qquad a2_{fde}.\ \varphi \wedge \psi \vdash \psi;$$
$$a3_{fde}.\ \varphi \vdash \varphi \vee \psi; \qquad a4_{fde}.\ \psi \vdash \varphi \vee \psi;$$
$$a5_{fde}.\ \sim(\varphi \wedge \psi) \dashv\vdash \sim\varphi \vee \sim\psi; \qquad a6_{fde}.\ \sim(\varphi \vee \psi) \dashv\vdash \sim\varphi \wedge \sim\psi;$$
$$a7_{fde}.\ \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi; \qquad a8_{fde}.\ \sim\sim\varphi \dashv\vdash \varphi;$$

2. axioms of strong implication:

$$a1_{\Rightarrow}.\ \varphi \wedge (\varphi \Rightarrow \psi) \vdash \psi;$$
$$a2_{\Rightarrow}.\ \chi \vdash (\varphi \Rightarrow \psi) \Rightarrow (\sim\psi \Rightarrow \sim\varphi);$$
$$a3_{\Rightarrow}.\ \chi \vdash (\sim\varphi \wedge \psi) \Rightarrow (\varphi \Rightarrow \psi);$$
$$a4_{\Rightarrow}.\ \chi \vdash \psi \Rightarrow (\sim\psi \vee (\varphi \Rightarrow \psi));$$
$$a5_{\Rightarrow}.\ \chi \vdash \sim\varphi \Rightarrow (\varphi \vee (\varphi \Rightarrow \psi));$$

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$$a6_{\Rightarrow}.\ \chi \vdash \varphi \vee (\sim\psi \vee (\varphi \Rightarrow \psi));$$
$$a7_{\Rightarrow}.\ \sim(\varphi \Rightarrow \psi) \dashv\vdash (\varphi \wedge \sim\psi);$$

3. and some modal axioms:

$$a1_{\Box}.\ \chi \vdash \Box(\varphi \Rightarrow \varphi);$$
$$a2_{\Box}.\ \chi \vdash \Box(\varphi \wedge \psi) \Leftrightarrow (\Box\varphi \wedge \Box\psi);$$
$$a3_{\Box}.\ \chi \vdash \Box(\varphi \vee \sim\psi) \Rightarrow (\Box\varphi \vee \sim\Box\psi);$$

where $\varphi \dashv\vdash \psi$ denotes two sequents $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

The rules of inference contain rules of FDE:

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}\ r1_{fde} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{(\varphi \vee \psi) \vdash \chi}\ r2_{fde} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash (\varphi \wedge \psi)}\ r3_{fde}$$

as well as two new modal rules:

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}\ r4_{\Box} \qquad \frac{\sim\varphi \vdash \sim\psi}{\sim\Box\varphi \vdash \sim\Box\psi}\ r5_{\Box}.$$

Notice that axioms $a1_{fde}$–$a7_{fde}$ together with rules $r1_{fde}$–$r3_{fde}$ constitute the axiom system R_{fde}, see, for example, Dunn (2000).

We say that a sequent S is $\mathsf{KN4}_{fde}$*-derivable* if there is a sequence $S_1, \ldots, S_n$ of sequents such that $S_n = S$ and for every $i \leq n$ one of the following holds: i) S_i is an axiom of $\mathsf{KN4}_{fde}$ or; ii) S_i is obtained from some previous sequents using one of the rules of inference of $\mathsf{KN4}_{fde}$.

For arbitrary sets of $\mathcal{L}^{\mathsf{KN4}}$-formulas Γ and $\Delta \neq \varnothing$ write $\Gamma \vdash_{\mathsf{KN4}} \Delta$ if there are $\mathcal{L}^{\mathsf{KN4}}$-formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ and $\psi_1, \ldots, \psi_k \in \Delta$ such that $\bigwedge_{i\leq n} \varphi_i \vdash \bigvee_{j\leq k} \psi_j$ is derivable in $\mathsf{KN4}_{fde}$. Here, we assume that brackets are gathered on the left for the purpose of defining $\bigwedge$ and $\bigvee$ and that $\bigwedge \varnothing$ denotes $p \Rightarrow p$. Observe again that $\mathsf{KN4}_{fde}$ does not have counter-theorems, so we cannot define $\bigvee \varnothing$ dually, which is why we restrict inferences to the ones with nonempty right-hand side. We adopt the usual notational conventions for $\vdash_{\mathsf{KN4}}$. Naturally, $\vdash_{\mathsf{KN4}}$ is a Scott consequence relation.

The following proposition contains some properties which will be helpful to us later on.

PROPOSITION 5.1. *For any* $\mathcal{L}^{\mathsf{KN4}}$*-formulas* φ, ψ:

1. $\varphi \vdash \varphi$ *is* $\mathsf{KN4}_{fde}$*-derivable*;
2. *if* $\chi \vdash \varphi \Rightarrow \psi$ *is* $\mathsf{KN4}_{fde}$*-derivable for every* χ*, then so is* $\varphi \vdash \psi$.[2]

Proof. 1. Apply the transitivity rule $r1_{fde}$ to the two parts of axiom $a8_{fde}$.

2. Take χ to be φ and assume that $\varphi \vdash \varphi \Rightarrow \psi$ is $\mathsf{KN4}_{fde}$-derivable. From this, by the previous item and rule $r3_{fde}$, we obtain that $\varphi \vdash \varphi \wedge (\varphi \Rightarrow \psi)$ is $\mathsf{KN4}_{fde}$-derivable. Apply transitivity to this sequent and axiom $a1_{\Rightarrow}$. □

Since the Scott consequence relation of $\mathsf{KN4}_{fde}$ is defined differently from the ones for system characterized by Hilbert-style axiom systems, we have to change the notion of a prime theory. Thus, we say that a set of $\mathcal{L}^{\mathsf{KN4}}$-formulas Γ is a *prime* $\mathsf{KN4}_{fde}$*-theory* if it is

- nontrivial;
- closed under taking conjunctions: $\varphi, \psi \in \Gamma$ then $\varphi \wedge \psi \in \Gamma$;

[2] See in particular the axioms of strong implication $a2_{\Rightarrow}$–$a6_{\Rightarrow}$ and all modal axioms.

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- closed under derivations: if $\varphi \in \Gamma$ and $\varphi \vdash \psi$ is $\mathsf{KN4}_{fde}$-derivable then $\psi \in \Gamma$;
- satisfies the disjunction property.

The following simple properties of prime $\mathsf{KN4}_{fde}$-theories can be established easily:

PROPOSITION 5.2. *Suppose* Γ *is a prime* $\mathsf{KN4}_{fde}$*-theory, then*:

1. $\varphi \in \Gamma$ *iff* $\psi \vdash \varphi$ *is* $\mathsf{KN4}_{fde}$*-derivable for some* $\psi \in \Gamma$;
2. *if* $\varphi \in \Gamma$ *and* $\varphi \Rightarrow \psi \in \Gamma$ *then* $\psi \in \Gamma$;
3. *if* $\chi \vdash \varphi \Rightarrow \psi$ *is* $\mathsf{KN4}_{fde}$*-derivable for all* χ*, then* $\varphi \Rightarrow \psi \in \Gamma$.

LEMMA 5.3 (Extension). *Suppose for some sets of* $\mathcal{L}^{\mathsf{KN4}}$*-formulas* Γ *and* $\Delta \neq \varnothing$ *we have* $\Gamma \nvdash_{\mathsf{KN4}} \Delta$. *Then there is a prime* $\mathsf{KN4}_{fde}$*-theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \nvdash_{\mathsf{KN4}} \Delta$.

Proof. The proof proceeds as usual. Let $\{\chi_i \mid i \in \mathbb{N}\}$ be some enumeration of all $\mathcal{L}^{\mathsf{KN4}}$-formulas. We define inductively a series of sets by putting $\Gamma_0 := \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\chi_n\}, & \text{if } \Gamma_n \cup \{\chi_n\} \nvdash_{\mathsf{KN4}} \Delta; \\ \Gamma_n, & \text{otherwise.} \end{cases}$$

Then put $\Gamma' := \bigcup_{i \in \mathbb{N}} \Gamma_i$.

First, we can show inductively that $\Gamma_i \nvdash_{\mathsf{KN4}} \Delta$ for all $i \in \mathbb{N}$ and hence that $\Gamma' \nvdash_{\mathsf{KN4}} \Delta$. This also immediately implies that Γ' is not trivial.

Suppose $\varphi, \psi \in \Gamma'$ but $\varphi \wedge \psi \notin \Gamma'$. Then by monotonicity and the definition above there is a natural number i such that $\Gamma_i, \varphi \wedge \psi \vdash_{\mathsf{KN4}} \Delta$ and $\varphi, \psi \in \Gamma_i$. Then $\Gamma_i \vdash_{\mathsf{KN4}} \varphi \wedge \psi$ and $\Gamma_i, \varphi \wedge \psi \vdash_{\mathsf{KN4}} \Delta$, hence by cut $\Gamma_i \vdash_{\mathsf{KN4}} \Delta$, which gives rise to a contradiction.

Suppose $\varphi \in \Gamma'$, $\varphi \vdash \psi$ is $\mathsf{KN4}_{fde}$-derivable and $\psi \notin \Gamma'$. Again we can assume that there is an i such that $\Gamma_i, \psi \vdash_{\mathsf{KN4}} \Delta$ and $\varphi \in \Gamma_i$, hence $\Gamma_i \vdash_{\mathsf{KN4}} \Delta$.

Finally, suppose $\varphi \vee \psi \in \Gamma'$, but $\varphi \notin \Gamma'$ and $\psi \notin \Gamma'$. Then there is a k such that $\Gamma_k, \varphi \vdash_{\mathsf{KN4}} \Delta$, $\Gamma_k, \psi \vdash_{\mathsf{KN4}} \Delta$ and $\varphi \vee \psi \in \Gamma_k$. By definition there are finite conjunctions χ_1 and χ_2 of formulas in Γ_k and finite disjunctions θ_1 and θ_2 of formulas in Δ such that $\chi_1 \wedge \varphi \vdash \theta_1$ and $\chi_2 \wedge \psi \vdash \theta_2$ are $\mathsf{KN4}_{fde}$-derivable. It is easy to show then that $\chi \wedge \varphi \vdash \theta$ and $\chi \wedge \psi \vdash \theta$ are $\mathsf{KN4}_{fde}$-derivable, where $\chi = \chi_1 \wedge \chi_2$ and $\theta = \theta_1 \vee \theta_2$. Then, by $r2_{fde}$, we obtain that $(\chi \wedge \varphi) \vee (\chi \wedge \psi) \vdash \theta$ is $\mathsf{KN4}_{fde}$-derivable and, by distributivity and transitivity, we have that $\chi \wedge (\varphi \vee \psi) \vdash \theta$ is $\mathsf{KN4}_{fde}$-derivable. But that means that $\Gamma_k, \varphi \vee \psi \vdash_{\mathsf{KN4}} \Delta$, which implies $\Gamma_k \vdash_{\mathsf{KN4}} \Delta$ since $\varphi \vee \psi \in \Gamma_k$. This concludes our proof. □

The *canonical* $\mathsf{KN4}_{fde}$*-model* $\mu_{\mathsf{KN4}} = \langle W_{\mathsf{KN4}}, R_{\mathsf{KN4}}, v^{+}_{\mathsf{KN4}}, v^{-}_{\mathsf{KN4}} \rangle$ is defined exactly as for $\mathsf{BK}^{\Box -}$, the only thing that has changed is the notion of a prime theory.

LEMMA 5.4. *Suppose* Γ *is a prime* $\mathsf{KN4}_{fde}$*-theory. Then*

$(\wedge+)$ $(\varphi \wedge \psi) \in \Gamma \iff \varphi \in \Gamma$ *and* $\psi \in \Gamma$;

$(\wedge-)$ $\sim(\varphi \wedge \psi) \in \Gamma \iff \sim\varphi \in \Gamma$ *or* $\sim\psi \in \Gamma$;

$(\vee+)$ $(\varphi \vee \psi) \in \Gamma \iff \varphi \in \Gamma$ *or* $\psi \in \Gamma$;

$(\vee-)$ $\sim(\varphi \vee \psi) \in \Gamma \iff \sim\varphi \in \Gamma$ *and* $\sim\psi \in \Gamma$;

$(\Rightarrow+)$ $\varphi \Rightarrow \psi \in \Gamma \iff (\varphi \notin \Gamma$ *or* $\psi \in \Gamma)$ *and* $(\sim\psi \notin \Gamma$ *or* $\sim\varphi \in \Gamma)$;

$(\Rightarrow-)$ $\sim(\varphi \Rightarrow \psi) \in \Gamma \iff \varphi \in \Gamma$ *and* $\sim\psi \in \Gamma$;

$(\Box+)$ $\Box\varphi \in \Gamma \iff \forall \Delta \in W_{\mathsf{KN4}}$ $(\Gamma R_{\mathsf{KN4}} \Delta$ *implies* $\varphi \in \Delta)$;

$(\Box-)$ $\sim\Box\varphi \in \Gamma \iff \exists \Delta \in W_{\mathsf{KN4}}$ $(\Gamma R_{\mathsf{KN4}} \Delta$ *and* $\sim\varphi \in \Delta)$.

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Proof. We only consider the cases of strong implication and necessity.

($\Rightarrow +$). Suppose $\varphi \Rightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$ by modus ponens. Additionally, by axiom $a2_{\Rightarrow}$, we have $(\varphi \Rightarrow \psi) \Rightarrow (\sim\psi \Rightarrow \sim\varphi) \in \Gamma$. Hence, by a couple of applications of modus ponens, if $\sim\psi \in \Gamma$ then $\sim\varphi \in \Gamma$.

For the other direction, assuming the right side of the equivalence holds, we have four possibilities: i) $\psi \in \Gamma$ and $\sim\varphi \in \Gamma$; ii) $\psi \in \Gamma$ and $\sim\psi \notin \Gamma$; iii) $\varphi \notin \Gamma$ and $\sim\varphi \in \Gamma$; iv) $\varphi \notin \Gamma$ and $\sim\psi \notin \Gamma$. We consider the second one as an example. By axiom $a4_{\Rightarrow}$ we know that $\psi \Rightarrow (\sim\psi \vee (\varphi \Rightarrow \psi)) \in \Gamma$, hence by modus ponens $\sim\psi \vee (\varphi \Rightarrow \psi) \in \Gamma$. Since $\sim\psi \notin \Gamma$, then by the disjunction property $\varphi \Rightarrow \psi \in \Gamma$.

($\Rightarrow -$). Use axiom $a7_{\Rightarrow}$, modus ponens and the case for conjunction.

($\Box +$). This is the only modal case we consider since they are pretty much the same as in Lemma 3.4. Suppose $\Box\varphi \notin \Gamma$ and assume that $\Gamma_{\Box} \vdash_L \{\varphi\} \cup \Gamma^{\sim\Box}$, where $\Gamma_{\Box}$ and $\Gamma^{\sim\Box}$ are defined as in the proof of Lemma 3.4. By definition there are $\varphi_1, \ldots, \varphi_n \in \Gamma_{\Box}$ and $\sim\psi_1, \ldots, \sim\psi_k \in \Gamma^{\sim\Box}$ such that $\bigwedge_{i\le n} \psi_i \vdash (\varphi \vee \bigvee_{j\le k} \sim\chi_j)$ is $\mathsf{KN4}_{fde}$-derivable. Then we have the following derivation schema (we skip details):

$$\dfrac{\dfrac{\dfrac{\dfrac{\bigwedge_{i\le n} \psi_i \vdash (\varphi \vee \bigvee_{j\le k} \sim\chi_j)}{\bigwedge_{i\le n} \psi_i \vdash (\varphi \vee \sim \bigwedge_{j\le k} \chi_j)}}{\Box \bigwedge_{i\le n} \psi_i \vdash \Box(\varphi \vee \sim \bigwedge_{j\le k} \chi_j)}}{\Box \bigwedge_{i\le n} \psi_i \vdash \Box\varphi \vee \sim\Box \bigwedge_{j\le k} \chi_j}}{\bigwedge_{i\le n} \Box\psi_i \vdash \Box\varphi \vee \sim\Box \bigwedge_{j\le k} \chi_j}.$$

From this we can conclude that $\sim\Box \bigwedge_{j\le k} \chi_j \in \Gamma$. It is not hard to show that the following sequent is $\mathsf{KN4}_{fde}$-derivable (use Proposition 5.1):

$$\sim\Box \bigwedge_{j\le k} \chi_j \vdash \bigvee_{j\le k} \sim\Box\chi_j,$$

from which we infer that $\bigvee_{j\le k} \sim\Box\chi_j \in \Gamma$.

Consequently, $\Gamma_{\Box} \nvdash_L \{\varphi\} \cup \Gamma^{\sim\Box}$. By the extension lemma there is a prime L-theory Δ such that $\Gamma_{\Box} \subseteq \Delta$ and $\Delta \nvdash_{\mathsf{KN4}} \{\varphi\} \cup \Gamma^{\sim\Box}$ and it is routine to show that $\Gamma R_{\mathsf{KN4}} \Delta$ and $\varphi \in \Delta$. □

From this we immediately obtain

LEMMA 5.5 (Canonical). *For every prime* $\mathsf{KN4}_{fde}$*-theory* Γ *and* $\mathcal{L}^{\mathsf{KN4}}$*-formula* φ *we have, where* $\mu_{\mathsf{KN4}} = \langle W_{\mathsf{KN4}}, R_{\mathsf{KN4}}, v^{+}_{\mathsf{KN4}}, v^{-}_{\mathsf{KN4}} \rangle$ *is the canonical* $\mathsf{KN4}_{fde}$*-model*:

$$\mu_{\mathsf{KN4}}, \Gamma \vDash^{+} \varphi \iff \varphi \in \Gamma; \qquad \mu_{\mathsf{KN4}}, \Gamma \vDash^{-} \chi \iff \sim\varphi \in \Gamma.$$

From this as before we derive a completeness result.

THEOREM 5.6 (Strong completeness). *For any sets of* $\mathcal{L}^{\mathsf{KN4}}$*-formulas* Γ *and* Δ *with* $\Delta \neq \varnothing$ *we have*:

$$\Gamma \vdash_{\mathsf{KN4}} \Delta \iff \Gamma \vDash_{\mathsf{KN4}} \Delta.$$

Notice that there does not seem to be a natural way of extracting a Hilbert-style axiom system for $\mathsf{KN4}$ from $\mathsf{KN4}_{fde}$.[3] Roughly speaking, the turnstile in $\mathsf{KN4}_{fde}$ corresponds to a first-degree instance of weak implication, while the language of $\mathsf{KN4}$ only contains strong

[3] We thank an anonymous referee for prompting this question.

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implication. Naturally, one could, having in mind the definition of weak implication via strong implication, attempt to replace every axiom $\varphi \vdash \psi$ with a formula $\psi \vee (\varphi \Rightarrow (\varphi \Rightarrow \psi))$, yet, as outlined at the beginning of the section, this is unlikely to result in an axiom system which is any better than Goble's.

As a final remark here, we outline how $\mathsf{BK}^{\Box-}$ can be characterized by modifying $\mathsf{KN4}_{fde}$ slightly. Denote by $\mathsf{BK}^{\Box-}_{fde}$ the system obtained by replacing the axioms for strong implication in $\mathsf{KN4}_{fde}$ with the following axioms for weak implication:

$$a1_{\rightarrow}.\ \varphi \wedge (\varphi \rightarrow \psi) \vdash \psi;$$
$$a2_{\rightarrow}.\ \chi \vdash \psi \rightarrow (\varphi \rightarrow \psi);$$
$$a3_{\rightarrow}.\ \chi \vdash \varphi \vee (\varphi \rightarrow \psi);$$
$$a4_{\rightarrow}.\ \chi \vdash {\sim}(\varphi \rightarrow \psi) \leftrightarrow (\varphi \wedge {\sim}\psi).$$

Completeness can be derived from the following, where a prime $\mathsf{BK}^{\Box-}_{fde}$-theory is defined exactly as for $\mathsf{KN4}_{fde}$:

LEMMA 5.7. *Suppose* Γ *is a prime* $\mathsf{BK}^{\Box-}_{fde}$*-theory, then*

$$(\rightarrow +) \qquad \varphi \rightarrow \psi \in \Gamma \iff (\varphi \notin \Gamma \text{ or } \psi \in \Gamma);$$
$$(\rightarrow -)\ {\sim}(\varphi \rightarrow \psi) \in \Gamma \iff \varphi \in \Gamma \text{ and } {\sim}\psi \in \Gamma.$$

§6. The sequent system G3KN4. In Arieli & Avron (1996), Ofer Arieli and Arnon Avron present the sequent calculus $GBL_{\supset}$ for the logic $BL_{\supset}$ of logical bilattices with weak implication.[4] The language of $BL_{\supset}$ contains, in addition to weak implication, $\sim$, $\wedge$, and $\vee$ also connectives $\otimes$ and $\oplus$ associated with the lattice meet and lattice join of the information order from the bilattice $FOUR_2$. As in other sequent calculi for logics with strong negation, $GBL_{\supset}$ contains left and right rules for double negations and left and right rules for compound formulas other than negations and for the strong negations of these compound formulas. The sequent rules use shared contexts, as in the G3cp calculus for classical propositional logic, cf. Negri & von Plato (2008); Troelstra & Schwichtenberg (2000). With respect to a choice between strong and weak implication, Arieli and Avron (Arieli & Avron, 1996, p. 52) explain that they prefer the latter "since the intuitive meaning of both is then clearer. Also, the corresponding proof systems are much simpler if we follow this choice." Since the support of falsity conditions for weak and strong implication coincide, when it comes to a sequent calculus presentation, the only modification that is required to obtain a system for the language with strong instead of weak implication concerns the left and the right rule for strong implication. Given the simple definition of strong implication in terms of weak implication, this modification is, however, complicated only insofar as it requires a sequent rule with *four* premise sequents instead of one or two.

We shall extend the $\{\sim, \wedge, \vee\}$-part of $GBL_{\supset}$ by suitable left and right rules for strong implications, strongly negated strong implications, the standard left-and-right introduction rule for formulas $\Box\varphi$ from the sequent calculus LK for the modal propositional logic K, and a left-and-right introduction rule for strongly negated formulas ${\sim}\Box\varphi$. Also, two more sequent rules, $L\,\Box{\sim}\Box$ and $R{\sim}\Box\,\Box$, introducing formulas of the form ${\sim}\Box\varphi$ and $\Box\varphi$ are

[4] Arieli and Avron denote the weak implication by '$\supset$' and they also define a sequent system $GBL_{\supset}(4)$ in the language that includes constants representing the four elements of the bilattice $FOUR_2$.

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required. The resulting system, G3KN4, is shown to be a contraction-free and cut-free sequent calculus for KN4. Although the calculus builds weakening (monotonicity) into the axioms, applications of weakening cannot, however, be eliminated from derivations in G3KN4 due to the sequent rules for formulas $\Box\varphi$ and $\sim\Box\varphi$. Moreover, since the rules for $\Rightarrow$ introduce formulas with a subformula appearing in antecedent position *and* a subformula appearing in succedent position of one and the same premise sequent, in order to obtain height-preserving invertibility of the rules for $\Rightarrow$, the usual left and right weakening rules are subsumed under a single left-and-right weakening rule, W, that allows weakening in antecedent and succedent position simultaneously.

Whilst Arieli and Avron give a semantical proof of cut-admissibility for $GBL_{\supset}$, we shall apply Dragalin's method from Dragalin (1988) to obtain a constructive proof of cut-admissibility for G3KN4, and we shall follow the presentation in Negri & von Plato (2008).[5]

From here on out, sequents have the form $\Gamma \rhd \Delta$, where Γ and Δ are finite, possibly empty multisets of $\mathcal{L}^{\mathsf{KN4}}$-formulas, $\sim\Delta = \{\sim\varphi \mid \varphi \in \Delta\}$, and $\Box\Delta = \{\Box\varphi \mid \varphi \in \Delta\}$. Instead of $\varnothing \rhd \Delta$ ($\Gamma \rhd \varnothing$), where $\varnothing$ is the empty multiset, we write $\rhd\Delta$ ($\Gamma\rhd$). In sequents the comma denotes multiset union, and instead of $\{\varphi\}$ we just write φ. The sequent system G3KN4 consists of the following axioms, logical rules, and structural rules:

Axioms:

$$p, \Gamma \rhd \Delta, p \qquad \sim p, \Gamma \rhd \Delta, \sim p \quad (p \in \mathsf{Prop})$$

Logical rules:

$$\frac{\varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}\, L\wedge \qquad \frac{\Gamma \rhd \Delta, \varphi \quad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}\, R\wedge$$

$$\frac{\sim\varphi, \Gamma \rhd \Delta \quad \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \wedge \psi), \Gamma \rhd \Delta}\, L\sim\wedge \qquad \frac{\Gamma \rhd \Delta, \sim\varphi, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \wedge \psi)}\, R\sim\wedge$$

$$\frac{\varphi, \Gamma \rhd \Delta \quad \psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta}\, L\vee \qquad \frac{\Gamma \rhd \Delta, \varphi, \psi}{\Gamma \rhd \Delta, \varphi \vee \psi}\, R\vee$$

$$\frac{\sim\varphi, \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \vee \psi), \Gamma \rhd \Delta}\, L\sim\vee \qquad \frac{\Gamma \rhd \Delta, \sim\varphi \quad \Gamma \rhd \Delta, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \vee \psi)}\, R\sim\vee$$

$$\frac{\Gamma \rhd \Delta, \varphi, \sim\psi \quad \sim\varphi, \Gamma \rhd \Delta, \varphi \quad \psi, \Gamma \rhd \Delta, \sim\psi \quad \sim\varphi, \psi, \Gamma \rhd \Delta}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta}\, L\Rightarrow$$

$$\frac{\varphi, \Gamma \rhd \Delta, \psi \quad \sim\psi, \Gamma \rhd \Delta, \sim\varphi}{\Gamma \rhd \Delta, \varphi \Rightarrow \psi}\, R\Rightarrow$$

$$\frac{\varphi, \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \Rightarrow \psi), \Gamma \rhd \Delta}\, L\sim\Rightarrow \qquad \frac{\Gamma \rhd \Delta, \varphi \quad \Gamma \rhd \Delta, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \Rightarrow \psi)}\, R\sim\Rightarrow$$

[5] Concerning the application of elementary proof methods in structural proof theory, Negri and von Plato (Negri & von Plato, 2008, p. xiv) write that "[t]he main difficulty, witnessed by the long development of structural proof theory, is to find the right rules." As to G3KN4, the problem was to find the rules $L\,\Box\sim\Box$ and $R\sim\Box\Box$, which essentially stem from axioms in Dunn (1995).

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$$\frac{\varphi, \Gamma \rhd \Delta}{\sim\sim\varphi, \Gamma \rhd \Delta}\ L{\sim\sim} \qquad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \sim\sim\varphi}\ R{\sim\sim}$$

$$\frac{\Gamma \rhd \varphi}{\Box\Gamma \rhd \Box\varphi}\ RL\Box \qquad \frac{\sim\varphi \rhd \sim\Gamma}{\sim\Box\varphi \rhd \sim\Box\Gamma}\ LR{\sim}\Box$$

$$\frac{\sim\varphi, \Gamma \rhd \sim\Delta}{\sim\Box\varphi, \Box\Gamma \rhd \sim\Box\Delta}\ L\Box{\sim}\Box \qquad \frac{\Gamma \rhd \varphi, \sim\Delta}{\Box\Gamma \rhd \Box\varphi, \sim\Box\Delta}\ R{\sim}\Box\,\Box$$

Structural rule:

$$\frac{\Gamma \rhd \Delta}{\Gamma, \Gamma' \rhd \Delta, \Delta'}\ W.$$

Notice that $RL\Box$ is a special case of $R{\sim}\Box\Box$ and $LR{\sim}\Box$ is a special case of $L\Box{\sim}\Box$. We chose to keep this redundancy to highlight (i) how G3KN4 can be extracted from $\mathsf{KN4}_{fde}$ and (ii) that rules $RL\Box$ and $LR{\sim}\Box$ are enough to derive axioms $a1_\Box$ and $a2_\Box$ (see Appendix 9).[6]

If

$$\frac{s_1, \ldots, s_k}{s}$$

is a sequent rule, then the sequents $s_1, \ldots, s_k$ are said to be the *premise sequents* of that rule and s its *conclusion sequent*. The compound formula introduced into the conclusion sequent of a logical rule is said to be the *principal* formula of the rule, so $\sim\Box\varphi$, for example, is principal in $LR{\sim}\Box$ and $L\Box{\sim}\Box$, whereas $\Box\varphi$ is principal in $RL\Box$ and $R{\sim}\Box\,\Box$.

DEFINITION 6.1. *Every axiom* $p, \Gamma \rhd \Delta, p$ *is a* derivation *of* $p, \Gamma \rhd \Delta, p$ *in* G3KN4*, and every axiom* $\sim p, \Gamma \rhd \Delta, \sim p$ *is a* derivation *of* $\sim p, \Gamma \rhd \Delta, \sim p$ *in* G3KN4*. Suppose* $d(s_1), \ldots, d(s_k)$ *are derivations of* $s_1, \ldots, s_k$ *in* G3KN4*, respectively, and*

$$\frac{s_1, \ldots, s_k}{s}$$

is a logical or structural rule of G3KN4*, then*

$$\frac{d(s_1), \ldots, d(s_k)}{s}$$

is a derivation of s *in* G3KN4*. A sequent* s *is* derivable *in* G3KN4 *if there is a derivation of* s *in* G3KN4*. We denote the derivability of a sequent* s *in* G3KN4 *by* $\Vdash_{\mathsf{KN4}} s$ *or just by* $\Vdash s$*.*

DEFINITION 6.2. *The* weight $w(\varphi)$ *of an* $\mathcal{L}^{\mathsf{KN4}}$*-formula* φ *and the height* $h(d)$ *of a derivation* d *in* G3KN4 *are inductively defined as follows*:

1. $w(p) = 1$*, for* $p \in$ Prop*,*
2. $w(\sharp\varphi) = w(\varphi) + 1$*, for* $\sharp \in \{\Box, \sim\}$*,*
3. $w(\varphi \sharp \psi) = w(\varphi) + w(\psi) + 1$*, for* $\sharp \in \{\Rightarrow, \wedge, \vee\}$*.*

The height *of an axiom is 0. If* $d(s_1), \ldots, d(s_k)$ *are derivations of* $s_1, \ldots, s_k$ *in* G3KN4*, and*

$$\frac{s_1, \ldots, s_k}{s}$$

[6] Again, we thank an anonymous referee for pointing this out.

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is a logical rule of G3KN4 *or W, then the height h(d) of the derivation d:*

$$\frac{d(s_1), \ldots, d(s_k)}{s}$$

is $max(h(d(s_1)), \ldots, h(d(s_k))) + 1$.

If a sequent s is derivable in G3KN4 with a height of derivation at most n, we write $\Vdash_n s$.

PROPOSITION 6.3. *For every* $\mathcal{L}^{\mathsf{KN4}}$*-formula* φ *and all finite sets of* $\mathcal{L}^{\mathsf{KN4}}$*-formulas* Γ, Δ *it holds that*

$$\Vdash_{\mathsf{KN4}} \varphi, \Gamma \rhd \varphi, \Delta.$$

Proof. By induction on the weight of φ. We present three cases.

If $\varphi = (\alpha \Rightarrow \beta)$, then we have

$$\dfrac{\dfrac{s_1 \quad s_2 \quad s_3 \quad s_4}{\alpha, \alpha \Rightarrow \beta, \Gamma \rhd \beta, \Delta}\, L\Rightarrow \quad \dfrac{s_5 \quad s_6 \quad s_7 \quad s_8}{\sim\beta, \alpha \Rightarrow \beta, \Gamma \rhd \sim\alpha, \Delta}\, L\Rightarrow}{\alpha \Rightarrow \beta, \Gamma \rhd \alpha \Rightarrow \beta, \Delta}\, R\Rightarrow,$$

where

$$\begin{aligned}
&s_1 = \alpha, \Gamma \rhd \beta, \alpha, \sim\beta, \Delta; && s_2 = \sim\alpha, \alpha, \Gamma \rhd \beta, \alpha, \Delta;\\
&s_3 = \beta, \alpha, \Gamma \rhd \beta, \sim\beta, \Delta; && s_4 = \sim\alpha, \beta, \alpha, \Gamma \rhd \beta, \Delta;\\
&s_5 = \sim\beta, \Gamma \rhd \sim\alpha, \alpha, \sim\beta, \Delta; && s_6 = \sim\alpha, \sim\beta, \Gamma \rhd \sim\alpha, \alpha, \Delta;\\
&s_7 = \beta, \sim\beta, \Gamma \rhd \sim\alpha, \sim\beta, \Delta; && s_8 = \sim\alpha, \beta, \sim\beta, \Gamma \rhd \sim\alpha, \Delta.
\end{aligned}$$

The premise sequents s_1–s_8 are derivable by the induction hypothesis.

If $\varphi = \Box\psi$ or $\varphi = \sim\Box\psi$, we have

$$\dfrac{\dfrac{\psi \rhd \psi}{\Box\psi \rhd \Box\psi}\, RL\Box}{\Box\psi, \Gamma \rhd \Box\psi, \Delta}\, W \qquad \dfrac{\dfrac{\sim\psi \rhd \sim\psi}{\sim\Box\psi \rhd \sim\Box\psi}\, LR\sim\Box}{\sim\Box\psi, \Gamma \rhd \sim\Box\psi, \Delta}\, W$$

□

The last two cases reveal that applications of the weakening rule W cannot be eliminated from derivations in G3KN4. In derivations we will sometimes tacitly assume Proposition 6.3 and treat all sequents $\varphi, \Gamma \rhd \Delta, \varphi$ as if they were axioms.

PROPOSITION 6.4. *If* $\Vdash \Gamma \rhd \Delta$*, then* Δ *is not empty.*

Proof. By induction on the derivation height, n. If $n = 0$, then $\Gamma \rhd \Delta$ is an axiom, and the claim holds. Suppose the claim holds for derivations of height n. It must be shown that it holds for derivations of height $n + 1$, but this is either obvious because a formula is introduced on the right hand side of $\rhd$, or it follows by the induction hypothesis. □

§7. Admissibility of structural rules. We show the admissibility of the structural rules *LC* and *RC* of left and right contraction

$$\frac{\varphi, \varphi, \Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta}\, LC \qquad \frac{\Gamma \rhd \varphi, \varphi\, \Delta}{\Gamma \rhd \varphi, \Delta}\, RC$$

in order to prove the eliminability and hence admissibility of the (cut)-rule

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$$\frac{\Gamma \rhd \Delta, \theta \qquad \theta, \Gamma' \rhd \Delta'}{\Gamma, \Gamma' \rhd \Delta, \Delta'}\ (\text{cut}).$$

To show the admissibility of contraction, we observe the invertibility of almost all logical rules.

PROPOSITION 7.1. *All logical rules of* G3KN4*, except for* $L\Box\sim\Box$ *and* $R\sim\Box\Box$*, are height-preserving invertible*:

1. *If* $\Vdash_n \varphi \wedge \psi, \Gamma \rhd \Delta$*, then* $\Vdash_n \varphi, \psi, \Gamma \rhd \Delta$.
2. *If* $\Vdash_n \Gamma \rhd \varphi \wedge \psi, \Delta$*, then* $\Vdash_n \Gamma \rhd \varphi, \Delta$ *and* $\Vdash_n \Gamma \rhd \psi, \Delta$.
3. *If* $\Vdash_n \varphi \vee \psi, \Gamma \rhd \Delta$*, then* $\Vdash_n \varphi, \Gamma \rhd \Delta$ *and* $\Vdash_n \psi, \Gamma \rhd \Delta$.
4. *If* $\Vdash_n \Gamma \rhd \varphi \vee \psi, \Delta$*, then* $\Vdash_n \Gamma \rhd \varphi, \psi, \Delta$.
5. *If* $\Vdash_n \sim(\varphi \wedge \psi), \Gamma \rhd \Delta$*, then* $\Vdash_n \sim\varphi, \Gamma \rhd \Delta$ *and* $\Vdash_n \sim\psi, \Gamma \rhd \Delta$.
6. *If* $\Vdash_n \Gamma \rhd \Delta, \sim(\varphi \wedge \psi)$*, then If* $\Vdash_n \Gamma \rhd \Delta, \sim\varphi, \sim\psi$.
7. *If* $\Vdash_n \sim(\varphi \vee \psi), \Gamma \rhd \Delta$*, then* $\Vdash_n \sim\varphi, \sim\psi, \Gamma \rhd \Delta$.
8. *If* $\Vdash_n \Gamma \rhd \Delta, \sim(\varphi \vee \psi)$*, then* $\Vdash_n \Gamma \rhd \Delta, \sim\varphi$ *and* $\Vdash_n \Gamma \rhd \Delta, \sim\psi$.
9. *If* $\Vdash_n \varphi \Rightarrow \psi, \Gamma \rhd \Delta$*, then* $\Vdash_n \Gamma \rhd \Delta, \varphi, \sim\psi$*,* $\Vdash_n \sim\varphi, \Gamma \rhd \Delta, \varphi$*,* $\Vdash_n \psi, \Gamma \rhd \Delta, \sim\psi$*, and* $\Vdash_n \sim\varphi, \psi, \Gamma \rhd \Delta$.
10. *If* $\Vdash_n \Gamma \rhd \Delta, \varphi \Rightarrow \psi$*, then* $\Vdash_n \varphi, \Gamma \rhd \Delta, \psi$ *and* $\Vdash_n \sim\psi, \Gamma \rhd \Delta, \sim\varphi$.
11. *If* $\Vdash_n \sim(\varphi \Rightarrow \psi), \Gamma \rhd \Delta$*, then* $\Vdash_n \varphi, \sim\psi, \Gamma \rhd \Delta$.
12. *If* $\Vdash_n \Gamma \rhd \Delta, \sim(\varphi \Rightarrow \psi)$*, then* $\Vdash_n \Gamma \rhd \Delta, \varphi$ *and* $\Vdash_n \Gamma \rhd \Delta, \sim\psi$.
13. *If* $\Vdash_n \sim\sim\varphi, \Gamma \rhd \Delta$*, then* $\Vdash_n \varphi, \Gamma \rhd \Delta$.
14. *If* $\Vdash_n \Gamma \rhd \Delta, \sim\sim\varphi$*, then* $\Vdash_n \Gamma \rhd \Delta, \varphi$.
15. *If* $\Vdash_n \Box\Gamma \rhd \Box\varphi$*, then* $\Vdash_n \Gamma \rhd \varphi$.
16. *If* $\Vdash_n \sim\Box\varphi \rhd \sim\Box\Gamma$*, then* $\Vdash_n \sim\varphi \rhd \sim\Gamma$.

Proof. By induction on the height of derivations, n. The proof follows the pattern of the proof of Theorem 3.1.1 in (Negri & von Plato, 2008, p. 49 f.), however, we have to take into account the presence of the weakening rule W. We present the cases of items 9, 10, and 15.

9: If $\varphi \Rightarrow \psi, \Gamma \rhd \Delta$ is an axiom, then the sequents (i) $\Gamma \rhd \Delta, \varphi, \sim\psi$, (ii) $\sim\varphi, \Gamma \rhd \Delta, \varphi$, (iii) $\psi, \Gamma \rhd \Delta, \sim\psi$ and (iv) $\sim\varphi, \psi, \Gamma \rhd \Delta$ are axioms as well, since $\varphi \Rightarrow \psi$ is neither a propositional variable nor a negated propositional variable. Suppose the claims hold for derivations of height up to n, and suppose further that $\Vdash_{n+1} \varphi \Rightarrow \psi, \Gamma \rhd \Delta$. If the derivation of $\varphi \Rightarrow \psi, \Gamma \rhd \Delta$ ends with an application of $L\Rightarrow$, every premise sequent has a derivation of height at most n, and we are done. If the last derivation step is neither an application of $L \Rightarrow$ nor of W, then in this step $\varphi \Rightarrow \psi, \Gamma \rhd \Delta$ has one or two premise sequents, $\varphi \Rightarrow \psi, \Gamma_1 \rhd \Delta_1$, respectively, $\varphi \Rightarrow \psi, \Gamma_2 \rhd \Delta_2$, each of derivation height at most n. By the induction hypothesis for claim 9, $\Vdash_n \Gamma_1 \rhd \Delta_1, \varphi, \sim\psi$, $\Vdash_n \sim\varphi, \Gamma_1 \rhd \Delta_1, \varphi$, $\Vdash_n \psi, \Gamma_1 \rhd \Delta_1, \sim\psi$, and $\Vdash_n \sim\varphi, \psi, \Gamma_1 \rhd \Delta_1$ and $\Vdash_n \Gamma_2 \rhd \Delta_2, \varphi, \sim\psi$, $\Vdash_n \sim\varphi, \Gamma_2 \rhd \Delta_2, \varphi$, $\Vdash_n \psi, \Gamma_2 \rhd \Delta_2, \sim\psi$, and $\Vdash_n \sim\varphi, \psi, \Gamma_2 \rhd \Delta_2$. One may then apply the rule of the last derivation step to obtain derivations of $\Gamma \rhd \Delta, \varphi, \sim\psi$ and $\sim\varphi, \Gamma \rhd \Delta, \varphi$ as well as $\psi, \Gamma \rhd \Delta, \sim\psi$ and $\sim\varphi, \psi, \Gamma \rhd \Delta$, each of derivation height $\leq n+1$. If the last derivation step is an application of W, we may just apply W, and since we have to derive $\sim\varphi, \Gamma \rhd \Delta, \varphi$ and $\psi, \Gamma \rhd \Delta, \sim\psi$ with displayed formulas on *both* sides of $\rhd$, we need the generalized weakening rule W to proceed from

$$\frac{\Gamma \rhd \Delta}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta}\ W$$

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to

$$\frac{\Gamma \rhd \Delta}{\sim\varphi, \Gamma \rhd \Delta, \varphi}\ W \qquad \text{and} \qquad \frac{\Gamma \rhd \Delta}{\psi, \Gamma \rhd \Delta, \sim\psi}\ W$$

thereby obtaining derivations of height $n+1$.

10: If $\Gamma \rhd \Delta, \varphi \Rightarrow \psi$ is an axiom, then so are $\Gamma \rhd \Delta, \psi$ and $\sim\psi, \Gamma \rhd \Delta, \sim\varphi$. If the derivation of $\Gamma \rhd \Delta, \varphi \Rightarrow \psi$ ends with an application of $R\Rightarrow$, every premise sequent has a derivation of height $\leq n$, and we are done. If the last derivation step is neither an application of $R\Rightarrow$ nor of W, then in this step $\Gamma \rhd \Delta, \varphi \Rightarrow \psi$ has one, two, or four premise sequents, $\Gamma_i \rhd \Delta_i, \varphi \Rightarrow \psi$, $i \in \{1, 2, 3, 4\}$, each of derivation height at most n. By the induction hypothesis for claim 10, $\Vdash_n \varphi, \Gamma_i \rhd \Delta_i, \psi$ and $\Vdash_n \sim\psi, \Gamma_i \rhd \Delta_i, \sim\varphi$. We can apply the rule of the last derivation step to obtain derivations of $\varphi, \Gamma \rhd \Delta, \psi$ and $\sim\psi, \Gamma \rhd \Delta, \sim\varphi$. If the last derivation step is an application of W, we may just apply W.

15: $\Box\Gamma \rhd \Box\varphi$ is not an axiom. If the derivation of $\Box\Gamma \rhd \Box\varphi$ ends with an application of $RL\Box$, the premise sequent has a derivation of height at most n. If the last derivation step is not an application of $RL\Box$, then it is an application of W with a premise sequent $\Box\Theta \rhd \Box\varphi$. By the induction hypothesis for claim 15, $\Vdash_n \Theta \rhd \varphi$. One application of W gives $\Vdash_{n+1} \Gamma \rhd \varphi$. Note that by Proposition 6.4, the derivation of $\Box\Gamma \rhd \Box\varphi$ cannot end with an application of W that introduces the succedent $\Box\varphi$. □

We now turn to the contraction rules.

THEOREM 7.2. *Rules LC and RC are height-preserving admissible*:

1. *If* $\Vdash_n \varphi, \varphi, \Gamma \rhd \Delta$, *then* $\Vdash_n \varphi, \Gamma \rhd \Delta$.
2. *If* $\Vdash_n \Gamma \rhd \varphi, \varphi, \Delta$, *then* $\Vdash_n \Gamma \rhd \varphi, \Delta$.

Proof. The proof is by simultaneous induction on the height of derivations for both claims and follows the pattern of the proof of Theorem 3.2.2 in (Negri & von Plato, 2008, p. 49 f.), the main differences being the presence of W, $RL\Box$, $LR{\sim}\Box$, $L\,\Box{\sim}\Box$, and $R{\sim}\Box\,\Box$ and the complication resulting from the use of strong instead of weak implication. If $n = 0$ and $\varphi, \varphi, \Gamma \rhd \Delta$ ($\Gamma \rhd \varphi, \varphi, \Delta$) is an axiom, then $\varphi, \Gamma \rhd \Delta$ ($\Gamma \rhd \varphi, \Delta$) is an axiom as well. Suppose the claim holds for derivations of height up to n.

If the contraction formula φ is *not principal* in the last rule applied to obtain $\varphi, \varphi, \Gamma \rhd \Delta$ or $\Gamma \rhd \varphi, \varphi, \Delta$, the cases where the contraction formula is introduced by W are particularly simple. We have:

$$\frac{\dfrac{\varphi, \Gamma \rhd \Delta}{\varphi, \varphi, \Gamma \rhd \Delta}\ W}{\varphi, \Gamma \rhd \Delta}\ W \qquad \frac{\dfrac{\Gamma \rhd \varphi, \Delta}{\Gamma \rhd \varphi, \varphi, \Delta}\ W}{\Gamma \rhd \varphi, \Delta}\ W$$

and obviously, $\Vdash_n \varphi, \Gamma \rhd \Delta$, respectively $\Vdash_n \Gamma \rhd \varphi, \Delta$. In the remaining subcases we can reverse the order of rule applications and apply the induction hypothesis. If the contraction formula φ has the shape $\Box\psi$, for example, then we have the following situations:

$$\frac{\dfrac{\psi, \psi, \Gamma \rhd \theta}{\Box\psi, \Box\psi, \Box\Gamma \rhd \Box\theta}\ RL\Box}{\Box\psi, \Box\Gamma \rhd \Box\theta}\ LC \qquad \frac{\dfrac{\psi, \psi, \Gamma \rhd \theta}{\psi, \Gamma \rhd \theta}\ LC}{\Box\psi, \Box\Gamma \rhd \Box\theta}\ RL\Box$$

$$\frac{\dfrac{\Theta, \chi, \chi \rhd \psi, {\sim}\Delta}{\Box\Theta, \Box\chi, \Box\chi \rhd \Box\psi, {\sim}\Box\Delta}\ R{\sim}\Box\,\Box}{\Box\Theta, \Box\chi \rhd \Box\psi, {\sim}\Box\Delta}\ LC \qquad \frac{\dfrac{\Theta, \chi, \chi \rhd \psi, {\sim}\Delta}{\Theta, \chi \rhd \psi, {\sim}\Delta}\ LC}{\Box\Theta, \Box\chi \rhd \Box\psi, {\sim}\Box\Delta}\ R{\sim}\Box\,\Box$$

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$$\dfrac{\dfrac{\sim\varphi, \chi, \chi, \Theta \rhd \sim\Delta}{\sim\Box\varphi, \Box\chi, \Box\chi, \Box\Theta \rhd \sim\Box\Delta}\, L\Box\sim\Box}{\sim\Box\varphi, \Box\chi, \Box\Theta \rhd \sim\Box\Delta}\, LC \qquad \dfrac{\dfrac{\sim\varphi, \chi, \chi, \Theta \rhd \sim\Delta}{\sim\varphi, \chi, \Theta \rhd \sim\Delta}\, LC}{\sim\Box\varphi, \Box\chi, \Box\Theta \rhd \sim\Box\Delta}\, L\Box\sim\Box$$

The case of formulas of the form $\sim\Box\psi$ is analogous.

If the contraction formula φ *is principal* in the last rule applied to obtain one of $\varphi, \varphi, \Gamma \rhd \Delta$ or $\Gamma \rhd \varphi, \varphi, \Delta$, we have to consider the shape of φ. The cases when φ is a conjunction or a disjunction are familiar, see (Negri & von Plato, 2008, p. 34, p. 53). The cases of negated conjunctions, negated disjunctions, negated strong implications, and double negations are analogous.

If the last rule applied is, for example, $L\sim\wedge$, assume that $\Vdash_n \sim\varphi, \sim(\varphi \wedge \psi), \Gamma \rhd \Delta$ and $\Vdash_n \sim\psi, \sim(\varphi \wedge \psi), \Gamma \rhd \Delta$. Then, by Theorem 7.1, Claim 5, $\Vdash_n \sim\varphi, \sim\varphi, \Gamma \rhd \Delta$ and $\Vdash_n \sim\psi, \sim\psi, \Gamma \rhd \Delta$. By the induction hypothesis we have $\Vdash_n \sim\varphi, \Gamma \rhd \Delta$ and $\Vdash_n \sim\psi, \Gamma \rhd \Delta$, so that by $L\sim\wedge$, we get $\Vdash_{n+1} \sim(\varphi \wedge \psi), \Gamma \rhd \Delta$.

If the last rule applied is $R\sim\wedge$, assume that $\Vdash_n \Gamma \rhd \sim(\varphi \wedge \psi), \sim\varphi, \sim\psi, \Delta$. By Theorem 7.1, Claim 6, $\Vdash_n \Gamma \rhd \sim\varphi, \sim\psi, \sim\varphi, \sim\psi, \Delta$. Applying the induction hypothesis twice gives $\Vdash_n \Gamma \rhd \sim\varphi, \sim\psi, \Delta$, and by $R\sim\wedge$, we obtain $\Vdash_{n+1} \Gamma \rhd \sim(\varphi \wedge \psi), \Delta$.

If the last rule applied is $L \Rightarrow$ assume the following:

(i) $\Vdash_n \Gamma, \varphi \Rightarrow \psi \rhd \Delta, \varphi, \sim\psi$, (ii) $\Vdash_n \sim\varphi, \Gamma, \varphi \Rightarrow \psi \rhd \Delta, \varphi$,
(iii) $\Vdash_n \psi, \Gamma, \varphi \Rightarrow \psi \rhd \Delta, \sim\psi$, (iv) $\Vdash_n \sim\varphi, \psi, \varphi \Rightarrow \psi, \Gamma \rhd \Delta$.

From this using Theorem 7.1, Claim 9, we can, respectively, obtain

(i) $\Vdash_n \Gamma \rhd \Delta, \varphi, \sim\psi, \varphi, \sim\psi$, (ii) $\Vdash_n \sim\varphi, \sim\varphi, \Gamma\rhd, \Delta, \varphi, \varphi$,
(iii) $\Vdash_n \psi, \psi, \Gamma \rhd \Delta, \sim\psi, \sim\psi$, (iv) $\Vdash_n \sim\varphi, \psi, \sim\varphi, \psi, \Gamma \rhd \Delta$.

By the induction hypothesis for both claims of Theorem 7.2, we, respectively, get

(i) $\Vdash_n \Gamma \rhd \Delta, \varphi, \sim\psi$, (ii) $\Vdash_n \sim\varphi, \Gamma\rhd, \Delta, \varphi$,
(iii) $\Vdash_n \psi, \Gamma \rhd \Delta, \sim\psi$, (iv) $\Vdash_n \sim\varphi, \psi, \Gamma \rhd \Delta$.

Finally, by one application of $L \Rightarrow$, we obtain $\Vdash_{n+1} \varphi \Rightarrow \psi, \Gamma \rhd \Delta$.

The case of rule $R \Rightarrow$ is again simpler. Suppose we have $\Vdash_n \varphi, \Gamma \rhd \Delta, \varphi \Rightarrow \psi, \psi$ and $\Vdash_n \sim\psi, \Gamma \rhd \Delta, \varphi \Rightarrow \psi, \sim\varphi$. By item 10 if Theorem 7.1 we conclude that $\Vdash_n \varphi, \varphi, \Gamma \rhd \Delta, \psi, \psi$ and $\Vdash_n \sim\psi, \sim\psi, \Gamma \rhd \Delta, \sim\varphi, \sim\varphi$. By the induction hypothesis for both claims of Theorem 7.2, we get $\Vdash_n \varphi, \Gamma \rhd \Delta, \psi$ and $\Vdash_n \sim\psi, \Gamma \rhd \Delta, \sim\varphi$, so that one application of $R \Rightarrow$ gives $\Vdash_{n+1} \Gamma \rhd \Delta, \varphi \Rightarrow \psi$.

If the contraction formula φ has the form $\Box\psi$ or $\sim\Box\psi$, inspection of the rules $RL\,\Box$, $LR\sim\Box$, $L\,\Box\sim\Box$, and $R\sim\Box\,\Box$ reveals that the contraction formula cannot be principal in the last derivation step so as to obtain the premise sequent of LC or RC. □

Next we consider (cut).

DEFINITION 7.3. *The* cut-height *of an instance of* (cut) *in a derivation is the sum of the derivation heights of the two premise sequents of that instance.*

THEOREM 7.4 (Cut elimination). *The* (cut) *rule is an admissible rule in* G3KN4.

Proof. The proof method is standard. It is shown that topmost applications of (cut) in derivations can be eliminated, so that successively all applications of (cut) can be elimi-

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nated from derivations. The proof is by induction on the weight of the cut-formula θ and a subinduction on the cut-height and makes use of an exhaustive case distinction.

Case 1. At least one of the premise sequents of (cut) is an axiom. If the left premise sequent $\Gamma \rhd \Delta, \theta$ is an axiom, then (i) the cut-formula θ occurs in Γ or (ii) Γ and Δ share a propositional variable or a negated propositional variable. In Subcase (i), $\Gamma, \Gamma' \rhd \Delta, \Delta'$ is derivable from $\theta, \Gamma' \rhd \Delta'$ by means of W. In Subcase (ii), $\Gamma, \Gamma' \rhd \Delta, \Delta'$ is an axiom. If the right premise sequent $\theta, \Gamma' \rhd \Delta'$ is an axiom, then (i) the cut-formula θ occurs in Δ' or (ii) Γ' and Δ' share a propositional variable or a negated propositional variable. In Subcase (i), $\Gamma, \Gamma' \rhd \Delta, \Delta'$ is derivable from $\Gamma \rhd \Delta, \theta$ by means of the weakening rule W. In Subcase (ii), $\Gamma, \Gamma' \rhd \Delta, \Delta'$ is an axiom.

Case 2. None of the premise sequents of (cut) is an axiom. Below we refer to rules $RL\Box$, $LR{\sim}\Box$, $L\Box{\sim}\Box$, and $R{\sim}\Box\,\Box$ as *modal* and to the rest as *nonmodal*.

Subcase 2.1. The left premise sequent was derived using a *nonmodal rule* and the cut formula on the left is *not principal*. Then every rule that has possibly been used to derive $\Gamma \rhd \Delta, \theta$ gives rise to a further subcase. The cases of the rules $L\wedge$, $R\wedge$, $L\vee$, $R\vee$ are dealt with in Negri & von Plato (2008). The cases of the rules $L{\sim}\wedge$, $R{\sim}\wedge$, $L{\sim}\vee$, $L{\sim}{\sim}$, $L \Rightarrow$, and $L{\sim} \Rightarrow$ are analogous. In each case the order of rule applications is inverted and induction on cut-height is used. We present the case when cut formula was introduced by an application of W. We replace derivation

$$\dfrac{\dfrac{\Gamma \rhd \Delta}{\Gamma, \Gamma' \rhd \theta, \Delta, \Delta'}\, W \qquad \Gamma'', \theta \rhd \Delta''}{\Gamma, \Gamma', \Gamma'' \rhd \Delta, \Delta', \Delta''}\,(\text{cut})$$

with the following application of W

$$\dfrac{\Gamma \rhd \Delta}{\Gamma, \Gamma', \Gamma'' \rhd \Delta, \Delta', \Delta''}\, W.$$

Next, we present the case of rule $L \Rightarrow$: The application of (cut) in

$$\dfrac{d \qquad \theta, \Gamma' \rhd \Delta'}{(\varphi \Rightarrow \psi), \Gamma'', \Gamma' \rhd \Delta, \Delta'}$$

where $d =$

$$\dfrac{\Gamma'' \rhd \Delta, \theta, \varphi, {\sim}\psi \quad {\sim}\varphi, \Gamma'' \rhd \Delta, \theta, \varphi \quad \psi, \Gamma'' \rhd \Delta, \theta, {\sim}\psi \quad {\sim}\varphi, \psi\Gamma'' \rhd \Delta, \theta}{(\varphi \Rightarrow \psi), \Gamma'' \rhd \Delta, \theta}$$

is replaced by four applications of (cut) with smaller cut-height:

$$\dfrac{\Gamma'' \rhd \Delta, \theta, \varphi, {\sim}\psi \qquad \theta, \Gamma' \rhd \Delta'}{\Gamma'', \Gamma' \rhd \Delta, \Delta', \varphi, {\sim}\psi} \qquad \dfrac{{\sim}\varphi, \Gamma'' \rhd \Delta, \theta, \varphi \qquad \theta, \Gamma' \rhd \Delta'}{{\sim}\varphi, \Gamma'', \Gamma' \rhd \Delta, \Delta', \varphi}$$

$$\dfrac{\psi, \Gamma'' \rhd \Delta, \theta, {\sim}\psi \quad \theta, \Gamma' \rhd \Delta'}{\psi, \Gamma'', \Gamma' \rhd \Delta, \Delta', {\sim}\psi} \qquad \dfrac{{\sim}\varphi, \psi, \Gamma'' \rhd \Delta, \theta \quad \theta, \Gamma' \rhd \Delta'}{{\sim}\varphi, \psi, \Gamma'', \Gamma' \rhd \Delta, \Delta'}.$$

One application of $L \Rightarrow$ then derives $(\varphi \Rightarrow \psi), \Gamma'', \Gamma' \rhd \Delta, \Delta'$.

Subcase 2.2. The right premise sequent was derived using a *nonmodal rule* and the cut formula on the right *not principal*. This subcase is analogous to the previous subcase.

Subcase 2.3. Both premises were derived using modal rules and in one of them, the cut formula is *not principal*. Observe that in this case it is impossible that in both premise sequents the cut formula is not principal. Accordingly, we present cases where the cut formula is principal in the right premise sequent, whereas in the left premise sequent

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the cut formula is not principal. Symmetric cases are analogous. $LR{\sim}\Box$ and $\theta = {\sim}\Box\chi$: There are two possibilities for deriving the right premise sequent of (cut) corresponding to applications of the rules $LR{\sim}\Box$ and $L\,\Box{\sim}\Box$. In both cases we invert the order of rule applications and use induction on cut-height. The derivation

$$\dfrac{\dfrac{{\sim}\varphi \rhd {\sim}\chi, {\sim}\Theta}{{\sim}\Box\varphi \rhd {\sim}\Box\chi, {\sim}\Box\Theta}\, LR{\sim}\Box \quad \dfrac{{\sim}\chi \rhd {\sim}\Theta'}{{\sim}\Box\chi \rhd {\sim}\Box\Theta'}\, LR{\sim}\Box}{{\sim}\Box\varphi \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\,(\text{cut})$$

can be replaced by

$$\dfrac{\dfrac{{\sim}\varphi \rhd {\sim}\chi, {\sim}\Theta \quad {\sim}\chi \rhd {\sim}\Theta'}{{\sim}\varphi \rhd {\sim}\Theta, {\sim}\Theta'}\,(\text{cut})}{{\sim}\Box\varphi \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\, LR{\sim}\Box\ .$$

The derivation

$$\dfrac{\dfrac{{\sim}\varphi \rhd {\sim}\chi, {\sim}\Theta}{{\sim}\Box\varphi \rhd {\sim}\Box\chi, {\sim}\Box\Theta}\, LR{\sim}\Box \quad \dfrac{{\sim}\chi, \Lambda \rhd {\sim}\Theta'}{{\sim}\Box\chi, \Box\Lambda \rhd {\sim}\Box\Theta'}\, L\,\Box{\sim}\Box}{{\sim}\Box\varphi, \Box\Lambda \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\,(\text{cut})$$

can be replaced by

$$\dfrac{\dfrac{{\sim}\varphi \rhd {\sim}\chi, {\sim}\Theta \quad {\sim}\chi, \Lambda \rhd {\sim}\Theta'}{{\sim}\varphi, \Lambda \rhd {\sim}\Theta, {\sim}\Theta'}\,(\text{cut})}{{\sim}\Box\varphi, \Box\Lambda \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\, L\,\Box{\sim}\Box$$

$L\Box{\sim}\Box$ and $\theta = {\sim}\Box\chi$: The two cases that must be distinguished are analogous. In the second case, where both premise sequents of (cut) are derived by means of $L\,\Box{\sim}\Box$ we have

$$\dfrac{\dfrac{{\sim}\varphi, \Lambda \rhd {\sim}\chi, {\sim}\Theta}{{\sim}\Box\varphi, \Box\Lambda \rhd {\sim}\Box\chi, {\sim}\Box\Theta}\, L\,\Box{\sim}\Box \quad \dfrac{{\sim}\chi, \Lambda' \rhd {\sim}\Theta'}{{\sim}\Box\chi, \Box\Lambda' \rhd {\sim}\Box\Theta'}\, L\,\Box{\sim}\Box}{{\sim}\Box\varphi, \Box\Lambda, \Box\Lambda' \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\,(\text{cut})\ .$$

This derivation can be replaced by

$$\dfrac{\dfrac{{\sim}\varphi, \Lambda \rhd {\sim}\chi, {\sim}\Theta \quad {\sim}\chi, \Lambda' \rhd {\sim}\Theta'}{{\sim}\varphi, \Lambda, \Lambda' \rhd {\sim}\Theta, {\sim}\Theta'}\,(\text{cut})}{{\sim}\Box\varphi, \Box\Lambda, \Box\Lambda' \rhd {\sim}\Box\Theta, {\sim}\Box\Theta'}\, L\,\Box{\sim}\Box$$

where Θ'' is the multiset union of Γ and Θ. $R{\sim}\Box\,\Box$: The two cases are again analogous.

Subcase 2.4. The cut-formula θ is principal in *both* premise sequents of (cut). Then the application of (cut) under consideration is replaced by one or more applications of (cut) with a cut formula of smaller weight. Note that θ cannot be of the form $\Box\chi$ or ${\sim}\Box\chi$. We present the case of $\theta = \varphi \Rightarrow \psi$ and consider the derivation of $\Gamma, \Gamma' \rhd \Delta, \Delta'$ by (cut) from the following two derivations, where we number the premise sequents as follows:

$$\dfrac{\overset{1}{\varphi, \Gamma \rhd \Delta, \psi} \quad \overset{2}{{\sim}\psi, \Gamma \rhd \Delta, {\sim}\varphi}}{\Gamma \rhd \Delta, (\varphi \Rightarrow \psi)}$$

$$\dfrac{\overset{3}{\Gamma' \rhd \Delta', \varphi, {\sim}\psi} \quad \overset{4}{{\sim}\varphi, \Gamma' \rhd \Delta,' \varphi} \quad \overset{5}{\psi, \Gamma' \rhd \Delta', {\sim}\psi} \quad \overset{6}{{\sim}\varphi, \psi, \Gamma' \rhd \Delta'}}{(\varphi \Rightarrow \psi), \Gamma' \rhd \Delta'}\ .$$

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We first present a derivation d of $\Gamma, \Gamma' \rhd \Delta, \Delta', \sim\varphi$:

$$\dfrac{\dfrac{\dfrac{\overset{3}{\Gamma' \rhd \Delta', \varphi, \sim\psi} \qquad \dfrac{\overset{1}{\varphi, \Gamma \rhd \Delta, \psi} \quad \overset{5}{\psi, \Gamma' \rhd \Delta', \sim\psi}}{\varphi, \Gamma, \Gamma' \rhd \Delta, \Delta', \sim\psi}\,(\mathrm{cut})}{\Gamma, \Gamma', \Gamma' \rhd \Delta, \Delta', \Delta', \sim\psi, \sim\psi}\,(\mathrm{cut})}{\Gamma, \Gamma', \Gamma' \rhd \Delta, \Delta', \Delta', \sim\psi}\,RC \qquad \overset{2}{\sim\psi, \Gamma \rhd \Delta, \sim\varphi}}{\Gamma, \Gamma, \Gamma', \Gamma' \rhd \Delta, \Delta, \Delta', \Delta', \sim\varphi}\,(\mathrm{cut})}{\vdots}$$
$$\Gamma, \Gamma' \rhd \Delta, \Delta', \sim\varphi$$

where here and in the following the dots indicate repeated applications of *LC* and *RC*. From d we proceed as follows to obtain a derivation d' of $\Gamma, \Gamma' \rhd \Delta, \Delta', \psi$:

$$\dfrac{\dfrac{d \qquad \overset{4}{\sim\varphi, \Gamma' \rhd \Delta', \varphi}}{\Gamma, \Gamma', \Gamma' \rhd \Delta, \Delta', \Delta', \varphi}\,(\mathrm{cut}) \qquad \overset{1}{\varphi, \Gamma \rhd \Delta, \psi}}{\Gamma, \Gamma, \Gamma', \Gamma' \rhd \Delta, \Delta, \Delta', \Delta', \psi}\,(\mathrm{cut})$$
$$\vdots$$
$$\Gamma, \Gamma' \rhd \Delta, \Delta', \psi \qquad .$$

Now we can make use of the so far unused sixth premise sequent:

$$\dfrac{\dfrac{d' \qquad \dfrac{d \qquad \overset{6}{\sim\varphi, \psi, \Gamma' \rhd \Delta'}}{\Gamma, \Gamma', \Gamma', \psi \rhd \Delta, \Delta', \Delta'}\,(\mathrm{cut})}{\Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \rhd \Delta, \Delta, \Delta', \Delta', \Delta'}\,(\mathrm{cut})}{\vdots}$$
$$\Gamma, \Gamma' \rhd \Delta, \Delta' \qquad . \qquad \square$$

Next we show that G3KN4 is a sequent calculus for KN4. If Γ and Δ are multisets of $\mathcal{L}^{\mathsf{KN4}}$-formulas and Γ', Δ' are the sets obtained from Γ and Δ by disregarding multiple occurrences of elements, we define $\Gamma \models_{\mathsf{KN4}} \Delta$ to mean that $\Gamma' \models_{\mathsf{KN4}} \Delta'$. Moreover, for finite Γ, $\bigwedge \Gamma$ ($\bigvee \Gamma$) is defined as the conjunction (disjunction) of the elements of Γ, where $\bigwedge \varnothing$ is defined as $p \Rightarrow p$ for some $p \in \mathsf{Prop}$.[7]

THEOREM 7.5. *For any finite multisets of $\mathcal{L}^{\mathsf{KN4}}$-formulas Γ, Δ with $\Delta \neq \varnothing$:*

$$\Vdash_{\mathsf{KN4}} \Gamma \rhd \Delta \iff \Gamma \models_{\mathsf{KN4}} \Delta.$$

Proof. Soundness, i.e., $\Vdash_{\mathsf{KN4}} \Gamma \rhd \Delta$ only if $\Gamma \models_{\mathsf{KN4}} \Delta$, can be shown by induction on the height of derivations in G3KN4. If the height is 0, then $\Gamma \rhd \Delta$ is an axiom $p, \Gamma' \rhd \Delta', p$ or $\sim p, \Gamma' \rhd \Delta', \sim p$, and obviously $\Gamma \models_{\mathsf{KN4}} \Delta$. If one of the logical rules is applied to derivations of height $\leq n$ to obtain a derivation of height $n+1$, one may just use the induction hypothesis and the definition of the relations $\models^+$ and $\models^-$ in KN4-models. We present the cases of the rules $L\Rightarrow$, $L\,\Box\sim\Box$, and $R\sim\Box\,\Box$.

$L\Rightarrow$: Suppose that (i) $\Gamma \models_{\mathsf{KN4}} \Delta, \varphi, \sim\psi$, (ii) $\sim\varphi, \Gamma \models_{\mathsf{KN4}} \Delta, \varphi$, (iii) $\psi, \Gamma \models_{\mathsf{KN4}} \Delta, \sim\psi$, (iv) $\sim\varphi, \psi, \Gamma \models_{\mathsf{KN4}} \Delta$, and suppose there is a KN4-model $\mu = \langle W, \leq, v^+, v^- \rangle$

[7] Once again, we cannot define $\bigvee \varnothing$ as there are no counter-theorems in KN4.

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and $x \in W$ such that (v) $\mu, x \models^+ (\varphi \Rightarrow \psi)$ and (vi) $\mu, x \models^+ \bigwedge \Gamma$. By (v), there are four jointly exhaustive cases: (a) $\mu, x \not\models^+ \varphi$ and $\mu, x \not\models^+ {\sim}\psi$, (b) $\mu, x \models^+ {\sim}\varphi$ and $\mu, x \not\models^+ \varphi$, (c) $\mu, x \models^+ \psi$ and $\mu, x \not\models^+ {\sim}\psi$, and (d) $\mu, x \models^+ {\sim}\varphi$ and $\mu, x \models^+ \psi$. If (a) holds, then by (i) and (vi), $\mu, x \models^+ \bigvee \Delta$; if (b) holds, then by (ii) and (vi), $\mu, x \models^+ \bigvee \Delta$; if (c) is the case, then by (iii) and (vi), $\mu, x \models^+ \bigvee \Delta$; if (d) holds, then by (iv) and (vi), we conclude that $\mu, x \models^+ \bigvee \Delta$.

$L\Box{\sim}\Box$: Suppose that ${\sim}\varphi, \Gamma \models_{\mathsf{KN4}} {\sim}\Delta$, and let there be a $\mathsf{KN4}$-model $\mu = \langle W, \leq, v^+, v^- \rangle$ and $x \in W$ such that $\mu, x \models^+ \bigwedge \Box\Gamma$ and $\mu, x \models^+ {\sim}\Box\varphi$. Then x is not a dead end, i.e., there is a $y \in W$ with $x \leq y$ and $\mu, y \models^+ {\sim}\varphi$. Since $\mu, x \models^+ \bigwedge \Box\Gamma$, for every $\psi \in \Gamma$, $\mu, y \models^+ \psi$. From ${\sim}\varphi, \Gamma \models_{\mathsf{KN4}} {\sim}\Delta$ it follows that $\mu, y \models^+ {\sim}\theta$ for at least one $\theta \in \Delta$, and thus $\mu, x \models^+ \bigvee {\sim}\Box\Delta$.

$R{\sim}\Box\,\Box$: Suppose that $\Gamma \models_{\mathsf{KN4}} \varphi, {\sim}\Delta$, and let there be a $\mathsf{KN4}$-model $\mu = \langle W, \leq, v^+, v^- \rangle$ and $x \in W$ such that $\mu, x \models^+ \bigwedge \Box\Gamma$. If x is a dead end, then $\mu, x \models^+ \Box\varphi$ and hence $\mu, x \models^+ \Box\varphi \vee \bigvee {\sim}\Box\Delta$. Thus, suppose that x is not a dead end. Then for every $y \in W$, if $x \leq y$, then $\mu, y \models^+ \bigwedge \Gamma$ and, by $\Gamma \models_{\mathsf{KN4}} \varphi, {\sim}\Delta$, $\mu, y \models^+ \varphi \vee \bigvee {\sim}\Delta$. Suppose for reductio that $\mu, x \not\models^+ \Box\varphi \vee \bigvee {\sim}\Box\Delta$. Then (i) $\mu, x \not\models^+ \Box\varphi$ and (ii) $\mu, x \not\models^+ \bigvee {\sim}\Box\Delta$. By (i), there exists $z \in W$ with $x \leq z$ and $\mu, z \not\models^+ \varphi$. Since $\mu, z \models^+ \varphi \vee \bigvee {\sim}\Delta$, $\mu, z \models^+ \bigvee {\sim}\Delta$. Therefore there exists $\theta \in \Delta$ with $\mu, z \models^+ {\sim}\theta$, i.e., $\mu, z \models^- \theta$. However, by (ii), for every $\theta \in \Delta$, $\mu, x \not\models^+ {\sim}\Box\theta$, whence for every $\theta \in \Delta$ and every $z \in W$ with $x \leq z$, $\mu, z \not\models^- \theta$; a contradiction.

To show completeness, i.e., $\Vdash_{\mathsf{KN4}} \Gamma \rhd \Delta$ if $\Gamma \models_{\mathsf{KN4}} \Delta$, we can use the fact that $\Gamma \models_{\mathsf{KN4}} \Delta$ iff $\Gamma \vdash_{\mathsf{KN4}} \Delta$ and we show that $\Gamma \vdash_{\mathsf{KN4}} \Delta$ only if $\Vdash_{\mathsf{KN4}} \Gamma \rhd \Delta$. The proof is thus by induction on the height of derivations in $\mathsf{KN4}_{fde}$, where the height of a derivation in $\mathsf{KN4}_{fde}$ is defined in analogy to the definition of derivation height in $\mathsf{G3KN4}$. We first consider the derivations of height 0. Clearly, for all eight FDE axioms ai_{fde} $(1 \leq i \leq 8)$ of the form $\theta \vdash \chi$ we have $\Vdash_{\mathsf{KN4}} \theta \rhd \chi$. The derivation of $a7_{\Rightarrow}$ is equally straightforward:

$$\dfrac{\dfrac{\varphi, {\sim}\psi \rhd \varphi \quad \varphi, {\sim}\psi \rhd {\sim}\psi}{\varphi{\sim}\psi \rhd (\varphi \wedge {\sim}\psi)}}{{\sim}(\varphi \Rightarrow \psi) \rhd (\varphi \wedge {\sim}\psi)} \qquad \dfrac{\dfrac{\varphi, {\sim}\psi \rhd \varphi}{(\varphi \wedge {\sim}\psi) \rhd \varphi} \quad \dfrac{\varphi, {\sim}\psi \rhd {\sim}\psi}{(\varphi \wedge {\sim}\psi) \rhd {\sim}\psi}}{(\varphi \wedge {\sim}\psi) \rhd {\sim}(\varphi \Rightarrow \psi)}\,.$$

Derivations of some of the remaining axioms of strong implication and the modal axioms of $\mathsf{KN4}_{fde}$ are given in the Appendix 9. If one of the rules of $\mathsf{KN4}_{fde}$ is applied to derivations of height $\leq n$ to obtain a derivation of height $n+1$, we use the induction hypothesis, logical rules of $\mathsf{G3KN4}$, and (cut) as an admissible rule. For rule $\mathrm{r1}_{fde}$ suppose that $\Vdash_{\mathsf{KN4}} \varphi \rhd \psi$ and $\Vdash_{\mathsf{KN4}} \psi \rhd \chi$. An application of (cut) delivers $\Vdash_{\mathsf{KN4}} \varphi \rhd \chi$. For the rules $\mathrm{r2}_{fde}$, $\mathrm{r3}_{fde}$, $\mathrm{r4}_{\Box}$, and $\mathrm{r5}_{\Box}$, use the rules $L\vee$, $R\wedge$, $RL\Box$, and $LR{\sim}\Box$, respectively. □

DEFINITION 7.6. *We define the weight $sw(s)$ of a sequent s, and first inductively define the sequent weight $sw(\varphi)$ of a formula φ*:

1. $sw(p) = 1$, *for* $p \in \mathsf{Prop}$,
2. $sw(\sharp\varphi) = sw(\varphi) + 1$, *for* $\sharp \in \{\Box, {\sim}\}$,
3. $sw(\varphi \sharp \psi) = sw(\varphi) \cdot 2 + sw(\psi) \cdot 2$, *for* $\sharp \in \{\Rightarrow, \wedge, \vee\}$.

If Γ, Δ are nonempty multisets of $\mathcal{L}^{\mathsf{KN4}}$-formulas, then $sw(\Gamma)$ is the sum of the sequent weights $sw(\varphi)$ with $\varphi \in \Gamma$, and $sw(\Gamma \rhd \Delta) = sw(\Gamma) + sw(\Delta)$.

Then, in particular, $sw(\Gamma \rhd \Delta, \varphi \Rightarrow \psi)$ is greater than the weights of the premise sequents of $R\Rightarrow$, i.e., $\varphi, \Gamma \rhd \Delta, \psi$ and ${\sim}\psi, \Gamma \rhd \Delta, {\sim}\varphi$.

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THEOREM 7.7. KN4 *is decidable.*

Proof. Inspection of the rules of G3KN4 reveals that root-first proof search terminates as it always leads to sequents of smaller sequent weight. □

§8. Conclusion. We presented the axiom systems $\mathsf{HBK}^{\Box-}$ and $\mathsf{H'BK}^{\Box-}$ for the paraconsistent modal logic $\mathsf{BK}^{\Box-}$. Moreover, we have shown that the system BK is an extension of the Fischer Servi-style logic $\mathsf{BK}^{\mathsf{FS}}$ by the axiom schema $\Box{\sim}p \leftrightarrow {\sim}\Diamond p$. Since the language $\mathcal{L}^{\Box-}$ of $\mathsf{HBK}^{\Box-}$ differs from the language $\mathcal{L}^{\mathsf{KN4}}$ of KN4 only with respect to the implication used, one might think that $\mathsf{HBK}^{\Box-}$ or $\mathsf{H'BK}^{\Box-}$ could form a suitable basis for obtaining a nice Hilbert-style axiomatization of KN4. However, the situation is not so simple, and we instead presented the FDE-style proof system $\mathsf{KN4}_{fde}$ and the sequent calculus G3KN4.

There are now four different kinds of proof systems for KN4 in the literature. Goble's (2006) axiom system is practically unusable because it comprises an infinite set of rules. The tableau calculi from Odintsov & Wansing (2017) and McGinnis (2006) are fine,[8] but they are systems of *semantic* tableaux that directly build on the relational semantics of KN4. The system $\mathsf{KN4}_{fde}$ of the present article has a finite set of rules, but it is still not extremely user-friendly because it contains several schematic axioms. The system $\mathsf{KN4}_{fde}$ has been instrumental in obtaining the sequent calculus G3KN4. The system G3KN4 is a cut-free standard sequent calculus and is user-friendly insofar as it permits root-first proof search, however, with the caveat that it makes use of a sequent rule with four premise sequents.

§9. Appendix A. Deriving the $\mathsf{KN4}_{fde}$ axioms of strong implication $a1_{\Rightarrow}, \ldots, a6_{\Rightarrow}$ in G3KN4 is simple. By way of example, we here present only derivations of $a2_{\Rightarrow}$, $a4_{\Rightarrow}$, and $a5_{\Rightarrow}$. Axiom $a2_{\Rightarrow}$:

$$\dfrac{\dfrac{d}{(\varphi \Rightarrow \psi) \rhd (\sim\psi \Rightarrow \sim\varphi)} \qquad \dfrac{\dfrac{\dfrac{\sim\psi, \varphi \rhd \varphi \qquad \sim\psi, \varphi \rhd \sim\psi}{\sim\psi, \varphi \rhd \sim(\varphi \Rightarrow \psi)}}{\sim\psi, \sim\sim\varphi \rhd \sim(\varphi \Rightarrow \psi)}}{\sim(\sim\psi \Rightarrow \sim\varphi) \rhd \sim(\varphi \Rightarrow \psi)}}{\dfrac{\rhd(\varphi \Rightarrow \psi) \Rightarrow (\sim\psi \Rightarrow \sim\varphi)}{\chi \rhd (\varphi \Rightarrow \psi) \Rightarrow (\sim\psi \Rightarrow \sim\varphi)}}\, R\Rightarrow$$

where $d =$

$$\dfrac{\sim\psi \rhd \sim\varphi, \varphi, \sim\psi \quad \sim\varphi, \sim\psi \rhd \sim\varphi, \varphi \quad \psi, \sim\psi \rhd \sim\varphi, \sim\psi \quad \sim\varphi, \psi, \sim\psi \rhd \sim\varphi}{\sim\psi, \varphi \Rightarrow \psi \rhd \sim\varphi}$$

Axiom $a4_{\Rightarrow}$:

$$\dfrac{\dfrac{\dfrac{\psi, \varphi \rhd \sim\psi, \psi \qquad \sim\psi, \psi \rhd \sim\psi, \sim\varphi}{\psi \rhd \sim\psi, (\varphi \Rightarrow \psi))}}{\psi \rhd (\sim\psi \vee (\varphi \Rightarrow \psi))} \qquad \dfrac{\dfrac{\sim\sim\psi, \varphi, \sim\psi \rhd \sim\psi}{\sim\sim\psi, \sim(\varphi \Rightarrow \psi) \rhd \sim\psi}}{\sim(\sim\psi \vee (\varphi \Rightarrow \psi)) \rhd \sim\psi}}{\dfrac{\rhd \psi \Rightarrow (\sim\psi \vee (\varphi \Rightarrow \psi))}{\chi \rhd \psi \Rightarrow (\sim\psi \vee (\varphi \Rightarrow \psi))}}$$

[8] The authors of Odintsov & Wansing (2017) were not aware of McGinnis (2006) when they wrote their article.

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Axiom $a5_{\Rightarrow}$:

$$\dfrac{\dfrac{\dfrac{\varphi, \sim\varphi \rhd \varphi, \psi \qquad \sim\psi, \sim\varphi \rhd \varphi, \sim\varphi}{\sim\varphi \rhd \varphi, (\varphi \Rightarrow \psi)}}{\sim\varphi \rhd (\varphi \vee (\varphi \Rightarrow \psi))} \qquad \dfrac{\dfrac{\dfrac{\sim\varphi, \varphi, \sim\psi \rhd \varphi}{\sim\varphi, \varphi, \sim\psi \rhd \sim\sim\varphi}}{\sim\varphi, \sim(\varphi \Rightarrow \psi) \rhd \sim\sim\varphi}}{\sim(\varphi \vee (\varphi \Rightarrow \psi)) \rhd \sim\sim\varphi}}{\dfrac{\rhd \sim\varphi \Rightarrow (\varphi \vee (\varphi \Rightarrow \psi))}{\chi \rhd \sim\varphi \Rightarrow (\varphi \vee (\varphi \Rightarrow \psi))}} \ .$$

The modal axioms $a1_{\Box}$, $a2_{\Box}$, and $a3_{\Box}$ of $\mathsf{KN4}_{fde}$ can be derived as follows, where we derive the two conjuncts of $a2_{\Box}$ separately:
Axiom $a1_{\Box}$:

$$\dfrac{\dfrac{\dfrac{\varphi \rhd \varphi \quad \sim\varphi \rhd \sim\varphi}{\rhd \varphi \Rightarrow \varphi}}{\rhd \Box(\varphi \Rightarrow \varphi)}}{\chi \rhd \Box(\varphi \Rightarrow \varphi)}$$

Axiom $a2_{\Box}$, one conjunct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi, \psi \rhd \varphi \quad \varphi, \psi \rhd \psi}{\varphi, \psi \rhd (\varphi \wedge \psi)}}{\Box\varphi, \Box\psi \rhd \Box(\varphi \wedge \psi)}}{(\Box\varphi \wedge \Box\psi) \rhd \Box(\varphi \wedge \psi)} \qquad \dfrac{\dfrac{\dfrac{\sim\varphi \rhd \sim\varphi, \sim\psi \quad \sim\psi \rhd \sim\varphi, \sim\psi}{\sim(\varphi \wedge \psi) \rhd \sim\varphi, \sim\psi}}{\sim\Box(\varphi \wedge \psi) \rhd \sim\Box\varphi, \sim\Box\psi}}{\sim\Box(\varphi \wedge \psi) \rhd \sim(\Box\varphi \wedge \Box\psi)}}{\dfrac{\rhd(\Box\varphi \wedge \Box\psi) \Rightarrow \Box(\varphi \wedge \psi)}{\chi \rhd (\Box\varphi \wedge \Box\psi) \Rightarrow \Box(\varphi \wedge \psi)}}$$

Axiom $a2_{\Box}$, the other conjunct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi, \psi \rhd \varphi}{(\varphi \wedge \psi) \rhd \varphi}}{\Box(\varphi \wedge \psi) \rhd \Box\varphi} \quad \dfrac{\dfrac{\varphi, \psi \rhd \psi}{(\varphi \wedge \psi) \rhd \psi}}{\Box(\varphi \wedge \psi) \rhd \Box\psi}}{\Box(\varphi \wedge \psi) \rhd (\Box\varphi \wedge \Box\psi)} \qquad \dfrac{\dfrac{\dfrac{\sim\varphi \rhd \sim\varphi, \sim\psi}{\sim\varphi \rhd \sim(\varphi \wedge \psi)}}{\sim\Box\varphi \rhd \sim\Box(\varphi \wedge \psi)} \quad \dfrac{\dfrac{\sim\psi \rhd \sim\varphi, \sim\psi}{\sim\psi \rhd \sim(\varphi \wedge \psi)}}{\sim\Box\psi \rhd \sim\Box(\varphi \wedge \psi)}}{\sim(\Box\varphi \wedge \Box\psi) \rhd \sim\Box(\varphi \wedge \psi)}}{\dfrac{\rhd \Box(\varphi \wedge \psi) \Rightarrow (\Box\varphi \wedge \Box\psi)}{\chi \rhd \Box(\varphi \wedge \psi) \Rightarrow (\Box\varphi \wedge \Box\psi)}} \ .$$

One application of $R\wedge$ then delivers a derivation of axiom $a2_{\Box}$.
Axiom $a3_{\Box}$:

$$\dfrac{\dfrac{\dfrac{\varphi \rhd \varphi, \sim\psi \quad \sim\psi \rhd \varphi, \sim\psi}{(\varphi \vee \sim\psi) \rhd \varphi, \sim\psi}}{\Box(\varphi \vee \sim\psi) \rhd \Box\varphi, \sim\Box\psi}\, R\sim\Box\Box}{\Box(\varphi \vee \sim\psi) \rhd (\Box\varphi \vee \sim\Box\psi)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\sim\varphi, \psi \rhd \psi}{\sim\varphi, \psi \rhd \sim\sim\psi}}{\sim\varphi, \psi \rhd \sim\varphi \quad \sim\varphi, \psi \rhd \sim\sim\psi}}{\sim\varphi, \psi \rhd \sim(\varphi \vee \sim\psi)}}{\sim\Box\varphi, \Box\psi \rhd \sim\Box(\varphi \vee \sim\psi)}\, L\Box\sim\Box}{\sim\Box\varphi, \sim\sim\Box\psi \rhd \sim\Box(\varphi \vee \sim\psi)} \Big}{\sim(\Box\varphi \vee \sim\Box\psi) \rhd \sim\Box(\varphi \vee \sim\psi)}}{\dfrac{\rhd \Box(\varphi \vee \sim\psi) \Rightarrow (\Box\varphi \vee \sim\Box\psi)}{\chi \rhd \Box(\varphi \vee \sim\psi) \Rightarrow (\Box\varphi \vee \sim\Box\psi)}} \ .$$

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