RANDOM MOTION AND ANALYTIC CONTINUED FRACTIONS

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In a previous note (Good (7)) it was shown that there is an intimate connexion between Legendre polynomials and 'trinomial' random walks, that is walks on a one-dimensional lattice with the same probabilities p_{-1}, p_0, p_1 at each point of taking a step of $-1, 0, 1$. The purpose of the present note is to point out that if the probabilities depend on the current position of the particle, then there is scope for applications of the theory of analytic continued fractions.

Consider then the random motion of a particle on the points $0, \pm 1, \pm 2, \ldots$, such that when the particle is at the point n it has probabilities $p_{n,-1}$, $p_{n,0}$, $p_{n,1}$ of taking steps of $-1, 0, 1$ ($n = 0, \pm 1, \pm 2, \ldots$; $p_{n,-1}+p_{n,0}+p_{n,1} = 1$). These probabilities do not depend on the previous history of the walk.

Let $c_{r,s}$ be the probability that, starting at the point r, the particle arrives again at r for the first time at step number s. Let $c^R_{r,s}$ (and $c^L_{r,s}$) be the corresponding probability subject to the further condition that in the meantime the particle has never been to the left (right) of the point r. We adopt the natural conventions

$$c_{r,0} = c^R_{r,0} = c^L_{r,0} = 0.$$

Let $d_{r,s}$ be the probability that, starting at the point r, the particle will arrive again at r on the sth step, not necessarily for the first time. Let $d^R_{r,s}$ (and $d^L_{r,s}$) be the corresponding probability subject to the further condition that in the meantime the particle has never been to the left (right) of the point r. We adopt the natural conventions

$$d_{r,0} = d^R_{r,0} = d^L_{r,0} = 1.$$

Define the six generating functions $C_r(z)$, $C^R_r(z)$, $C^L_r(z)$, $D_r(z)$, $D^R_r(z)$, $D^L_r(z)$ ($r = 0, \pm 1, \pm 2, \ldots$), where, for example,

$$C^R_r(z) = \sum_{s=1}^{\infty} c^R_{r,s} z^s, \quad D_r(z) = \sum_{s=0}^{\infty} d_{r,s} z^s.$$

We also write $C_0(z)$ as $C(z)$, etc.

Now we see readily that

$$c^R_{r,1} = p_{r,0},$$

$$c^R_{r,s} = p_{r,1} p_{r+1,-1} d^R_{r+1,s-2} \quad (s \geqslant 2). \tag{1}$$

Therefore

$$C^R_r(z) = p_{r,0} z + p_{r,1} p_{r+1,-1} z^2 D^R_{r+1}(z). \tag{2}$$

But $D^R_r(z)$ is the sum for $k = 0, 1, 2, \ldots$ of the generating functions corresponding to returns for the zeroth, first, second, ... times, so that

$$D^R_r(z) = \sum_{k=0}^{\infty} (C^R_r(z))^k = \frac{1}{1 - C^R_r(z)}. \tag{3}$$

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(Compare Kac (10), Feller (3), Good (7) and especially Harris (8), § 5.) Thus

$$D_r^R(z) = \frac{1}{1-p_{r,0}z-p_{r,1}p_{r+1,-1}z^2D_{r+1}^R(z)}, \tag{4}$$

$$D^R(z) = \cfrac{1}{1-p_{0,0}z-\cfrac{p_{0,1}p_{1,-1}z^2}{1-p_{1,0}z-\cfrac{p_{1,1}p_{2,-1}z^2}{1-p_{2,0}z-\dots}}}, \tag{5}$$

at least formally. In fact the continued fraction is uniformly convergent when $|z|$ is sufficiently small, provided that $\limsup p_{r,0} < 1$. This statement follows from p. 50 of Wall (13). Therefore ((13), Theorem 54·1) the corresponding power series expansion is also uniformly convergent for sufficiently small $|z|$.

The remaining generating functions can now be determined in principle from the equations

$$D^L(z) = \cfrac{1}{1-p_{0,0}z-\cfrac{p_{0,-1}p_{-1,1}z^2}{1-p_{-1,0}z-\cfrac{p_{-1,-1}p_{-2,1}z^2}{1-p_{-2,0}z-\dots}}}, \tag{6}$$

$$C^R(z) = 1-1/D^R(z) \tag{7}$$

$$= p_{0,0}z+\cfrac{p_{0,1}p_{1,-1}z^2}{1-p_{1,0}z-\cfrac{p_{1,1}p_{2,-1}z^2}{1-p_{2,0}z-\dots}}, \tag{8}$$

$$C^L(z) = 1-1/D^L(z), \tag{9}$$

$$C(z) = C^R(z)+C^L(z)-p_{0,0}z, \tag{10}$$

$$D(z) = \frac{1}{1-C(z)}. \tag{11}$$

The probability of ultimate return to the origin if $p_{0,1} = 1$ is

$$C^R(1) = \cfrac{p_{1,-1}}{p_{1,-1}+p_{1,1}-\cfrac{p_{1,1}p_{2,-1}}{p_{2,-1}+p_{2,1}-\cfrac{p_{2,1}p_{3,-1}}{p_{3,-1}+p_{3,1}-\dots}}},$$

and this, by a formula of Euler's (see, for example, Euler (2), or Pringsheim (11), p. 134), is equal to $S/(1+S)$, where

$$S = \frac{p_{1,-1}}{p_{1,1}}+\frac{p_{1,-1}p_{2,-1}}{p_{1,1}p_{2,1}}+\frac{p_{1,-1}p_{2,-1}p_{3,-1}}{p_{1,1}p_{2,1}p_{3,1}}+\dots.$$

This formula agrees with § 4 of Harris (8), who used an entirely different method of proof. (He assumed $p_{r,0} = 0$ but this was perhaps inessential.)

Example (i). Take

$$p_{r,0} = 0 \quad (r = 0, 1, 2, \dots),$$

$$p_{2r,1} = \frac{\alpha+r}{\gamma+2r} \quad (r = 0, 1, 2, \dots),$$

$$p_{2r+1,1} = \frac{\beta+r+1}{\gamma+2r+1} \quad (r = 0, 1, 2, \dots),$$

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where $\gamma \geqslant \alpha \geqslant 0$, $\gamma \geqslant \beta \geqslant -1$. (If $\beta = \alpha - \frac{1}{2}$, we have $p_{r,1} = (\alpha + \frac{1}{2}r)/(\gamma + r)$, $r = 0, 1, 2, \ldots$.) The origin may be regarded as a point of repulsion. We find that $D_r^R(z)$ is equal to Gauss's continued fraction (see, for example, Gauss (5); Uspensky (12), p. 53; Wall (13), p. 339) which is expressible as a ratio of two hypergeometric functions:

$$D^R(z) = \frac{F(\alpha, \beta+1, \gamma+1, z^2)}{F(\alpha, \beta, \gamma, z^2)}. \tag{12}$$

(Compare Gillis (6), p. 144.) By equating coefficients of powers of z in the equation

$$D^R(z) . F(\alpha, \beta, \gamma, z^2) = F(\alpha, \beta+1, \gamma+1, z^2)$$

we could write down a recurrence relation for $d^R_{0,s}$ ($s = 0, 1, 2, \ldots$).

Example (ii). In example (i) we can allow all of α, β and γ to be negative, say,

$$\alpha = -n - \delta_1, \quad \beta = -n - \tfrac{1}{2} - \delta_2, \quad \gamma = -2n - \delta_3,$$

where n is a non-negative integer and

$$0 \leqslant \delta_1 \leqslant \delta_3 < 1, \quad 0 \leqslant \delta_2 \leqslant \tfrac{1}{2}, \quad 0 < \delta_3 \leqslant \tfrac{1}{2} + \delta_2.$$

Then the sequence $\{p_{r,1}\}$ ($r = 0, 1, 2, \ldots$) is

$$\frac{n+\delta_1}{2n+\delta_3}, \quad \frac{n-\frac{1}{2}+\delta_2}{2n-1+\delta_3}, \quad \ldots, \quad \frac{\frac{3}{2}+\delta_2}{3+\delta_3}, \quad \frac{1+\delta_1}{2+\delta_3}, \quad \frac{\frac{1}{2}+\delta_2}{1+\delta_3}, \quad \frac{\delta_1}{\delta_3}, \quad \frac{\frac{1}{2}-\delta_2}{1-\delta_3}, \quad \ldots.$$

There is a peculiar region in the neighbourhood of $r = n$. By suitable choice of $\delta_1, \delta_2, \delta_3$ this region can be made into a barrier (a sort of potential barrier—Joos (9), index) that is difficult or impossible to penetrate from the left (δ_1/δ_3 small or zero), or from the right (δ_1/δ_3 nearly or exactly equal to 1), or from either side (δ_1 and δ_2 small, $\delta_3 = \frac{1}{2}$). The case of elastic motion, $p_{r,1} = \frac{1}{2}(1 - r/r_0)$ where r_0 is a positive integer, is not covered by our example. (See Wang and Uhlenbeck (14), p. 327, and Kac (10), p. 379.)

Example (iii). In addition to the assumptions of example (i) or (ii) assume that

$$p_{-2r,-1} = \frac{\alpha' + r}{\gamma' + 2r} \quad (r = 0, 1, 2, \ldots),$$

$$p_{-2r-1,-1} = \frac{\beta' + r + 1}{\gamma' + 2r + 1} \quad (r = 0, 1, 2, \ldots),$$

where α', β', γ' satisfy the conditions laid down for α, β, γ, and in addition, in order to have consistency at the origin,

$$\frac{\alpha}{\gamma} + \frac{\alpha'}{\gamma'} = 1.$$

Then we can write down $D^L(z)$ as before and can then determine $C^R(z)$, $C^L(z)$, $C(z)$ and $D(z)$ from equations (8), (9), (10) and (11). Recurrence relations can then be written down for $d^R_{0,s}$, $d^L_{0,s}$, $c^R_{0,s}$, $c^L_{0,s}$, $c_{0,s}$, $d_{0,s}$. The same method can be applied to get $D^R_r(z), \ldots, D_r(z), d^R_{r,s}, \ldots, d_{r,s}$.

If $\gamma > \alpha + \beta + n$, $\gamma' > \alpha' + \beta' + n$, then the first n moments of the time taken for the first return to the origin can be expressed in terms of a finite number of gamma functions. This can be done with the aid of Gauss's formula for $F(a, b, c, 1)$. Moments not so expressible will I think be infinite.

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Example (iv). In example (iii) take $\gamma = 2\alpha$, $\beta = \alpha - \frac{1}{2}$, $\gamma' = 2\alpha'$, $\beta' = \alpha' - \frac{1}{2}$. Then $p_{r,-1} = p_{r,1} = \frac{1}{2}$ for all r and we have the classical random walk for which

$$D(z) = (1-z^2)^{-\frac{1}{2}}.$$

(See, for example, Feller (4), Chapter 12.) Therefore, by our equations (11), (10), (9) and (7), together with $D^R(z) = D^L(z)$, we have that

$$D^R(z) = 2z^{-2}(1-(1-z^2)^{\frac{1}{2}}),$$

and

$$(1-z^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}z^2 \cdot \frac{F(\alpha, \alpha+\frac{1}{2}, 2\alpha+1, z^2)}{F(\alpha, \alpha-\frac{1}{2}, 2\alpha, z^2)}.$$

This relation can be checked by means of the formula

$$F(a-\tfrac{1}{2}, a, 2a, z^2) = \left[\frac{1+(1-z^2)^{\frac{1}{2}}}{2}\right]^{1-2a}$$

(Bateman Project Staff (1), § 2·8, (6)) by putting $a = \alpha + \frac{1}{2}$ and $a = \alpha$ in turn and taking the quotient.

Example (v). Take $p_{r,-1} = p_{-1}$, $p_{r,0} = p_0$, $p_{r,1} = p_1$ whereupon we have the trinomial random walk. Equation (4) gives

$$D^R(z) = \frac{1-p_0 z - (1-2p_0 z + D^2 z^2)^{\frac{1}{2}}}{2p_{-1}p_1 z^2},$$

where $D^2 = p_0^2 - 4p_{-1}p_1$, and therefore, from (3) and (10),

$$C(z) = 1 - (1-2p_0 z + D^2 z^2)^{\frac{1}{2}},$$

as in Good (7).

Example (vi). In example (i) take $\alpha = \gamma$. Then $p_{0,1} = 1$, so that our motion is restricted (almost certainly) to the points $0, 1, 2, \ldots$. Then

$$D(z) = D^R(z) = \frac{F(\alpha, \beta+1, \alpha+1, z^2)}{F(\alpha, \beta, \alpha, z^2)}$$
$$= (1-z^2)^{\beta} F(\alpha, \beta+1, \alpha+1, z^2).$$

Therefore

$$d_{0,2s+1} = 0 \quad (s = 0, 1, 2, \ldots),$$

$$d_{0,2s} = \sum_{\nu=0}^{s} (-1)^{\nu} \binom{\beta}{\nu} \binom{\beta+s-\nu}{\beta} \frac{\alpha}{\alpha+s-\nu}.$$

(The sum of the series on the right must be non-negative if $\alpha > \beta$, an inequality that is not obvious directly.) If $\beta = 0$,

$$d_{0,2s} = \frac{\alpha}{\alpha+s}.$$

Example (vii). Take $\beta = 0$ in example (i). Then

$$D^R(z) = F(\alpha, 1, \gamma+1, z^2)$$

and

$$d^R_{0,2s} = \frac{\alpha(\alpha+1)\ldots(\alpha+s-1)}{(\gamma+1)(\gamma+2)\ldots(\gamma+s)}.$$

Example (viii). Take $\alpha = \gamma = \frac{1}{2}$ and $\beta = 0$ in example (vi). We get

$$D(z) = F(\tfrac{1}{2}, 1, \tfrac{3}{2}, z^2) = \frac{1}{2z}\log\frac{1+z}{1-z}.$$

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Thus if $$p_{r,1} = \frac{2r+1}{4r+1} \quad (r = 0, 1, 2, \ldots),$$

then $$d_{0,2s+1} = 0, \quad d_{0,2s} = \frac{1}{2s+1} \quad (s = 0, 1, 2, \ldots).$$

In an eternity the expected number of returns is infinite but only just. Moreover, return is almost certain since $C(1) = 1$. (Cf. Feller (4), p. 244.)

Example (ix). In example (vi) take $\alpha = \gamma = 1$, $\beta = 0$. Then

$$D(z) = F(1, 1, 2, z^2) = -z^{-2}\log(1-z^2).$$

Thus if $$p_{2r,1} = \frac{r+1}{2r+1}, \quad p_{2r+1,1} = 0 \quad (r = 0, 1, 2, \ldots),$$

then $$d_{0,2s+1} = 0, \quad d_{0,2s} = \frac{1}{s+1} \quad (s = 0, 1, 2, \ldots).$$

Example (x). In example (vi) take $\alpha = \gamma = \frac{1}{2}$, $\beta = -\frac{1}{2}$. Then

$$p_{2r,1} = \frac{2r+1}{4r+1}, \quad p_{2r+1,1} = \frac{2r+1}{4r+3} \quad (r = 0, 1, 2, \ldots),$$

$$D(z) = \frac{F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z^2)}{F(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, z^2)} = \frac{1}{z}\frac{\arcsin z}{\sqrt{(1-z^2)}},$$

$$d_{0,2s} = \sum_{\nu=0}^{s}\binom{-\frac{1}{2}}{2s-\nu}\binom{-\frac{1}{2}}{\nu}\frac{1}{2\nu+1} = \frac{1}{2^{4s}}\sum_{\nu=0}^{s}\binom{4s-2\nu}{2s-\nu}\binom{2\nu}{\nu}\frac{1}{2\nu+1}.$$

Further examples can be derived from Wall (13), chapter 18.

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