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Cutsets in Infinite Graphs

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We answer three questions posed in a paper by Babson and Benjamini. They introduced a parameter C_G for Cayley graphs G that has significant application to percolation. For a minimal cutset of G and a partition of this cutset into two classes, take the minimal distance between the two classes. The supremum of this number over all minimal cutsets and all partitions is C_G. We show that if it is finite for some Cayley graph of the group then it is finite for any (finitely generated) Cayley graph. Having an exponential bound for the number of minimal cutsets of size n separating o from infinity also turns out to be independent of the Cayley graph chosen. We show a 1-ended example (the lamplighter group), where C_G is infinite. Finally, we give a new proof for a question of de la Harpe, proving that the number of n-element cutsets separating o from infinity is finite unless G is a finite extension of $\mathbb{Z}$.

1. Introduction

In their paper [1], Babson and Benjamini introduce a parameter C_G for an infinite Cayley graph G in the following way. Let G_+ be the set of vertices in G together with the set of ends of G. Given a subset Y in $V(G) \cup E(G)$, let

$$C(Y) = \sup_{Y_1 \cup Y_2 = Y} \{\mathrm{dist}(Y_1, Y_2)\}.$$

If $C(Y) \leqslant t$, then Y is said to be *t-close*. Let

$$C_G = \sup C(\Pi),$$

where Π ranges over all minimal cutsets between points $x, y \in G_+$.

For example, it is easy to check that for the square-grid we have $C_{\mathbb{Z}^2} = 2$, while $C_G = 3$ for the hexagonal grid G. The so-called lamplighter group will turn out to be such that its C_G is infinity (see Section 4).

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An exponential bound on the number of minimal cutsets of size n provides one with a nontrivial upper bound on the percolation critical probability p_c of G. Moreover, if C_G is bounded for some one-ended graph G then the critical probability p_u for having a unique infinite cluster is smaller than 1, as shown in [1]. There it is also shown that for finitely presented groups G (*i.e.*, groups with finitely many defining relators) C_G is finite. We shall give a shorter and elementary proof of that in Section 5. This answered Question 3 in [2] positively for finitely presented groups, showing that a finitely presented group with one end has $p_u < 1$. The original question, whether p_u is smaller than 1 in finitely generated groups with one end, is still open, so it is natural to ask whether C_G is finite for any Cayley graph G.

At the end of [1] there are three questions.

Question 1: Does having some exponential bound on the number of minimal cutsets of size n in a Cayley graph depend only on the group and not on the choice of generators?

Question 2: Does the property 'C_G is finite' for a Cayley graph depend only on the group and not on the choice of generators?

Question 3: Are there finitely generated groups with one end so that C_G is not finite?

In Sections 2 and 3 we give positive answers to the first and second questions of [1] respectively. In both cases we prove the invariance not only for different Cayley graphs of the same group but also for any two graphs that have bounded degrees and that are quasi-isometric under a bijection. Similar, but lengthier, arguments would show that the same conclusions hold for any two quasi-isometric bounded degree graphs.

In Section 4 we show that the lamplighter group is an example of a group with one end and infinite C_G, hence answering Question 3 negatively.

Babson and Benjamini conjecture that, for any Cayley graph, the number of minimal cutsets separating a fixed vertex o from infinity is exponentially bounded in the size of the cutset. They prove it for finitely presented groups. Problem VI.19 from [4] is weaker: 'In a finitely generated group which is not almost cyclic, does the size of spheres tend to infinity?' (A group is almost cyclic if it is a finite extension of $\mathbb{Z}$.) We give a short proof in Section 6 that the sizes of any sequence of (distinct) cutsets tend to infinity. This has already been shown by Anna Erschler (see the appendix of the second edition of [4]), using a result of Coulhon and Saloff-Coste [3].

For any set X of vertices of a graph, we denote by $N_n(X)$ the n-neighbourhood of X, that is, the set of points at distance $\leqslant n$ from X in the graph. Note that although $N_n(X)$ will be used for different graphs, it is the set X that indicates which graph is understood.

In what follows, for simpler phrasing, 'cutset' will always mean a set of edges whose deletion disconnects the graph. Everything remains valid if we use cutsets consisting of vertices.

We do not always distinguish between vertex sets and subgraphs induced by them.

A *bijective quasi-isometry* (or bi-Lipschitz map) *with constant* m between the graphs $G = (V, E)$ and $G' = (V', E')$, is a bijection f from $V(G)$ to $V(G')$ such that the following holds. For any $x, y \in V(G)$,

$$1/m \cdot \mathrm{dist}_G(x, y) \leqslant \mathrm{dist}_{G'}(f(x), f(y)) \leqslant m \cdot \mathrm{dist}_G(x, y).$$

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If there is such a map between G and G', we shall say that they are *bijectively quasi-isometric*. Note that different Cayley graphs of the same group are always bijectively quasi-isometric.

Given a subset X of G, ∂X will denote its external boundary, that is, the set of vertices not in X but adjacent to some vertex of X. We use $\partial_i X$ for the inner boundary.

2. The invariance of the exponential bound

Given a graph G and $o \in V(G)$, we say that there is an *exponential bound for the number of minimal cutsets* separating o from infinity if there is a constant α such that the number of minimal cutsets of size n separating o from infinity is at most α^n.

In this section we prove that having an exponential bound for the number of minimal cutsets in o is invariant under bijective quasi-isometries.

We shall need the following lemma, which is basically Lemma 6 in [1]; a stronger bound is given in [5].

Lemma 2.1. *Let G be a graph of degrees at most d and o a fixed vertex of it. The number of subsets of vertices of size n that contain o and induce a connected subgraph in G is at most d^{2n}.*

Proof. Choose a spanning tree in an induced subgraph as above and define a 'depth-first walk' in it: a walk that visits every vertex and goes through each edge at most twice. The walk determines the set of vertices, it has length $\leqslant 2n$, and in each step there are at most d ways to continue such a walk. □

Theorem 2.2. *Let ι be a bijective quasi-isometry from a graph G to G' with constant m. Suppose that the degrees in G and G' are bounded by d. Let o be some fixed vertex of G. Then there is an exponential bound for the number of minimal cutsets in G separating o from infinity if and only if there is an exponential bound in G' for the number of cutsets separating $\iota(o)$ from infinity.*

Proof. Denote by C_n and C'_n the set of minimal cutsets of size n separating o and $o' := \iota(o)$ from infinity in G and G' respectively. Let K_n (K'_n) be the set of connected subgraphs that arise as the connected component containing o (o') after removing an element of C_n (C'_n) from G (G'). We shall define a map ϕ from K_n to $H_n := \bigcup_{j=1}^{nd^{2m}} K'_j$. The map ϕ will have the property that, for every $\chi \in H_n$, the set $\{\kappa \in K_n : \phi(\kappa) = \chi\}$ has at most c^n elements for some constant c independent of n. The existence of ϕ shows that if $|K'_n|$ (and hence $|H_n|$) is at most exponential, then $|K_n| \leqslant |H_n| c^n$ is also at most exponential. Hence the theorem will follow.

So, let κ be an element in K_n. Define $\phi(\kappa)$ to be $N_m(\iota(\kappa))$. If $x, y \in V(\kappa)$ are adjacent in G then there is a path of length $\leqslant m$ in G' between $\iota(x)$ and $\iota(y)$. So the vertices of $\iota(\kappa)$ are in one component of $\phi(\kappa)$, and since any other vertex in $\phi(\kappa)$ is in the same component of $\phi(\kappa)$ as some vertex of $\iota(\kappa)$, we see that $\phi(\kappa)$ is connected. Now $\phi(\kappa)$ contains o'. Moreover, for any (external) boundary vertex y of $\iota(\kappa)$ in G', $\iota^{-1}(y)$ is at distance at

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most m from a boundary vertex of κ in G. The set of vertices that are at distance $\leqslant m$ from $\partial\kappa$ in G have cardinality $\leqslant d^m|\partial\kappa|$, so we get that $|\partial\iota(\kappa)| \leqslant d^m|\partial\kappa|$. Since $\phi(\kappa)$ is the m-neighbourhood of $\iota(\kappa)$ in G', the boundary of $\phi(\kappa)$ is in the m-neighbourhood of $\partial\iota(\kappa)$ in G', so $|\partial\phi(\kappa)| \leqslant d^m|\partial\iota(\kappa)|$. We get $|\partial\phi(\kappa)| \leqslant d^{2m}|\partial\kappa|$ from these two inequalities. Hence $\phi(\kappa)$ is indeed in H_n.

What remains to be shown is that $|\{\kappa \in K_n : \phi(\kappa) = \chi\}| \leqslant c^n$ for any $\chi \in H_n$.

Fix $\chi \in H_n$ and let $\tau := \iota^{-1}(\chi)$. If $\phi(\kappa) = \chi$ for some $\kappa \in K_n$ then $\kappa \subset \tau$. Furthermore, the m^2-neighbourhood of κ in G contains τ (by $N_m(\iota(\kappa)) = \chi$ and the definition of m). Thus $\tau\backslash\kappa$ is contained in $N_{m^2}(\partial\kappa)$. Since $N_{m^2}(\partial\kappa)$ has $\leqslant d^{m^2}|\partial\kappa| = d^{m^2}n$ elements, $|\tau\backslash\kappa| \leqslant d^{m^2}n$. Now $G\backslash\kappa$ has only infinite connected components by minimality of the cutsets. Hence we can get κ from τ by removing a subgraph S of τ of size $\leqslant d^{m^2}n$ and such that any connected component of S contains some element of $\partial_i\tau$. It thus suffices to show that there is an exponential bound on the number of such Ss.

So let $\mathscr{S}$ be the set of subgraphs of τ of size $\leqslant d^{m^2}n$ with the property that each component contains a vertex of $\partial_i\tau$. Any element S of $\mathscr{S}$ can be described as follows.

Let $S_1, \ldots, S_k$ be the components of S and fix an element r_i of $\partial_i\tau$ in S_i for each i; let $R := \{r_1, \ldots, r_k\}$. Now, if we first choose R as a subset of $\partial_i\tau$, then choose the sizes of the S_i, and finally choose the actual subgraphs S_i of τ of the given sizes and each incident to the corresponding element of R, then we obtain any possible $S \in \mathscr{S}$. One can choose R as a subset of $\partial_i\tau$ (where $|\partial_i\tau| \leqslant d^{m^2}n$), in at most $2^{d^{m^2}n}$ ways. Once we have $\{r_1, \ldots, r_k\} = R$, we choose $|S_i|$ for each i so that $\sum_{i=1}^k |S_i| \leqslant d^{m^2}n$ (by the definition of $\mathscr{S}$). This can be done in at most $\binom{d^{m^2}n}{k} \leqslant 2^{d^{m^2}n}$ ways. Finally, we choose the particular S_is, knowing their sizes. By 2.1, there are at most $d^{2(|S_1|+\ldots+|S_k|)} \leqslant d^{2d^{m^2}n}$ ways to do so.

We have shown that there are $\leqslant 2^{d^{m^2}n} \cdot 2^{d^{m^2}n} \cdot d^{2d^{m^2}n} = (4d^2)^{d^{m^2}n}$ ways to choose S, thus $|\mathscr{S}|$ is exponentially bounded. This finishes the proof. □

3. The invariance of finiteness of C_G

In this section we show that if C_G is infinite for a graph G, then $C_{G'}$ is infinite for any graph G' that is bijectively quasi-isometric to G. This means that the answer to Question 2 of [1] is positive.

Note that a minimal cutset Π between two vertices x and y is also a minimal cutset between one of them and an end. Otherwise there are paths from x and from y to an end that do not intersect Π, and these could be used (by the definition of ends) to find a path between x and y that does not intersect Π, giving a contradiction. Hence the supremum in the definition of C_G remains the same with the extra hypothesis that y is in $G_+\backslash G$.

Consider a minimal cutset that separates a finite subgraph X from infinity, and any boundary vertex v of X. There is an infinite path starting from v and going to infinity without intersecting X in any point other than v, since any component of $G\backslash X$ is infinite by minimality.

Lemma 3.1. *Suppose that a minimal cutset S that separates the connected subgraph X of G from $\xi \in G_+\backslash G$ is not l-close. Then the set S_n of edges that separate $N_n(X)$ from ξ is a minimal cutset that is not $(l-2n)$-close.*

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Proof. We may assume that $l \geqslant 2n$. Let A and B partition S so that $\mathrm{dist}(A, B) > l$. Then we have $N_n(A) \cap N_n(B) = \emptyset$. The set of vertices in $N_n(X)$ that are incident to some edge in S_n is in $N_n(A) \cup N_n(B)$, and it has nonempty intersection with both $N_n(A)$ and $N_n(B)$ because of our condition about the paths. Hence the partition generated on S_n by $N_n(A)$ and $N_n(B)$ shows that it is not $(l - 2n)$-close. □

Theorem 3.2. *Suppose that C_G is infinite for the graph G. Then $C_{G'}$ is infinite for any graph G' that is bijectively quasi-isometric to G.*

Proof. Let $\iota : V(G) \to V(G')$ be a bijective quasi-isometry with constant m. For each k, let G_k be a connected subgraph in G whose boundary is not k-close, and such that from any point of ∂G_k there is a path to infinity not intersecting G_k. Such subgraphs exist by our assumption on G and the remark about the definition of C_G. As in the previous section, the m-neighbourhood $N_m(\iota(G_k))$ is connected. This implies that the set S_k of edges that separate $N_m(\iota(G_k))$ from infinity is a minimal cutset. By 3.1, S_k is not $(k/m - 2m)$-close. So the S_ks provide us with a sequence of minimal cutsets where the distances for certain partitions tend to infinity. This shows that $C_{G'}$ is infinite. □

Remark. The assumption that G and G' are bijectively quasi-isometric was not necessary in the last two sections. Basically similar arguments show that the conclusions remain true for any two quasi-isometric bounded degree graphs.

4. The lamplighter group has infinite C_G

In this section we answer Question 3 of [1].

The *lamplighter group* is defined as the semidirect product of $\mathbb{Z}$ with $\sum_{x\in\mathbb{Z}} \mathbb{Z}_2$. For elements $p_1, p_2 \in \mathbb{Z}$ and $l_1, l_2 \in \sum_{x\in\mathbb{Z}} \mathbb{Z}_2$, the product is defined as

$$(p_1, l_1)(p_2, l_2) := (p_1 + p_2, l_1 \oplus S^{-m_1} l_2),$$

where S is the left shift, $S(l)(i) = l(i + 1)$ and $\oplus$ is componentwise addition mod 2. One can think of the elements of the lamplighter group as configurations where at each integer there is a lamp, either switched on or off*, and there is a lamplighter standing at one of the integers. A possible set of generators is $\{(1, \omega), (0, \lambda)\}$, where ω stands for the sequence of all zeros, and λ for the sequence of all zeros but a 1 in the 0th position. The first generator corresponds to that the lamplighter moves one step to the right, and the second one to that he switches the lamp in his current position.

Theorem 4.1. *If G is a Cayley graph of the lamplighter group then C_G is infinite.*

The Diestel–Leader graph $DL(k, n)$ is constructed as follows. Let T and T' be a $k + 1$-regular and an $n + 1$-regular tree respectively, and suppose that they are rooted at infinity so that their vertices are arranged into levels corresponding to the integers. Do it so that a vertex of T on the ith level has k children on the $(i + 1)$th level and the parent on the

* Only finitely many being on.

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$(i-1)$th level; a vertex of T' on the ith level has n children on the $(i-1)$th level and the parent on the $(i+1)$th level. Let the level of v in T (resp. T') be denoted by $l_T(v)$ (resp. $l_{T'}(v)$). $DL(k,n)$ is defined on the vertex set $\{(x,x') \in V(T) \times V(T') : l_T(x) = l_{T'}(x')\}$. There is an edge between (x,x') and (y,y') if and only if x and y are connected in T and x' and y' are connected in T'.

It is well known that $DL(2,2)$ is isomorphic to a Cayley graph of the lamplighter group. Briefly, fix a bi-infinite path R in T and a bi-infinite path R' in T' so that these paths intersect each level in exactly one vertex. For each vertex in T and T', call the edge that goes to one of its children a 0-edge, and the other one a 1-edge. Do it so that the paths R and R' contain only 0-edges. Now, the level of a vertex (x,x') in $DL(2,2)$ is the position of the lamplighter, and what the lamplighter sees on his left (right) is just the sequence of zeros and ones on the edges of the infinite path connecting x (x') to the root in infinity in T (T'), meaning the infinite path that always goes towards parents. For more details, see [8].

Fix a vertex o on the 0th level of T and a vertex o' on the kth level of T'. Let the subtree F_k (F'_k) consist of the offspring of o (o') of distance at most k from it. Let H_k be the subgraph $\{(x,x') : x \in F_k, x' \in F'_k\}$ in $DL(2,2)$ and let C_k be the set of its boundary edges. Notice that C_k is a minimal cutset and that it is the disjoint union of edges incident to $A_k = \{(x,x') : \deg_{F_k}(x) = 1\}$ and $B_k = \{(x,x') : \deg_{F_k}(x') = 1\}$ respectively. The distance of A_k and B_k is obviously k. Hence the sequence C_k shows that $C_{DL(2,2)}$ is infinite.

5. C_G in finitely presented groups

The result in this section is the key in [1] to proving $p_u < 1$ for finitely presented groups with one end. We present a shorter proof here, using elementary arguments and terminology, as opposed to their argument using cohomology groups.

We use the obvious correspondence between subsets of $E(G)$ and the elements of $\{0,1\}^{E(G)}$ regarded as vectors, where mod 2 addition on the latter corresponds to symmetric differences in the case of the subsets. Given a set K of cycles in a graph G, we say that a cycle C in G is *generated by* K if C can be written as a mod 2 sum of cycles from K. Note that any cycle in a Cayley graph of a finitely presented group is generated by the set of cycles of length at most t, where t is the maximal length of relators in this presentation of the group.

Theorem 5.1. *Let G be a graph such that any cycle is generated by a set K of cycles in G. Suppose that any cycle in K has length at most t. Let Π be a minimal cutset separating a vertex x from $y \in V(G) \cup \{\infty\}$. Then, for any nontrivial partition $\Pi_1 \cup \Pi_2$ of Π, there are vertices $x_i \in \Pi_i$ $(i = 1,2)$ such that $\mathrm{dist}(x_1,x_2) \leqslant t/2$. In other words, $C_G \leqslant t/2$.*

Proof. It is enough to show that there is a cycle in K that intersects both Π_1 and Π_2. By minimality of the cutset Π, there are paths P_i between x and y, $i = 1,2$, such that P_i does not intersect Π_{3-i}. We may write $P_1 - P_2$ as a mod 2 sum of cycles from K: $P_1 - P_2 = \sum_{c \in K'} c$, for a certain $K' \subset K$. Let K'_1 be the set of those cycles in K' that intersect Π_1, and $K'_2 := K' \backslash K'_1$. Define

$$\theta := P_1 - \sum_{c \in K'_1} c = P_2 + \sum_{c \in K'_2} c.$$

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The right-hand side is the sum of cycles and paths that do not intersect Π_1, hence θ does not contain any edge of Π_1. On the other hand, since the only odd degrees that θ has are in x and y, these two have to be in the same connected component of θ. Thus, there is a path from x to y that intersects Π_2. Since P_1 does not, we deduce that some cycle in K_1' does. This cycle intersects both Π_1 and Π_2. □

6. Sizes of cutsets tend to infinity

By the growth of a Cayley graph we mean the function that takes in n the size of a ball of radius n.

For a rooted tree T and $x \in V(T)$ let the subtree T_x be defined as the set of all descendants of x (including x itself) and the edges induced by them.

A *lexicographically minimal spanning tree* T of a Cayley graph G is a subtree rooted at the origin and defined in the following way. Fix a linear ordering of the generators of G and their inverses. For any vertex v of the Cayley graph choose the word representing v that is lexicographically minimal among all such words. There is a path in G that represents this word; define T to be the union of all these paths (as v ranges through every vertex). The graph we get is indeed a spanning tree. Moreover, it is *subperiodic*, that is, for any x in $V(T)$ there is an embedding of T_x into T that maps x to o. The growth rates of T and G are the same. These are straightforward corollaries of the definition of T; for more details about lexicographically minimal spanning trees, see, for example, [6].

Lemma 6.1. *If a subperiodic tree T has finitely many infinite rays then it has linear growth.*

Proof. For $x \in V(G)$ denote by F_x the union of the finite components of $T\backslash\{x\}$ not containing o. Define $S := \{x \in V(T) : |F_x| > |F_y| \text{ for every } y \text{ where } \mathrm{dist}(o, y) < \mathrm{dist}(o, x)\}$. By subperiodicity of T, for any $x \in V(T)$ there is an embedding ϕ of T_x into T such that $\phi(x) = o$. Fix $x \neq o$ and a corresponding ϕ. If $z \in S \cap T_x$ then $\phi(F_z)$ can not be a subset of $F_{\phi(z)}$ by the definition of S. So $\phi(F_z)$ has the property that some of its vertices are mapped into vertices of an infinite ray starting from $\phi(z)$ and not intersecting o. This infinite ray contains no other vertices from $\phi(T_x)$ except those few from $\phi(F_z)$. So there are at most as many different 'F_zs' as pairwise disjoint infinite rays, that is, $|S \cap T_x|$ is finite. This can hold for any $x \neq o$ if and only if $|S|$ is finite. Hence $|F_x|$ is bounded for every x, and T has linear growth. □

Theorem 6.2. *In any Cayley graph G and for $n > 0$, there are only finitely many cutsets of size n separating a fixed vertex o from infinity, unless G is a finite extension of $\mathbb{Z}$.*

Proof. Let S_r stand for the sphere of radius r around o, and let B_r be the ball. By a cutset we always mean a cutset separating o from infinity.

If G has infinitely many ends then it is nonamenable and has an exponential bound for the number of minimal cutsets of size n by 2.1, as shown in [1]. So we may assume that G has one end. Fix $n > 0$. Choose a lexicographically minimal spanning tree T in G. If T has finitely many ends then the group grows linearly by 6.1. Hence it is a finite

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extension of $\mathbb{Z}$. (For a proof that groups of linear growth are finite extensions of $\mathbb{Z}$, see, for example, Corollary 3.18 in [7].) If T has infinitely many ends then there is a ball around o such that any minimal cutset of size n intersects it, namely a ball such that there are at least $n+1$ disjoint infinite rays starting from its boundary. Choose X to be a set of edges that occurs in infinitely many minimal cutsets of size n (to prove by contradiction), and maximal with this property. (So $0 < |X| < n$.) Since X is not a cutset and G has one end, there are numbers R and r such that any vertex in S_r is connected to o by a path in $G\backslash X$ and not intersecting S_R. But then a cutset that has no edge in B_R but those of X cannot be minimal (it is necessarily a cutset without X too). This shows, by the choice of X, that there cannot be infinitely many minimal cutsets of size n and containing X. This contradiction finishes the proof. □

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