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Non-linear interactions, causality and the R-matrix

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Abstract. The problem of causality in non-relativistic scattering where a bounded region of non-linear interaction occurs is re-examined in the context of a paper by Ebel. In the model considered it is shown that the acausal behaviour cannot be accounted for in the straightforward manner that is possible for an energy-dependent but linear potential which gives rise to a similar R-matrix.

1. *Introduction.* A mathematical result of ours(3) concerning the analytical structure of the R-matrix for a simple model, which is exactly soluble, of a non-linear but bounded potential added to a square well potential has given rise to discussions(1) concerning the physical meaning of this result. We would like to re-examine its physical interpretation, for, of course, the mathematical result still stands.

First it is necessary to distinguish two separate problems:

(a) the problem of the non-linear interaction *per se* which gives rise to a derivative matrix

$$R = \frac{\mathrm{sn}\,(\kappa a)}{\kappa a\,\mathrm{cn}\,(\kappa a)\,\mathrm{dn}\,(\kappa a)}, \tag{1}$$

where a is the radius of the interaction and κ is related to the wave number q through the equation

$$\kappa = \frac{q}{\sqrt{2}}\left[1+\left(1-\frac{2\xi}{q^4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{2}$$

ξ being proportional to the non-linear coupling strength λ and also a function of the intensity. (For details of the notation see (3).) The modulus k of the elliptic function appearing in equation (1) is also a function of the wave number and of the intensity by the equation

$$k^2 = \tfrac{1}{2}\xi\kappa^{-4}; \tag{3}$$

(b) the energy-dependent, but linear, interaction potential scattering problem, where R has the value

$$R = \frac{\mathrm{sn}\,(qa)}{qa\,\mathrm{cn}\,(qa)\,\mathrm{dn}\,(qa)}, \tag{4}$$

where now the elliptic functions have constant modulus. The appropriate (very strange) potential for this case is proportional to

$$q^2k^2[\mathrm{sn}^2\,(qr)-\mathrm{cn}^2\,(qr)] \quad (r<a). \tag{5}$$

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2. *The related linear problem.* The discussion of Ebel(1) centres on the second of these problems, where causality, in the sense of van Kampen(2), can be applied, since wave packets, without sharp fronts, however, can be constructed and therefore R, if the theory is causal, must satisfy the Wigner–Neumann(6) conditions for an R-function. One of these conditions is

$$\frac{dR}{dq} > 0 \tag{6}$$

which is the first, for $q > 0$, of the hierarchy of inequalities

$$\left.\begin{aligned} &\det|W_{ij}| \geqslant 0, \\ W_{ij} &\equiv (E_i - E_j)^{-1}[R(E_i) - R(E_j)] \quad (i \neq j), \\ &\equiv \frac{dR(E_i)}{dE_i} \qquad\qquad\qquad (i = j). \end{aligned}\right\} \tag{7}$$

Equations (7) imply that, for a causal interaction, R has its poles on the real axis; therefore the R of equation (4) is not an R-function, since the zeros of $\mathrm{cn}\,(qa)$ form a two-dimensional array ($k \neq 0$). The energy dependence chosen for the potential (5) thus ensures that this potential scattering is acausal. In(3) we showed that equation (6) is satisfied by (4). Ebel(1) showed by direct calculation that (4) does not satisfy the second of the set of equations (7).

It is easy to see that any energy-dependent potential may given rise to an R-matrix, or its corresponding S-matrix, with an analytical structure different from the causal one implied by equations (7). For starting from the Schrödinger equation for the problem (for s-waves),

$$\frac{d^2u_q}{dr^2} + [q^2 - 2mV(r,q)]\,u_q(r) = 0, \tag{8}$$

it is straightforward to show that, if $\eta = \eta_1 + i\eta_2$ denotes the position of an S-matrix pole, then, away from the imaginary axis ($\Re\,\eta \neq 0$), one has

$$\Im\,\eta = \frac{1}{2\int_0^a |u_\eta(r)|^2\,dr}\left\{\frac{2m}{\Re\eta}\int_0^a |u_\eta(r)|^2\,\Im V(r,\eta)\,dr - |u_\eta(a)|^2\right\}. \tag{9}$$

Therefore,

$$\Im\,V(r,\eta) = 0 \tag{10}$$

is a sufficient condition for these poles to lie on the lower half q-plane. This is the usual quantum mechanical case, where V is a q-independent real function of r. This condition, however, will not in general be satisfied by q-dependent potentials, for which class of potentials equation (9) gives them a necessary and sufficient restriction to ensure that the poles of the R-matrix will all lie on the real axis

$$\frac{1}{\Re\,\eta}\int_0^a |u_\eta(r)|^2\,\Im\,V(r,\eta)\,dr < \frac{|u_\eta(a)|^2}{2m}. \tag{11}$$

3. *The non-linear problem.* The first of the two above-mentioned problems (problem (a)) presents deeper results: a non-linear equation has no superposition principle and therefore causality cannot be considered from a wave-packet point of view.

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This statement is tantamount to saying that in a non-linear problem causality is not (at least *a priori*) expressed by equations (7) as these equations have been obtained precisely starting from the wave-packet approach (4). This emphasizes the basic difference between problems (*a*) and (*b*), a difference overlooked by Ebel (1).

In the Appendix of (3) we showed that equation (6) can still be deduced from van Kampen's causality condition, when applied to a mono-wave number incident wave; none of the remaining equations (7) could however be analogously obtained, as the superposition of two solutions of close wave number will not in this case be another solution. The superposition principle is thus essential. One of us ((5)) has demonstrated by construction how the completeness relation in a linear, and thus superposable, theory gives rise to the analytical structure deducible from the inequalities (7).

Now, in (3) we showed that equation (6) can still be satisfied by the R-matrix obtained for the non-linear problem (1), at least for small intensities. A similar statement can be made with regards to the question of unitarity: the R-matrix (1) depends on the intensity of the incident beam, as one would expect, since the output of a non-linear black box depends on the intensity of the input, and only for certain ranges of intensity does the normal unitary condition

$$SS^* = 1 \tag{12}$$

hold. Thus the example with the R-matrix of equation (1) is not one for which causality and unitarity hold and R has singularities off the real axis.

For the non-linear equation

$$\frac{d^2u_q}{dr^2} + (q^2 - \lambda u_q^2)\, u_q = 0 \tag{13}$$

arguments similar to those leading to (11) give the inequality

$$\frac{1}{\Re\eta}\int_0^a |u_\eta(r)|^2\, \Im\{\lambda u_\eta^2(r)\}\, dr < |u_\eta(a)|^2, \tag{14}$$

which for the solution of (13), namely

$$u_\eta = \frac{c}{\kappa}\,\mathrm{sn}\,\kappa r$$

may be well violated. To see this, notice that $|c|^2$ cancels on both sides of equation (14), leaving a term depending on c^2 on the left-hand side. Now, if the left-hand side is negative, the inequality is trivially satisfied; however, if it is positive, then, by making c^2, which is related to the intensity, sufficiently large, the inequality will be surely violated, and poles of S will exist in the upper half plane ($\Re\eta \neq 0$).

A final remark concerns the time dependence of the wave. Equation (2) of (3) is time independent, and thus the claim that the interaction is non-local in time reads into the problem a Schrödinger-like time-dependence behaviour which we were careful to avoid, as it cannot be introduced in the non-linear problem we considered. For, if

$$\Psi(r,t) = \phi(r)\, e^{-iEt} \quad (\hbar = 1), \tag{15}$$

where $\phi(r)$ satisfies

$$\left(-\frac{\nabla^2}{2m} + \bar{\lambda} r^2 \phi^2(r)\,\Theta\right)\phi(r) = E\phi(r), \tag{16}$$

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then Ψ does not obey the non-linear Schrödinger-like time-dependent equation

$$\left(-\frac{\nabla^2}{2m}+\bar{\lambda}r^2\Psi^2\Theta\right)\Psi(r,t)=i\frac{\partial\Psi}{\partial t}. \tag{17}$$

On the other hand, in a time-dependent equation such as

$$\left(-\frac{\nabla^2}{2m}+\bar{\lambda}r^2\Psi^*\Psi\Theta\right)\Psi(r,t)=i\frac{\partial\Psi}{\partial t}, \tag{18}$$

which admits monochromatic solutions, a separation of variables as given by (15) leads to an equation where the spacial non-linearity is $\lambda\phi^*\phi$, and this can be shown by the method of Rosenfeld(4) to give rise to an R-matrix which is an R-function, as explicitly remarked in a footnote of (3). This can also be seen immediately from the inequality (14) since the left-hand side vanishes. It is the same type of product, ϕ^2, which restricts all the discussion to spherical symmetric spacial dependence. Of course, in problem (*b*), with an energy-dependent potential of the form (5), the Schrödinger-like time dependence will always be necessary to deduce the wave-packet causality conditions.

To summarize: an energy-dependent potential will often be acausal and give poles of R off the real axis, but this behaviour can be readily accounted for; on the other hand, the problem of causality in the non-linear case stands as a completely open question. To paraphrase a remark of Fermi's: it does not matter what sort of lions are inside $r = a$—provided they are not non-linear lions.

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