Notes

99.01 Intercepts and turning points of polynomial functions

Introduction

One powerful use of calculus is finding the turning points of a function. Sometimes calculus can be avoided and, instead, symmetry can be exploited as in the case of quadratic functions that intersect the x-axis in two points A and B. The x-value of the turning point is the x-coordinate of the midpoint of $\overline{AB}$. A student in Colin Foster's high school class asked [1] whether a cubic polynomial function could be found with this same property that we shall now call the *Intercept-turning point property* (ITPP).

Definition: If the graph of a polynomial function $y = f(x)$ intersects the x-axis at points with coordinates $(a,0)$ and $(b,0)$ and at no points in between and if the graph has a single turning point on $[a, b]$ whose x-value is at the midpoint $x = \frac{1}{2}(a + b)$ then we say that the polynomial satisfies the ITPP on the interval $[a, b]$.

In [1], Foster establishes the fact that a cubic polynomial never satisfies the ITPP, but also shows that there exist polynomials of degree two and degree four and larger that do. In this note we shall continue the inquiry by establishing the following.

Section 1: Not only does a cubic polynomial not satisfy the ITPP on any $[a, b]$, but the midpoint of $[a, b]$ is the only x-value on $[a, b]$ where the turning point cannot appear.

Section 2: A polynomial function has at most one interval $[a, b]$ where the ITPP holds and the graph is symmetric about the line with equation $x = \frac{1}{2}(a + b)$.

Section 3: The curve of a cubic polynomial function is a special case of a cubic algebraic curve whose points satisfy

$$x^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Jy + K = 0.$$

We shall modify the ITPP for these curves.

Modified intercept-turning point property (MITPP)

If the graph of a polynomial equation intersects the x-axis at points with coordinates $(a,0)$ and $(b,0)$ and at no points in between and if the graph has at least one turning point on $[a, b]$ whose x-value is at the midpoint $\frac{1}{2}(a + b)$ then we say that the polynomial satisfies the MITPP on the interval $[a, b]$.

We shall show the following:

(a) there exist cubic algebraic curves that satisfy the MITPP where, on one interval $[a, b]$, there are either one or three turning points;

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(b) there exist cubic algebraic curves that satisfy the MITPP on two intervals $[a, b]$ and $[b, c]$ and there are at most two turning points on each.

Section 1

We now prove that if a cubic polynomial function $y = f(x)$ intersects the x-axis at points $(a,0)$ and $(b,0)$ and at no points in between and if the graph has a single turning point on $[a, b]$, then the only x-value on $[a, b]$ where the turning point cannot appear is at $x = \frac{1}{2}(a + b)$.

Consider the cubic polynomial function $f(x) = p(x - a)(x - b)(x - q)$ with $a \neq b$ and $p \neq 0$. Then

$$f'(x) = p(x - a)(x - b)D_x(x - q) + (x - q)D_x p(x - a)(x - b)$$

so that at a turning point

$$0 = p(x - a)(x - b) + (x - q)p(2x - a - b),$$

that is,

$$q = \frac{(x - a)(x - b)}{2x - a - b} + x \tag{1}$$

provided $2x - a - b \neq 0$. The x-value of the turning point gives the q-value for the cubic polynomial. However, the only x-value that does not produce a q-value is $x = \frac{1}{2}(a + b)$, where the right-hand side of (1) is singular.

Section 2

If a polynomial function $y = f(x)$ is symmetric about the line with equation $x = d$ then we can shift the graph so that $y = g(x) = f(x + d)$ is symmetric about the y-axis. This implies that $g(x) = g(-x)$. Since g is a polynomial, it can contain only even powers of x and this characterises g as an even function. Furthermore, the graph of g can be symmetric about at most one vertical line. We have the following result:

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function. If there exists a real number d, such that $y = g(x) = f(x + d)$ is an even function with at least two real roots, then f satisfies the ITPP on exactly one interval $[a, b]$ and is symmetric about the line $x = \frac{1}{2}(a + b)$.

Proof: Since $y = g(x) = f(x + d)$ is an even function with at least two real roots, we know that g is symmetric about the y-axis and so the roots come in pairs as $-r$ and r. Let r be the smallest positive root, and so the graph of g is symmetric on the interval $[-r, r]$. Since $g(x)$ is an even polynomial function, then $g'(x)$ is an odd function with no constant term and so $g'(0) = 0$. Thus a turning point of g is at $x = 0$ and the polynomial satisfies the ITPP.

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Section 3(a)

We consider a cubic algebraic curve that intersects the x-axis in three points $(-p,0)$, $(p,0)$ and $(q,0)$ where $0 < p < q$. The form of the cubic is given below,

$$ay^3 + (bx + c)y^2 + (dx^2 + ex + f)y = x^3 - qx^2 - p^2x + qp^2, \quad (2)$$

with $a, b, c, d, e, f \in \mathbb{R}$. We consider one interval $[-p, p]$ and the special form

$$ay^3 + cy^2 + ey + f = x^2. \quad (3)$$

Readers familiar with elliptic curves will recognize this as a standard form of an elliptic curve with the x and y variables swapped. Elliptic curves are an important area of current mathematical research ([2] gives a nice introduction). The graphs in Figure 1 below show examples of cubic equations where there is one turning point and three turning points. The term in x^2 on the right-hand side of equation (3) ensures that the curve is symmetric about the y-axis and that any turning points occur at $x = 0$. The cubic expression in y implies that there can be no more than three turning points.

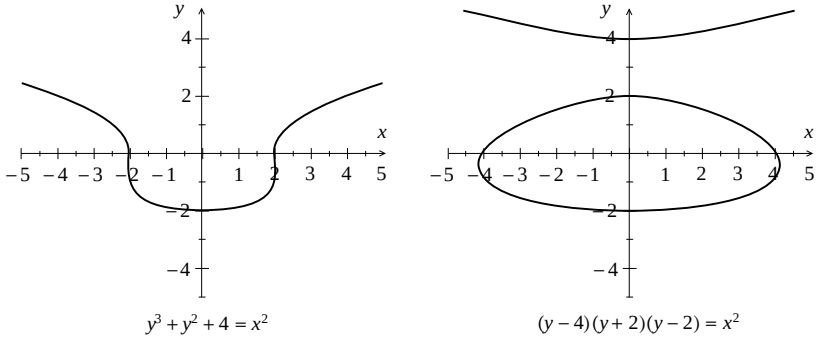


FIGURE 1

Interestingly, (3) does not correspond to a curve with exactly two turning points. If the equation has this form then it looks like $(y-\alpha)^2(y-\beta) = x^2$, where the turning points are at $(0, \alpha)$ and $(0, \beta)$. But

$$\frac{dy}{dx} = \frac{2x}{(y-\alpha)^2 + 2(y-\beta)(y-\alpha)}$$

is undefined at $(0, \alpha)$ and so this situation cannot occur.

Section 3(b)

We look at the MITPP being satisfied on both intervals $[-p, p]$ and $[p, q]$. Consider the cubic equation

$$ay^3 + (bx + c)y^2 + (dx^2 + ex + f)y = x^3 - qx^2 - p^2x + p^2q,$$

for which

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$$\frac{dy}{dx} = \frac{3x^2 - 2qx - p^2 - by^2 - 2dxy - ey}{3ay^2 + 2y(bx + c) + dx^2 + ex + f} = 0$$

$$3x^2 - 2qx - p^2 - by^2 - 2dxy - ey = 0. \quad (4)$$

We now proceed with an example to illustrate how to determine values of the coefficients of a cubic algebraic curve that will give a maximum number of turning points.

Example: Let $p = 2$, $q = 6$, $b = 1$ and $e = -5$, then equations (2) and (4) become

$$ay^3 + (x + c)y^2 + \left(dx^2 - 5x + f\right)y = x^3 - 6x^2 - 4x + 24 \text{ and } \quad (5)$$

$$3x^2 - 12x - 4 - y^2 - 2dxy + 5y = 0. \quad (6)$$

A point (x, y) satisfying both (5) and (6) lies on the cubic curve and is a turning point. Equation (6) implies that at any x-value there can be at most two turning points. We want turning points at $x = 0$ and so substitute $x = 0$ into (5) and (6) giving $ay^3 + cy^2 + fy - 24 = 0$ and $y^2 - 5y + 4 = 0$. Solutions to (6) will also be solutions to (5) if the division

$$\frac{ay^3 + cy^2 + fy - 24}{y^2 - 5y + 4} = ay + (c + 5a) + \frac{(21a + 5c + f)y + (-24 - 4c - 20a)}{y^2 - 5y + 4}$$

yields no remainder, that is, if $21a + 5c + f = 0$ and $-24 - 4c - 20a = 0$. Solving these in terms of a we get $c = -5a - 6$ and $f = 4a + 30$. We also want turning points at $x = 4$ and so substitute $x = 4$ and the values for c and f into (5) and (6), giving $ay^3 + (5a - 2)y^2 + (16d + 4a + 10)y + 24 = 0$ and $y^2 + (8d - 5)y + 4 = 0$. Two solutions will be common if the division

$$\frac{ay^3 + (5a - 2)y^2 + (16d + 4a + 10)y + 24}{y^2 + (8d - 5)y + 4}$$

$$= -ay + (-8ad - 2) + \frac{\left(64ad^2 - 40ad + 32d\right)y + 32ad + 32}{y^2 + (8d - 5)y + 4}$$

yields no remainder, that is, if $64ad^2 - 40ad + 32d = 0$ and $32ad + 32 = 0$. Substituting $d = -\frac{1}{a}$ into the first equation, we get

$$64a\left(-\frac{1}{a}\right)^2 - 40a\left(-\frac{1}{a}\right) + 32\left(-\frac{1}{a}\right) = 0,$$

so that $a = -\frac{4}{5}$. We then have $c = -2$, $d = \frac{5}{4}$ and $f = \frac{134}{5}$. The general cubic equation becomes

$$-\frac{4}{5}y^3 + (x - 2)y^2 + \left(\frac{5}{4}x^2 - 5x + \frac{134}{5}\right)y = x^3 - 6x^2 - 4x + 24,$$

whose graph is displayed in Figure 2. Two turning points at $x = 0$ and two more at $x = 4$ is the most that we can expect with the MITPP under these circumstances.

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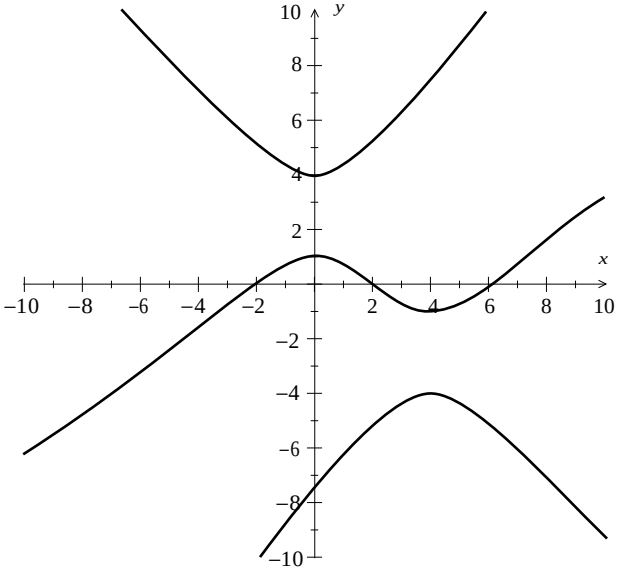


FIGURE 2

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99.02 Series involving fixed-column decimal representations

Introduction

As an end-of-term piece of homework, I set the students in one of my senior classes the task of devising their own question, along with its solution, related to at least one of the mathematical topics we had covered in the previous few months. My intention was to give the students the opportunity to tackle each other's questions. One that immediately caught my eye went, with a little bit of tidying up, as follows:

> Let a_0, a_1, a_2, a_3, … be the increasing and infinite sequence of all positive integers such that when each of these numbers is written in decimal notation, there is an 8 in the hundreds column. Work out the sum of the infinite series
>
> $$\frac{a_0}{2^0} + \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots .$$

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