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Continuation and bifurcation associated to the dynamical spectral sequence

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Abstract. In this paper we consider a filtered chain complex C and its differential given by a connection matrix Δ which determines an associated spectral sequence (E^r, d^r). We present an algorithm which sweeps the connection matrix in order to span the modules E^r in terms of bases of C and gives the differentials d^r. In this process a sequence of similar connection matrices and associated transition matrices are produced. This algebraic procedure can be viewed as a continuation, where the transition matrices give information about the bifurcation behavior. We introduce directed graphs, called flow and bifurcation schematics, that depict bifurcations that could occur if the sequence of connection matrices and transition matrices were realized in a continuation of a Morse decomposition, and we present a dynamic interpretation theorem that provides conditions on a parameterized family of flows under which such a continuation could occur.

1. *Introduction*

Let M be an n-dimensional compact Riemannian manifold, $f : M \to \mathbb{R}$ a Morse function which is Morse–Smale, and φ the gradient flow of f. Let $\mathcal{D}(M)$ be a Morse decomposition of M, that is, a collection $\mathcal{D}(M) = \{M(p)\}_{p\in P}$ of mutually disjoint compact invariant subsets of M called *Morse sets*, indexed by a finite set P, on which it is possible to impose a partial order $>$ such that: if $\gamma \in M \backslash \bigcup_{p\in P} M(p)$, then there exist $q > p$ such that $\alpha(\gamma) \subset M(q)$ and $\omega(\gamma) \subset M(p)$. Any partial order on P which satisfies these properties is called *admissible order*. We work with a finite Morse decomposition $\bigcup M(p)$, $p \in P = \{1, \ldots, m\}$, such that there are distinct critical values c_p with $f^{-1}(c_p) \supset M(p)$. Then

$$\{F_p\}_{p=1}^m = \{f^{-1}(-\infty, c_p + \epsilon)\}_{p=1}^m.$$

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This defines an admissible order on M called the *filtration order*. In our case, each Morse set, $M(p)$, is a non-degenerate singularity of the gradient flow φ, and hence the Conley index, $h(M(p))$, of each Morse set is a pointed n-sphere. We denote by $h_k^{(p)}$ the index k singularity in $F_p \backslash F_{p-1}$. A map $g : M \to M$ is filtration preserving if for every $p \in P$ we have $g(F_p) \subset F_{p-1}$.

The collection of orbits $M \backslash \bigcup_{p \in P} M(p)$ are the *connecting orbits* of the Morse decomposition. The set of the connecting orbits with $\alpha(\gamma) \subset M(q)$ and $\omega(\gamma) \subset M(p)$ is denoted by $C(M(q), M(p))$. There is a minimal admissible ordering which we call *flow defined* ordering and denote by $>_\varphi$. It is defined by $q >_\varphi p$ if and only if there exists a sequence $p = p_0, p_1, \ldots, p_m = q$ such that $C(M(p_{i+1}), M(p_i)) \neq \emptyset$.

Given a partial ordered set $(P, <)$, we define an *interval* in P to be a subset $I \subset P$ such that: if $p, r \in I$ and $p < q < r$ then $q \in I$.

Let $\{\varphi_\lambda, \lambda \in \Lambda\}$ be a continuous family of flows, where $\Lambda \subset \mathbb{R}$ is connected. This is equivalent to the continuity of the *parameterized flow*

$$\Phi : \mathbb{R} \times M \times \Lambda \to M \times \Lambda$$

given by

$$\Phi(t, x, \lambda) = (\varphi_\lambda(t, x), \lambda).$$

If $N \subset M \times \Lambda$ is an isolating neighborhood for Φ and $N_\lambda := N \cap M \times \lambda$ then, for all $\lambda \in \Lambda$, N_λ is an isolating neighborhood for φ_λ. Two isolated invariant sets S_{λ_0} and S_{λ_1} for φ_{λ_0} and φ_{λ_1} respectively are related by continuation if there exists an isolating neighborhood N of Φ such that $S_{\lambda_0} = \mathrm{Inv}(N_{\lambda_0})$ and $S_{\lambda_1} = \mathrm{Inv}(N_{\lambda_1})$.

We will use the notion of *order preserving continuation of a Morse decomposition*: let $U \subset \Lambda$ be connected. A Morse decomposition continues over U preserving order if there is an isolated invariant set $\mathbf{S}$ in $M \times U$ with Morse decomposition $\{\mathbf{M}(p), p \in P\}$. It implies that there is a partial ordering $<$ which is valid for all U, that is, there is an admissible order for the decomposition on the level of the parameterized flow. This definition is stronger than assuming that $\{M_\lambda(p), p \in P\}$ is a Morse decomposition for each $\lambda \in U$, that is, only assuming that we have a Morse decomposition for each parameter λ is not enough to guarantee that these orders are compatible and that there exists a partial order over all U.

For more details on the Conley index theory used within this paper, see [**Co, Fr1, Fr2, Fr3**].

Example 1.1. Consider the example in Figure 1. We can point out that $1 < 2 < 3 < 4$ continues across but $1 < 2 < 3, 4$ with 3 and 4 non-comparable does not because of the 4–3 connection in the middle figure.

Given a Morse decomposition, a connection matrix associated to it is the upper triangular boundary operator

$$\Delta : \bigoplus_{p \in P} H_*(h(M(p))) \to H_*(h(M(p))),$$

where $H_*(h(M(p)))$ is the homology of the Conley index of $M(p)$.

In the case where each $M(p)$ is a non-degenerate singularity and the stable and unstable manifolds intersect transversally, the connection matrix Δ associated to $\mathcal{D}(M)$ is unique

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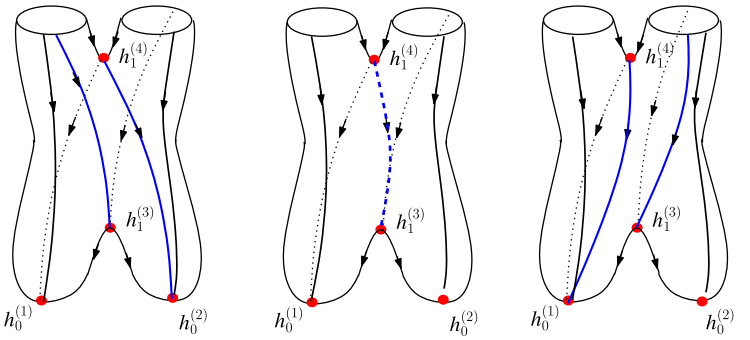


FIGURE 1. An example of continuation.

(see [**R2, R3**]). It can also be defined as the differential of the graded Morse chain complex (C, Δ), where C is generated by the singularities and graded by their indices, that is, $C = \mathbb{Z}_2\langle \operatorname{crit} f\rangle$†, and Δ is determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \operatorname{crit}_{k-1} f} a(x, y)\langle y\rangle,$$

where $a(x, y)$ is the number of connecting orbits counted mod 2 for non-degenerate singularities x and y of indices k and $k-1$, respectively. Moreover, Δ is an upper triangular matrix with $\Delta \circ \Delta = 0$. Non-zero entries of these matrices imply the existence of connecting orbits, hence connection matrices provide information concerning global dynamics.

It need not be the case that the columns of the matrix Δ be ordered with respect to k. We only require that the map Δ_k be filtration preserving, that is, $\Delta_k(F_pC_k) \subset F_{p-1}C_{k-1}$, and hence the matrix Δ_k is upper triangular.

In a Morse decomposition that consists of non-degenerate singularities, if the connection matrices are not unique, then there must be non-transversal intersections between stable and unstable manifolds of the singularities. For example, there could be a saddle–saddle connection as in Figure 1. Via algebraic transition matrices possible non-transversal intersections occurring in a Morse decomposition can be identified [**Fr3**].

Algebraically, the idea behind transition matrices is straightforward. Specifically, if a connection matrix is associated with an admissible ordering of a Morse decomposition, and T is upper triangular with respect to the admissible ordering and such that $T(p) : H_*(h(M(p))) \to H_*(h(M(p)))$ is the identity for all p, then T is an isomorphism and $\Delta' = T^{-1}\Delta T$ is also a connection matrix for the admissible ordering of the Morse decomposition. Such a map T is called an (algebraic) transition matrix (between Δ and Δ').

Thus, algebraic transition matrices are, in some sense, partial-order-respecting change of basis transformations for the connection matrices associated to an admissible ordering of the Morse decomposition. This idea of change of basis is central to the sweeping method,

† Without loss of generality, we assume a unique singularity of index n and index 0.

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introduced in §3, where the initial input is a connection matrix and a set of other connection matrices is iteratively constructed by change of basis operations.

The algebraic transition matrices not only give us an indication of the non-uniqueness of the connection matrices, but also potentially identify the non-transversal stable and unstable manifold intersections that occur in a continuation of a Morse decomposition. Specifically, under appropriate circumstances, if an off-diagonal transition matrix entry is non-zero, then there is a connection between the corresponding Morse sets at some parameter value in the continuation.

In §§2–4 we introduce the sweeping method, an iterative process that, given a connection matrix Δ, generates a collection of connection matrices, $\Delta = \Delta^1, \ldots, \Delta^F$, and transition matrices, $T^1, \ldots, T^{F-1}$, such that $\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r$ for all $r = 1, \ldots, F-1$. The connection matrices Δ^r are the connection matrices associated with the spectral sequence determined by Δ.

We would like to view the connection matrices Δ^r and transition matrices T^r as arising from a continuation in a parameterized family of flows for which Δ is the connection matrix for a Morse decomposition at an initial parameter value. In this way, Δ^r and T^r then track bifurcations that occur across the continuation.

To that end, in §5 we introduce a set of graphs, called flow schematics, bifurcation schematics, local continuation schematics, and continuation schematics, associated to connection matrices and transition matrices generated by the sweeping method. Specifically, associated to each connection matrix there is a flow schematic whose vertices correspond to Morse sets and whose edges correspond to potential connections between Morse sets that are identified by non-trivial connection matrix entries. From the flow schematics, the bifurcation schematics associated to the transition matrices and the local continuation schematics are constructed using a process that mirrors the sweeping method steps for generating transition matrices and connection matrices. It is shown that each successive flow schematic is obtained via this process of modifying flow schematics and bifurcation schematics. Combining all of these schematics into the continuation schematic, we obtain a graph that is a full representation of actual connecting orbits and bifurcation orbits that could arise under a continuation in a parameterized family of flows for which the Δ^r and T^r are actual connection matrices and transition matrices, respectively.

The graphs presented in §5 are schematic, representing structure we could observe in an actual parameterized family of flows. In §6, we prove that, under appropriate conditions, in a parameterized family of flows with connection matrices Δ^r and transition matrices T^r, the bifurcation orbits identified in the schematics must arise.

2. *Dynamical spectral sequences*

For a filtered graded chain (C, ∂) complex we can define a *spectral sequence*. In fact, given a chain complex (C, ∂) endowed with an increasing filtration F_pC, a *spectral sequence* $\{E^r, d^r\}$, $r \geq 0$, is a sequence of chain complexes, where:

- E^r is bigraded module over a principal ideal domain R. In this paper we work with $R = \mathbb{Z}_2$ and hence the bigraded modules E^r are actually vector spaces over $\mathbb{Z}_2$.
- d^r is a differential with bidegree $(-r, r-1)$ in E^r, that is, a collection of homomorphisms $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ for all p and q, such that $d^r \circ d^r = 0$.

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- for $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$, where the *homology module* $H(E^r)$ is the bigraded module

$$H_{p,q}(E^r) = \frac{\mathrm{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\mathrm{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

The algebraic formulas for the vector spaces are

$$E^r_{p,q} = Z^r_{p,q}/(Z^{r-1}_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}),$$

where

$$Z^r_{p,q} = \{c \in F_p C_{p+q} \mid \partial c \in F_{p-r} C_{p+q-1}\}.$$

For more details on spectral sequences see [**Sp**].

Since each Morse set is a singularity of index k, the only q such that $E^r_{p,q}$ is non-zero is $q = k - p$. Hence, we will omit reference to q, that is, E^r_p is in fact $E^r_{p,k-p}$.

The module Z^r_p consists of chains in $F_p C$ with boundary in $F_{p-r} C$. This makes it natural to look at chains associated to the columns of the connection matrix to the left of and including the $(p+1)$th column. This guarantees that any linear combination of chains respects the filtration. Furthermore, since the boundary of the chains must be in $F_{p-r} C$ we must consider columns or linear combinations that respect the filtration and that have the property that the entries in rows $i > (p - r + 1)$ are all zeros. Hence, the significant entry in the connection matrix is determined by the element on the rth diagonal on the $(p - r + 1)$th row and $(p + 1)$th column.

In [**CRS, MRS**] we proved that the connection matrix Δ determines a spectral sequence, that is, it determines the spaces E^r and how it induces the differentials d^r. To achieve this we apply the sweeping method to the connection matrix Δ. This method singles out important non-zero entries, which we will refer to as primary pivots and change of basis pivots, of the rth diagonal of Δ^r in order to define a matrix Δ^{r+1}. At each step, Δ^{r+1} is obtained from Δ^r by a change of basis.

Moreover, as r increases, the $\mathbb{Z}_2$-modules E^r_p change generators. In practice, the generators of the chain complex C mentioned above are very specific: singularities in the Morse case. The domain of d^r, E^r, is a certain quotient of a subgroup of C. Elements in this domain are represented by elements of C whose appropriate classes are in the kernels of all previous differentials d^s, $s < r$.

Note that the rth diagonal of Δ which intersects Δ_k has entries $\Delta_{p+1-r,p+1}$ that represent the number of intersections mod 2 between the unstable and stable spheres determined by the connections between unstable and stable manifolds of $M(p+1)$ and $M(p+1-r)$ for $p \in \{r, \ldots, m-1\}$. Clearly, if the $(p+1)$th column intersects the submatrix Δ_k, then $M(p+1)$ and $M(p+1-r)$ are respectively singularities of Morse index k and $k-1$ which we denote by h_k and h_{k-1}. These singularities are in the filtrations $F_p \backslash F_{p-1}$ and $F_{p-r} \backslash F_{p-r-1}$, respectively. Hence, we say that the pair (h_k, h_{k-1}) has *gap* r. In summary, the rth diagonal when intersected with Δ_k is registering information of numerically consecutive singularities of Morse indices k and $k-1$ with gap r. We will use the same notation to indicate an elementary chain of C.

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3. *Sweeping method*

In this section we present the sweeping method, which constructs recursively a family of matrices Δ^r for $r \geq 0$, where $\Delta^0 = \Delta$, by considering at each stage the rth diagonal. This family of matrices will be used to determine the spectral sequence (E^r, d^r).

For a fixed diagonal r the method described below must be applied for all k simultaneously.

A—Initial step

(1) Consider all columns h_k together with all rows h_{k-1} in Δ. Let $\Delta_{k_{i,j}}$ be the entries in Δ, where the ith row is h_{k-1} and the jth column is h_k.

Let ξ_1 be the first diagonal of Δ which contains non-zero entries $\Delta_{k_{i,j}}$, which will be denoted as *index k primary pivots*. It follows that for each non-zero $\Delta_{k_{i,j}}$ on ξ_1 the entries $\Delta_{k_{s,j}}$ for $s > i$ are all zero. These entries must be zero otherwise they would have been detected as primary pivots on a ξ diagonal for $\xi < \xi_1$.

We end this first step by defining Δ^{ξ_1} as Δ with the index k primary pivots on the ξ_1th diagonal marked.

(2) Consider the matrix Δ^{ξ_1} and let $\Delta^{\xi_1}_{k_{i,j}}$ be the entries in Δ^{ξ_1}, where the ith row is h_{k-1} and the jth column h_k. Let ξ_2 be the first diagonal greater than ξ_1 which contains non-zero entries $\Delta^{\xi_1}_{k_{i,j}}$. We now construct a matrix Δ^{ξ_2} following the procedure.

Given a non-zero entry $\Delta^{\xi_1}_{k_{i,j}}$ on the ξ_2th diagonal of Δ^{ξ_1}:

(a) If there are no primary pivots on the ith row and the jth column, mark it as an index k primary pivot and the numerical value of the entry remains the same, $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

(b) If this is not the case, consider the entries in the jth column and in the sth row with $s > i$ in Δ^{ξ_1}.

(b1) If there is an index k primary pivot in an entry in the jth column and in a row s, with $s > i$, then the numerical value remains the same and the entry is left unmarked, $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

(b2) If there are no primary pivots in the jth column below $\Delta^{\xi_1}_{k_{i,j}}$ then there is an index k primary pivot on the ith row, say in the tth column of Δ^{ξ_1}, with $t < j$. In this case the entry remains the same, but the entry $\Delta^{\xi_1}_{k_{i,j}}$ is marked as a *change of basis pivot*, $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

Note that we have defined a matrix Δ^{ξ_2} which is actually equal to Δ^{ξ_1} except that the ξ_2th diagonal is marked with primary and change of basis pivots. See Figure 2.

B—Intermediate step

In this step we consider a matrix Δ^r with the primary and change of basis pivots marked on the ξth diagonal for all $\xi \leq r$. We will now describe how Δ^{r+1} is defined. If there is no change of basis pivot on the rth diagonal we go directly to step B.2, that is, $\Delta^{r+1} = \Delta^r$ with the $(r+1)$th diagonal marked with primary and change of basis pivots as in B.2.

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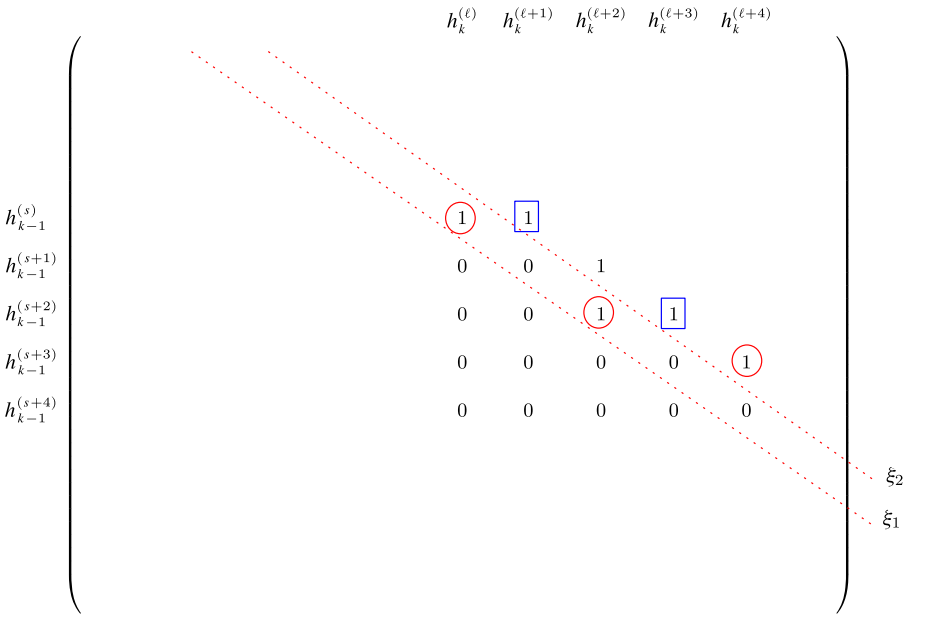


FIGURE 2. Diagonals ξ_1 and ξ_2.

B.1—Change of basis

Suppose that $\Delta^r_{k_{i,j}}$ is a change of basis pivot on the rth diagonal. Since we have a change of basis pivot in the ith row, there is a column, namely the tth column, associated to a k-chain such that $\Delta^r_{k_{i,t}}$ is a primary pivot. Then perform a change of basis on Δ^r by adding the tth column of Δ^r to the jth column of Δ^r, in order to zero out the entry $\Delta^r_{k_{i,j}}$ without introducing non-zero entries in $\Delta^r_{k_{s,j}}$ for $s > i$. The notation $h_k^{(\ell)}$ indicates the elementary k-chain associated to the ℓth column of Δ.

Once this is done, we obtain a k-chain associated to the jth column of Δ^{r+1}. It is a linear combination over $\mathbb{Z}_2$ of the tth h_k columns of Δ^r and the jth column of Δ^r such that $\Delta^{r+1}_{k_{i,j}} = 0$. It is also a linear combination of h_k columns of Δ on and to the left of the jth column.

Observe that if the $\overline{\ell}$th column of Δ^r is an h_k column, it corresponds to a linear combination over $\mathbb{Z}_2$, $\sigma_k^{(\overline{\ell}),r} = \sum_{\ell=\kappa}^{\overline{\ell}} c_\ell^{\overline{\ell},r} h_k^{(\ell)}$, of h_k columns of Δ, where the κth h_k column is the first column in Δ associated to a k-chain. The notation of $\sigma_k^{(\overline{\ell}),r}$ indicates the Morse index k and the $\overline{\ell}$th column of Δ^r. Hence if the jth column of Δ^{r+1} is an h_k column, it will be

$$\begin{aligned}\sigma_k^{(j),r+1} &= \underbrace{\sum_{\ell=\kappa}^{j} c_\ell^{j,r} h_k^{(\ell)}}_{\sigma_k^{(j),r}} + q_t \underbrace{\sum_{\ell=\kappa}^{t} c_\ell^{t,r} h_k^{(\ell)}}_{\sigma_k^{(t),r}} \\ &= c_\kappa^{j,r+1} h_k^{(\kappa)} + c_{\kappa+1}^{j,r+1} h_k^{(\kappa+1)} + \cdots + c_{j-1}^{j,r+1} h_k^{(j-1)} + c_j^{j,r+1} h_k^{(j)}. \qquad (1)\end{aligned}$$

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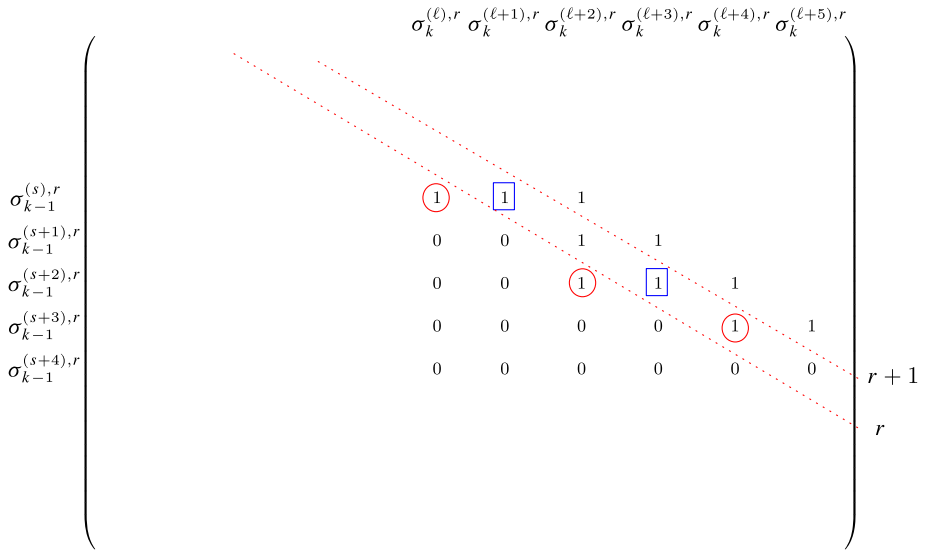


FIGURE 3. Sweeping method: Δ^r.

It is clear that the first column of any Δ_k cannot undergo any change of basis since there is no column to its left.

Once the above procedure is done for all change of basis pivots of the rth diagonal of Δ^r we can define a change of basis matrix.

Therefore the matrix Δ^{r+1} has numerical values determined by the change of basis over $\mathbb{Z}_2$ of Δ^r. In particular, all the change of basis pivots on the rth diagonal Δ^r are zero in Δ^{r+1}. See Figures 3 and 4.

B.2—Marking the $(r+1)$th diagonal of Δ^{r+1}

Consider the matrix Δ^{r+1} defined in the previous step. We mark the $(r+1)$th diagonal with primary and change of basis pivots as follows.

Given a non-zero entry $\Delta^{r+1}_{k_{i,j}}$:

(1) If there are no primary pivots on the ith row and the jth column, mark it as an index k primary pivot.

(2) If this is not the case, consider the entries in the jth column and in the sth row with $s > i$ in Δ^{r+1}.

  (b1) If there is an index k primary pivot in the entries in the jth column below $\Delta^{r+1}_{k_{i,j}}$ then leave the entry unmarked.

  (b2) If there are no primary pivots in the jth column below $\Delta^{r+1}_{k_{i,j}}$ then there is an index k primary pivot on the ith row, say in the tth column of Δ^{r+1}, with $t < j$. In this case, mark it as a change of basis pivot. See Figure 4.

C—Final step

We repeat the above procedure until all diagonals have been considered.

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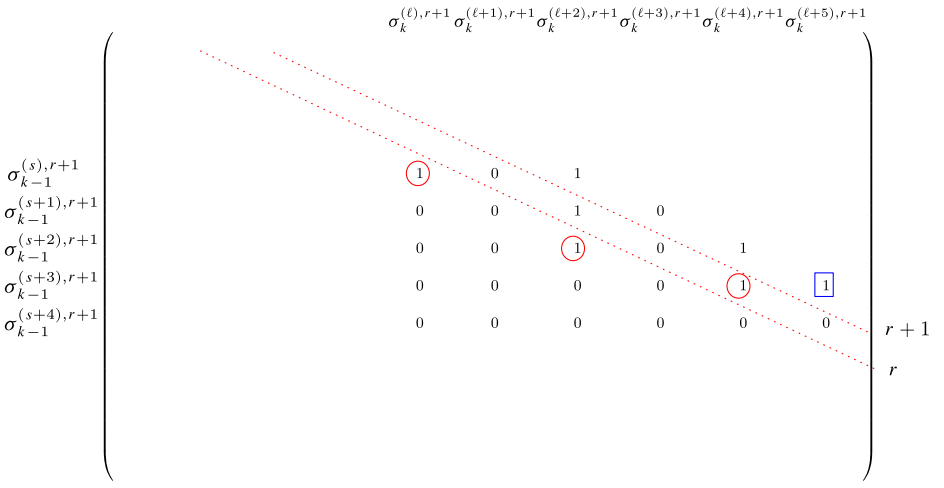


FIGURE 4. Sweeping method: Δ^{r+1}.

It is easy to see that all Δ^r are upper triangular and $\Delta^r \circ \Delta^r = 0$ since they are recursively obtained from the initial connection matrix Δ by change of basis over $\mathbb{Z}_2$. Note that if the entry $\Delta^r_{k_{p-r+1,p+1}}$ has been identified by the sweeping method as a primary pivot or a change of basis pivot then $\Delta^r_{k_{s,p+1}} = 0$ for all $s > p - r + 1$.

In [**MRS**] it is shown how the $\mathbb{Z}_2$-vector spaces E^r_p are determined by the connection matrix Δ. In order to do this, a formula for $Z^r_{p,k-p}$ is established in terms of the $\sigma_k^{r,p}$ determined by the sweeping method:

$$Z^r_{p,k-p} = \mathbb{Z}_2[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}],$$

where κ is the first column in Δ associated to a k-chain and $\mu^{(j),\zeta} = 0$ whenever the primary pivot of the jth column is below the $(p - r + 1)$th row and $\mu^{(j),\zeta} = 1$ otherwise. Moreover, the sweeping method also shows how Δ induces the differentials d^r_p in the spectral sequence. Whenever E^r_p and E^r_{p-r} are both non-zero, the map $d^r_p : E^r_p \to E^r_{p-r}$ is multiplication by the entry $\Delta^r_{p-r+1,p+1}$ which is either a primary pivot or a zero with a column of zero entries below it. Otherwise, d^r_p is zero.

4. *Transition matrix associated to the spectral sequence*

Let Δ be a connection matrix over $\mathbb{Z}_2$ associated to the Morse decomposition $\mathcal{D}(M)$, where each Morse set is a non-degenerate singularity of the gradient flow φ, and let $\Delta^1, \ldots, \Delta^F$ be the matrices obtained from the sweeping method applied to Δ. These matrices are also connection matrices (see [**Fr3**, p. 897]).

Let T^r be the change of basis matrix at the rth step of the sweeping method; that is, T^r is such that

$$\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r.$$

Each matrix T^r is upper triangular. In fact, the pth column of Δ^{r+1} is a linear combination of columns on and to the left of the pth column of Δ^r. Moreover, recall that we

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have a collection of chain complexes (C^r, Δ^r), where $C_k^r = \mathbb{Z}_2[\sigma_k^{(k_1),r}, \ldots, \sigma_k^{(k_f),r}] = \mathbb{Z}_2[h_k^{(k_1)}, \ldots, h_k^{(k_f)}]$ and

$$\sigma_k^{(p),r+1} = 1 \cdot \sigma_k^{(p),r} + 1 \cdot \sigma_k^{(t),r},$$

where Δ_{it}^r is the primary pivot, that is, the coefficient of the k-chain associated to the pth column of Δ^r on the change of basis is equal to one. Hence, T^r is upper triangular and its main diagonal has entries all equal to one. Moreover, each T^r is an isomorphism and

$$\Delta^{r+1} = (T^r)^{-1} \Delta^r T^r,$$

that is, $T^r \Delta^{r+1} = \Delta^r T^r$ and hence

$$\Delta^r T^r + T^r \Delta^{r+1} = 0.$$

We have thus proven the following proposition.

PROPOSITION 4.1. *Each matrix T^r associated to the spectral sequence satisfies the following properties:*

(1) $\Delta T + T\Delta' = 0$;
(2) *T is an isomorphism;*
(3) *T is an upper triangular matrix.*

The matrices T^r are called *transition matrices associated to the spectral sequence* and, as shown above, they are determined by the sweeping algorithm.

Note that, as a consequence of the sweeping method, if $T_{ij}^r = 1$ then $T_{ij}^\ell = 0$ for all $\ell \neq r$.

Let $T^1, \ldots, T^f$ be the transition matrices associated to the sweeping method and let $T = T^1 \cdots T^f$ be the product matrix. Hence we have $f = F - 1$ and

$$\Delta^F = T^{-1} \Delta^1 T.$$

5. *Continuation associated to the sweeping method*

In the remaining sections, we examine to what extent the algebra associated to the sweeping method reflects changes that occur under the continuation of a Morse decomposition whose connection matrix changes according to the sweeping method. As we have seen, the sweeping method performs algebraic cancelations. From the point of view of the dynamics, what happens is a change in the connecting orbits: longer orbits appear due to cancelations of shorter ones. To see this more clearly we will introduce in §5.2 a schematic which is a directed graph associated to the connection matrices and transition matrices. The vertices represent the singularities and the directed edges represent the connecting orbits mod 2. For each case we construct directed graphs representing the connection matrices Δ^r and Δ^{r+1} to describe the situation prior to and after the use of the sweeping method. We refer to these as flow schematics. We also construct a bifurcation schematic which represents the intermediary situation between Δ^r and Δ^{r+1}. For now, we analyze a two-dimensional example.

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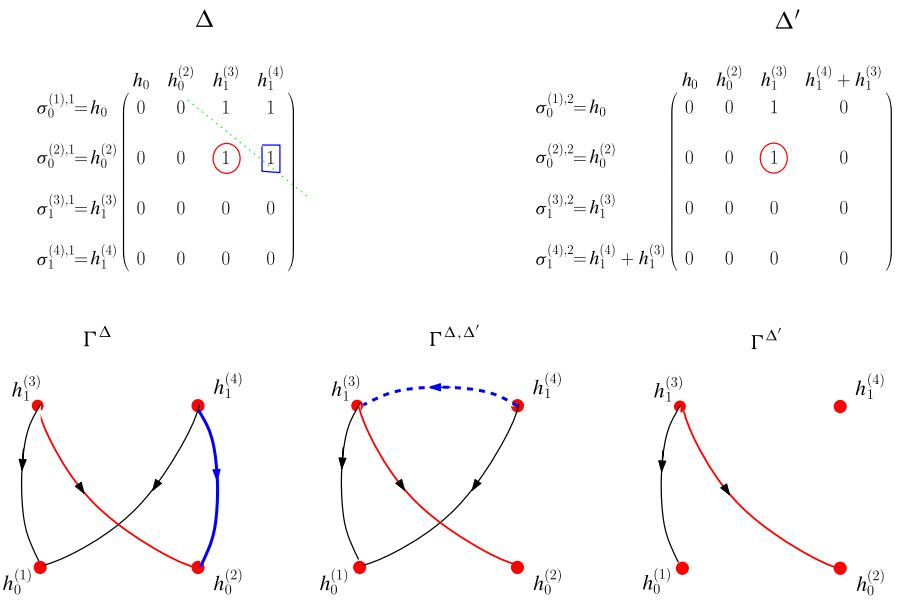


FIGURE 5. Sweeping: connection matrices and associated schematics.

5.1. *Continuation example in dimension two.* We start analyzing the case where we have two sinks and two saddles. This particular case will help in understanding the general case. Let Δ be a connection matrix over $\mathbb{Z}_2$ associated to the Morse decomposition $\mathcal{D}(M)$, where each Morse set is a non-degenerate singularity of the gradient flow φ, and let Δ' be the matrix obtained when we apply the sweeping method to Δ. The matrix Δ' is also a connection matrix (see [**Fr3**, p. 897]) and is similar to Δ.

In Figure 5 we present flow schematics representing the connection matrices Δ and Δ' and also the bifurcation schematic $\Gamma^{\Delta,\Delta'}$ which represents a bifurcation situation. The bifurcation schematic $\Gamma^{\Delta,\Delta'}$ is constructed by removing from the flow schematic Γ^{Δ} the directed edge joining $h_1^{(4)}$ and $h_0^{(2)}$ corresponding to the change of basis pivot $\Delta_{2,4}$ and inserting a directed edge (bifurcation orbit—dashed edge) connecting two vertices representing singularities of the same index $h_1^{(3)}$ and $h_1^{(4)}$. This reflects a change of basis in the sweeping method which adds the column containing the primary pivot to the column containing the change of basis pivot and thus is identified in the change of basis matrix as $T_{3,4} = 1$. The flow schematic representing the connection matrix Δ' can also be obtained from the bifurcation schematic by counting mod 2 all paths joining vertices representing consecutive index singularities, including the paths which contain bifurcation edges. The only exceptions are paths which contain the bifurcation orbit and orbits corresponding to the primary pivots since these paths are equivalent to orbits corresponding to change of basis pivots. In Figure 5 this can be seen in the paths $(h_1^{(4)}, h_1^{(3)}, h_0^{(1)})$ and $(h_1^{(4)}, h_0^{(1)})$ in $\Gamma^{\Delta,\Delta'}$. Since we are counting this as 0 mod 2 paths, there is no path connecting $h_1^{(4)}$ to $h_0^{(1)}$ in $\Gamma^{\Delta'}$. The exceptional case is the one counting the bifurcation orbit $(h_1^{(4)}, h_1^{(3)})$ and $(h_1^{(3)}, h_0^{(2)})$. The path $(h_1^{(4)}, h_1^{(3)}, h_0^{(2)})$ is equivalent to the path $(h_1^{(4)}, h_0^{(2)})$ which

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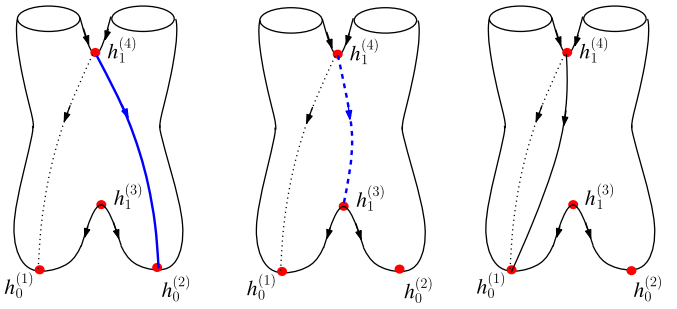


FIGURE 6. Parameterized family of flows realizing connection matrices Δ and Δ' and the bifurcation between them.

corresponds to a change of basis pivot. This is why there is no edge connecting $h_1^{(4)}$ to $h_0^{(2)}$ in $\Gamma^{\Delta'}$. We will describe this in more detail in §5.2.

While the schematics in Figure 5 are constructed from the matrices Δ and Δ', they represent connections that occur in a parameterized family of flows in which Δ and Δ' are connection matrices. We illustrate such a parameterized family in Figure 6.

The connection matrix Δ is associated with the flow on the left and the connection matrix Δ' is associated with the flow on the right. The flow in the middle is the bifurcation situation where the $4 \to 2$ connecting orbit is broken, introducing a $4 \to 3$ saddle–saddle connection, which then leads to the second $4 \to 1$ connection.

In §6 we prove a general theorem describing conditions under which, given connection matrices Δ and Δ' and a transition matrix T between them, and given a parameterized family of flows where Δ and Δ' are the connection matrices at the beginning and end parameters, the transition matrix T identifies bifurcations that must occur in the transition across the parameter values. In the setting we consider there, the transition matrix T is interpreted as a so-called singular transition matrix between Δ and Δ', arising as a block in a larger connection matrix obtained by putting a product-flow structure on the parameterized family of flows. In the product flow, the matrices Δ, Δ', and T combine to form the 'larger connection matrix' as illustrated in Figure 7.

In Figure 8, we combine the schematics in Figure 5 into one schematic which represents the continuation situation occurring in the parameterized family of flows. We illustrate there that the four Morse sets continue across in the parameterized family, and we see the bifurcation that occurs, starting with a $4 \to 2$ connection, transitioning to the $4 \to 3$ connection, and finally resulting in 0 (mod 2) connections from 4 to 1.

5.2. *Schematics associated to the sweeping method.* Our goal is to describe the sweeping method in terms of schematics: for each connection matrix we have associated a unique flow schematic.

Let Δ be a connection matrix over $\mathbb{Z}_2$ associated to the Morse decomposition $\mathcal{D}(M)$, and let $\Delta^1, \ldots, \Delta^F$ be the connection matrices obtained by applying the sweeping method to Δ. Let $T^1, \ldots, T^f$ be the associated transition matrices, where $f = F - 1$, such that $\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r$. We construct a family of schematics representing the connection

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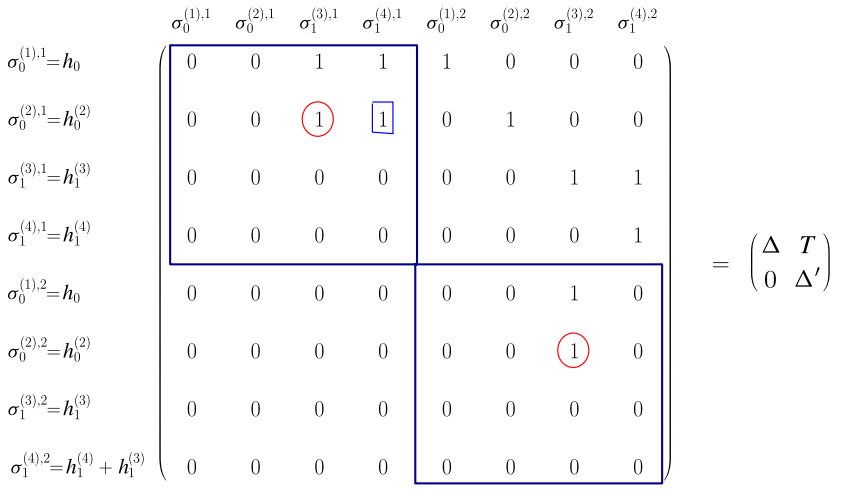


FIGURE 7. A product flow connection matrix defined by Δ, Δ' and T.

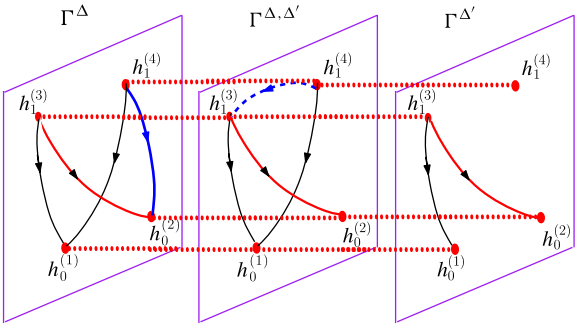


FIGURE 8. Combining schematics into one for the parameterized family.

matrices and also the changes of basis that are performed in order to obtain one matrix from the previous one. A *flow schematic* for Δ^r is a directed graph Γ^r such that:

- each vertex in Γ^r represents a singularity associated to the connection matrix Δ^r;
- each non-zero entry Δ^r_{ij} in Δ^r is represented in Γ^r by a directed edge denoted by Γ^r_{ij}.

For clarity, the vertices representing singularities in the schematics appear with decreasing index from top to bottom and those of the same index are represented on the same level appearing from left to right respecting the order in the index k block of Δ^r from the leftmost column to the rightmost column.

A *bifurcation schematic* $\Gamma^{r,r+1}$ associated to T^r is a directed graph obtained from Γ^r by removing all edges corresponding to the change of basis pivots of Δ^r and inserting edges corresponding to the non-zero off-diagonal entries of T^r (Figure 9). Consider the collection of all the directed paths in $\Gamma^{r,r+1}$ joining vertices representing singularities of consecutive indices. These paths may use the bifurcation edge. However, they must

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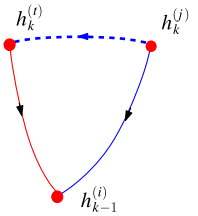


FIGURE 9. $\Gamma^{r,r+1}$.

always join vertices representing consecutive singularities, that is, start at an h_k and end at a h_{k-1}. A *local schematic continuation* is $(\Gamma^r, \Gamma^{r,r+1}, \Gamma^{r+1})$, whenever Γ^{r+1} is obtained from $\Gamma^{r,r+1}$ counting all of these directed paths mod 2 with the exception of the path containing the bifurcation edge followed by the edge corresponding to the primary pivot. A *schematic continuation* associated to the sweeping method is the collection $\mathcal{C} = (\Gamma^1, \Gamma^{1,2}, \Gamma^2, \Gamma^{2,3}, \ldots, \Gamma^{F-1,F}, \Gamma^F)$ if each triple $(\Gamma^r, \Gamma^{r,r+1}, \Gamma^{r+1})$ is a local schematic continuation.

Remark 5.1. We could have defined a bifurcation schematic as a directed graph obtained from Γ^r by only inserting edges corresponding to the non-zero off-diagonal entries of T^r and not removing the edge corresponding to the change of basis pivot. In this case $(\Gamma^r, \Gamma^{r,r+1}, \Gamma^{r+1})$ would be a local schematic continuation whenever Γ^{r+1} is obtained from $\Gamma^{r,r+1}$ counting all the directed paths mod 2 with no exceptions. In reality, both definitions are equivalent, since whenever Δ^r_{ij} is a change of basis pivot corresponding to the edge $\Gamma^r_{ij} = (h_k^{(j)}, h_{k-1}^{(i)})$, there is also a path $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(i)})$ in $\Gamma^{r,r+1}$ containing the bifurcation edge $(h_k^{(j)}, h_{k-1}^{(t)})$ and the edge corresponding to the primary pivot $(h_k^{(t)}, h_{k-1}^{(i)})$. Hence, in this situation, there are always two paths connecting $h_k^{(j)}$ and $h_{k-1}^{(i)}$. We chose the former definition due to its similarity to the behavior of orbits in a flow undergoing bifurcation.

Our goal is to provide a dynamical interpretation for the sweeping method by proving the existence of a schematic continuation as in the following theorem.

THEOREM 5.2. *Let Δ^1 and Δ^F be the initial and final connection matrices obtained from the sweeping method and let T be the transition matrix such that $\Delta^F = T^{-1}\Delta^1 T$. Then there exists a schematic continuation $\mathcal{C} = (\Gamma^1, \Gamma^{1,2}, \Gamma^2, \Gamma^{2,3}, \ldots, \Gamma^{F-1,F}, \Gamma^F)$ such that the total order $<$ associated to the sweeping method filtration continues.*

Proof. All we have to prove is that the diagram $(\Gamma^r, \Gamma^{r,r+1}, \Gamma^{r+1})$ is a local schematic continuation. Let $\widetilde{\Gamma}^r$ be the flow schematic where each edge between two vertices representing singularities is the number of paths mod 2. A directed edge $\widetilde{\Gamma}^r_{sp}$ in $\widetilde{\Gamma}^r$ between two vertices representing singularities with consecutive indices $h_k^{(p)}$ and $h_{k-1}^{(s)}$ exists if the number of paths mod 2 in $\Gamma^{r,r+1}$ between the $h_k^{(p)}$ and $h_{k-1}^{(s)}$ is equal to one. This holds except when $\Delta^r_{sp} = \Delta^r_{ij}$ is a change of basis pivot represented by the edge Γ^r_{ij}. In this case, we count the path starting at $h_k^{(j)}$ and ending at $h_{k-1}^{(i)}$ together with Γ^r_{ij}. This results in zero

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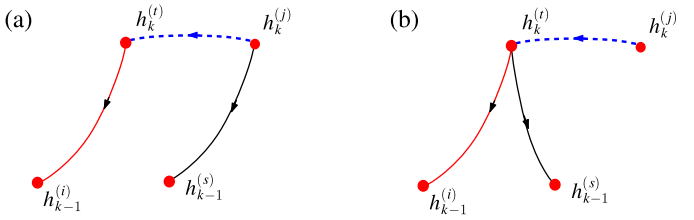


FIGURE 10. $\Gamma^{r,r+1}$ Case 1A.

mod 2. Hence, there is no edge $\widetilde{\Gamma}^r_{ij}$. We will prove that $\widetilde{\Gamma}^r$ is equal to the flow schematic Γ^{r+1} for the connection matrix Δ^{r+1}.

Recall that the bifurcation schematic $\Gamma^{r,r+1}$ by definition has the same edges as Γ^r_{ij} except for the ones corresponding to change of basis pivots. These edges corresponding to the change of basis pivots are replaced by edges between two vertices representing singularities of the same index. In fact, let $\Delta^r_{k_{ij}}$ be a change of basis pivot and Γ^r_{ij} be the edge between the two vertices representing singularities with consecutive indices $h_k^{(j)}$ and $h_{k-1}^{(i)}$. If Γ^r_{it} is the edge corresponding to the primary pivot on the ith row joining $h_k^{(t)}$ and $h_{k-1}^{(i)}$, the bifurcation schematic $\Gamma^{r,r+1}$ has an edge $\Gamma^{r,r+1}_{tj}$ which joins two vertices representing singularities of the same index $h_k^{(j)}$ and $h_k^{(t)}$. Note that the edges in $\Gamma^{r,r+1}$ between two singularities of the same index $h_k^{(j)}$ and $h_k^{(t)}$ correspond to a change of basis performed on the connection matrix Δ^r. The change of basis is given by $\sigma_k^{(j),r+1} = \sigma_k^{(j),r} + \sigma_k^{(t),r}$ and hence it corresponds to a non-zero entry T^r_{tj} in the transition matrix T^r. Our motivation is that this represents dynamically a bifurcation that has a connection between singularities of the same index and thus we refer to this edge as the bifurcation edge.

This proof will be divided into cases, and for each case it is shown that Γ^{r+1}_{sj} exists if and only if $\widetilde{\Gamma}^r_{sj}$ exists.

Case 1. There is a bifurcation edge starting at $h_k^{(j)}$. In this case, there is a change of basis pivot in the jth column. Let Δ^r_{ij} be this change of basis pivot and Δ^r_{it} the primary pivot in the ith row.

A. Suppose that Γ^{r+1}_{sj} exists, that is, $\Delta^{r+1}_{sj} = 1$. In this case, we must have $s \neq i$ since Δ^r_{ij} is a change of basis pivot and hence $\Delta^{r+1}_{ij} = 0$, that is, Γ^{r+1}_{ij} does not exist. From the sweeping method we know that $\sigma_k^{(j),r+1} = \sigma_k^{(j),r} + \sigma_k^{(t),r}$ and then, for each s, $\Delta^{r+1}_{sj} = \Delta^r_{sj} + \Delta^r_{st}$. Hence, there are two possibilities.

(1) $\Delta^r_{sj} = 1$ and $\Delta^r_{st} = 0$. In this case, we know that Γ^r_{sj} exists and Γ^r_{st} does not and hence $\Gamma^{r,r+1}_{sj}$ exists and $\Gamma^{r,r+1}_{st}$ does not. See Figure 10(a). Thus, there is just one path between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ in $\Gamma^{r,r+1}$, namely $(h_k^{(j)}, h_{k-1}^{(s)})$, which means that $\widetilde{\Gamma}^r_{sj}$ exists.

(2) $\Delta^r_{sj} = 0$ and $\Delta^r_{st} = 1$. In this case, we know that Γ^r_{sj} does not exist and Γ^r_{st} does and hence $\Gamma^{r,r+1}_{sj}$ does not exist and $\Gamma^{r,r+1}_{st}$ does. See Figure 10(b). Thus, there is just

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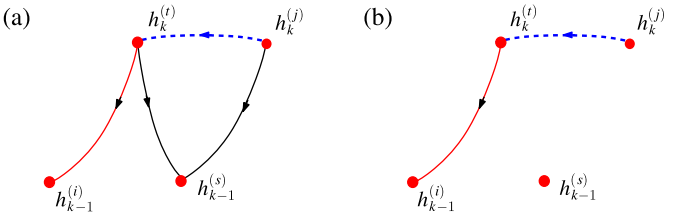


FIGURE 11. $\Gamma^{r,r+1}$ Case 1B.

one path between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ in $\Gamma^{r,r+1}$, namely $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(s)})$, which means that $\widetilde{\Gamma}_{sj}^r$ exists.

B. Now suppose that Γ_{sj}^{r+1} does not exist. Then, by definition, $\Delta_{sj}^{r+1} = 0$. Now we have to consider both cases: when $s \neq i$ and when $s = i$. As in the previous case, $\Delta_{sj}^{r+1} = \Delta_{sj}^r + \Delta_{st}^r$ for each s. Hence, there are three possibilities.

(1) $\Delta_{sj}^r = 1$ and $\Delta_{st}^r = 1$ for $s \neq i$. In this case, we know that both Γ_{sj}^r and Γ_{st}^r exist. Since Δ_{st}^r is not a change of basis pivot and, for $s \neq i$, neither is Δ_{sj}^r, then $\Gamma_{sj}^{r,r+1}$ and $\Gamma_{st}^{r,r+1}$ exist. See Figure 11(a). Thus, there are two paths between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ in $\Gamma^{r,r+1}$, namely $(h_k^{(j)}, h_{k-1}^{(s)})$ and $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(s)})$. Counting mod 2, we have that $\widetilde{\Gamma}_{sj}^r$ does not exist.

(2) $\Delta_{sj}^r = 0$ and $\Delta_{st}^r = 0$ for $s \neq i$. In this case, Γ_{sj}^r and Γ_{st}^r do not exist, and hence neither do $\Gamma_{sj}^{r,r+1}$ and $\Gamma_{st}^{r,r+1}$. See Figure 11(b). Thus, there is no path between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ in $\Gamma^{r,r+1}$, which means that $\widetilde{\Gamma}_{sj}^r$ does not exist.

(3) $s = i$. By definition of $\widetilde{\Gamma}^r$, we know that $\widetilde{\Gamma}_{ij}^r$ does not exist. This is the exceptional case of the definition $\widetilde{\Gamma}^r$.

Case 2. There is a bifurcation edge ending at $h_{k-1}^{(s)}$. In this case, there is a primary pivot in the sth column, namely, Δ_{qs}^r, and a change of basis pivot in this same row, that is, the qth row, and in the rth diagonal. Let $\Delta_{qs'}^r$ be this change of basis pivot.

Also, by the sweeping method, we have that $\sigma_k^{(j),r+1} = \sigma_k^{(j),r}$ and $\partial(\sigma_k^{(j),r+1}) = \partial(\sigma_k^{(j),r})$. Equivalently,

$$\Delta_{sj}^r \sigma_{k-1}^{(s),r} + \Delta_{s'j}^r \sigma_{k-1}^{(s'),r} = \Delta_{sj}^{r+1} \sigma_{k-1}^{(s),r+1} + \Delta_{s'j}^{r+1} \sigma_{k-1}^{(s'),r+1}.$$

Since

$$\sigma_{k-1}^{(s),r+1} = \sigma_{k-1}^{(s),r}$$

and

$$\sigma_{k-1}^{(s'),r+1} = \sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r},$$

it follows that

$$\Delta_{sj}^r \sigma_{k-1}^{(s),r} + \Delta_{s'j}^r \sigma_{k-1}^{(s'),r} = \Delta_{sj}^{r+1} \sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1} (\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}). \tag{2}$$

We now prove in this case that Γ_{sj}^{r+1} exists if and only if $\widetilde{\Gamma}_{sj}^r$ exists. Let us suppose that $\widetilde{\Gamma}_{sj}^r$ exists. This is equivalent to saying that there is a path between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ in $\Gamma^{r,r+1}$.

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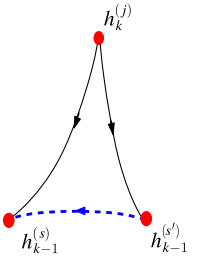


FIGURE 12. $\Gamma^{r,r+1}$ Case 2.

Since Δ_{sj}^r is not a change of basis pivot there are two kinds of possible paths in $\Gamma^{r,r+1}$ between $h_k^{(j)}$ and $h_{k-1}^{(s)}$: the path $(h_k^{(j)}, h_{k-1}^{(s)})$ which is the directed edge $\Gamma_{sj}^{r,r+1}$ and the path $(h_k^{(j)}, h_{k-1}^{(s')}, h_{k-1}^{(s)})$. See Figure 12. In the first case, $\Gamma_{sj}^{r,r+1}$ exists and $\Gamma_{s'j}^{r,r+1}$ does not and in the latter case $\Gamma_{s'j}^{r,r+1}$ exists and $\Gamma_{sj}^{r,r+1}$ does not. Let us analyze each case in more detail. The two possibilities can be seen in Figure 12 by removing the corresponding edges each case requires.

(1) $\Gamma_{sj}^{r,r+1}$ exists and $\Gamma_{s'j}^{r,r+1}$ does not. Since $\Delta_{s'j}^r$ is not a change of basis pivot, this is equivalent to saying that Γ_{sj}^r exists and $\Gamma_{s'j}^r$ does not: $\Delta_{sj}^r = 1$ and $\Delta_{s'j}^r = 0$. Replacing this in (2), we have

$$\sigma_{k-1}^{(s),r} = \Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}),$$

which is possible if and only if $\Delta_{sj}^{r+1} = 1$ and $\Delta_{s'j}^{r+1} = 0$, that is, Γ_{sj}^{r+1} exists.

(2) $\Gamma_{sj}^{r,r+1}$ does not exist and $\Gamma_{s'j}^{r,r+1}$ does. Since Δ_{sj}^r is not a change of basis pivot, then $\Gamma_{s'j}^r$ exists and Γ_{sj}^r does not, that is, $\Delta_{s'j}^r = 1$ and $\Delta_{sj}^r = 0$. Replacing this in (2) we have

$$\sigma_{k-1}^{(s'),r} = \Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}),$$

which is possible if and only if $\Delta_{sj}^{r+1} = \Delta_{s'j}^{r+1} = 1$, that is, Γ_{sj}^{r+1} exists.

Case 3. There is a bifurcation edge starting at $h_k^{(j)}$ and a bifurcation edge ending at $h_{k-1}^{(s)}$. In this case, there is a change of basis pivot in the jth column. Let Δ_{ij}^r be this change of basis pivot and Δ_{it}^r the primary pivot in the ith row. Also, there is a primary pivot in the sth column, namely, Δ_{qs}^r and a change of basis pivot in this same row, that is, the qth row, and in the rth diagonal. Let $\Delta_{qs'}^r$ be this change of basis pivot.

In this case, there is a bifurcation edge in $\Gamma^{r,r+1}$ starting at $h_k^{(j)}$ and another bifurcation edge in $\Gamma^{r,r+1}$ ending at $h_{k-1}^{(s)}$. Also, in this case, $s \neq i$, since otherwise we would have a primary pivot in the ith row and one in the ith column and this is not possible by [**CRS**, Proposition 3.2]. Let Δ_{it}^r be the primary pivot in the ith row and $\Delta_{qs'}^r$ be the change of basis pivot in the qth row. It follows by the sweeping method that $\sigma_k^{(j),r+1} = \sigma_k^{(j),r} + \sigma_k^{(t),r}$ and $\partial(\sigma_k^{(j),r+1}) = \partial(\sigma_k^{(j),r} + \sigma_k^{(t),r})$. Equivalently,

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r+1} + \Delta_{s'j}^{r+1}\sigma_{k-1}^{(s'),r+1} = \Delta_{sj}^r\sigma_{k-1}^{(s),r} + \Delta_{s'j}^r\sigma_{k-1}^{(s'),r} + \Delta_{st}^r\sigma_{k-1}^{(s),r} + \Delta_{s't}^r\sigma_{k-1}^{(s'),r}. \quad (3)$$

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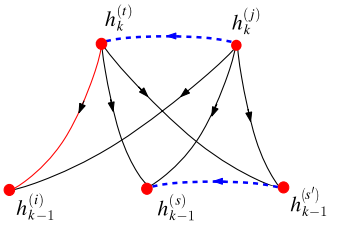


FIGURE 13. $\Gamma^{r,r+1}$ Case 3.

By the sweeping method $\sigma_{k-1}^{(s),r+1} = \sigma_{k-1}^{(s),r}$ and $\sigma_{k-1}^{(s'),r+1} = \sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}$. Substituting this in equation (3), we have

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma^{(s),r}) = \Delta_{sj}^{r}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r}\sigma_{k-1}^{(s'),r} + \Delta_{st}^{r}\sigma_{k-1}^{(s),r} + \Delta_{s't}^{r}\sigma_{k-1}^{(s'),r}. \tag{4}$$

Suppose that $\widetilde{\Gamma}_{sj}^{r}$ exists. This is equivalent to saying that there is a path in $\Gamma^{r,r+1}$ between $h_k^{(j)}$ and $h_{k-1}^{(s)}$. Let us first assume that $s' \neq i$. Since Δ_{ij}^{r} is a change of basis pivot and Δ_{qs}^{r} is a primary pivot, there are four kinds of possible paths between $h_k^{(j)}$ and $h_{k-1}^{(s)}$: the path $(h_k^{(j)}, h_{k-1}^{(s)})$ which is the directed edge $\Gamma_{sj}^{r,r+1}$, the path $(h_k^{(j)}, h_{k-1}^{(t)}, h_{k-1}^{(s)})$, the path $(h_k^{(j)}, h_{k-1}^{(s')}, h_{k-1}^{(s)})$ and the path $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(s')}, h_{k-1}^{(s)})$. See Figure 13. We will consider an analysis which can be seen in Figure 13 by removing the corresponding edges each case requires. We have the following cases.

(1) $\Gamma_{sj}^{r,r+1}$ exists and $\Gamma_{st}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$, $\Gamma_{s'j}^{r,r+1}$ do not. In this case, Γ_{sj}^{r} exists and since Δ_{it}^{r} is a primary pivot in the tth column there are no change of basis pivots in the tth column of Δ^{r}, that is, Δ_{st}^{r} and $\Delta_{s't}^{r}$ are not change of basis pivots. Hence, since $\Gamma_{st}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$ do not exist, Γ_{st}^{r} and $\Gamma_{s't}^{r}$ do not exist either. It follows that $\Delta_{st}^{r} = \Delta_{s't}^{r} = 0$. Thus, we have $\Delta_{st}^{r} = \Delta_{s't}^{r} = 0$, $\Delta_{sj}^{r} = 1$. Note that and $\Delta_{s'j}^{r}$ can either be zero or one. However, if $\Delta_{s'j}^{r} = 1$ then $\Gamma_{s'j}^{r}$ exists and $\Gamma_{s'j}^{r,r+1}$ does not, which just happens if $\Delta_{s'j}^{r}$ is a change of basis pivot and, in this case, we would have $s' = i$. Substituting in equation (4), we have

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}) = \sigma_{k-1}^{(s),r},$$

which is possible if and only if $\Delta_{sj}^{r+1} = 1$ and $\Delta_{s'j}^{r+1} = 0$, that is, Γ_{sj}^{r+1} exists.

(2) $\Gamma_{st}^{r,r+1}$ exists and $\Gamma_{sj}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$, $\Gamma_{s'j}^{r,r+1}$ do not. Since $s \neq i$, Δ_{sj}^{r} is not a change of basis pivot and hence Γ_{sj}^{r} does not exist. Using the same arguments as those in the previous item, we have that $\Gamma_{s't}^{r}$, $\Gamma_{s'j}^{r}$ also do not exist. Thus, we have $\Delta_{st}^{r} = 1$, $\Delta_{sj}^{r} = \Delta_{s't}^{r} = \Delta_{s'j}^{r} = 0$. Substituting in equation (4), we have

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}) = \sigma_{k-1}^{(s),r},$$

which is possible if and only if $\Delta_{sj}^{r+1} = 1$ and $\Delta_{s'j}^{r+1} = 0$, that is, Γ_{sj}^{r+1} exists.

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(3) $\Gamma^{r,r+1}_{s't}$ exists and $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{s'j}$ do not. Using the same arguments as those in the previous items, Γ^{r}_{st}, Γ^{r}_{sj} and $\Gamma^{r}_{s'j}$ do not exist. Thus, we have $\Delta^{r}_{st} = \Delta^{r}_{sj} = \Delta^{r}_{s'j} = 0$, $\Delta^{r}_{s't} = 1$. Substituting in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} exists.

(4) $\Gamma^{r,r+1}_{s'j}$ exists and $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{s't}$ do not. Using the same arguments as in items (1) and (2), we have that Γ^{r}_{sj}, Γ^{r}_{st}, $\Gamma^{r}_{s't}$ do not exist and hence $\Delta^{r}_{sj} = \Delta^{r}_{st} = \Delta^{r}_{s't} = 0$. In this case, equation (4) is equal to

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} exists.

(5) $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s'j}$ and $\Gamma^{r,r+1}_{st}$ exist and $\Gamma^{r,r+1}_{s't}$ does not. As we have seen in item (1), $\Delta^{r}_{s't}$ is not a change of basis pivot and thus, if $\Gamma^{r,r+1}_{s't}$ does not exist, then neither does $\Gamma^{r}_{s't}$. It follows that $\Delta^{r}_{s't} = 0$. In this case,

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1} = \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} exists.

(6) $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s'j}$ and $\Gamma^{r,r+1}_{s't}$ exist and $\Gamma^{r,r+1}_{st}$ does not. Analogously to the previous case, Δ^{r}_{st} is not a change of basis pivot, and hence Γ^{r}_{st} does not exist. It follows that $\Delta^{r}_{st} = 0$ and

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} + \sigma^{(s'),r}_{k-1} = \sigma^{(s),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = 1$ and $\Delta^{r+1}_{s'j} = 0$, that is, Γ^{r+1}_{sj} exists.

(7) $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s't}$ and $\Gamma^{r,r+1}_{st}$ exist and $\Gamma^{r,r+1}_{s'j}$ does not. Since $s' \neq i$, as in item (1) we have that $\Delta^{r}_{s'j}$ is not a change of basis pivot and hence $\Gamma^{r}_{s'j}$ does not exist. It follows that $\Delta^{r}_{s'j} = 0$ and, substituting in equation (4),

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1} = \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} exists.

(8) $\Gamma^{r,r+1}_{s't}$, $\Gamma^{r,r+1}_{s'j}$ and $\Gamma^{r,r+1}_{st}$ exist and $\Gamma^{r,r+1}_{sj}$ does not. Since $s \neq i$, Δ^{r}_{sj} is not a change of basis pivot, and hence Γ^{r}_{sj} does not exist. It follows that $\Delta^{r}_{sj} = 0$ and

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} + \sigma^{(s'),r}_{k-1} = \sigma^{(s),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = 1$ and $\Delta^{r+1}_{s'j} = 0$, that is, Γ^{r+1}_{sj} exists.

Let us now analyze the case when $s' = i$. This is the specific case where the edge $\Gamma^{r}_{s't}$ corresponds to the primary pivot Δ^{r}_{it} and the edge $\Gamma^{r}_{s'j}$ corresponds to the change of basis pivot Δ^{r}_{ij}. Hence, $\Gamma^{r,r+1}_{s'j}$ does not exist and $\Gamma^{r,r+1}_{s't}$ is not used to construct paths by

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definition. Then there are just two kinds of possible paths between $h_k^{(j)}$ and $h_{k-1}^{(s)}$: the path $(h_k^{(j)}, h_{k-1}^{(s)})$ which is the directed edge $\Gamma_{sj}^{r,r+1}$ and the path $(h_k^{(j)}, h_{k-1}^{(t)}, h_{k-1}^{(s)})$.

Since $\widetilde{\Gamma}_{sj}^{r}$ exists, we have two possibilities: either $\Gamma_{sj}^{r,r+1}$ exists and $\Gamma_{st}^{r,r+1}$ does not or $\Gamma_{st}^{r,r+1}$ exists and $\Gamma_{sj}^{r,r+1}$ does not. Since Δ_{st}^{r} and Δ_{sj}^{r} are not change of basis pivots, either Γ_{sj}^{r} exists and Γ_{st}^{r} does not or Γ_{st}^{r} exists and Γ_{sj}^{r} does not, that is, either $\Delta_{sj}^{r} = 1$ and $\Delta_{st}^{r} = 0$ or $\Delta_{st}^{r} = 1$ and $\Delta_{sj}^{r} = 0$. Moreover, since $\Delta_{s't}^{r}$ is a primary pivot and $\Delta_{s'j}^{r}$ is a change of basis pivot, $\Delta_{s't}^{r} = \Delta_{s'j}^{r} = 1$ and $\Delta_{s'j}^{r+1} = 0$. It follows that equation (4) is equivalent to

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} = \Delta_{sj}^{r}\sigma_{k-1}^{(s),r} + \sigma_{k-1}^{(s'),r} + \Delta_{st}^{r}\sigma_{k-1}^{(s),r} + \sigma_{k-1}^{(s'),r} = \Delta_{sj}^{r}\sigma_{k-1}^{(s),r} + \Delta_{st}^{r}\sigma_{k-1}^{(s),r},$$

where the right-hand side of the equation is always equal to $\sigma_{k-1}^{(s),r}$. This is only possible if $\Delta_{sj}^{r+1} = 1$, that is, Γ_{sj}^{r+1} exists.

Now suppose that $\widetilde{\Gamma}_{sj}^{r}$ does not exist. This is equivalent to saying that there are zero paths mod 2 in $\Gamma^{r,r+1}$ between $h_k^{(j)}$ and $h_{k-1}^{(s)}$. Again, we will first assume that $s' \neq i$. Recall that there are four kinds of possible paths between $h_k^{(j)}$ and $h_{k-1}^{(s)}$: the path $(h_k^{(j)}, h_{k-1}^{(s)})$ which is the directed edge $\Gamma_{sj}^{r,r+1}$, the path $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(s)})$, the path $(h_k^{(j)}, h_{k-1}^{(s')}, h_{k-1}^{(s)})$ and the path $(h_k^{(j)}, h_k^{(t)}, h_{k-1}^{(s')}, h_{k-1}^{(s)})$. We have the following cases.

(1) $\Gamma_{sj}^{r,r+1}$, $\Gamma_{st}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$ and $\Gamma_{s'j}^{r,r+1}$ do not exist. Since Δ_{it}^{r} is a primary pivot in the tth column, it follows that, by the sweeping method, there are no change of basis pivots in the tth column of Δ^{r}, that is, Δ_{st}^{r} and $\Delta_{s't}^{r}$ are not change of basis pivots. Hence, if $\Gamma_{st}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$ do not exist, neither do Γ_{st}^{r} and $\Gamma_{s't}^{r}$. It follows that $\Delta_{st}^{r} = \Delta_{s't}^{r} = 0$. Also, since $s \neq i$ and $s' \neq i$, $\Delta_{s'j}^{r}$ and Δ_{sj}^{r} are not change of basis pivots and thus, $\Gamma_{s'j}^{r}$ and Γ_{sj}^{r} do not exist, that is, $\Delta_{s'j}^{r} = \Delta_{sj}^{r} = 0$. Thus, we have $\Delta_{st}^{r} = \Delta_{s't}^{r} = \Delta_{s'j}^{r} = \Delta_{sj}^{r} = 0$ and, substituting this in equation (4),

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}) = 0,$$

which is possible if and only if $\Delta_{sj}^{r+1} = \Delta_{s'j}^{r+1} = 0$, that is, Γ_{sj}^{r+1} does not exist.

(2) $\Gamma_{st}^{r,r+1}$, $\Gamma_{s't}^{r,r+1}$ exist and $\Gamma_{sj}^{r,r+1}$, $\Gamma_{s'j}^{r,r+1}$ do not. Using the same arguments as those in the previous item, we have that, since $s \neq i$ and $s' \neq i$, Γ_{sj}^{r} and $\Gamma_{s'j}^{r}$ do not exist. Thus, we have that Γ_{st}^{r}, $\Gamma_{s't}^{r}$ exist and Γ_{sj}^{r}, $\Gamma_{s'j}^{r}$ do not, that is, $\Delta_{st}^{r} = \Delta_{s't}^{r} = 1$ and $\Delta_{sj}^{r} = \Delta_{s'j}^{r} = 0$. Substituting this in equation (4), we have

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}) = \sigma_{k-1}^{(s),r} + \sigma_{k-1}^{(s'),r},$$

which is possible if and only if $\Delta_{sj}^{r+1} = 0$ and $\Delta_{s'j}^{r+1} = 1$, that is, Γ_{sj}^{r+1} does not exist.

(3) $\Gamma_{s't}^{r,r+1}$, $\Gamma_{s'j}^{r,r+1}$ exist and $\Gamma_{st}^{r,r+1}$, $\Gamma_{sj}^{r,r+1}$ do not. Using the same arguments as in item (1), we have that Γ_{st}^{r} and Γ_{sj}^{r} do not exist. Thus, we have that $\Gamma_{s't}^{r}$, $\Gamma_{s'j}^{r}$ exist and Γ_{st}^{r}, Γ_{sj}^{r} do not, that is, $\Delta_{s't}^{r} = \Delta_{s'j}^{r} = 1$ and $\Delta_{st}^{r} = \Delta_{sj}^{r} = 0$. Substituting this in equation (4), we have

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} + \Delta_{s'j}^{r+1}(\sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s),r}) = \sigma_{k-1}^{(s'),r} + \sigma_{k-1}^{(s'),r} = 0,$$

which is possible if and only if $\Delta_{sj}^{r+1} = \Delta_{s'j}^{r+1} = 0$, that is, Γ_{sj}^{r+1} does not exist.

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(4) $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{s'j}$ exist and $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s't}$ do not. Using the same arguments as in item (1), we have that $\Delta^r_{st} = \Delta^r_{s'j} = 1$ and $\Delta^r_{sj} = \Delta^r_{s't} = 0$. Substituting this in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = 0$ and $\Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} does not exist.

(5) $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s't}$ exist and $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{s'j}$ do not. Using the same arguments as in item (1), we have that $\Delta^r_{sj} = \Delta^r_{s't} = 1$ and $\Delta^r_{st} = \Delta^r_{s'j} = 0$. Substituting this in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = 0$ and $\Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} does not exist.

(6) $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{sj}$ exist and $\Gamma^{r,r+1}_{s't}$, $\Gamma^{r,r+1}_{s'j}$ do not. Using the same arguments as in the previous items, we have $\Delta^r_{st} = \Delta^r_{sj} = 1$ and $\Delta^r_{s't} = \Delta^r_{s'j} = 0$. Substituting this in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s),r}_{k-1} = 0,$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 0$, that is, Γ^{r+1}_{sj} does not exist.

(7) $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s'j}$ exist and $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{s't}$ do not. Using the same arguments as in the previous items, we have $\Delta^r_{sj} = \Delta^r_{s'j} = 1$ and $\Delta^r_{st} = \Delta^r_{s't} = 0$. Substituting this in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1},$$

which is possible if and only if $\Delta^{r+1}_{sj} = 0$ and $\Delta^{r+1}_{s'j} = 1$, that is, Γ^{r+1}_{sj} does not exist.

(8) $\Gamma^{r,r+1}_{st}$, $\Gamma^{r,r+1}_{sj}$, $\Gamma^{r,r+1}_{s't}$, and $\Gamma^{r,r+1}_{s'j}$ exist. Then $\Delta^r_{st} = \Delta^r_{sj} = \Delta^r_{s't} = \Delta^r_{s'j} = 1$ and, substituting this in equation (4), we have

$$\Delta^{r+1}_{sj}\sigma^{(s),r+1}_{k-1} + \Delta^{r+1}_{s'j}(\sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1}) = \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} + \sigma^{(s),r}_{k-1} + \sigma^{(s'),r}_{k-1} = 0,$$

which is possible if and only if $\Delta^{r+1}_{sj} = \Delta^{r+1}_{s'j} = 0$, that is, Γ^{r+1}_{sj} does not exist.

Let us now analyze the case where $s' = i$, that is, when the edge $\Gamma^r_{s't}$ corresponds to the primary pivot $\Delta^r_{it} = 1$ and the edge $\Gamma^r_{s'j}$ corresponds to the change of basis pivot $\Delta^r_{ij} = 1$. In this case, $\Gamma^{r,r+1}_{s'j}$ does not exist and $\Gamma^{r,r+1}_{s't}$ is not used to construct paths by definition and hence there are just two kinds of possible paths between $h^{(j)}_k$ and $h^{(s)}_{k-1}$: the path $(h^{(j)}_k, h^{(s)}_{k-1})$ which is the directed edge $\Gamma^{r,r+1}_{sj}$, and the path $(h^{(j)}_k, h^{(t)}_k, h^{(s)}_{k-1})$.

If $\widetilde{\Gamma}^r_{sj}$ does not exist, we have two possibilities: either both $\Gamma^{r,r+1}_{st}$ and $\Gamma^{r,r+1}_{sj}$ exist or $\Gamma^{r,r+1}_{st}$ and $\Gamma^{r,r+1}_{sj}$ do not. Since Δ^r_{st} and Δ^r_{sj} are not change of basis pivots then either both Γ^r_{st} and Γ^r_{sj} exist or Γ^r_{st} and Γ^r_{sj} do not, that is, either $\Delta^r_{st} = \Delta^r_{sj} = 1$ or $\Delta^r_{st} = \Delta^r_{sj} = 0$. Moreover, since $\Delta^r_{s't}$ is a primary pivot and $\Delta^r_{s'j}$ is a change of basis pivot, $\Delta^r_{s't} = \Delta^r_{s'j} = 1$

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and $\Delta_{s'j}^{r+1} = 0$. It follows that equation (4) is equivalent to

$$\Delta_{sj}^{r+1}\sigma_{k-1}^{(s),r} = \Delta_{sj}^{r}\sigma_{k-1}^{(s),r} + \sigma_{k-1}^{(s'),r} + \Delta_{st}^{r}\sigma_{k-1}^{(s),r} + \sigma_{k-1}^{(s'),r} = \Delta_{sj}^{r}\sigma_{k-1}^{(s),r} + \Delta_{st}^{r}\sigma_{k-1}^{(s),r},$$

where the right-hand side of the equation is always equal to zero. This is only possible if $\Delta_{sj}^{r+1} = 0$, that is, Γ_{sj}^{r+1} does not exist.

Case 4. There are no bifurcation edges starting at $h_k^{(j)}$ and also there are no bifurcation edges ending at $h_{k-1}^{(s)}$. In this case, Δ_{ij}^{r} is not a change of basis pivot for any i and Δ_{qs}^{r} is not a primary pivot for any s. Note that the only possible path between $h_k^{(j)}$ and $h_{k-1}^{(s)}$ is the directed edge $\Gamma_{sj}^{r,r+1}$. Hence, $\widetilde{\Gamma}_{sj}^{r}$ exists if and only if $\Gamma_{sj}^{r,r+1}$ exists. Since Δ_{sj} is not a change of basis pivot, $\Gamma_{sj}^{r,r+1}$ exists if and only if Γ_{sj}^{r} exists. Also, since there are no changes of basis involving the pth or the sth row, we have $\Delta_{sj}^{r} = \Delta_{sj}^{r+1}$ and hence Γ_{sj}^{r+1} exists if and only if Γ_{sj}^{r} exists. □

Example 5.3. Let Δ and $\Delta^1, \ldots, \Delta^8$ be respectively the connection matrices from the example given in Appendix A. The family of connection matrices $\Delta^1, \ldots, \Delta^8$ are obtained from Δ by using the sweeping method. We present in Appendix A the flow and bifurcation schematics corresponding to each matrix. The collection of schematics $\mathcal{C} = (\Gamma^1, \Gamma^{1,2}, \Gamma^2, \Gamma^{2,3}, \ldots, \Gamma^{7,8}, \Gamma^8)$ is a schematic continuation.

In order to construct the spectral sequence associated to the Morse complex we fixed the finest filtration. This filtration determines a total ordering which is an admissible ordering for all of the connection matrices and transition matrices constructed by the sweeping method. Hence, we can say that the schematic continuation associated to the sweeping method described in §5.2 respects order.

We have constructed the schematic continuation $\mathcal{C} = (\Gamma^1, \Gamma^{1,2}, \Gamma^2, \Gamma^{2,3}, \ldots, \Gamma^{F-1,F}, \Gamma^F)$ such that the total order $<$ associated to the sweeping method filtration continues. We will now prove that for each non-zero entry of T, where $\Delta^F = T^{-1}\Delta^1 T$, there exists a chain of edges in the corresponding continuation schematic.

THEOREM 5.4. *Let Δ^1 and Δ^F be the initial and final connection matrices obtained from the sweeping method and let T be the transition matrix such that $\Delta^F = T^{-1}\Delta^1 T$. Let $\mathcal{C}$ be the schematic continuation such that the total order $<$ associated to the sweeping method filtration continues. Then for every p and p^*, if $T(p, p^*)$ is a non-trivial entry in the kth block of T, then there exists a chain of edges between $h_k^{(p)}$ and $h_k^{(p^*)}$ in $\mathcal{C}$, that is, there exist $0 \le r_0 < r_1 < \cdots < r_\ell \le f$ and $p = p_0, p_1, \ldots, p_\ell, p_{\ell+1} = p^*$ such that there exist edges between $h_k^{(p_i)}$ and $h_k^{(p_{i+1})}$ in Γ^{r_i,r_i+1} for $i = 0, 1, \ldots, \ell$.*

Proof. Note that an edge between two vertices representing singularities of the same index in Γ^{r_i,r_i+1} corresponds to a non-zero entry on the transition matrix T^{r_i}. Hence, the proof of this theorem will follow directly from the following lemma. □

LEMMA 5.5. *Let $T^1, \ldots, T^f$ be the transition matrices associated to the sweeping method and let $T = T^1 T^2 \cdots T^f$ be the product matrix. If an entry T_{pp^*} of T is non-zero then there exist $0 \le r_0 < r_1 < \cdots < r_\ell \le f$ and $p = p_0, p_1, \ldots, p_\ell, p_{\ell+1} = p^*$ such that for $i = 0, 1, \ldots, \ell$, $T_{p_i p_{i+1}}^{r_i}$ is non-zero.*

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Proof. For each $r = 1, \ldots, f$, let $\widetilde{T}^r = T^1 T^2 \cdots T^r$. Hence, T can be written as a product of two matrices $T = \widetilde{T}^{f-1} T^f$ and each entry T_{ij} is

$$T_{ij} = \sum_{\ell=1}^{n} \widetilde{T}^{f-1}_{i\ell} T^f_{\ell j}. \tag{5}$$

Since both matrices are upper triangular, $\widetilde{T}^{f-1}_{i\ell} = 0$ for $\ell < i$ and $T^f_{\ell j} = 0$ for $\ell > j$, and hence (5) can be written as

$$T_{ij} = \sum_{\ell=i}^{j} \widetilde{T}^{f-1}_{i\ell} T^f_{\ell j}. \tag{6}$$

Suppose $T_{pp^*} = 1$. As we have seen, this entry of the product matrix T can be written as

$$\begin{aligned} T_{pp^*} &= \sum_{\ell=p}^{p^*} \widetilde{T}^{f-1}_{p\ell} T^f_{\ell p^*} \\ &= \widetilde{T}^{f-1}_{pp} T^f_{pp^*} + \widetilde{T}^{f-1}_{pp+1} T^f_{p+1p^*} + \cdots + \widetilde{T}^{f-1}_{pp^*-1} T^f_{p^*-1p^*} + \widetilde{T}^{f-1}_{pp^*} T^f_{p^*p^*}. \end{aligned} \tag{7}$$

Since $T_{pp^*} = 1$ then at least one of the summands above has to be non-zero, that is, there exists p_1, $p < p_1 < p^*$, such that $\widetilde{T}^{f-1}_{pp_1} T^f_{p_1 p^*} = 1$ and hence $\widetilde{T}^{f-1}_{pp_1} = T^f_{p_1 p^*} = 1$.

Now we have

$$\widetilde{T}^{f-1}_{pp_1} = \sum_{\ell=p}^{p_1} \widetilde{T}^{f-2}_{p\ell} T^{f-1}_{\ell p_1} = 1,$$

and once again at least one of the summands above has to be non-zero. Hence, there exists p_2, $p \le p_2 \le p_1$, such that $\widetilde{T}^{f-2}_{pp_2} T^{f-1}_{p_2 p_1} = 1$ and hence $\widetilde{T}^{f-2}_{pp_2} = T^{f-1}_{p_2 p_1} = 1$. Following the same line of argument, we find a sequence $p^* \ge p_1 \ge p_2 \ge \cdots \ge p_{f-1} \ge p$ such that

$$T^1_{pp_{f-1}} = T^2_{p_{f-1}p_{f-2}} = \cdots = T^{f-1}_{p_2 p_1} = T^f_{p_1 p^*} = 1.$$

Note that whenever $p_i = p_{i+1}$, this means that $T^r_{p_{i+1}p_i}$ is an entry on the main diagonal of the transition matrix T^r and this entry does not represent a bifurcation edge in $\Gamma^{r_i, r_{i+1}}$. Hence, we will take a subsequence of $p_1, \ldots, p_{f-1}$ considering just the distinct elements of the sequence. For simplicity we will keep the same notation. □

Remark 5.6. Whenever the non-zero entry of T is in the first diagonal, that is, p and p^* are adjacent, there exists an edge between $h^{(p)}$ and $h^{(p^*)}$. In fact, if $p^* = p + 1$ then the sum in (7) is

$$T_{pp+1} = \widetilde{T}^{f-1}_{pp} T^f_{pp+1} + \widetilde{T}^{f-1}_{pp+1} T^f_{p+1p+1} \tag{8}$$

and hence we have that either T^f_{pp+1} or $\widetilde{T}^{f-1}_{pp+1}$ is non-zero. If T^f_{pp+1} is non-zero then the result follows. Otherwise, we write

$$1 = \widetilde{T}^{f-1}_{pp+1} = \widetilde{T}^{f-2}_{pp} T^{f-1}_{pp+1} + \widetilde{T}^{f-2}_{pp+1} T^{f-1}_{p+1p+1}$$

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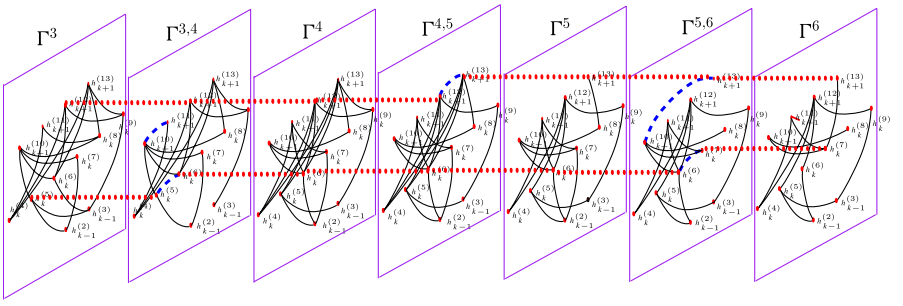


FIGURE 14. Local schematic continuation for the example in Appendix A.

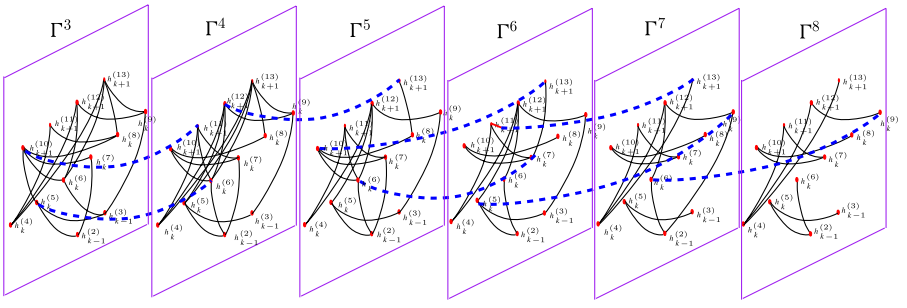


FIGURE 15. Schematic continuation seen with bifurcation edges across the parameter space.

and either T^{f-1}_{pp+1} or $\widetilde{T}^{f-2}_{pp+1}$ is non-zero. Following the same line of argument, after a finite number of steps, we find r such that T^{r}_{pp+1} is non-zero and the result follows.

Example 5.7. Let us consider the schematics presented in Appendix A and the product matrix T. Observe that $T_{57} = 1$ and there is no directed edge connecting $h_k^{(7)}$ to $h_k^{(5)}$ in the local schematic continuation. However, there is a chain of edges connecting $h_k^{(7)}$ to $h_k^{(5)}$ in the local schematic continuation. See the chain $h_k^{(7)}$, $h_k^{(6)}$, $h_k^{(5)}$ in Figure 14.

The schematic continuation can also be represented as in Figure 15 where the bifurcation orbits go across the parameter space rather than staying confined in the bifurcation schematics $\Gamma^{r,r+1}$.

6. *Sweeping method and bifurcation orbits*

The schematics presented in the previous section detail possible changes in behavior that could occur under a continuation of a Morse decomposition that has connection matrices that change according to the sequence of connection matrices and transition matrices output by the sweeping method.

In this section, we develop a dynamic interpretation theorem in a setting of a parameterized family of flows, where the sweeping method output of $n \times n$ connection matrices and transition matrices, $\Delta^1, T^1, \Delta^2, T^2, \ldots, T^{F-1}, \Delta^F$, reveals connecting

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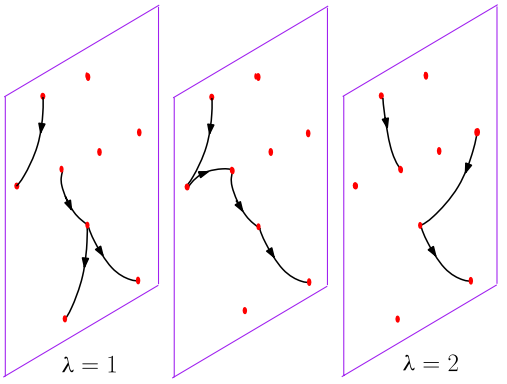


FIGURE 16. A parametrized family of flows.

orbits that arise under the continuation. Thus, we identify a setting where the bifurcations suggested by the schematic continuation actually do arise.

Here, we make the following assumptions.

- There is a parameterized family of flows on M with parameter space $[1, F] \subseteq R$.
- There is an invariant set S_λ continuing across the parameter interval.
- There is a Morse decomposition $\mathcal{D}(M)_\lambda = \{M(p)_\lambda | p = 1, \ldots, n\}$ of the invariant set S_λ continuing across the parameter interval, and the sweeping method filtration ordering is an admissible ordering that continues across the parameter space.
- For each $r = 1, \ldots, F$, the sweeping method connection matrix Δ^r is the connection matrix for the Morse decomposition at parameter value r.

With these assumptions, we interpret the sweeping method transition matrices as singular transition matrices for the parameterized family of flows over $[1, F]$, and we present a result where non-trivial entries in the transition matrix imply the existence of orbits that arise under continuation and that connect sets in the Morse decompositions that continue across the parameter interval. Our main connecting orbit existence result (Theorem 6.1) addresses the parameter subinterval $[1, 2]$. It can be applied over each parameter subinterval $[r, r + 1]$ to yield a result for the whole parameter space $[1, F]$. The setting and results we present here follow closely those of Reineck [**R1**]. It was there that the singular transition matrix idea was initially formalized, but the name 'singular' was not used until later (see [**McMi, McMi2**]). Thus, assume that we have a parameterized family of flows, invariant set continuation, and Morse decomposition continuation, as described above. We focus on the parameter interval $[1, 2]$. We assume that Δ^1 and Δ^2 are connection matrices for the Morse decompositions at parameter values 1 and 2, respectively. The situation is represented in the diagrams of Figure 16.

By copying the flow at parameter value 1 to parameter values just less than 1 and copying the flow at parameter value 2 to parameter values just greater than 2, we can assume that our parameter interval is $(1 - \phi, 2 + \phi)$ for some small ϕ. Now put a flow on the parameter space $Y = (1 - \phi, 2 + \phi)$, via $y' = \varepsilon(y - 1)(y - 2)$ for $\varepsilon > 0$.

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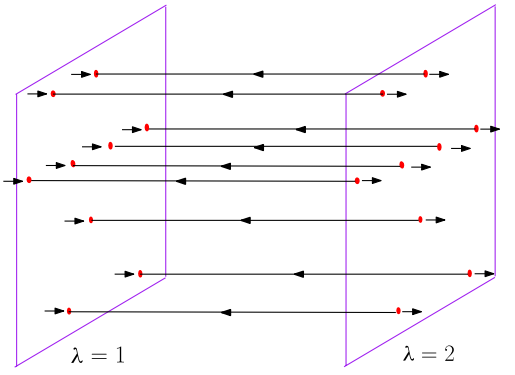


FIGURE 17. The product flow.

This yields a product flow on $M \times Y$ in which $S = \bigcup_{\lambda\in[1,2]} S_\lambda$ is an invariant set with Morse decomposition $\mathcal{D} = \{M(1)_1, \ldots, M(n)_1, M(1)_2, \ldots, M(n)_2\}$. (See Figure 17.) With $<_\lambda$ representing the flow ordering at parameter λ in the parameterized family of flows, every admissible ordering $<^\times$ for this Morse decomposition satisfies the following properties:

- for $\lambda = 1, 2$, if $p <_\lambda q$, then $(p, \lambda) <^\times (q, \lambda)$;
- for all p, $(p, 1) <^\times (p, 2)$;
- if $(p, 1) <^\times (q, 2)$, then $p < q$ (where $<$ is the sweeping method filtration ordering).

Now, for all λ, we let $H_k(h(M(p)_\lambda))$ denote the dimension-k Conley homology index of the invariant set $M(p)_\lambda$ in the flow on M with parameter λ, and for $\lambda = 1, 2$, we let $H_k(h(M(p)_\lambda))^\times$ denote the dimension-k Conley homology index of the invariant set $M(p)_\lambda$ in the product flow on $M \times Y$. It follows that $H_k(h(M(p)_1))^\times = H_k(h(M(p)_1))$, and $H_k(h(M(p)_2))^\times = H_{k-1}(h(M(p)_2))$ for all k and p. The latter equality holds because, for each p, the Conley index of $M(p)_2$ as an invariant set in the product flow is equal to the suspension with a circle of the Conley index of $M(p)_2$ as an invariant set in the flow on M at parameter value 2.

Thus, the connection matrix Δ^1 is a map on the sum of Conley homology complexes, $\oplus_p H_k(h(M(p)_1))^\times$, and the connection matrix Δ^2 defines a map $\Delta^{2^\times}$ on the sum of Conley homology complexes, $\oplus_p H_k(h(M(p)_2))^\times$, where each map in the matrix $\Delta^{2^\times}$ is obtained by shifting the corresponding map in Δ^2 up a dimension in the homology complex. For simplicity, we drop the superscript $\times$ in $\Delta^{2^\times}$, and leave it up to the context to determine which map Δ^2 refers to.

Similarly, the transition matrix T^1 naturally defines a map from $\oplus_p H_k(h(M(p)_2))^\times$ to $\oplus_p H_k(h(M(p)_1))^\times$, and we denote the resulting map by T^1 as well.

Now, we can combine the maps Δ^1, Δ^2, and T^1 to form a matrix,

$$\begin{pmatrix} \Delta^1 & T^1 \\ 0 & \Delta^2 \end{pmatrix},$$

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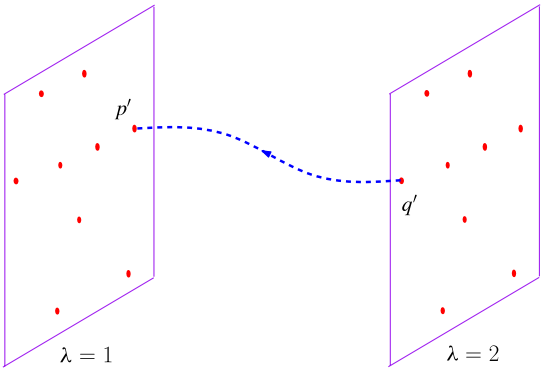


FIGURE 18. A connecting orbit exists between $M(q')_1$ and $M(p')_0$ in the product flow.

mapping $(\oplus_p H_k(h(M(p)_1))^\times) \oplus (\oplus_p H_k(h(M(p)_2))^\times)$ to itself. We are interested in the situation where Δ persists as a connection matrix for the Morse decomposition $\mathcal{D}$ of the invariant set S in the product flows obtained as we slow down the parameter space component of the flow (that is, as we let $\varepsilon \to 0$ in $y' = \varepsilon(y-1)(y-2)$). Under such a circumstance, T^1 is called a *singular transition matrix* between the connection matrices Δ^1 and Δ^2. Our interest now is in determining what implications there are for the structure of the parameterized family of flows, given that T^1 is a singular transition matrix between the connection matrices Δ^1 and Δ^2. Specifically, suppose that T^1_{pq} is non-trivial. Then this corresponds to a non-trivial entry in Δ, implying that in the flow ordering $<^{\times f}$ on $\mathcal{D}$, we have $(p, 1) <^{\times f} (q, 2)$. Consequently, there exist p' and q' such that $p \leq_1 p'$, $p' < q'$, $q' \leq_2 q$ and there is a connecting orbit from $M(q')_2$ to $M(p')_1$ in the product flow. See Figure 18.

It is the above-mentioned connecting orbit that interests us most here, but before we address it further, we introduce assumptions from which it follows that $p = p'$ and $q = q'$. Under that circumstance, T^1_{pq} non-trivial implies that there is a connecting orbit from $M(q)_2$ to $M(p)_1$ in the product flow. First, note that if we assume that each Morse set is a non-degenerate singularity, then in the product flow, if $M(p)_1$ has index k, then T^1_{pq} non-trivial implies that $M(q)_2$ has index $k+1$. Therefore, $M(q)_2$ has index k in the flow at parameter 2. It follows that, with p' and q' as above, $M(q')_2$ and $M(p')_1$ also have index k in the flows at the corresponding parameter values. If we further assume that at parameter values 1 and 2, the unstable and stable manifolds of the singularities intersect transversally, then at those parameter values there are no connecting orbits between singularities of the same index, and therefore we must have $p = p'$ and $q = q'$.

Now, under the above circumstances, if T^1 is a singular transition matrix between the connection matrices Δ^1 and Δ^2, and T^1_{pq} is non-trivial, then for ε sufficiently small, there is a connecting orbit from $M(q)_2$ to $M(p)_1$ in the product flow. Also, as ε approaches 0, the flow in the parameter space slows down so that, for instance, the connecting orbit runs 'slow' in the parameter direction and 'fast' in the direction of M, causing it to run close to

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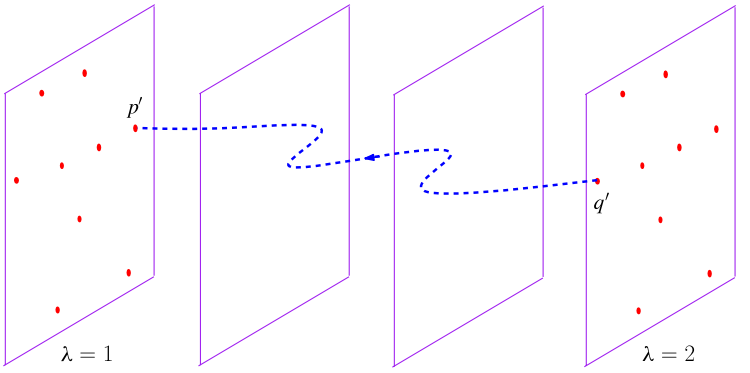


FIGURE 19. As $\epsilon \to 0$, the connecting orbit runs slow in the parameter direction and fast in the X direction.

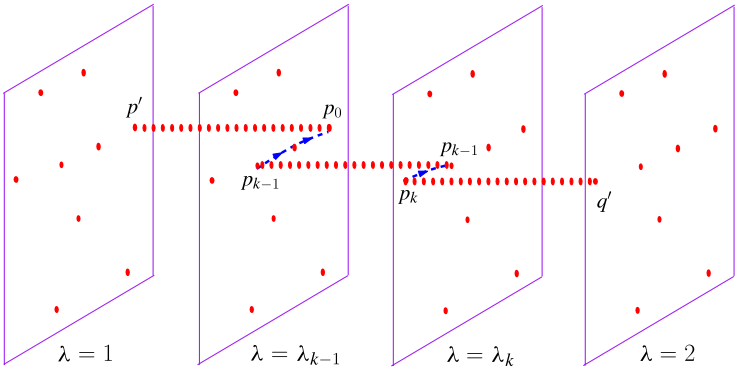


FIGURE 20. The connecting orbit converges to an invariant set made up of Morse sets at changing parameter values and connecting orbits at fixed parameter values.

connecting orbits that occur in M at parameter values between 1 and 2 in the parameterized family of flows. See Figures 19 and 20.

As Figure 20 suggests, in the limit as $\varepsilon \to 0$, the connecting orbit converges to an invariant set that contains Morse sets over an interval of parameter values in [1, 2] and connecting orbits between Morse sets at specific parameter values between 1 and 2. We make that more explicit in the following theorem. This is our dynamic interpretation result where, under appropriate assumptions on a parameterized family of flows, a non-trivial entry T^1_{pq} in a transition matrix implies the existence of connecting orbits 'going from $M(q)$ to $M(p)$' in the parameterized family of flows.

THEOREM 6.1. *Assume the following:*

- *there is a parameterized family of flows M with parameter space* [1, 2]*;*
- *there is an invariant set S_λ continuing across the parameter interval;*

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- *there is a Morse decomposition* $\mathcal{D}(M)_\lambda = \{M(p)_\lambda | p = 1, \ldots, n\}$ *of the invariant set* S_λ *continuing across the parameter interval, and the sweeping method filtration ordering,* $<$*, is an admissible ordering that continues across the parameter space;*
- *each* $M(p)_\lambda$ *is a non-degenerate singularity, and in the flows at parameter values* 1 *and* 2*, the stable and unstable manifolds of the singularities intersect transversally;*
- Δ^1 *and* Δ^2 *are the connection matrices at parameter values* 1 *and* 2*, respectively, and* T^1 *is a singular transition matrix between them;*
- T^1_{pq} *is non-trivial.*

Then there exist $p = p_0, p_1, \ldots, p_k = q$*, and* $1 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 2$ *such that for each* j*,* $p_{j-1} <_{\lambda_j} p_j$ *where* $<_{\lambda_j}$ *is the flow ordering for the Morse decomposition* M_{λ_j}*.*

The significance of $p_{j-1} <_{\lambda_j} p_j$ is that in the flow at parameter λ_j there are connecting orbits running from $M(p_j)_{\lambda_j}$, possibly through some intermediary Morse sets, to $M(p_{j-1})_{\lambda_j}$. (See Figure 20.)

Proof. We construct product flows, as described above, with flow on the parameter space defined by $y' = \varepsilon(y-1)(y-2)$. As argued above, given that Δ^1 and Δ^2 are the connection matrices at parameter values 1 and 2, respectively, T^1 is a singular transition matrix between them, and T^1_{pq} is non-trivial, it follows that for sufficiently small ε there is a connecting orbit from $M(q)_2$ to $M(p)_1$ in the product flow. By [**R1**, Theorem 3.13] the existence of $p = p_0, p_1, \ldots, p_k = q$ and $1 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 2$ such that, for each j, $p_{j-1} <_{\lambda_j} p_j$ follows. □

Note that if p and q are adjacent in the ordering $<$, then this result simply implies that for some parameter value $\lambda \in (1, 2)$ there is a connecting orbit from $M(q)_\lambda$ to $M(p)_\lambda$.

Via Theorem 6.1 we have created a general setting of a parameterized family of flows where the information in the sweeping method connection matrices, transition matrices, and schematic continuation is reflected in the actual structure of bifurcation orbits that arise under the transition across the family of flows.

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A. *Appendix*

Example A.1. Let Δ be as in Figure A.1. Applying the sweeping method to Δ we obtain the matrices $\Delta^1, \ldots, \Delta^9$ given by Figures A.2–A.10, respectively.

We present in Figures A.11–A.22 the transition matrix T and all the flow and bifurcation schematics, Γ^r and $\Gamma^{r,r+1}$ respectively, associated to the connection matrices Δ^r and transition matrices T^r obtained from the sweeping method applied to Δ in the above example.

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	F_0 h_0	F_1 $h_{k-1}^{(2)}$	F_2 $h_{k-1}^{(3)}$	F_3 $h_k^{(4)}$	F_4 $h_k^{(5)}$	F_5 $h_k^{(6)}$	F_6 $h_k^{(7)}$	F_7 $h_k^{(8)}$	F_8 $h_k^{(9)}$	F_9 $h_{k+1}^{(10)}$	F_{10} $h_{k+1}^{(11)}$	F_{11} $h_{k+1}^{(12)}$	F_{12} $h_{k+1}^{(13)}$	F_{13} h_n
F_0 h_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_1 $h_{k-1}^{(2)}$	0	0	0	0	1	0	1	0	0	0	0	0	0	0
F_2 $h_{k-1}^{(3)}$	0	0	0	0	1	1	0	0	1	0	0	0	0	0
F_3 $h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	1	1	1	0
F_4 $h_k^{(5)}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0
F_5 $h_k^{(6)}$	0	0	0	0	0	0	0	0	0	1	0	1	1	0
F_6 $h_k^{(7)}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0
F_7 $h_k^{(8)}$	0	0	0	0	0	0	0	0	0	1	1	0	1	0
F_8 $h_k^{(9)}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0
F_9 $h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{10} $h_{k+1}^{(11)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{11} $h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{12} $h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{13} h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE A.1. Δ.

	h_0	$h_{k-1}^{(2)}$	$h_{k-1}^{(3)}$	$h_k^{(4)}$	$h_k^{(5)}$	$h_k^{(6)}$	$h_k^{(7)}$	$h_k^{(8)}$	$h_k^{(9)}$	$h_{k+1}^{(10)}$	$h_{k+1}^{(11)}$	$h_{k+1}^{(12)}$	$h_{k+1}^{(13)}$	h_n
h_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k-1}^{(2)}$	0	0	0	0	1	0	1	0	0	0	0	0	0	0
$h_{k-1}^{(3)}$	0	0	0	0	1	1	0	0	1	0	0	0	0	0
$h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	1	1	1	0
$h_k^{(5)}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$h_k^{(6)}$	0	0	0	0	0	0	0	0	0	1	0	1	1	0
$h_k^{(7)}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$h_k^{(8)}$	0	0	0	0	0	0	0	0	0	1	1	0	1	0
$h_k^{(9)}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(11)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE A.2. Δ^1. Marking primary pivots.

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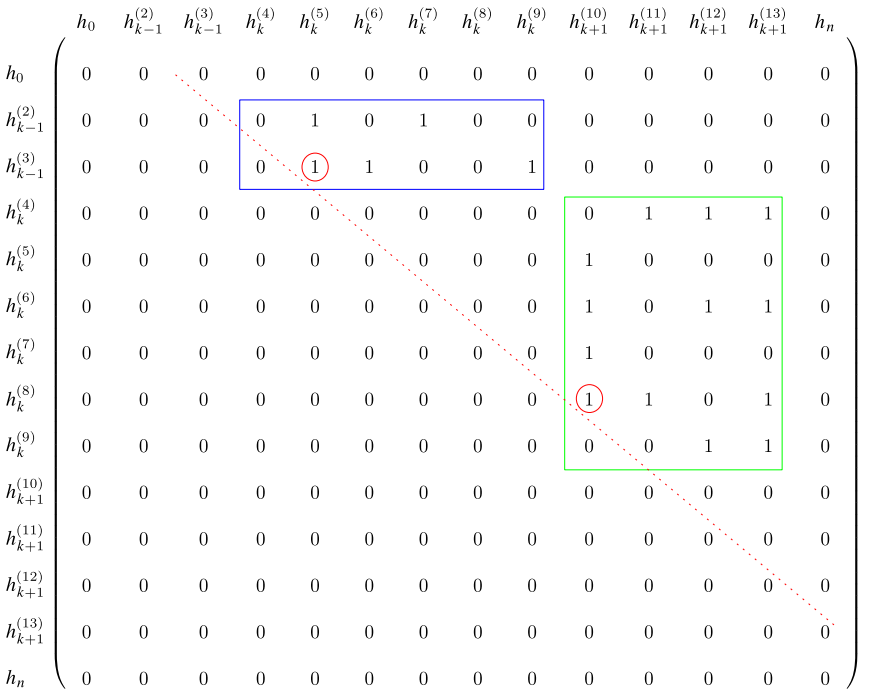


FIGURE A.3. Δ^2. Marking primary and change of basis pivots.

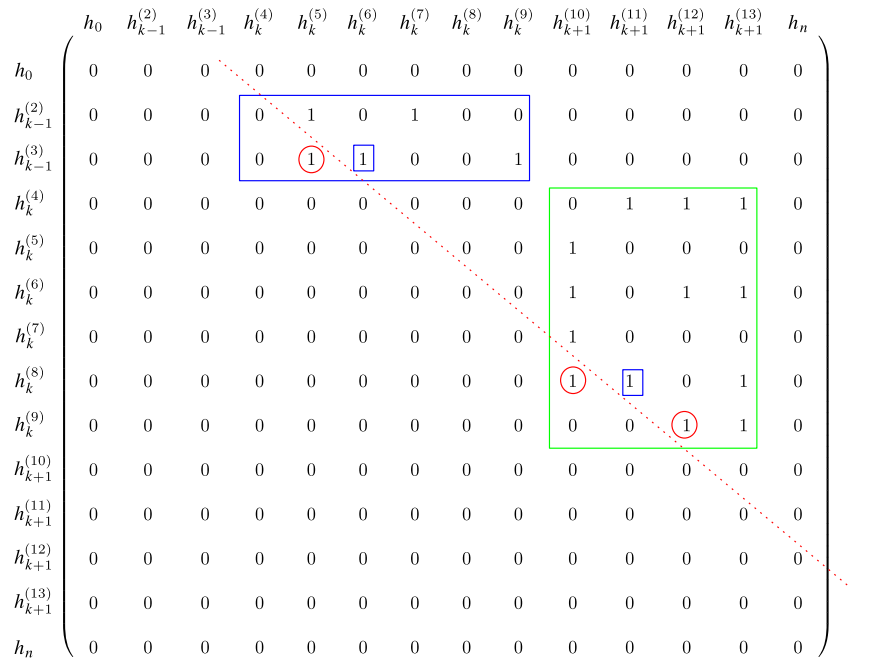


FIGURE A.4. Δ^3. Change of basis and marking pivots.

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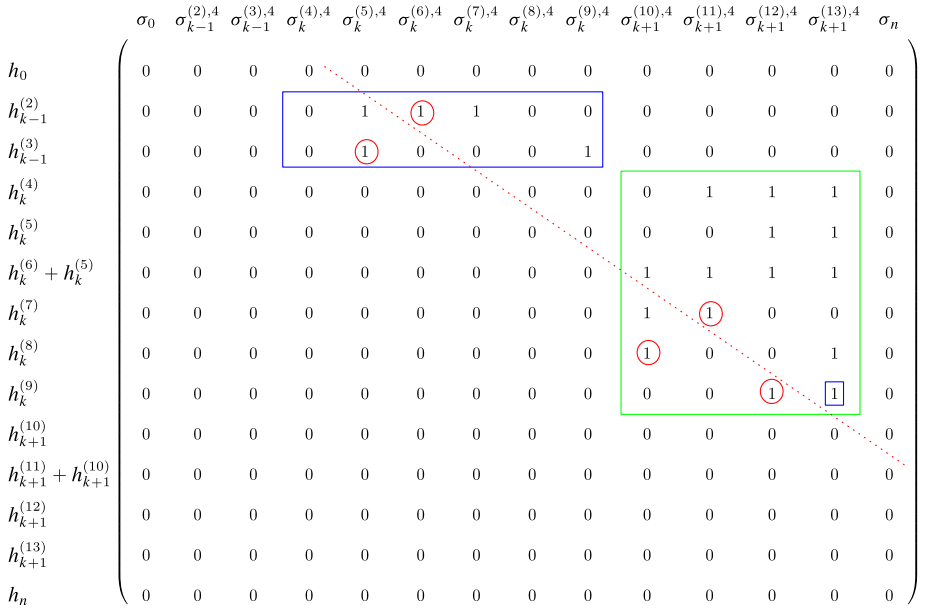


FIGURE A.5. Δ^4.

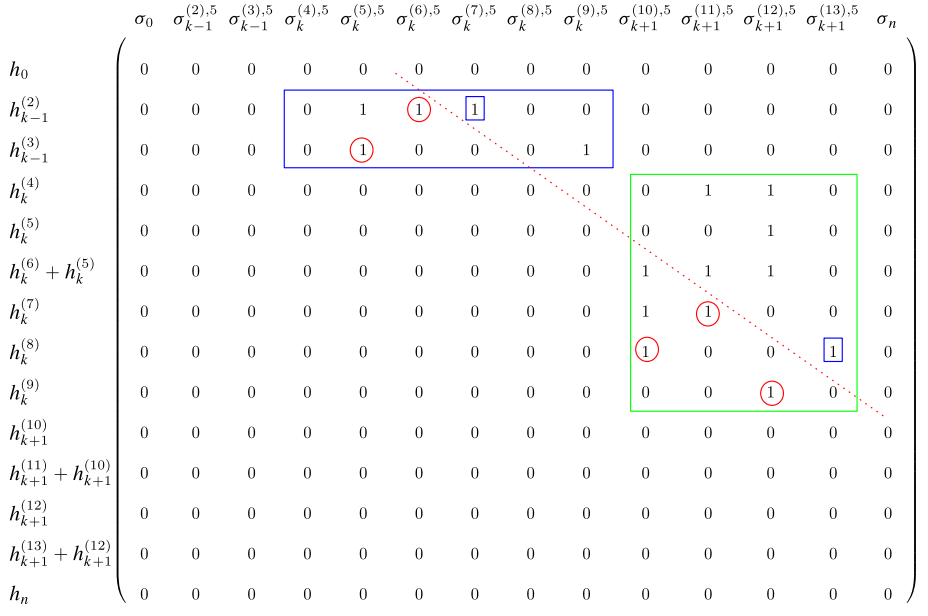


FIGURE A.6. Δ^5.

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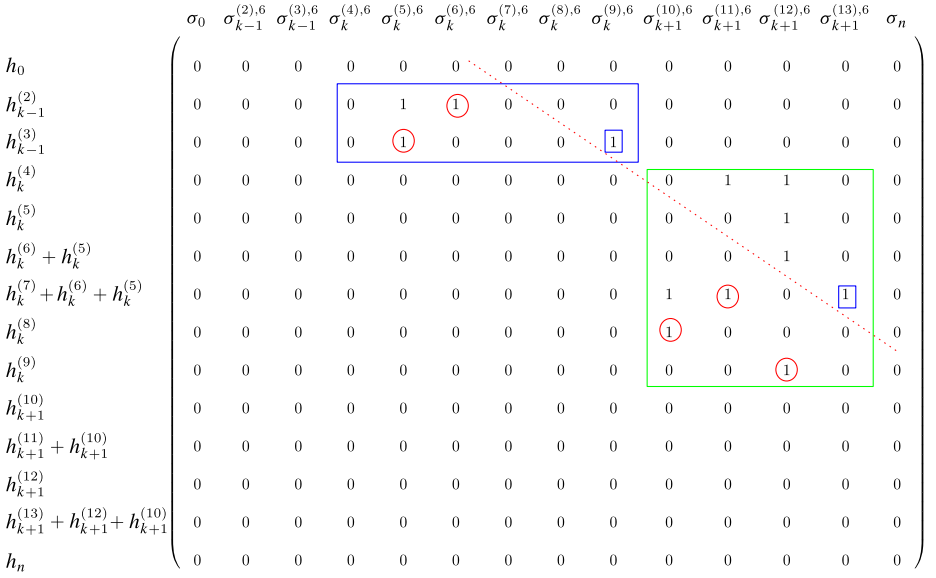


FIGURE A.7. Δ^6.

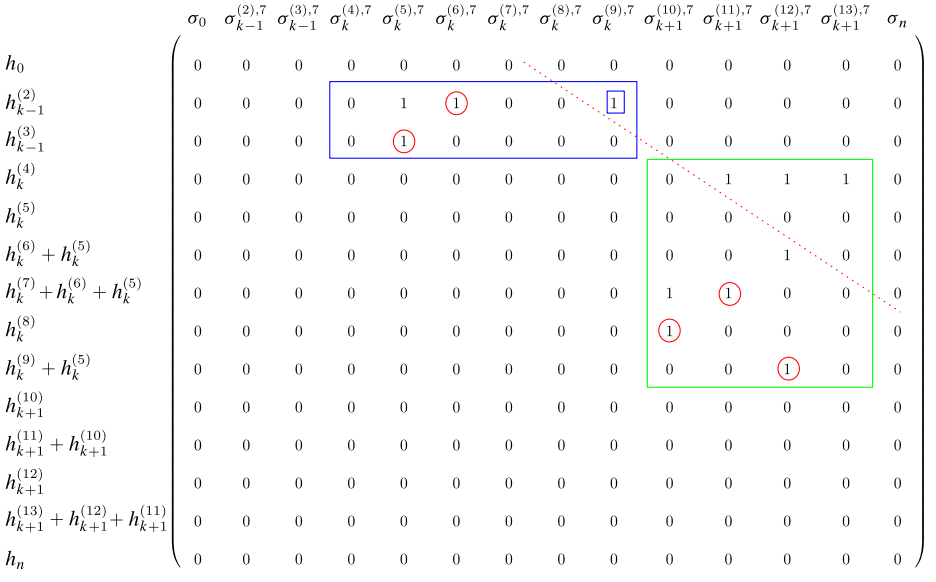


FIGURE A.8. Δ^7.

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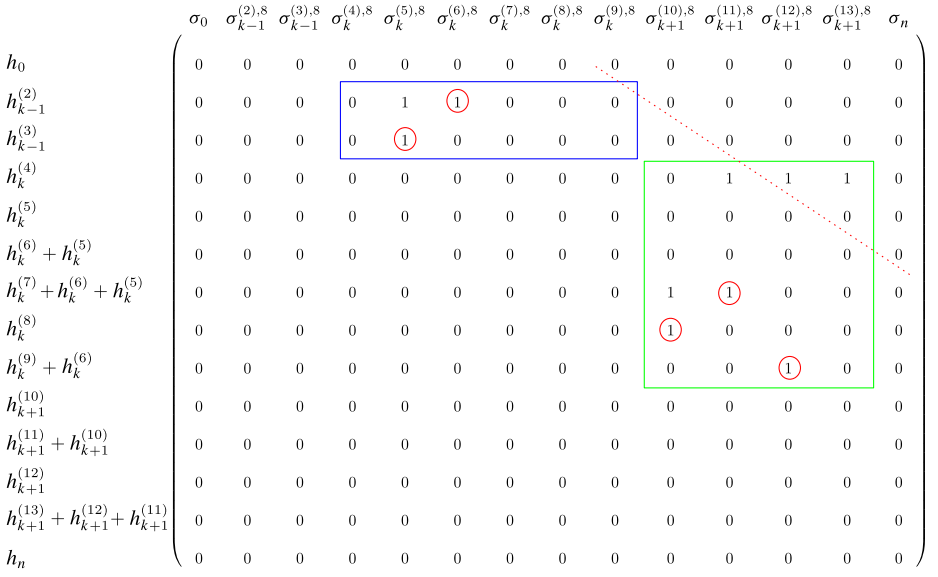


FIGURE A.9. Δ^8.

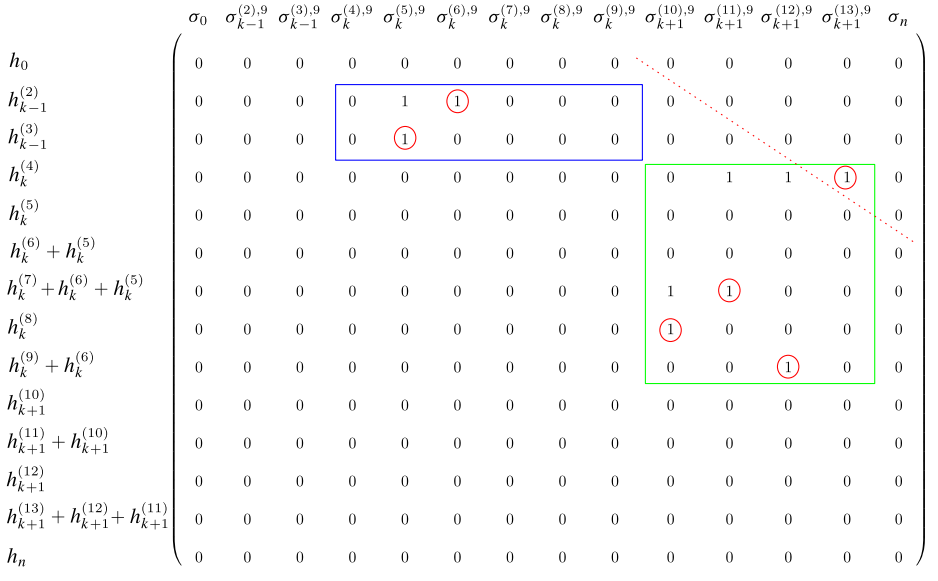


FIGURE A.10. Δ^9.

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	σ_0	$\sigma_{k-1}^{(2),9}$	$\sigma_{k-1}^{(3),9}$	$\sigma_k^{(4),9}$	$\sigma_k^{(5),9}$	$\sigma_k^{(6),9}$	$\sigma_k^{(7),9}$	$\sigma_k^{(8),9}$	$\sigma_k^{(9),9}$	$\sigma_{k+1}^{(10),9}$	$\sigma_{k+1}^{(11),9}$	$\sigma_{k+1}^{(12),9}$	$\sigma_{k+1}^{(13),9}$	σ_n
h_0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k-1}^{(2)}$	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k-1}^{(3)}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$h_k^{(4)}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$h_k^{(5)}$	0	0	0	0	1	1	1	0	0	0	0	0	0	0
$h_k^{(6)}$	0	0	0	0	0	1	1	0	1	0	0	0	0	0
$h_k^{(7)}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$h_k^{(8)}$	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$h_k^{(9)}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	1	1	0	0	0
$h_{k+1}^{(11)}$	0	0	0	0	0	0	0	0	0	0	1	0	1	0
$h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0
h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	1

FIGURE A.11. $T = T^1 \cdots T^f$.

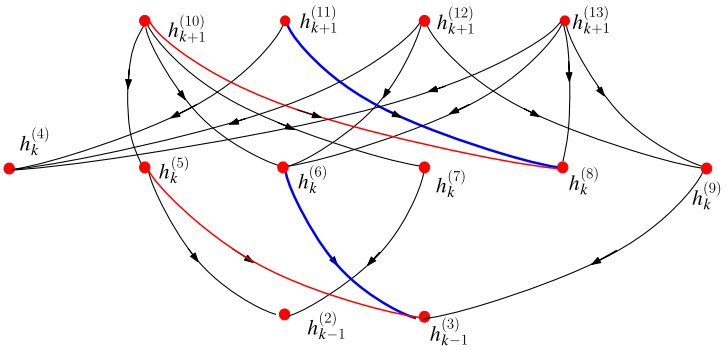


FIGURE A.12. $\Gamma^1 = \Gamma^2 = \Gamma^3$.

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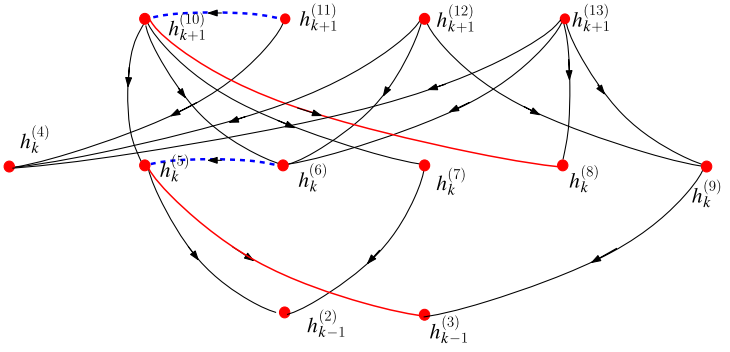


FIGURE A.13. $\Gamma^{3,4}$, Δ^3 and T^3.

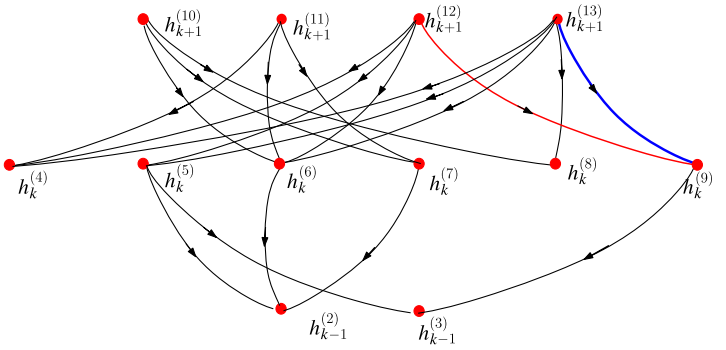


FIGURE A.14. Γ^4, Δ^4.

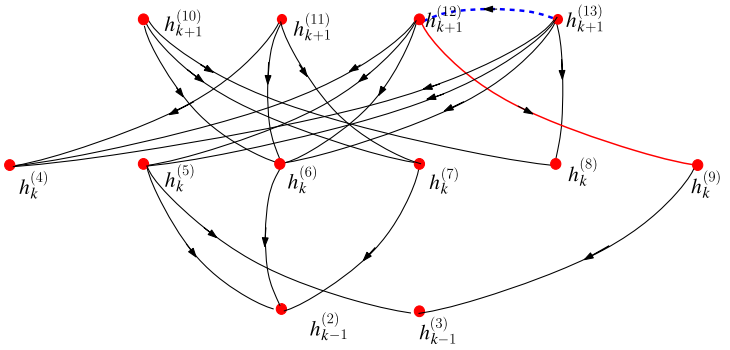


FIGURE A.15. $\Gamma^{4,5}$, Δ^4 and T^4.

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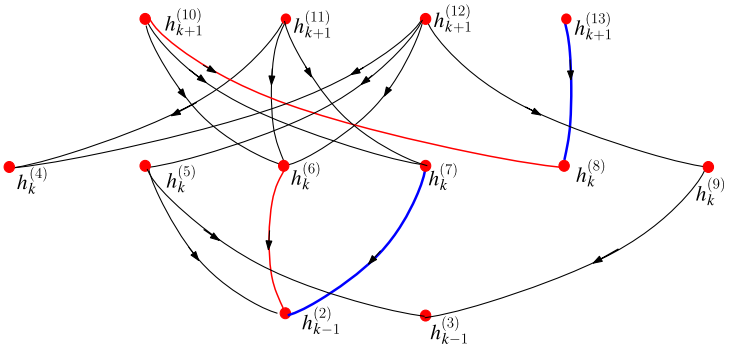


FIGURE A.16. Γ^5, Δ^5.

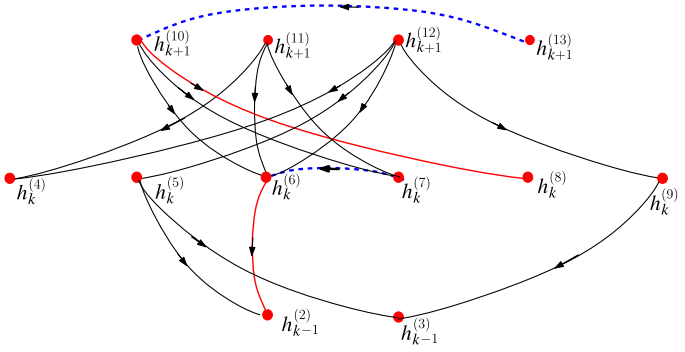


FIGURE A.17. $\Gamma^{5,6}$, Δ^5 and T^5.

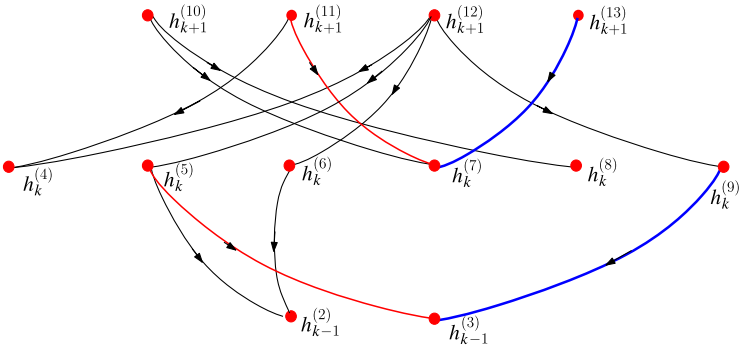


FIGURE A.18. Γ^6, Δ^6.

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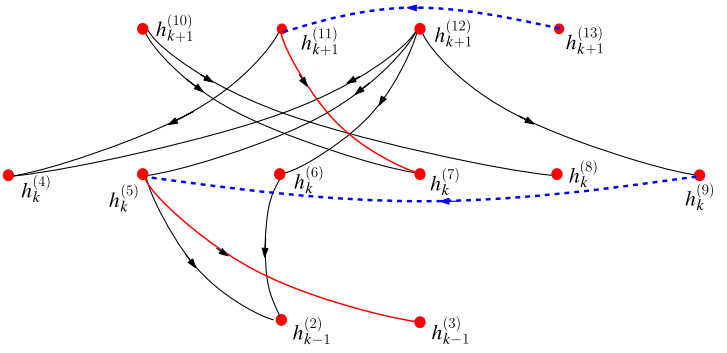


FIGURE A.19. $\Gamma^{6,7}$, Δ^6 and T^6.

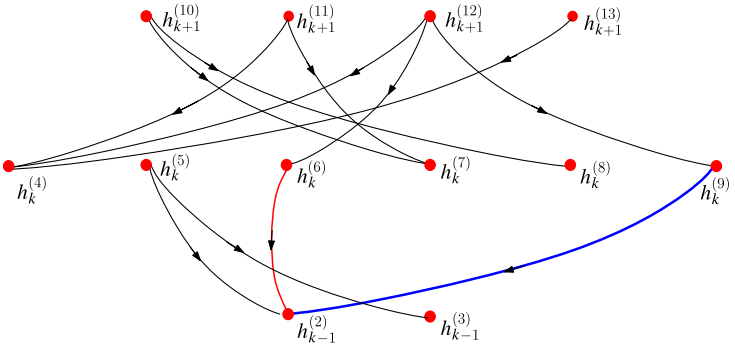


FIGURE A.20. Γ^7, Δ^7.

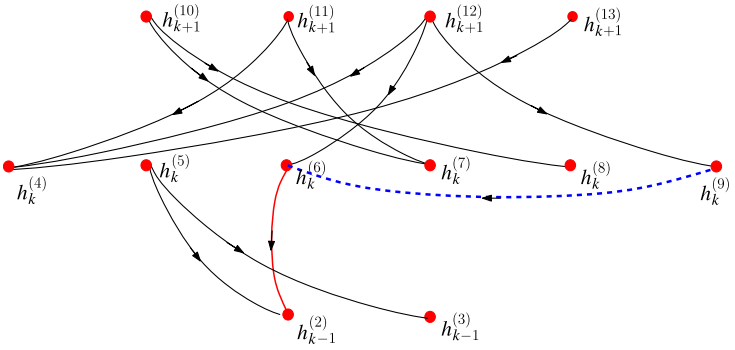


FIGURE A.21. $\Gamma^{7,8}$, Δ^7 and T^7.

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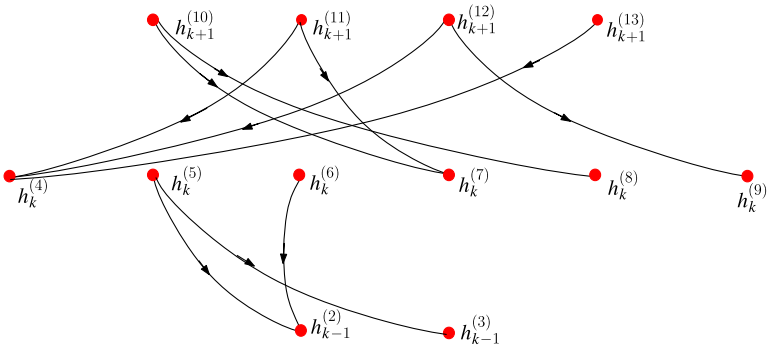


FIGURE A.22. Γ^8.

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