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# Multi-Symplectic Method for the Zakharov-Kuznetsov Equation

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**Abstract.** A new scheme for the Zakharov-Kuznetsov (ZK) equation with the accuracy order of $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$ is proposed. The multi-symplectic conservation property of the new scheme is proved. The backward error analysis of the new multi-symplectic scheme is also implemented. The solitary wave evolution behaviors of the Zakharov-Kunetsov equation is investigated by the new multi-symplectic scheme. The accuracy of the scheme is analyzed.

**AMS subject classifications**: 65D17

**Key words**: The Zakharov-Kuznetsov equation, multi-symplectic method, backward error analysis.

## 1 Introduction

The two-dimensional generalization of the KDV equation, or the ZK equation

$$u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}=0 \tag{1.1}$$

was first derived by Zakharov and Kuznetsov (1974) [26] in three dimensional form to describe nonlinear ion acoustic waves in a magnetized plasma [13, 16]

$$u_t+uu_x+u_{xxx}+(\Delta u)_x=0, \quad \Delta=\partial_x^2+\partial_y^2+\partial_z^2. \tag{1.2}$$

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A variety of physical phenomena, in the purely dispersive limit, are governed by this type of equation; for example, the Rossby waves in rotating atmosphere [22], and the isolated vortex of the drift waves in three dimensional plasma [21]. Although Eq. (1.1) is not even integrable, quite a lot is now known about its nonlinear wave and soliton solutions. Numerical and analytical results of Eq. (1.1) have been investigated in [14, 15].

Recently, Chen [9], from the Preissman scheme for multi-symplectic equations, derived a multi-symplectic numerical scheme for the ZK equation that can be simplified to an implicit 36-point scheme. In this paper, we proposed a new multi-symplectic Euler-box scheme to solve the two-dimensional ZK equation.

The paper is organized as follows: in Section 2, the multi-symplectic structure for the ZK equation is introduced and we propose a new multi-symplectic scheme for the ZK equation and prove its discrete multi-symplectic conservation law. In Section 3, we implement the backward error analysis for the new multi-symplectic scheme of the ZK equation. In Section 4, the solitary wave behaviors of the ZK equation are investigated by the new multi-symplectic scheme and the accuracy of the scheme is analyzed. We finish the paper with conclusion remarks in Section 5.

## 2 A new multi-symplectic scheme for the ZK equation

Introducing the potential $\varphi_x = u$, Eq. (1.1) is equivalent to

$$\varphi_{xxt} + \varphi_x \varphi_{xx} + \varphi_{xxxx} + \varphi_{xxyy} = 0. \tag{2.1}$$

Now, we introduce some variables: $u = \varphi_x$, $v = \varphi_{xx}$, $w = \varphi_{xy}$, $p = -\varphi_{xt}/2$.

According to the covariant De Donder-Weyi Hamilton function theories and the multi-symplectic concept introduced by Bridges [2–7, 12], the ZK equation can be reformulated as a system of five first-order partial differential equations which can be written in the form:

$$M\partial_t z + K\partial_x z + L\partial_y z = \nabla_z S(z), \quad z = (p,u,\varphi,v,w)^T \in R^5, \tag{2.2}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

and $S(z) = up - (v^2 + w^2)/2 - u^3/6$. For details, we refer to [7], $\nabla_z S(z)$ is the gradient of $S(z)$ with respect to the standard inner product on $R^5$. The system (2.2) is a Hamiltonian formulation of the ZK equation on a multi-symplectic structure, where $M, K, L \in R^{n\times n}$ are skew-symmetric matrices and $S(z): R^n \to R$ is a smooth function of the $z(x,y,t)$.

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For Eq. (2.2), one of the most important characteristic is that it satisfies the multi-symplectic conservation law [1,6,7,10,11,18,20,24]

$$\frac{\partial}{\partial t}\omega+\frac{\partial}{\partial x}\kappa+\frac{\partial}{\partial y}q=0, \tag{2.3}$$

where

$$\omega=\frac{1}{2}dz\wedge Mdz,\quad \kappa=\frac{1}{2}dz\wedge Kdz,\quad q=\frac{1}{2}dz\wedge Ldz, \tag{2.4}$$

are differential two-forms. So, when a numerical scheme is developed, we expect that the multi-symplectic conservation law (2.3) should be preserved. Bridges and Reich defined a numerical scheme as a multi-symplectic scheme if the scheme preserves a discrete multi-symplectic conservation law [7]. Specifically, if we discretize Hamiltonian PDEs (2.2) as follows

$$M\partial_t^{i,j,n}z_{i,j}^n+K\partial_x^{i,j,n}z_{i,j}^n+L\partial_y^{i,j,n}z_{i,j}^n=\nabla_z S(z_{i,j}^n), \tag{2.5}$$

where $z_{i,j}^n=z(x_i,y_j,t_n)$, $\partial_t^{i,j,n}$, $\partial_x^{i,j,n}$ and $\partial_y^{i,j,n}$ are the discretizations of the derivatives $\partial_t$, $\partial_x$ and $\partial_y$ respectively, then the scheme is multi-symplectic provided that it can preserve the following discrete conservation law

$$\partial_t^{i,j,n}\omega_{i,j}^n+\partial_x^{i,j,n}\kappa_{i,j}^n+\partial_y^{i,j,n}q_{i,j}^n=0, \tag{2.6}$$

where

$$\omega_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Mdz_{i,j}^n),\quad \kappa_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Kdz_{i,j}^n),\quad q_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Ldz_{i,j}^n). \tag{2.7}$$

Set $t_n$, $n=0,1,2,\cdots,N_1$; $x_i$, $i=1,2,\cdots,N_2$; $y_j$, $j=1,2,\cdots,N_3$ be the regular grids of the integral domain, $z_{i,j}^n$ is an approximation to $z(x_i,y_j,t_n)$, $\Delta t$ is the time-step, $\Delta x$ is the $x$ direction step, $\Delta y$ is the $y$ direction step, and

$$\delta_{\frac{t}{2}}^{\pm}z_{i,j}^n=\pm\frac{z_{i,j}^{n\pm\frac{1}{2}}-z_{i,j}^n}{\frac{1}{2}\Delta t},\quad z_{i,j}^{n\pm\frac{1}{2}}=\frac{z_{i,j}^n+z_{i,j}^{n\pm1}}{2},\quad \delta_x^{\pm}z_{i,j}^n=\pm\frac{z_{i\pm1,j}^n-z_{i,j}^n}{\Delta x},\quad \delta_x^{\pm}z_{i,j}^n=\pm\frac{z_{i,j\pm1}^n-z_{i,j}^n}{\Delta y}.$$

We propose a new scheme for Eq. (2.2). It can be written as

$$\begin{aligned}&M_+\delta_{\frac{t}{2}}^+z_{i,j}^{n+\frac{1}{2}}+M_-\delta_{\frac{t}{2}}^-z_{i,j}^{n+\frac{1}{2}}+K_+\delta_x^+z_{i,j}^{n+\frac{1}{2}}+K_-\delta_x^-z_{i,j}^{n+\frac{1}{2}}\\&\quad+L_+\delta_y^+z_{i,j}^{n+\frac{1}{2}}+L_-\delta_y^-z_{i,j}^{n+\frac{1}{2}}=\nabla_z S(z_{i,j}^{n+\frac{1}{2}}),\end{aligned} \tag{2.8}$$

where $M_+$, $M_-$, $K_+$, $K_-$ and $L_+$, $L_-$ are matrix splitting for the matrices $M$, $K$ and $L$, respectively, s.t. [8,17,24],

$$M=M_++M_-,\qquad M_+^T=-M_-, \tag{2.9a}$$

$$K=K_++K_-,\qquad K_+^T=-K_-, \tag{2.9b}$$

$$L=L_++L_-,\qquad L_+^T=-L_-. \tag{2.9c}$$

Scheme (2.8) satisfies the discrete multi-symplectic conservation law.

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**Theorem 2.1.** *The new scheme* (2.8) *is a multi-symplectic scheme with the following discrete multi-symplectic conservation law*

$$\delta_{\frac{t}{2}}^{+}\omega_{i,j}^{n+\frac{1}{2}}+\delta_{x}^{+}\kappa_{i,j}^{n+\frac{1}{2}}+\delta_{y}^{+}q_{i,j}^{n+\frac{1}{2}}=0, \tag{2.10}$$

*where*

$$\omega_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}},\quad \kappa_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_{+}dz_{i,j}^{n+\frac{1}{2}},\quad q_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_{+}dz_{i,j}^{n+\frac{1}{2}}.$$

*Proof.* Consider the variational equation of (2.8)

$$\begin{aligned}&M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}+K_{+}\delta_{x}^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}\delta_{x}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&\quad+L_{+}\delta_{y}^{+}dz_{i,j}^{n+\frac{1}{2}}+L_{-}\delta_{y}^{-}dz_{i,j}^{n+\frac{1}{2}}=S_{zz}(z_{i,j}^{n+\frac{1}{2}})dz_{i,j}^{n+\frac{1}{2}}.\end{aligned} \tag{2.11}$$

Taking the wedge product with $dz_{i,j}^{n+1/2}$ and the variation equation (2.11), since $dz_{i,j}^{n+1/2}\wedge S_{zz}(z_{i,j}^{n+1/2})dz_{i,j}^{n+1/2}=0$, we have

$$\begin{aligned}&dz_{i,j}^{n+\frac{1}{2}}\wedge(M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}+K_{+}\delta_{x}^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}\delta_{x}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&\quad+L_{+}\delta_{y}^{+}dz_{i,j}^{n+\frac{1}{2}}+L_{-}\delta_{y}^{-}dz_{i,j}^{n+\frac{1}{2}})=0.\end{aligned} \tag{2.12}$$

Considering the items containing $\delta_{t/2}^{+}$ or $\delta_{t/2}^{-}$ in Eq. (2.12), we have

$$\begin{aligned}&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&=dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+M_{-}^{T}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&=dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}-M_{+}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&=dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}\\&=dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\Big[\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+1}-dz_{i,j}^{n+\frac{1}{2}})\Big]+\Big[\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i,j}^{n})\Big]\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}\\&=\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}dz_{i,j}^{n+1}-dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}})\\&=\delta_{\frac{t}{2}}^{+}(dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}).\end{aligned} \tag{2.13}$$

Considering the items containing $\delta_{x}^{+}$ or $\delta_{x}^{-}$ in Eq. (2.12), we have

$$\begin{aligned}&dz_{i,j}^{n+\frac{1}{2}}\wedge K_{+}\delta_{x}^{+}dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge K_{-}\delta_{x}^{-}dz_{i,j}^{n+\frac{1}{2}}\\&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_{+}\delta_{x}^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}^{T}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_{x}^{-}dz_{i,j}^{n+\frac{1}{2}}\end{aligned}$$

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$$
\begin{aligned}
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\delta_x^+dz_{i,j}^{n+\frac{1}{2}}-K_+dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_x^-dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\delta_x^+dz_{i,j}^{n+\frac{1}{2}}+\delta_x^-dz_{i,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\left[\frac{1}{\Delta x}(dz_{i+1,j}^{n+\frac{1}{2}}-dz_{i,j}^{n+\frac{1}{2}})\right]+\left[\frac{1}{\Delta x}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i-1,j}^{n+\frac{1}{2}})\right]\wedge K_+dz_{i,j}^{n+\frac{1}{2}}\\
&=\frac{1}{\Delta x}(dz_{i,j}^{n+\frac{1}{2}}\wedge K_+dz_{i+1,j}^{n+\frac{1}{2}}-dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}})\\
&=\delta_x^+(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}}).
\end{aligned}\tag{2.14}
$$

Considering the items containing $\delta_y^+$ or $\delta_y^-$ in Eq. (2.12), we have

$$
\begin{aligned}
&dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge L_-\delta_y^-dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+dz_{i,j}^{n+\frac{1}{2}}+L_-^Tdz_{i,j}^{n+\frac{1}{2}}\wedge\delta_y^-dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+dz_{i,j}^{n+\frac{1}{2}}-L_+dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_y^-dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+dz_{i,j}^{n+\frac{1}{2}}+\delta_y^-dz_{i,j}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\left[\frac{1}{\Delta y}(dz_{i,j+1}^{n+\frac{1}{2}}-dz_{i,j}^{n+\frac{1}{2}})\right]+\left[\frac{1}{\Delta y}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i,j-1}^{n+\frac{1}{2}})\right]\wedge L_+dz_{i,j}^{n+\frac{1}{2}}\\
&=\frac{1}{\Delta y}(dz_{i,j}^{n+\frac{1}{2}}\wedge L_+dz_{i,j+1}^{n+\frac{1}{2}}-dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}})\\
&=\delta_y^+(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}}).
\end{aligned}\tag{2.15}
$$

Taking Eqs. (2.13)-(2.15) into Eq. (2.12), we have

$$
\delta_{\frac{t}{2}}^+(dz_{i,j}^n\wedge M_+dz_{i,j}^{n+\frac{1}{2}})+\delta_x^+(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}})+\delta_y^+(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}})=0.\tag{2.16}
$$

The proof is completed. □

We start from the new scheme (2.8). Note that the matrix splitting (2.9) is not unique [25]. We can obtain different schemes with different splitting methods. Now we take $M_+$, $K_+$ and $L_+$ as upper triangle matrices. They are

$$
M_+=\begin{pmatrix}0&0&0&0&0\\0&0&\frac{1}{2}&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad K_+=\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad L_+=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}.
$$

Submitting the above matrices into the multi-symplectic scheme (2.8), we get the discrete form of the multi-symplectic PDEs (2.2)

$$
\delta_x^+\varphi_{i,j}^{n+\frac{1}{2}}=u_{i,j}^{n+\frac{1}{2}},\tag{2.17a}
$$

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$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=(p-\frac{1}{2}u^2)_{i,j}^{n+\frac{1}{2}}, \tag{2.17b}$$

$$-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}=0, \tag{2.17c}$$

$$-\delta_x^{-}u_{i,j}^{n+\frac{1}{2}}=-v_{i,j}^{n+\frac{1}{2}}, \tag{2.17d}$$

$$-\delta_y^{-}u_{i,j}^{n+\frac{1}{2}}=-w_{i,j}^{n+\frac{1}{2}}. \tag{2.17e}$$

Applying $\delta_x^-$ to Eq. (2.17b), noting that the finite difference operators mutually commute, we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.18}$$

Note that

$$-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}=\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}, \tag{2.19}$$

we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.20}$$

Substituting (2.17d) and (2.17e) into (2.20), we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.21}$$

If we submit the index $i$ by $i+1$ in Eq. (2.21), then we obtain

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i+1,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i+1,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.22}$$

Note that

$$\delta_x^{-}\varphi_{i+1,j}^{n+\frac{1}{2}}=\delta_x^{+}\varphi_{i,j}^{n+\frac{1}{2}}=u_{i,j}^{n+\frac{1}{2}}. \tag{2.23}$$

We obtain the following multi-symplectic scheme of the ZK equation:

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}u_{i,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i+1,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_x^{-}(u_{i+1,j}^{n+\frac{1}{2}})^2+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i+1,j}^{n+\frac{1}{2}}=0. \tag{2.24}$$

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In finite difference format the scheme is given as follows:

$$\begin{aligned}
&\frac{1}{2\Delta t}(u_{i+1,j}^{n+1}+u_{i,j}^{n+1})+\frac{1}{2(\Delta x)^3}(u_{i+2,j}^{n+1}-3u_{i+1,j}^{n+1}+3u_{i,j}^{n+1}-u_{i-1,j}^{n+1})\\
&\quad+\frac{1}{2\Delta x(\Delta y)^2}(u_{i+1,j+1}^{n+1}-2u_{i+1,j}^{n+1}+u_{i+1,j-1}^{n+1}-u_{i,j+1}^{n+1}+2u_{i,j}^{n+1}-u_{i,j-1}^{n+1})\\
=&\frac{1}{2\Delta t}(u_{i+1,j}^{n}+u_{i,j}^{n})-\frac{1}{2(\Delta x)^3}(u_{i+2,j}^{n}-3u_{i+1,j}^{n}+3u_{i,j}^{n}-u_{i-1,j}^{n})\\
&\quad-\frac{1}{2\Delta x(\Delta y)^2}(u_{i+1,j+1}^{n}-2u_{i+1,j}^{n}+u_{i+1,j-1}^{n}-u_{i,j+1}^{n+1}+2u_{i,j}^{n}-u_{i,j-1}^{n})\\
&\quad-\frac{1}{2\Delta x}\Big(\Big(\frac{u_{i+1,j}^{n+1}+u_{i+1,j}^{n}}{2}\Big)^2-\Big(\frac{u_{i,j}^{n+1}+u_{i,j}^{n}}{2}\Big)^2\Big).
\end{aligned}\tag{2.25}$$

**Theorem 2.2.** *The discrete multi-symplectic scheme* (2.24) *for the ZK equation* (1.1) *satisfies the discrete multi-symplectic conservation law*

$$\begin{aligned}
&\frac{1}{2}\delta_{\frac{t}{2}}^{+}(du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\delta_{x}^{+}(dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})\\
&\qquad+\delta_{y}^{+}(du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})=0.
\end{aligned}\tag{2.26}$$

*Proof.* From Eq. (2.8), we can get

$$\begin{aligned}
&\delta_{\frac{t}{2}}^{+}(dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}})+\delta_{x}^{+}(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_{+}dz_{i,j}^{n+\frac{1}{2}})+\delta_{y}^{+}(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_{+}dz_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{2}\delta_{\frac{t}{2}}^{+}(du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\delta_{x}^{+}(dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})+\delta_{y}^{+}(du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{\Delta t}(du_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+1}-du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\frac{1}{\Delta x}(dp_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i+1,j}^{n+\frac{1}{2}}-dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge dv_{i+1,j}^{n+\frac{1}{2}}-du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})+\frac{1}{\Delta y}(du_{i,j}^{n+\frac{1}{2}}\wedge dw_{i,j+1}^{n+\frac{1}{2}}-du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{2}du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\frac{1}{2}\Delta t}(d\varphi_{i,j}^{n+1}-d\varphi_{i,j}^{n+\frac{1}{2}})\Big)+\frac{1}{2}\Big(\frac{1}{\frac{1}{2}\Delta t}(du_{i,j}^{n+\frac{1}{2}}-du_{i,j}^{n})\Big)\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+dp_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta x}(d\varphi_{i+1,j}^{n+\frac{1}{2}}-d\varphi_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta x}(dp_{i,j}^{n+\frac{1}{2}}-dp_{i-1,j}^{n+\frac{1}{2}})\Big)\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta x}(dv_{i+1,j}^{n+\frac{1}{2}}-dv_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta x}(du_{i,j}^{n+\frac{1}{2}}-du_{i-1,j}^{n+\frac{1}{2}})\Big)\wedge dv_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta y}(dw_{i,j+1}^{n+\frac{1}{2}}-dw_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta y}(du_{i,j}^{n+\frac{1}{2}}-du_{i,j-1}^{n+\frac{1}{2}})\Big)\wedge dw_{i,j}^{n+\frac{1}{2}}\\
=&\frac{1}{2}du_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{+}d\varphi_{i,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_{\frac{t}{2}}^{-}du_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+dp_{i,j}^{n+\frac{1}{2}}\wedge\delta_{x}^{+}d\varphi_{i,j}^{n+\frac{1}{2}}+\delta_{x}^{-}dp_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\delta_{x}^{+}dv_{i,j}^{n+\frac{1}{2}}+\delta_{x}^{-}du_{i,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}}+du_{i,j}^{n+\frac{1}{2}}\wedge\delta_{y}^{+}dw_{i,j}^{n+\frac{1}{2}}+\delta_{y}^{-}du_{i,j}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}}.
\end{aligned}\tag{2.27}$$

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Differentiating Eqs. (2.17a)-(2.17e) respectively, we can get

$$\delta_x^+ d\varphi_{i,j}^{n+\frac{1}{2}} = du_{i,j}^{n+\frac{1}{2}}, \tag{2.28a}$$

$$\frac{1}{2}\delta_{\frac{t}{2}}^+ d\varphi_{i,j}^{n+\frac{1}{2}} + \delta_x^+ dv_{i,j}^{n+\frac{1}{2}} + \delta_y^+ dw_{i,j}^{n+\frac{1}{2}} = dp_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}} du_{i,j}^{n+\frac{1}{2}}, \tag{2.28b}$$

$$-\frac{1}{2}\delta_{\frac{t}{2}}^- du_{i,j}^{n+\frac{1}{2}} - \delta_x^- dp_{i,j}^{n+\frac{1}{2}} = 0, \tag{2.28c}$$

$$-\delta_x^- du_{i,j}^{n+\frac{1}{2}} = -dv_{i,j}^{n+\frac{1}{2}}, \tag{2.28d}$$

$$-\delta_y^- du_{i,j}^{n+\frac{1}{2}} = -dw_{i,j}^{n+\frac{1}{2}}. \tag{2.28e}$$

Taking Eqs. (2.28) into Eq. (2.27), we can get

$$\begin{aligned}
&\delta_{\frac{t}{2}}^+ (dz_{i,j}^n \wedge M_+ dz_{i,j}^{n+\frac{1}{2}}) + \delta_x^+ (dz_{i-1,j}^{n+\frac{1}{2}} \wedge K_+ dz_{i,j}^{n+\frac{1}{2}}) + \delta_y^+ (dz_{i,j-1}^{n+\frac{1}{2}} \wedge L_+ dz_{i,j}^{n+\frac{1}{2}}) \\
=&du_{i,j}^{n+\frac{1}{2}} \wedge (dp_{i,j}^{n+\frac{1}{2}} -) + u_{i,j}^{n+\frac{1}{2}} du_{i,j}^{n+\frac{1}{2}} - \delta_x^- dp_{i,j}^{n+\frac{1}{2}} \wedge d\varphi_{i,j}^{n+\frac{1}{2}} + dp_{i,j}^{n+\frac{1}{2}} \wedge du_{i,j}^{n+\frac{1}{2}} \\
&+\delta_x^- dp_{i,j}^{n+\frac{1}{2}} \wedge d\varphi_{i,j}^{n+\frac{1}{2}} + dv_{i,j}^{n+\frac{1}{2}} \wedge dv_{i,j}^{n+\frac{1}{2}} + dw_{i,j}^{n+\frac{1}{2}} \wedge dw_{i,j}^{n+\frac{1}{2}} \\
=&0.
\end{aligned} \tag{2.29}$$

The proof is completed. □

## 3 Backward error analysis for the new multi-symplectic scheme

We now assume $z$ is a sufficiently smooth function that, when evaluated at the lattice points, satisfies Eq. (2.8) [19, 23]. Expanding $z$ in a Taylor series about $t_{n+1/2}$, we obtain

$$z_{i,j}^{n+1} = z + \frac{\Delta t}{2} z_t + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 z_{tt} + \frac{1}{6}\left(\frac{\Delta t}{2}\right)^3 z_{ttt} + \frac{1}{24}\left(\frac{\Delta t}{2}\right)^4 z_{tttt} + \cdots, \tag{3.1a}$$

$$z_{i,j}^{n} = z - \frac{\Delta t}{2} z_t + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 z_{tt} - \frac{1}{6}\left(\frac{\Delta t}{2}\right)^3 z_{ttt} + \frac{1}{24}\left(\frac{\Delta t}{2}\right)^4 z_{tttt} - \cdots, \tag{3.1b}$$

where $z = z(x_i, y_j, t_{n+1/2})$. We have

$$z_{i,j}^{n+\frac{1}{2}} = \frac{z_{i,j}^n + z_{i,j}^{n+1}}{2} = z + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 z_{tt} + \frac{1}{24}\left(\frac{\Delta t}{2}\right)^4 z_{tttt} + \mathcal{O}(\Delta t^5). \tag{3.2}$$

So we have

$$\frac{z_{i,j}^{n+1} - z_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = z_t + \frac{\Delta t^2}{24} z_{ttt} + \mathcal{O}(\Delta t^4), \tag{3.3a}$$

$$\frac{z_{i,j}^{n+\frac{1}{2}} - z_{i,j}^{n}}{\frac{1}{2}\Delta t} = z_t + \frac{\Delta t^2}{24} z_{ttt} + \mathcal{O}(\Delta t^4). \tag{3.3b}$$

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From Eq. (3.2), we have

$$z_{i,j}^{n+\frac{1}{2}}=z+\mathcal{O}(\Delta t^2). \tag{3.4}$$

Expanding $z$ in a Taylor series about $x_i$, we obtain

$$z_{i+1,j}^{n+1}=z_{i,j}^{n+1}+\Delta x(z_x)_{i,j}^{n+1}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n+1}+\cdots, \tag{3.5a}$$

$$z_{i+1,j}^{n}=z_{i,j}^{n}+\Delta x(z_x)_{i,j}^{n}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n}+\cdots. \tag{3.5b}$$

From Eqs. (3.5a) and (3.5b), we have

$$\begin{aligned}z_{i+1,j}^{n+\frac{1}{2}}&=\frac{z_{i+1,j}^{n}+z_{i+1,j}^{n+1}}{2}=z_{i,j}^{n+\frac{1}{2}}+\Delta x(z_x)_{i,j}^{n+\frac{1}{2}}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n+\frac{1}{2}}+\mathcal{O}(\Delta x^3)\\&=z_{i,j}^{n+\frac{1}{2}}+\Delta x(z_x)+\frac{\Delta x^2}{2}z_{xx}+\mathcal{O}(\Delta x^3)+\mathcal{O}(\Delta x\Delta t^2).\end{aligned}$$

So we can get

$$\frac{z_{i+1,j}^{n+\frac{1}{2}}-z_{i,j}^{n+\frac{1}{2}}}{\Delta x}=z_x+\frac{\Delta x}{2}z_{xx}+\mathcal{O}(\Delta x^2+\Delta t^2). \tag{3.6}$$

In the same way, we can get

$$\frac{z_{i,j}^{n+\frac{1}{2}}-z_{i-1,j}^{n+\frac{1}{2}}}{\Delta x}=z_x-\frac{\Delta x}{2}z_{xx}+\mathcal{O}(\Delta x^2+\Delta t^2), \tag{3.7a}$$

$$\frac{z_{i,j+1}^{n+\frac{1}{2}}-z_{i,j}^{n+\frac{1}{2}}}{\Delta y}=z_y+\frac{\Delta y}{2}z_{yy}+\mathcal{O}(\Delta y^2+\Delta t^2), \tag{3.7b}$$

$$\frac{z_{i,j}^{n+\frac{1}{2}}-z_{i,j-1}^{n+\frac{1}{2}}}{\Delta y}=z_y-\frac{\Delta y}{2}z_{yy}+\mathcal{O}(\Delta y^2+\Delta t^2). \tag{3.7c}$$

Substituting Eqs. (3.3a)-(3.4) and Eqs. (3.6)-(3.7c) into Eq. (2.8) yields the modified PDE

$$M(z_t+\frac{\Delta t^2}{24}z_{ttt})+Kz_x+\frac{\Delta x}{2}(K_+-K_-)z_{xx}+Lz_y+\frac{\Delta y}{2}(L_+-L_-)z_{yy}=\nabla_z S(z). \tag{3.8}$$

Substituting $M$, $K$, $K_+$, $K_-$, $L$, $L_+$, $L_-$ and $z$ into modified Eq. (3.8) gives

$$\varphi_x+\frac{1}{2}\Delta x\varphi_{xx}=u, \tag{3.9a}$$

$$\frac{1}{2}\varphi_t+\frac{1}{48}(\Delta t)^2\varphi_{ttt}+v_x+\frac{1}{2}\Delta xv_{xx}+w_y+\frac{1}{2}\Delta yw_{yy}=p-\frac{1}{2}u^2, \tag{3.9b}$$

$$-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-p_x+\frac{1}{2}\Delta xp_{xx}=0, \tag{3.9c}$$

$$-u_x+\frac{1}{2}\Delta xu_{xx}=-v, \tag{3.9d}$$

$$-u_y+\frac{1}{2}\Delta yu_{yy}=-w. \tag{3.9e}$$

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Substituting Eq. (3.9d) and (3.9e) into Eq. (3.9b), we have

$$p=\frac{1}{2}\varphi_t+\frac{1}{48}(\Delta t)^2\varphi_{ttt}+u_{xx}-\frac{1}{4}(\Delta x)^2u_{xxxx}+u_{yy}-\frac{1}{4}(\Delta y)^2u_{yyyy}+\frac{1}{2}u^2. \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9c), we have

$$\begin{aligned}&-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-\frac{1}{2}\varphi_{xt}-\frac{1}{48}(\Delta t)^2\varphi_{xttt}-u_{xxx}+\frac{1}{4}(\Delta x)^2u_{xxxxx}-u_{xyy}\\&+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{2}(u^2)_x+\frac{1}{4}\Delta x\varphi_{xxt}+\frac{1}{96}\Delta x(\Delta t)^2\varphi_{xxttt}+\frac{1}{2}\Delta xu_{xxxx}\\&-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\frac{1}{2}\Delta xu_{xxyy}-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}+\frac{1}{4}\Delta x(u^2)_{xx}=0.\end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned}&\varphi_{xt}=u_t-\frac{1}{2}\Delta x\varphi_{xxt}, &&\varphi_{xxt}=u_{xt}-\frac{1}{2}\Delta x\varphi_{xxxt},\\&\varphi_{xttt}=u_{ttt}-\frac{1}{2}\Delta x\varphi_{xxttt}, &&\varphi_{xxttt}=u_{xttt}-\frac{1}{2}\Delta x\varphi_{xxxttt},\end{aligned}$$

we have

$$\begin{aligned}&-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-\frac{1}{2}u_t+\frac{1}{4}\Delta xu_{xt}-\frac{1}{8}(\Delta x)^2\varphi_{xxxt}-\frac{1}{48}(\Delta t)^2u_{ttt}\\&+\frac{1}{96}\Delta x(\Delta t)^2u_{xttt}-\frac{1}{192}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}-u_{xxx}+\frac{1}{4}(\Delta x)^2u_{xxxxx}-u_{xyy}\\&+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{2}(u^2)_x+\frac{1}{4}\Delta xu_{xt}-\frac{1}{8}(\Delta x)^2\varphi_{xxxt}+\frac{1}{96}\Delta x(\Delta t)^2u_{xttt}\\&-\frac{1}{192}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}+\frac{1}{2}\Delta xu_{xxxx}-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\frac{1}{2}\Delta xu_{xxyy}\\&-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}+\frac{1}{4}\Delta x(u^2)_{xx}=0.\end{aligned} \tag{3.12}$$

So we have

$$\begin{aligned}&u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\\&=\frac{1}{2}\Delta x\left(u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\right)_x-\frac{1}{24}(\Delta t)^2u_{ttt}-\frac{1}{4}(\Delta x)^2\varphi_{xxxt}\\&\quad+\frac{1}{48}\Delta x(\Delta t)^2u_{xttt}-\frac{1}{96}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}+\frac{1}{4}(\Delta x)^2u_{xxxxx}\\&\quad+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{8}(\Delta x)^3u_{xxxxxx}-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}\\&=\frac{1}{2}\Delta x\left(u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\right)_x+\frac{1}{48}\Delta x(\Delta t)^2u_{xttt}\\&\quad-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2),\end{aligned} \tag{3.13}$$

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which is an $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$ perturbation of the ZK equation (1.1).

The modified equation (3.8) can be written in the form of a standard multi-symplectic PDE

$$\tilde{M}\tilde{z}_t+\tilde{K}\tilde{z}_x+\tilde{L}\tilde{z}_y=\nabla_{\tilde{z}}\tilde{S}(\tilde{z}) \tag{3.14}$$

for $\tilde{z}=(z,z_t,z_{tt},z_x,z_y)^T$, and

$$\tilde{x}=S(z)+\frac{\Delta t^2}{24}z_{tt}^T M z_t-\frac{\Delta x}{2}z_x^T P z_x-\frac{\Delta y}{2}z_y^T Q z_y,$$

with the skew-symmetric matrices

$$\tilde{M}=\begin{pmatrix} M & 0 & \frac{\Delta t^2}{24}M & 0 & 0\\ 0 & -\frac{\Delta t^2}{24}M & 0 & 0 & 0\\ \frac{\Delta t^2}{24}M & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad \tilde{K}=\begin{pmatrix} K & 0 & 0 & \Delta x P & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\Delta x P & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{L}=\begin{pmatrix} L & 0 & 0 & 0 & \Delta y Q\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\Delta y Q & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $P=(K_+-K_-)/2$, $Q=(L_+-L_-)/2$.

## 4 Numerical simulations

In this section, we test the new derived schemes on the solitary wave of the ZK equation. We consider the ZK equation with exact boundary condition. For fixed $n$, we give the definition of maxerror$(n)$:

$$\text{maxerror}(n)=\max_{i,j}|u_{i,j}^n-u(x_i,y_j,t_n)|, \tag{4.1}$$

where $u_{i,j}^n$ is the numerical solution while $u(x_i,y_j,t_n)$ is the exact solution.

### 4.1 Numerical simulation 1

The steady progressive wave solutions of the form $u=U(x-ct,y)$ satisfy the following equation:

$$\Delta U=cU-\frac{1}{2}U^2,\quad \Delta\equiv\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial y^2},\quad X\equiv x-ct, \tag{4.2}$$

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where $c$ represents the wave velocity to be determined by solving Eq. (4.2). It is easy to see that a steady progressive wave solution of the form

$$U(x,y,t)=3c\text{sech}^2\left[\frac{1}{2}\sqrt{c}(X\cos\theta+y\sin\theta)\right] \tag{4.3}$$

is an exact solution. This solution represents an oblique one-dimensional solitary wave with an inclined angle $\theta$ with respect to the x-axis. We carry out our numerical computation on the domain $[0,34]\times[0,2]$ with the parameters $c=2$, $\theta=\pi/3$, and choose $\Delta x=0.2$, $\Delta y=0.1$, $\Delta t=0.1$. We take the following initial conditions

$$U(x,y,5)=3c\text{sech}^2\left[\frac{1}{2}\sqrt{c}((x-5c)\cos\theta+y\sin\theta)\right] \tag{4.4}$$

just for computing convenience, and it has nothing to do with the scheme and the results. Fig. 1 shows the initial condition at $t=5$. Figs. 2 and 3 show the numerical solution at $t=8$ and $t=11$ respectively. We can see the moving of wave. Fig. 4 shows the error between the numerical solution and the exact solution at $t=11$. Fig. 5 shows the trend of the maxerror$(n)$ as time evolves. From that, we can see that the scheme has the good numerical performance.

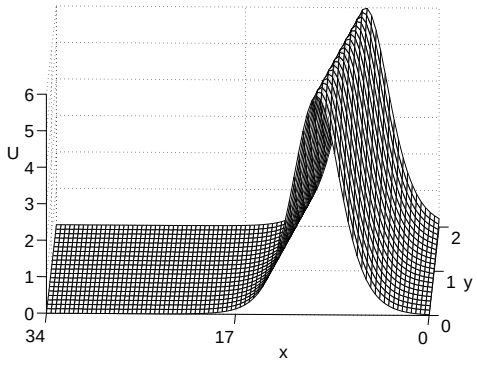



Figure 1: The wave form of the solitary wave at $t=5$.

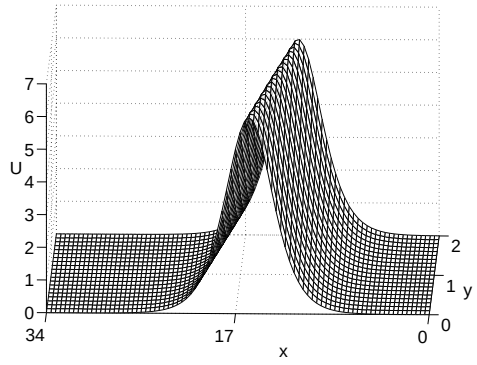



Figure 2: Numerical solution of the solitary wave at $t=8$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

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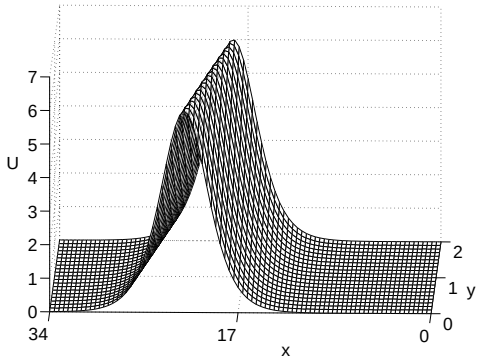



Figure 3: Numerical solution of the solitary wave at $t=11$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

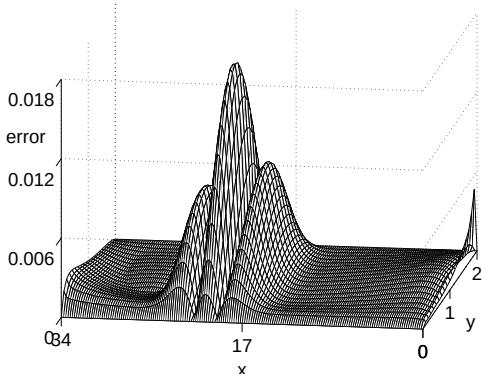



Figure 4: The error between the numerical solution and the exact solution of the solitary wave at $t=11$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

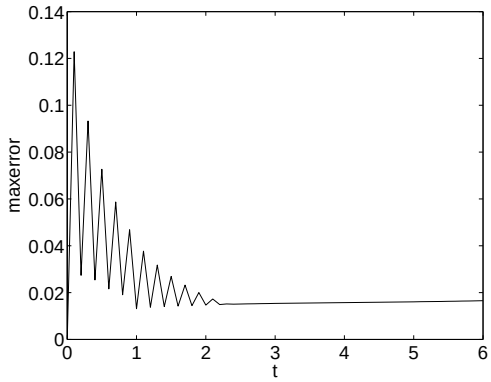



Figure 5: The trend of the maxerror$(n)$ of the solitary wave as time evolves with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

## 4.2 Numerical simulation 2

Next we try the cylindrical solitary wave of the ZK equation. The cylindrical solition of the ZK equation can be expressed as

$$U(x,y,t)=3\text{sech}^2\left[\frac{1}{2}\sqrt{(x-ct-x_0)^2+(y-y_0)^2}\right]. \tag{4.5}$$

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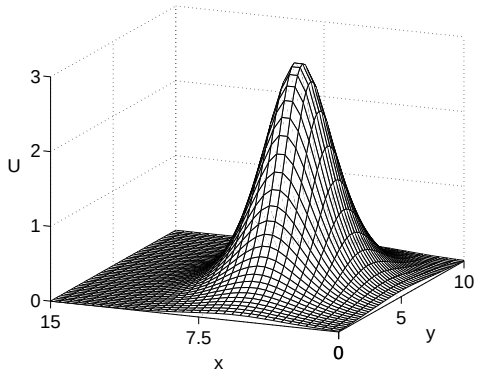



Figure 6: The wave form of the cylindrical solitary wave at $t=0$.

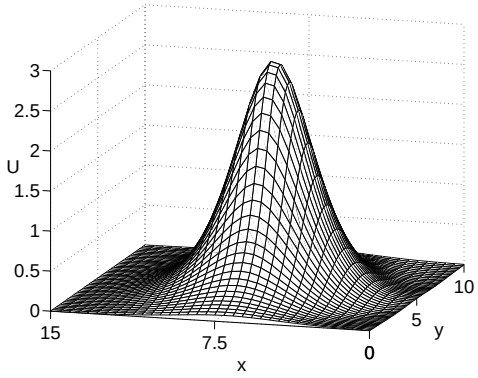



Figure 7: Numerical solution of the cylindrical solitary wave at $t=2.5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

We take the following initial condition:

$$U(x,y,0)=3\text{sech}^2\left[\frac{1}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}\right]. \tag{4.6}$$

We compute in a rectangle $[0,15]\times[0,10]$ with the parameters $c=0.5$, $x_0=5.0$, $y_0=5.0$, and choose $\Delta x=0.2$, $\Delta y=0.2$, $\Delta t=0.1$.

Fig. 6 shows the initial condition at $t=0$. Figs. 7 and 8 show the numerical solution at $t=2.5$ and $t=5$ respectively. We can see the moving of wave from the graph. Fig. 9 shows the error between the numerical solution and the exact solution at $t=5$. The error can be diminished by reducing the spacial step and the time step.

## 5 Conclusions

In this paper, we propose a new scheme for the ZK equation with the accuracy order of $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$. The new scheme is a multi-symplectic scheme that preserves the intrinsic geometry property of the equation. Numerical results show that the new multi-symplectic scheme can well simulate the solitary evolution behaviors of the ZK equation.

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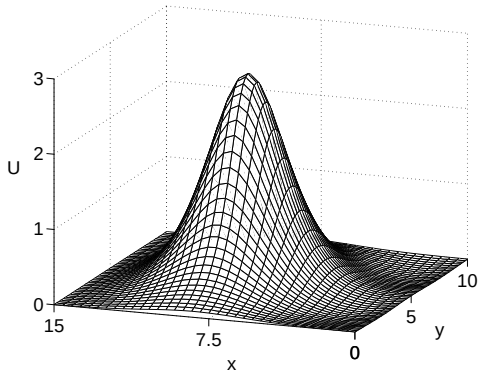



Figure 8: Numerical solution of the cylindrical solitary wave at $t=5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

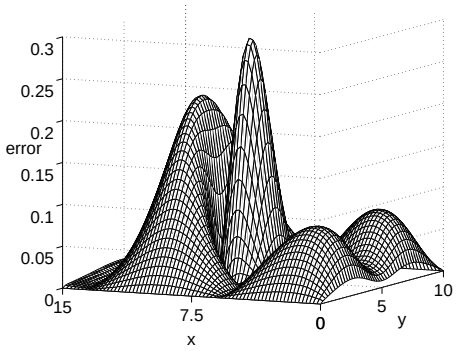



Figure 9: The error between the numerical solution and the exact solution of the cylindrical solitary wave at $t=5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

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