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A 3-manifold which is not a product

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The purpose of this note is to construct a non-compact 3-manifold M with the following properties:

(i) M is orientable and irreducible,

(ii) ∂M is connected and compact and its inclusion in M is a homotopy equivalence,

(iii) M is not homeomorphic to $\partial M \times R_+$.

(Throughout this paper, all manifolds will be PL and all embeddings will be PL and locally flat.)

This is a counter example to a natural question which is related to the following question, raised by A. Marden in connexion with the study of Kleinian groups:

If a 3-manifold M satisfies (i) and (ii) above and has R^3_+ as its universal covering space, then is M homeomorphic to $\partial M \times R_+$?

The restriction that M be irreducible, i.e. every 2-sphere embedded in M bounds a 3-ball, is necessary to avoid the Poincaré conjecture. Note that condition (ii) implies that M has one end. For clearly $H_2(M, \partial M) = 0$, and by Poincaré duality this implies that $H^1_f(M) = 0$. The exact sequence of (2) now shows that M has one end. T. W. Tucker has an example showing that if we omit the restriction that ∂M be compact then the answer to our question is negative. Finally we note that our example M does not help us with this new question as our proof depends on the fact that M does not have R^3_+ as universal covering space.

The construction of M is by glueing together $F \times R_+$ (F a compact surface) and Whitehead's example (3) of an open, contractible, irreducible 3-manifold W not homeomorphic to R^3. We learnt the right way to glue together two non-compact manifolds from C. T. C. Wall. We first give a new proof of Whitehead's result which seems to be simpler than that in (3). The methods in our proof are also needed to prove our result about M. The only hard theorem we quote is the fact that R^3 is irreducible (1).

The construction of W. We first express the 3-sphere S^3 as the union of two solid tori S and T with disjoint interiors. Let $\phi\colon T \to T$ be an embedding as in Fig. 1, i.e. ϕ is null-homotopic, $\phi(T)$ is not contained in a 3-ball in T but $\phi(T)$ is isotopic to T in S^3. Let ϕ^n denote the n-fold composite of ϕ and let ϕ^0 be the identity map. Then we define $W = S^3 - \bigcap_{n=1}^{\infty} \phi^n(T)$, which is an open 3-manifold.

Let $W_n = S^3 - \operatorname{int}(\phi^n(T))$. By our choice of ϕ, $W_n \cong S^1 \times D^2$. Again by our choice of

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ϕ, $\phi^{n+1}(T)$ is null-homotopic in $\phi^n(T) = S^3 - W_n$. As linking numbers are symmetric it follows that W_n is null-homotopic in $S^3 - \phi^{n+1}(T) = W_{n+1}$. We can now see that $\pi_i(W) = 0$ for all $i \geqslant 1$ as any sphere in W lies in some W_n and so is null-homotopic in W. Hence W is contractible by Whitehead's theorem.

It also follows that W is irreducible. For, if we have a 2-sphere S embedded in W, it must bound a compact submanifold B of W as $\pi_2(W) = 0$. B must be a 3-ball as B lies in R^3 which is irreducible (1). Note that this argument shows that any 3-manifold X which embeds in R^3 and has $\pi_2(X) = 0$ is irreducible. Hence in particular, R^3_+, $S^1 \times D^2$ and $F \times R_+$ are all irreducible, where F is any closed orientable surface not S^2. We shall be using these examples later.

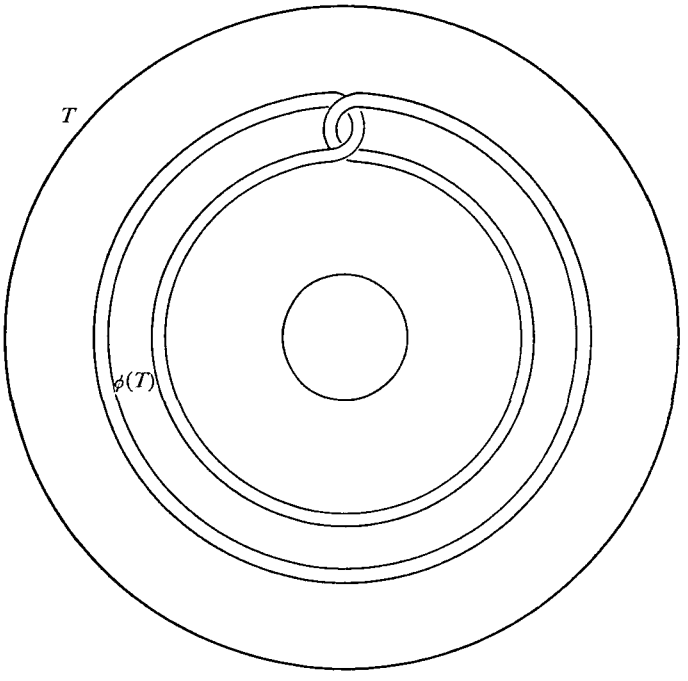


Fig. 1

Finally we must show that W is not R^3. If it were, then any compact subset of W would be contained in a 3-ball in W. The following result, originally proved by Whitehead (3), shows that this is false.

LEMMA. *The core circle C of the solid torus S is not contained in a 3-ball in W.*

Proof. Suppose C is contained in a 3-ball B in W. By transversality we can suppose that $\partial B \cap \left(\bigcup_{n=0}^{\infty} \partial W_n \right)$ is a collection of $n(B)$ disjoint circles, where $n(B)$ is some integer. We will show below how to replace B by a 3-ball B_1 containing C, with $n(B_1) < n(B)$. By repeating this process, we will eventually obtain a 3-ball B containing C, with $n(B) = 0$. Thus B is contained in S, which contradicts our choice of C.

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We now show how to carry out the reduction step. There must be a circle L in $\partial B \cap \left(\bigcup_{n=0}^{\infty} \partial W_n\right)$ which bounds a 2-disc D in ∂B which contains no other circles of the collection. We will have $L \subset \partial B \cap \partial W_n$, for some n, and hence $D \subset W_{n+1} - W_n$ or $D \subset W_n - W_{n-1}$, where W_{-1} means the empty set.

In either case, we claim that L must bound a 2-disc D' in ∂W_n. Suppose $D \subset W_{n+1} - W_n$ and let U be a regular neighbourhood of D. If L does not bound a 2-disc in ∂W_n, then $(\partial W_n - U \cap \partial W_n) \cup \partial U$ is a 2-sphere in $W_{n+1} - \operatorname{int} W_n = \phi^n(T) - \phi^{n+1}(T)$. As $\phi^n(T)$ is irreducible, this 2-sphere bounds a 3-ball in $\phi^n(T)$ which must contain $\phi^{n+1}(T)$. This contradicts our choice of ϕ. If $D \subset W_n - W_{n-1}$ and $n \geqslant 1$, a similar argument gives a 3-ball in $\phi^{n-1}(T)$ containing $\phi^n(T)$ which is a contradiction. If $D \subset W_0 = S$, a similar argument gives a 3-ball in S containing C which is again a contradiction.

So far we have shown that L bounds 2-discs D and D' in ∂B and ∂W_n respectively. We choose a circle M of $\partial B \cap D'$ which bounds a 2-disc D'' in D' containing no other circle of the collection. We may have $M = L$, of course.

If $D'' \subset B$, then we cut B along D'' to obtain two 3-balls B' and B''. One of these say B' must contain C and we choose $B_1 = B'$. If D'' is not contained in B, then

$$D'' \cap B = \partial D'' = M.$$

Now M separates ∂B into two 2-discs X and Y. The 2-spheres $X \cup D''$ and $Y \cup D''$ bound 3-balls B' and B'' in W. As D'' is not contained in B, neither of B' and B'' is contained in B. Therefore $B' \supset B$ or $B' \cap B = X$. In the second case, we must have $B'' = B' \cup B$, for otherwise $B' \cup B \cup B''$ would be a 3-sphere in W. Thus one of B' and B'', say B', must contain B. We choose $B_1 = B'$. Of course, in each of the cases above $B_1 \cap \partial W_n$ contains the 2-disc D'', but a small isotopy of B_1 will get rid of D''. Clearly, $B_1 \cap \left(\bigcup_{n=0}^{\infty} \partial W_n\right) \subset B \cap \left(\bigcup_{n=0}^{\infty} \partial W_n\right)$ and we have got rid of M. Hence $n(B_1) < n(B)$. This completes the proof of the lemma.

The construction of M. Let $N = F \times R_+$, where F is any closed orientable surface not S^2. We remove from N and from W the interior of a regular neighbourhood of a properly embedded copy of R_+. That is to say, the copy of R_+ runs from an interior point 'out to infinity' in each case. The new manifolds N' and W' are also irreducible. M is obtained by glueing N' to W' along the new boundary components which are both planes. M satisfies conditions (i) and (ii) by construction. To see that M is irreducible, we choose a 2-sphere S in M and consider $S \cap \partial W'$. If this intersection is empty then S bounds a 3-ball in M. If not we choose a circle in $S \cap \partial W'$, innermost in $\partial W'$ and argue rather as in the proof of our lemma, using induction on the number of circles in $S \cap \partial W'$.

Suppose that M is homeomorphic to $F \times R_+$, then M has universal covering space R^3_+. As W' is simply connected, we have a proper embedding of W' in the interior of R^3_+. Hence the core circle C of the solid torus S is contained in a 3-ball B in R^3_+. By applying the methods of the above lemma to $B \cap \partial W'$, we can arrange that B is contained in W', which contradicts the lemma. Note that we can go straight to a

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circle of $B \cap \partial W'$ innermost in $\partial W'$, which simplifies the argument. Thus M does have all the properties stated at the beginning of this paper.

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