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Three-wave coupling and parametric decay of kinetic Alfvén waves

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The dynamic equation and coupling coefficient of the three-wave interaction among kinetic Alfvén waves (KAWs) are derived by use of plasma kinetic theory. Linear and nonlinear effects of finite ion Larmor radius are kept for arbitrary value of the 'kinetic variable' $\kappa = k_\perp \rho_i$. The parametric decay KAW $\rightarrow$ KAW + KAW is investigated and the threshold amplitude for decay instability in a Maxwellian plasma is calculated. The growth rate of decay instability varies as $k_\perp^2$ in both limits $\kappa^2 \ll 1$ and $\kappa^2 \gg 1$. The main tendency of KAWs is towards nonlinear destabilization at very low wave amplitudes $B_k/B_0 \lesssim 10^{-3}$. Two applications concerning KAW dynamics in the magnetosphere and in the solar corona show that three-wave resonant interaction among KAWs may be responsible for the turbulent character of their behaviour, often observed in space plasmas.

1. Introduction

Three-wave resonant interaction is a basic process resulting in various nonlinear plasma phenomena (see Stenflo (1994) and references therein), and the coupling coefficient (matrix element) of the three-wave interaction process is a key mathematical object in the nonlinear wave theory (Galeev and Sagdeev 1979). Thus the main problem of the theory is the derivation of the coupling coefficient; once this coefficient has been found, one can use theoretical methods developed earlier to study nonlinear properties of the considered modes.

Nonlinear properties of kinetic Alfvén waves (KAWs) are of great interest from both theoretical and practical points of view (Shukla and Stenflo 1995). The first nonlinear mechanism for KAW excitation (by LH pump wave) was considered by Shukla and Mamedow (1978), and many articles on nonlinear excitation mechanisms for KAWs have appeared since then (e.g. Sharma and Tripathi 1988; Hasan and Guha 1990; Stenflo and Shukla 1995; Yukhimuk *et al.* 1998).

At the same time, in contrast to ideal-MHD Alfvén waves, three KAWs can effectively interact among themselves. Three-wave resonant interaction among KAWs have been studied in the past (Erokhin *et al.* 1978; Voitenko 1989; Volokitin and Dubinin 1989; Gang and Leboeuf 1993; Yukhimuk and Kucherenko 1993), but the common approach so far has been two-fluid MHD theory in the limit of weak wave dispersion, $k_\perp \rho_i \ll 1$. In fact, even the linear dispersion of $k_\perp \rho_i \ll 1$ Alfvén waves – known from kinetic theory – cannot be obtained from two-fluid MHD. Therefore the results of MHD nonlinear theory also cannot be considered as exact, even in the limit $k_\perp \rho_i \ll 1$.

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Moreover, KAWs with $k_\perp \rho_i \approx 1$ and $k_\perp \rho_i > 1$ may be easily excited (see e.g. Voitenko 1996), for which previous results are not applicable. This is because two-fluid MHD is not appropriate for describing wave motions of ions in the plane perpendicular to $\mathbf{B}_0$ if transverse wavelengths are comparable to the gyroradius of thermal ions. It is also evident that linear kinetic effects of wave–particle interaction, responsible for KAW excitation and dissipation, cannot be self-consistently included using the MHD approach. To study the excitation/dissipation of KAWs and nonlinear interaction among them self-consistently, kinetic plasma theory using the Maxwell–Vlasov set of equations is required. Kinetic theory has been used previously to study the coupling of MHD Alfvén waves with magnetosonic waves (Elvsén 1992) and of kinetic Alfvén waves with ion-Bernstein waves (Sharma and Tripathi 1988).

In the present paper, the kinetic theory formulation (Sec. 2) is used to derive the nonlinear equation describing excitation of KAWs and three-wave interaction among them, accounting for the linear and nonlinear ion-Larmor-radius effects. The linear (Sec. 4) and nonlinear (Sec. 5) responses of the plasma components with arbitrary velocity distribution functions are calculated for arbitrary value of the 'kinetic variable' $\kappa = k_\perp \rho_i$. Dynamical equations and coupling coefficients (matrix elements) of the three-wave resonant interaction of KAWs with arbitrary dispersion are derived (Sec. 6). Parametric decay instability of KAWs is investigated, and general expressions for the decay growth rate and threshold are obtained (Sec. 7). Limiting cases of strong ($k_\perp \rho_i \gg 1$) and weak ($k_\perp \rho_i \ll 1$) dispersion are examined, and the threshold of the decay in Maxwellian plasma is found (Sec. 7). The results of kinetic theory in $k_\perp \rho_i \ll 1$ limit are compared with those previously obtained using MHD theory (Sec. 8). A summary follows in Sec. 10.

2. Basic equations and theoretical background

Let us consider a fully ionized uniform hydrogen plasma in a background magnetic field B_0. Consideration is restricted to plasma β (plasma/magnetic pressure ratio) in the range $m_e/m_i < \beta < 1$, typical of many laboratory and space plasmas.

To describe low-frequency ($\omega < \Omega_i$), magnetically non-compressible ($\tilde{B}_z = 0$) KAWs, one can use two fluctuating electromagnetic wave potentials: a scalar $\tilde{\phi}$ and a (z-component) vector $\tilde{A}_z$ (Hasegawa and Chen 1976). These potentials induce in the αth plasma species charge and current-density perturbations $\tilde{q}_\alpha$ and $\tilde{j}_\alpha$, obeying the quasineutrality equation

$$\sum_\alpha \tilde{q}_\alpha = \sum_\alpha e_\alpha \tilde{n}_\alpha = 0 \tag{2.1}$$

and Ampère's law in the z direction:

$$\sum_\alpha \tilde{j}_\alpha = -\frac{c}{4\pi} \nabla^2 \tilde{A}_z. \tag{2.2}$$

The difference between the MHD and kinetic theories here is in the calculation of the number-density and current-density perturbations $\tilde{n}_\alpha$ and $\tilde{j}_\alpha$. In order to take into account important kinetic properties of KAWs, we calculate $\tilde{n}_\alpha$ and $\tilde{j}_\alpha$ using the perturbed part $\tilde{f}_\alpha$ of the corresponding velocity distribution function ($F_\alpha = F_\alpha^0 + \tilde{f}_\alpha$):

$$\tilde{j} = e \int d^3\mathbf{V}\, V_z \tilde{f}, \qquad \tilde{n} = \int d^3\mathbf{V}\, \tilde{f}. \tag{2.3}$$

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In these expressions and below, where it will not cause misunderstanding, the subscript α, labelling the plasma species, is dropped.

We consider the processes on time scales much less than the Coulomb collisional time scale, in which case the distribution function F satisfies the 'collisionless' Vlasov equation

$$d_t F = 0, \tag{2.4}$$

where the Vlasov operator d_t, has the form of a total time derivative and depends on the set of independent variables $\{\mathbf{X}\}$:

$$d_t = \sum_i \frac{\partial X_i}{\partial t} \frac{\partial}{\partial X_i}. \tag{2.5}$$

The coefficients $\partial X_i/\partial t$ are determined by the standard equations of particle motion.

The solution to the Vlasov equation (2.4) may be found using perturbation theory by expansion of the function F and operator d_t as

$$F = F^0 + f^L + f^{NL}, \tag{2.6}$$

$$d_t = d_t^0 + d_t^L, \tag{2.7}$$

where the superscripts 0, L and NL indicate the unperturbed, linear and nonlinear (in the wave amplitudes) parts respectively.

Keeping second-order terms in (2.4), we get an equation for the nonlinear response:

$$d_t^0 f^{NL} + d_t^L f^L = 0. \tag{2.8}$$

The linear response, entering this equation, is to be found from the linearized Vlasov equation

$$d_t^0 f^L + d_t^L F^0 = 0, \tag{2.9}$$

where the evolution of the function F^0, slowly varying under the action of wave–particle collisions, is determined by the time-averaged Vlasov equation

$$\overline{d_t^L f^L} + d_t^0 F^0 = 0. \tag{2.10}$$

The set of equations (2.1)–(2.10) describes the excitation of KAW turbulence and relaxation of the plasma with unstable initial (and/or boundary) conditions. For the sake of simplicity, in studying the nonlinear interaction among KAWs, we restrict ourselves to the case of a given background distribution F^0, neglecting its slow (quasilinear) evolution described by (2.10).

Let us decompose the non-zero electromagnetic potentials $\tilde{\phi}$ and $\tilde{A}_z$ in KAWs as Fourier series

$$\left\{ \begin{array}{c} \tilde{\phi}(\mathbf{r},t) \\ \tilde{A}_z(\mathbf{r},t) \end{array} \right\} = \sum_{\mathbf{k}} \left\{ \begin{array}{c} \phi_k \\ A_k \end{array} \right\} \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t), \tag{2.11}$$

where $\phi_k \equiv \phi(\mathbf{k})$ and $A_k \equiv A_z(\mathbf{k})$ are Fourier amplitudes.

To investigate finite-Larmor-radius effects, it is convenient to use the guiding-centre variables $(t; \mathbf{r}_g; V_z; V_\perp^2; \theta)$ in (2.8) and (2.9). In this case, (2.11) may be rewritten as

$$\left\{ \begin{array}{c} \tilde{\phi}(\mathbf{r},t) \\ \tilde{A}_z(\mathbf{r},t) \end{array} \right\} = \sum_{\mathbf{k}} \left\{ \begin{array}{c} \phi_k \\ A_k \end{array} \right\} \sum_s J_s \exp[-i(\vartheta - \chi)s] \exp(i\mathbf{k}\cdot\mathbf{r}_g - i\omega_k t), \tag{2.12}$$

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where $J_s \equiv J_s(k_\perp V_\perp/\Omega)$ is the Bessel function and

$$\tan\chi = \frac{k_y}{k_x}, \qquad \tan\vartheta = \frac{V_y}{V_x}.$$

The equations of motion (coefficients in (2.5)) read as

$$\left(\frac{d\mathbf{V}_\perp}{dt}\right)^L = \frac{-ie}{m}\sum_{\mathbf{k}} \mathbf{k}_\perp \left(\phi_k - \frac{V_z}{c}A_k\right), \tag{2.13a}$$

$$\left(\frac{d\mathbf{r}_g}{dt}\right)^L = \frac{-ie}{m\Omega}\sum_{\mathbf{k}} (\mathbf{k}_\perp \times \mathbf{z}) \left(\phi_k - \frac{V_z}{c}A_k\right), \tag{2.13b}$$

$$\left(\frac{dV_z}{dt}\right)^L = \frac{-ie}{m}\sum_{\mathbf{k}} \left[k_z\left(\phi_k - \frac{V_z}{c}A_k\right) + \frac{\mathbf{k}_\perp\cdot\mathbf{V}_\perp}{c}A_k\right], \tag{2.13c}$$

$$\left(\frac{d\theta}{dt}\right)^L = \frac{-ie}{m}\sum_{\mathbf{k}} \left[-\frac{(\mathbf{k}_\perp \times \mathbf{V}_\perp)_z}{|\mathbf{V}_\perp|^2}\left(\phi_k - \frac{V_z}{c}A_k\right)\right]. \tag{2.13d}$$

3. Linear response

For the perturbed distribution function, we adopt a Fourier representation in gyrocentre coordinates $\mathbf{r}_g$:

$$\tilde{f}^L = \sum_{\mathbf{k}} f^L_{gk} \exp(i\mathbf{k}\cdot\mathbf{r}_g - i\omega_k t). \tag{3.1}$$

Expanding the coefficients f_{gk}, (3.1) may be rewritten as

$$\tilde{f}^L = \sum_{\mathbf{k}}\sum_{n} f^L_{kn} \exp[-in(\theta-\chi)] \exp(-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}_g), \tag{3.2}$$

where χ is the angle between the x axis and $\mathbf{k}_\perp$. Substituting the equations of motion in explicit form into (2.9) and using the expansions (2.12) and (3.2), after some algebra we obtain an equation for f^L_{kn} as

$$\begin{aligned}\sum_n \exp[-in(\theta-\chi)]\exp(-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}_g)\Bigg\{&(-\omega_k + k_z V_z + n\Omega) f^L_{kn} \\ -\frac{e}{m}J_n\left(\frac{k_\perp V_T}{\Omega}\right)\Bigg\{&\left[k_z\left(\phi_k - \frac{\omega_k}{k_z c}A_k\right) + \frac{k_\perp V_\perp}{c}\cos(\vartheta-\chi)\,A_k\right]\frac{\partial F^0}{\partial V_z} \\ +&\left[k_\perp V_\perp \cos(\vartheta-\chi)\left(\phi_k - \frac{V_z}{c}A_k\right)\right]\frac{1}{V_\perp}\frac{\partial F^0}{\partial V_\perp}\Bigg\}\Bigg\}.\end{aligned} \tag{3.3}$$

By use of recurrence relations for the Bessel functions, we can substitute $k_\perp V_\perp \cos(\vartheta-\chi) J_n \to n\Omega J_n$ in this equation, excluding the angle dependence. Thus we obtain the following expression for the coefficients f^L_{kn}:

$$\begin{aligned}f^L_{kn} = \frac{e}{m}\frac{J_n(\kappa\zeta)}{-\omega_k + k_z V_z + n\Omega}\Bigg\{&k_z\left(\phi_k - \frac{\omega_k}{k_z c}A_k\right)\frac{\partial}{\partial V_z} \\ &+n\Omega\left[\left(\phi_k - \frac{V_z}{c}A_k\right)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + \frac{1}{c}A_k\frac{\partial}{\partial V_z}\right]\Bigg\}F^0\end{aligned} \tag{3.4}$$

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where

$$\zeta = \frac{V_\perp}{V_T}, \qquad \kappa = k_\perp \rho = \frac{k_\perp V_T}{\Omega},$$

that is $k_\perp V_\perp/\Omega = \kappa\zeta$.

The expression (3.2) with (3.4) for f^L_{kn} gives the plasma linear response to magnetically incompressible electromagnetic waves with $B_z = 0$. For low-frequency ($\omega_k < \Omega$) KAWs, f^L_{kn} may be approximated as

$$f^L_{kn} = \frac{e}{m} J_n(\kappa\zeta)[\delta_{n,0}\hat{U} + (1-\delta_{n,0})\hat{W}]F^0 \frac{V_k}{c} A_k, \tag{3.5}$$

where $V_k = \omega_k/k_z$ is the wave phase velocity along $\mathbf{B}_0$, and the following operators have been introduced:

$$\hat{U} = \frac{V_k}{V_z - V_k}\left(\frac{\phi_k}{A_k}\frac{c}{V_k} - 1\right)\frac{1}{V_k}\frac{\partial}{\partial V_z}, \tag{3.6}$$

$$\hat{W} = \left(\frac{\phi_k}{A_k}\frac{c}{V_k} - \frac{V_z}{V_k}\right)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + \frac{1}{V_k}\frac{\partial}{\partial V_z}. \tag{3.7}$$

In the case of a Maxwellian F^0, (3.4) reproduces that obtained by Sharma and Tripathi (1988).

The Fourier amplitude f^L_k may now be expressed using f^L_{kn}:

$$f^L_k = \sum_{n,l} \exp[-i(\vartheta - \chi)(l-n)] J_l(\kappa\zeta) f^L_{kn}. \tag{3.8}$$

Averaging this over the gyroangle ϑ and using the low-frequency expression (3.5) for f^L_{kn}, we get

$$\int_0^{2\pi} d\vartheta\, f^L_k = 2\pi \frac{e}{m}\{J_0^2(\kappa\zeta)\hat{U} + [1 - J_0^2(\kappa\zeta)]\hat{W}\}F^0 \frac{V_k}{c} A_k. \tag{3.9}$$

4. Nonlinear response

We use the same expansion as (3.2) for the nonlinear response f^{NL}:

$$\tilde{f}^{NL} = \sum_{\mathbf{k}} \sum_n f^{NL}_{kn} \exp[-in(\theta - \chi)] \exp(-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}_g). \tag{4.1}$$

Substituting (4.1) for $\tilde{f}^{NL}$ into the second-order equation (2.8), we get

$$\begin{aligned}
&\sum_{\mathbf{k}}\sum_n (-\omega_k + k_z V_z + n\Omega) f^{NL}_{kn} \exp[-in(\theta-\chi)] \exp(-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}_g) \\
&\quad = \frac{e}{m} \sum_{\mathbf{k}_1,\mathbf{k}_2} \sum_{s,l} \exp[-is(\theta - \chi_1)] \exp[-i(\omega_1+\omega_2)t + i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}_g] J_s(\kappa_1\zeta) \\
&\qquad \times \left[-\frac{i}{\Omega}G_1 + \mathbf{k}_{1\perp}\cdot\mathbf{V}_\perp \hat{W}_1 - \omega_1 Q_1 \frac{1}{V_1}\frac{\partial}{\partial V_z} - (\mathbf{k}_1 \times \mathbf{V}_\perp)_z P_1 \frac{1}{V_\perp^2}\frac{\partial}{\partial\vartheta}\right] \\
&\qquad \times \frac{V_1}{c} A_1 \exp[-il(\vartheta - \chi_2)] f^L_{2l},
\end{aligned} \tag{4.2}$$

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where

$$G_k = (\mathbf{k}_1 \times \mathbf{k}_2)_z P_k, \qquad P_k = 1 - \frac{V_z}{V_k} - Q_k, \qquad Q_k = \frac{T_e}{T_i}\frac{\kappa^2}{K_k^2},$$

and subscripts k, 1 and 2 denote dependence on k, k_1 and k_2 respectively. The linear relation between potentials, $\phi_k = (1 - Q_k)(V_k/c)A_k$, is substituted for ϕ_k; then $\hat{U}$ and $\hat{W}$ become

$$\hat{U} = \frac{V_k}{V_z - V_k}(-Q_k)\frac{\partial}{V_k \partial V_z}, \tag{4.3}$$

$$\hat{W} = P_k \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + \frac{1}{V_k}\frac{\partial}{\partial V_z}. \tag{4.4}$$

Again, by use of the recursion formulae for the Bessel functions, it is possible to make the following substitutions in (4.2) (the prime $'$ indicates the derivative):

$$\mathbf{k}_{1\perp} \cdot \mathbf{V}_\perp J_s \to s\Omega J_s, \qquad (\mathbf{k}_1 \times \mathbf{V}_\perp)_z J_s \to i k_1 V_\perp J_s', \qquad \frac{\partial}{\partial \theta} \to -il.$$

Then, solving with respect to the series amplitude, f_{kn}^{NL}, one obtains

$$\begin{aligned} f_{kn}^{NL} = {} & \frac{\Omega}{k_z(V_k - V_z) - n\Omega} \sum_{\mathbf{k}_1,\mathbf{k}_2} \sum_{s,l} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta^0_{n-s-l} \\ & \times \exp[i(\omega_k - \omega_1 - \omega_2)t] \exp[-i(\chi - \chi_1)n] \exp[-i(\chi_2 - \chi_1)l] J_s(\kappa_1\zeta) J_l(\kappa_2\zeta) \\ & \times \left[iG_1 - s\Omega^2 \hat{W}_1 + \frac{\omega_1}{\Omega} Q_1 \Omega^2 \frac{1}{V_1}\frac{\partial}{\partial V_z} + P_1 l \frac{J_s'(\kappa_1\zeta)}{J_s(\kappa_1\zeta)} \frac{k_{1\perp}\Omega}{V_\perp} \right] \\ & \times [\delta_{l,0}\hat{U}_2 + (1 - \delta_{l,0})\hat{W}_2] F^0 \frac{A_1 A_2}{B_0^2} V_1 V_2. \end{aligned} \tag{4.5}$$

We can now carry out the summations over l and s. By use of the summation theorem for the Bessel functions (Abramowitz and Stegun 1965), one can check the following relations:

$$\exp[i(\chi - \chi_1)n] \sum_{s,j} \delta^0_{n-s-l} J_s(\kappa_1\zeta) J_l(\kappa_2\zeta) \exp[-i(\chi_2 - \chi_1)l] = J_n(\kappa\zeta),$$

$$\begin{aligned} \exp[i(\chi - \chi_1)n] \sum_{s,l} \delta^0_{n-s-l} s J_s(\kappa_1\zeta) J_l(\kappa_2\zeta) \exp[-i(\chi_2 - \chi_1)l] \\ = \frac{\kappa_1}{\kappa} n J_n(\kappa\zeta) \cos(\chi - \chi_1) - i\kappa_1\zeta J'(\kappa\zeta) \sin(\chi - \chi_1), \end{aligned}$$

$$\begin{aligned} \exp[i(\chi - \chi_1)n] \frac{2}{\kappa_1} \sum_{s,l} \delta^0_{n-s-l} s J_s(\kappa_1\zeta) J_l'(\kappa_2\zeta) \exp[-i(\chi_2 - \chi_1)l] \\ = i J_n(\kappa\zeta) \sin(\chi_2 - \chi_1) - i[J_n(\kappa\zeta) + 2J_n''(\kappa\zeta)] \sin(2\chi - \chi_2 - \chi_1) \\ + 2n \left[\frac{J_n(\kappa\zeta)}{\kappa}\right]' \cos(2\chi - \chi_2 - \chi_1), \end{aligned}$$

$$\begin{aligned} \frac{2}{\kappa_1} \sum_{s,l} \delta^0_{n-s-l} l J_s'(\kappa_1\zeta) J_l(\kappa_2\zeta) \exp(i\chi_2 l + i\chi_1 s - in\chi) = -i J_n(\kappa\zeta) \sin(\chi_2 - \chi_1) \\ -i[J_n(\kappa\zeta) + 2J_n''(\kappa\zeta)]) \sin(2\chi - \chi_2 - \chi_1) + 2n \left[\frac{J_n(\kappa\zeta)}{\kappa}\right]' \cos(2\chi - \chi_2 - \chi_1). \end{aligned}$$

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With these, (4.5) gives

$$
\begin{aligned}
f_{kn}^{NL} = {} & \frac{\boldsymbol{\Omega}}{k_z(V_k - V_z) - n\boldsymbol{\Omega}} \sum_{\mathbf{k}_1,\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \exp[i(\omega_k - \omega_1 - \omega_2)t] \\
& \times \left\{ \left[iG_1 - n\boldsymbol{\Omega}^2 \hat{W}_1 - nP_1 \kappa_2 \zeta \frac{J_0'(\kappa_2\zeta)}{J_0(\kappa_2\zeta)} + Q_1 \boldsymbol{\Omega}^2 \frac{\omega_1}{\boldsymbol{\Omega}} \frac{1}{V_1} \frac{\partial}{\partial V_z} \right] \right. \\
& \times \exp[-i(\chi - \chi_1)n] J_n(\kappa_1\zeta) J_0(\kappa_2\zeta) \hat{U}_2 \\
& + \left\{ \{J_n(\kappa_1\zeta) - J_n(\kappa_1\zeta) J_0(\kappa_2\zeta) \exp[-i(\chi - \chi_1)n]\} \left(iG_1 + \omega_1 \boldsymbol{\Omega} Q_1 \frac{1}{V_k} \frac{\partial}{\partial V_z} \right) \right. \\
& - \left[\frac{\kappa_1}{\kappa} n J_n(\kappa\zeta) \cos(\chi - \chi_1) - i\kappa_1 \zeta J'(\kappa\zeta) \sin(\chi - \chi_1) - n J_0(\kappa_2\zeta) J_n(\kappa_1\zeta) \right. \\
& \times \exp[-i(\chi - \chi_1)n] \boldsymbol{\Omega}^2 \hat{W}_1 - [ik_{1\perp} k_{2\perp} J_n(\kappa\zeta) \sin(\chi_2 - \chi_1) - n J_0'(\kappa_2\zeta) J_n(\kappa_1\zeta)] \\
& \left. \left. \left. \times \exp[-i(\chi - \chi_1)n] \frac{k_{2\perp}\boldsymbol{\Omega}}{V_\perp} \right] P_1 \right\} \hat{W}_2 \right\} F^0 \frac{A_1 A_2}{B_0^2} V_1 V_2. \qquad (4.6)
\end{aligned}
$$

Using the above expression for f_{kn}^{NL}, we find the Fourier amplitude averaged over the gyroangle ϑ:

$$
\begin{aligned}
\int_0^{2\pi} d\vartheta \, f_k^{NL} &= 2\pi \sum_n J_n(\kappa\zeta) f_{kn}^{NL} \\
&= 2\pi i \sum_{\mathbf{k}_1,\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \exp[i(\omega_k - \omega_1 - \omega_2)t] \frac{A_1 A_2}{B_0^2} V_1 V_2 \hat{M}_{12} F^0, \qquad (4.7)
\end{aligned}
$$

where the dominant terms in the operator $\hat{M}_{12}$ are

$$
\begin{aligned}
\hat{M}_{12} = {} & (\mathbf{k}_1 \times \mathbf{k}_2)_z \frac{\boldsymbol{\Omega}_i}{k_z(V_k - V_z)} [J_0(\kappa\zeta) J_0(\kappa_1\zeta) J_0(\kappa_2\zeta) P_1 (\hat{U}_2 - \hat{W}_2) \\
& + V_T^2 (\zeta/\kappa) J_0(\kappa\zeta) J_0'(\kappa\zeta) \hat{W}_1 \hat{W}_2].
\end{aligned}
$$

The terms dropped here are of order β and $(\omega/\boldsymbol{\Omega})^2$.

5. Nonlinear dispersion equation for KAWs

The expressions derived above provide a full kinetic description of excitation and three-wave interaction among KAWs in a low-β plasma. With the help of the nonlinear response (4.7), it is possible to calculate perturbed currents and charges in any plasma component α, provided the quasisteady distribution functions F_α^0 are known. Since the resonant (with waves) particle effect is primarily on the γ^L and has only a minor influence on the $\hat{M}_{12}$, we restrict ourselves to isotropic functions F_α^0 with thermal spreading $V_{T\alpha}$. In what follows we consider the waves with phase velocities $V_{Ti}^2 \ll V_k^2 \ll V_{Te}^2$, because strong electron Landau damping prevents the existence of the KAW eigenmode at $V_k^2 \approx V_{Te}^2$.

Taking the equilibrium distribution functions to be Maxwellian for ions and electrons, the calculations of the charge density and current density as velocity-space integrals by use of (3.9) and (4.7) in (2.3) are straightforward. The resulting Fourier

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components of the linear charge and current perturbations are:

$$q_{ek}^L = \frac{\omega_{pe}^2}{4\pi}\left(-\frac{1}{V_{Te}^2}+i\delta_e\right)\left(\phi_k-\frac{V_k}{c}A_k\right), \tag{5.1}$$

$$q_{ik}^L = \frac{\omega_{pi}^2}{4\pi}\left[\left(\frac{\Lambda_0}{V_k^2}+i\delta_i\right)\left(\phi_k-\frac{V_k}{c}A_k\right)-\frac{1-\Lambda_0}{V_{Ti}^2}\phi_k\right], \tag{5.2}$$

$$j_{ek}^L = \frac{\omega_{pe}^2}{4\pi}V_k\left(-\frac{1}{V_{Te}^2}+i\delta_e\right)\left(\phi_k-\frac{V_k}{c}A_k\right), \tag{5.3}$$

$$j_{ik}^L = \frac{\omega_{pi}^2}{4\pi}V_k\left(\frac{\Lambda_0}{V_k^2}+i\delta_i\right)\left(\phi_k-\frac{V_k}{c}A_k\right). \tag{5.4}$$

From the quasineutrality equation (2.1) and Ampère's law in the z direction (2.2), we obtain the nonlinear equation for KAWs:

$$\left(1-i\pi\frac{\kappa^2}{K_k^2}\frac{T_e}{T_i}V_{Te}^2\sum_\alpha\frac{n_\alpha}{n_0}\frac{m_e}{m_\alpha}\left.\frac{\partial\overline{F}_{0\alpha}}{\partial V_z}\right|_{V_z=V_k}-\frac{\omega_k^2}{k_z^2V_A^2K_k^2}\right)A_k$$
$$=\frac{4\pi}{k^2c}\left[j_k^{NL}-V_k\left(1-\frac{T_e}{T_i}\frac{\kappa^2}{K_k^2}\right)q_k^{NL}\right], \tag{5.5}$$

In the linear approximation, $j^{NL}=q^{NL}=0$, we get the well-known KAW dispersion relation

$$\omega_k = |k_z|\,V_AK_k, \tag{5.6}$$

and the (quasilinear) increment (decrement)

$$\gamma_k = \frac{\pi}{2}\omega_k\frac{\kappa^2}{K_k^2}\frac{T_e}{T_i}V_{Te}^2\sum_\alpha\frac{n_\alpha}{n_0}\frac{m_e}{m_\alpha}\left.\frac{\partial\overline{F}_{0\alpha}}{\partial V_z}\right|_{V_z=V_k}. \tag{5.7}$$

The function

$$K_k = K(\kappa) = \kappa\left(\frac{1}{1-\Lambda_0}+\frac{T_e}{T_i}\right)^{1/2}$$

determines the dispersive properties of KAWs, $\Lambda_0 = I_0(\kappa^2)\exp(-\kappa^2)$,

$$\overline{F}_\alpha^0 = 2\pi\int dV_\perp V_\perp J_0^2(k_\perp V_\perp/\Omega_\alpha)F_\alpha^0(V_z,V_\perp),$$

$\kappa = k_\perp\rho_i$ and $I_0(\kappa^2)$ is the modified Bessel function. The dispersion function K_k has the following asymptotics:

$$K_k \sim \begin{cases} 1+\dfrac{1}{2}\left(\dfrac{3}{4}+\dfrac{T_e}{T_i}\right)\kappa^2 & (\kappa^2\ll 1), \quad (5.8)\\[2ex] \left(1+\dfrac{T_e}{T_i}\right)^{1/2}\kappa & (\kappa^2\gg 1). \quad (5.9)\end{cases}$$

In the above expressions we retain the possibility that in the resonant regions of velocity space the distribution functions can deviate from the equilibrium ones, which does not affect the wave dispersion but can change the wave growth (damping) rate due to Čerenkov resonance, which is important for KAW excitation and dissipation. Also, the restriction to phase velocities $V_k < V_{Te}$ imposes the upper

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limit $\kappa_{\max} \approx V_{Te}/V_A$ on the possible perpendicular wavenumber of the Alfvén eigenmode in a $m_e/m_i < \beta < 1$ plasma.

As follows from (5.1), the density response of electrons is Boltzmannian owing to their motion parallel to $\mathbf{B}_0$ in the parallel (electromagnetic) potential $\phi_k - (V_k/c)A_k$. In the large-wavenumber domain, $1 < \kappa < \kappa_{\max}$, the function $\Lambda_0 \to 0$ and the ion density response (5.2) becomes Boltzmannian too, but due to ion motion perpendicular to $\mathbf{B}_0$ in the perpendicular (electrostatic) potential ϕ_k. In this latter case, one can use the approximation (5.9) for K_k in the KAW dispersion relation (5.6).

6. Canonical equation and coupling coefficients of the three-wave KAW interaction

In a Maxwellian plasma, the nonlinear (second-order) perturbations of charge density and current density due to KAWs are calculated as velocity-space integrals by use of (4.7) in (2.3). Taking into account that the phase velocity of the μth wave, $V_\mu = \omega_\mu/k_{\mu z}$, can be positive ($V_\mu > 0$ with $k_{\mu z} > 0$) or negative ($V_\mu < 0$ with $k_{\mu z} < 0$), we find the nonlinear current and density response at $(\mathbf{k}, \omega_k)$:

$$(j_k^\sigma)^{NL} = n_0 e V_k \frac{iV_A^2}{2V_{Ti}^2}\frac{\Omega_i}{\omega_k}\sum_{\mathbf{k}_1,\mathbf{k}_2}\sum_{\sigma_1,\sigma_2}\delta(\sigma\mathbf{k}-\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2)\exp[i(\sigma\omega_k-\sigma_1\omega_1-\sigma_2\omega_2)t]$$
$$\times(s_2K_2-s_1K_1)\left(s_1\frac{\kappa_1^2}{K_1}+s_2\frac{\kappa_2^2}{K_2}+K_k\right)\sigma_1\sigma_2\,(\mathbf{k}_1\times\mathbf{k}_2)_z\,\frac{A_1^{\sigma_1}A_2^{\sigma_2}}{B_0^2}, \qquad (6.1)$$

$$(q_k^\sigma)^{NL} = n_0 e \frac{iV_A^2}{2V_{Ti}^2}\frac{\Omega_i}{\omega_k}\sum_{\mathbf{k}_1,\mathbf{k}_2}\sum_{\sigma_1,\sigma_2}\delta(\sigma\mathbf{k}-\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2)\exp[i(\sigma\omega_k-\sigma_1\omega_1-\sigma_2\omega_2)t]$$
$$\times K_k\,(s_2K_2-s_1K_1)\,\sigma_1\sigma_2\,(\mathbf{k}_1\times\mathbf{k}_2)_z\,\frac{A_1^{\sigma_1}A_2^{\sigma_2}}{B_0^2}, \qquad (6.2)$$

where $s_\mu = \mp 1$ for $k_{\mu z} \lessgtr 0$ ($\mu = k, 1, 2$), and k_z is chosen to be > 0. The summation over $\sigma_1, \sigma_2 = \pm 1$ introduces complex conjugate terms $A_\mu^{+1} \equiv A_\mu$ and $A_\mu^{-1} \equiv A_\mu^*$, because only real parts should be taken for physical quantities entering binary combinations. Subsequent substitution of (6.1) and (6.2) into (5.5) leads to the dynamic equations for the Fourier amplitudes:

$$\left(\frac{\partial}{\partial t}-\gamma_k\right)A_k^\sigma$$
$$=\sum_{\mathbf{k}_1,\mathbf{k}_2}\sum_{\sigma_1,\sigma_2}\delta(\sigma\mathbf{k}-\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2)\delta\left(\sigma\omega_k-\sigma_1\omega_1-\sigma_2\omega_2\right)U(k,k_1,k_2)A_1^{\sigma_1}A_2^{\sigma_2}, \qquad (6.3)$$

with the matrix element of three-wave interaction

$$U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \frac{V_A}{4B_0}(s_1K_1-s_2K_2)\frac{K_k}{\kappa^2}\left(s_1\frac{\kappa_1^2}{K_1}+s_2\frac{\kappa_2^2}{K_2}+\frac{\kappa^2}{K_k}\right)\sigma_1\sigma_2\,(\mathbf{k}_1\times\mathbf{k}_2)_z\,. \qquad (6.4)$$

In equation (6.3) we have restored the time derivative, making the inverse substitution $\omega_k \to -i\partial/\partial t$.

The expression (6.4) for the coupling coefficients is the main mathematical result of the present paper. We see from (6.4) that three-wave resonant interaction among KAWs is 'local' in k space, i.e. the wavenumbers of effectively interacting KAWs are of the same order, $k_\perp \approx k_{1\perp} \approx k_{2\perp}$. Note that a non-zero value of $U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$

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and a nonlinear (three-wave) coupling among KAW triads become possible owing to the finite value of the 'kinetic factor' κ. In MHD Alfvén waves $\kappa_\mu = 0$, making $K = K_1 = K_2 = 1$, so that $U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 0$ for resonant triads, in agreement with the well-known fact that three-wave resonant interaction is impossible among ideal-MHD Alfvén waves.

The matrix element $U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ is the key mathematical object in the nonlinear wave theory. Once the matrix element has been found, one can follow theoretical methods developed earlier to study the nonlinear properties of a given mode. Using the expressions (6.3) and (6.4), in the next section we study the parametric decay of KAWs, which is very important in the processes of nonlinear saturation of KAW instabilities and in KAW spectral spreading.

7. Parametric decay of KAWs

The decay instability KAW $\rightarrow$ KAW + KAW in a hot plasma has been examined by Erokhin *et al.* (1978), followed by other authors, but the common approach so far has been some kind of two-fluid MHD. Here we present the first study of KAW decay instability for an arbitrary value of the kinetic parameter κ, based on the coupling coefficient (6.4) found from plasma kinetic theory.

Let us consider a pump KAW with frequency ω_k, wave vector $\mathbf{k}$, and amplitude A_k, propagating in the z-axis direction, i.e. $k_z > 0$. It follows from (6.3) that the waves $(\omega_1, \mathbf{k}_1, A_1)$ and $(\omega_2, \mathbf{k}_2, A_2)$ can effectively interact with the pump provided that the resonant conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}, \tag{7.1a}$$

$$\omega_1 + \omega_2 = \omega_k \tag{7.1b}$$

are satisfied. The dynamical equations for waves 1 and 2, coupled via the parametric pump wave, are

$$\left(\frac{\partial}{\partial t} - \gamma_1\right) A_1 = -U(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}_2) A_k A_2^*; \tag{7.2}$$

$$\left(\frac{\partial}{\partial t} - \gamma_2\right) A_2^* = -U(-\mathbf{k}_2, -\mathbf{k}, \mathbf{k}_1) A_k^* A_1. \tag{7.3}$$

Acting with the operator $\partial/\partial t - \gamma_2$ on (7.2) and excluding A_2^* by use of (7.3), we get an equation for the amplitude A_1 (the same equation for the amplitude A_2 is obtained in the same way):

$$\left[\frac{\partial^2}{\partial t^2} - (\gamma_1 + \gamma_2)\frac{\partial}{\partial t} + \gamma_1\gamma_2\right] \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\} = U(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}_2) U(-\mathbf{k}_2, -\mathbf{k}, \mathbf{k}_1) \, |A_k|^2 \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\} A_1 A_2. \tag{7.4}$$

An exponential, $\propto \exp(\delta t)$, solution to (7.4) has

$$\delta = \frac{\gamma_1 + \gamma_2}{2} \pm \left[\left(\frac{\gamma_1 - \gamma_2}{2}\right)^2 + \gamma_{NL}^2\right]^{1/2},$$

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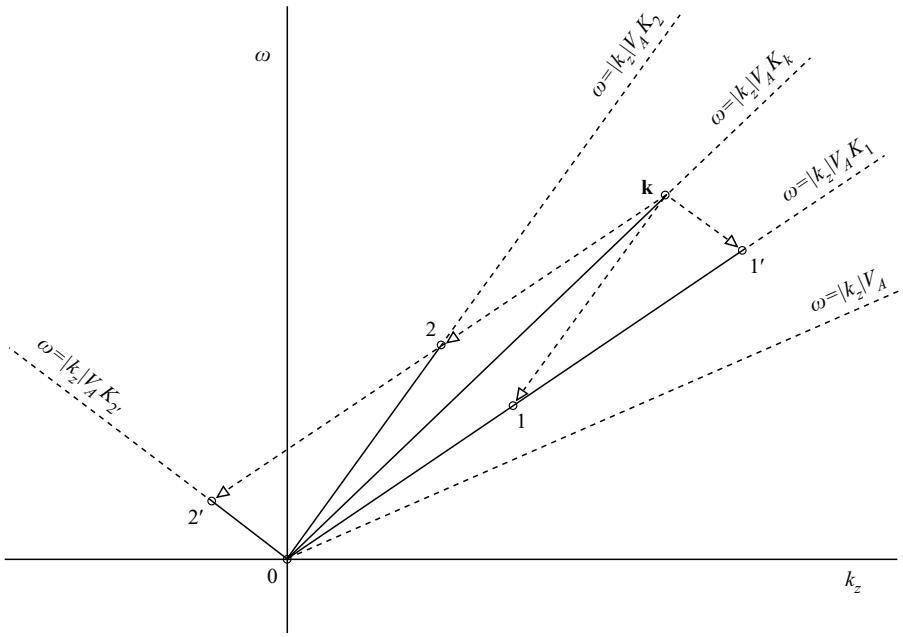


Figure 1. Two possible channels for decay of the pump KAW k: (i) decay into two parallel-propagating KAWs $k \to 1+2$ ($k_{1z} > 0$, $k_{2z} > 0$); (ii) decay into two counterstreaming KAWs $k \to 1' + 2'$ ($k_{1z} > 0$, $k_{2z} < 0$), where we have assumed $1' = 1$ for simplicity. The dispersion of the MHD Alfvén wave $\omega = k_z V_A$ is shown for reference.

where the rate of nonlinear interaction

$$\gamma_{NL}^2 = \Omega_i^2 \left(\frac{V_A}{4V_{Ti}}\right)^2 \frac{K_1 K_2 (K_2 - s_2 K_k)(s_1 K_k - K_1)}{\kappa^2 \kappa_1^2 \kappa_2^2} \left(s_1 \frac{\kappa_1^2}{K_1} + s_2 \frac{\kappa_2^2}{K_2} + \frac{\kappa^2}{K_k}\right)^2 \times |\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2|_z^2 \left|\frac{B_k}{B_0}\right|^2, \tag{7.5}$$

and waves 1 and 2 grow if

$$\delta > 0. \tag{7.6}$$

For the rate of nonlinear interaction γ_{NL} to be real, i.e. $\gamma_{NL}^2 > 0$, the dispersion functions K_μ must satisfy the following decay condition:

$$(K_2 - s_2 K_k)(s_1 K_k - K_1) > 0. \tag{7.7}$$

In the case of weak wave–particle interaction, $\gamma_1, \gamma_2 \ll \gamma_{NL}$, the growth rate of waves 1 and 2 is determined by the rate of nonlinear interaction: $\delta \approx \gamma_{NL}$. A discussion of decay conditions and a presentation of the decay increment in general form (Stenflo 1994) will be given in Sec. 8.

The decay condition (7.7) implies that at least one dispersion function of the decay products should be less than the dispersion function of the pump wave ($\gamma_{NL}^2 < 0$ for arbitrary s_1 and s_2 if both K_1 and $K_2 > K_k$). In what follows we choose $K_1 < K_k$; then decay is possible (i) into parallel-propagating decay KAWs, $k_{1z} > 0$, $k_{2z} > 0$ ($s_1 = s_2 = 1$), and (ii) into counterstreaming decay KAWs, $k_{1z} > 0$, $k_{2z} < 0$ ($s_1 = 1$, $s_2 = -1$) (here $k_z > 0$ is chosen for the pump wave). The corre-

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sponding parallelograms in the (ω, k_z) plane, based on triplets in the KAW spectra, are shown in Fig. 1.

Let as consider these two cases separately.

7.1. Parallel-propagating decay KAWs ($k_{1z} > 0$, $k_{2z} > 0$)

As follows from (7.7) with $s_1 > 0$ and $s_2 > 0$, the decay process involving parallel-propagating KAWs $(\omega_k, \mathbf{k}, A_k) \to (\omega_1, \mathbf{k}_1, A_1) + (\omega_2, \mathbf{k}_2, A_2)$ can take place if the dispersion functions of the decay products K_1 and K_2 bound the dispersion function K_k of the pump KAW:

$$K_1 < K_k < K_2. \tag{7.8}$$

Since the function K is a monotonic function of its argument κ, we come to the conclusion that the wavenumbers of the interacting waves must satisfy the following ordering:

$$0 < k_{1\perp} < k_\perp < k_{2\perp} < 2k_\perp. \tag{7.9}$$

The last inequality here follows from combination of the resonant condition for wave vectors with the second inequality.

For KAWs with weak dispersion, $\kappa^2, \kappa_1^2, \kappa_2^2 \ll 1$, we use the asymptotic (5.8) in (7.5):

$$\gamma_{NL} = \Omega_i \frac{V_A}{8V_{Ti}} \left(\frac{3}{4} + \frac{T_e}{T_i}\right) \frac{(\kappa^2 + \kappa_1^2 + \kappa_2^2)[(\kappa_2^2 - \kappa_k^2)(\kappa_k^2 - \kappa_1^2)]^{1/2}}{\kappa\kappa_1\kappa_2} \times |\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2|_z \left|\frac{B_k}{B_0}\right| \tag{7.10}$$

For KAWs with strong dispersion, $\kappa^2, \kappa_1^2, \kappa_2^2 \gg 1$, we use the asymptotic (5.8) in (7.5):

$$\gamma_{NL} = \Omega_i \frac{V_A}{4V_{Ti}} \left(1 + \frac{T_e}{T_i}\right)^{1/2} \times \frac{\kappa + \kappa_1 + \kappa_2}{\kappa} \left[\frac{(\kappa_2 - \kappa_k)(\kappa_k - \kappa_1)}{\kappa_1\kappa_2}\right]^{1/2} |\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2|_z \left|\frac{B_k}{B_0}\right|. \tag{7.11}$$

Maximizing the above expressions for γ_{NL} with respect to the wavenumbers of the decay products, one can obtain the rate of resonant decay instability of KAWs, giving a good approximation for the rate of three-wave interaction in $\mathbf{k}$-space. The result is

$$\gamma_k^{NL} = 0.375\Omega_i \frac{V_A}{V_{Ti}} \kappa^3 \left|\frac{B_k}{B_0}\right|, \tag{7.12}$$

with $\kappa_1 = 0.717\kappa$ and $\kappa_2 = 1.49\kappa$ for $\kappa^2 \ll 1$, and

$$\gamma_k^{NL} = 0.25\Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left|\frac{B_k}{B_0}\right|, \tag{7.13}$$

with $\kappa_1 = 0.726\kappa$ and $\kappa_2 = 1.465\kappa$ for $\kappa^2 \gg 1$. Here we take $T_e = T_i$ for simplicity. The expressions (7.12) and (7.13) show a strong dependence of the interaction rate on the perpendicular wavenumber.

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7.2. Counterstreaming decay KAWs ($k_{1z} > 0$, $k_{2z} < 0$)

In this case for weakly dispersing waves, $\kappa^2 \ll 1$, instead of (7.10) and (7.12), we have

$$\gamma_{NL} = \Omega_i \frac{V_A}{4V_{Ti}} \left(\frac{3}{4} + \frac{T_e}{T_i}\right)^{1/2} \frac{(\kappa^2 + \kappa_1^2 - \kappa_2^2)(\kappa^2 - \kappa_1^2)^{1/2}}{\kappa\kappa_1\kappa_2} \times |\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2|_z \left|\frac{B_k}{B_0}\right|, \quad (7.14)$$

with maximum growth rate

$$\gamma_k^{NL} = 0.28\Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left|\frac{B_k}{B_0}\right| \quad (7.15)$$

at $\kappa_1 = 0.776\kappa$ and $\kappa_2 = 0.442\kappa$.

For strong wave dispersion, $\kappa^2 \gg 1$,

$$\gamma_{NL} = \Omega_i \frac{V_A}{4V_{Ti}} \left(1 + \frac{T_e}{T_i}\right)^{1/2} \times \frac{\kappa + \kappa_1 - \kappa_2}{\kappa} \left[\frac{(\kappa_2 + \kappa_k)(\kappa_k - \kappa_1)}{\kappa_1\kappa_2}\right]^{1/2} |\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2|_z \left|\frac{B_k}{B_0}\right| \quad (7.16)$$

with maximum growth rate

$$\gamma_k^{NL} = 0.18\Omega_i \frac{V_A}{V_{Ti}} \kappa^2 \left|\frac{B_k}{B_0}\right| \quad (7.17)$$

at $\kappa_1 = 0.707\kappa$ and $\kappa_2 = 0.704\kappa$.

Comparing (7.12) with (7.15), and (7.13) with (7.17), we come to the conclusion that in the long-wavelength domain decay is much more effective into counterstreaming KAWs than into parallel-propagating KAWs In contrast, in the short-wavelength domain decay into parallel-propagating KAWs dominates, although the difference is not so large (by a factor of 1.6).

Let us estimate the threshold amplitude for decay instability in a Maxwellian plasma when the growth of decay products due to nonlinear coupling via the pump is balanced by their electron Landau damping. In a uniform plasma, the threshold amplitude for the pump to excite waves 1 and 2 is obtained from the marginal decay condition (7.6), $\gamma_1\gamma_2 = \gamma_{NL}^2$, where γ_1 and γ_2 are to be determined from (5.7). In a Maxwellian plasma, the electron Landau damping

$$\gamma_k^L = -\omega_k \left(\frac{\pi}{8}\right)^{1/2} \frac{V_A}{V_{Te}} \frac{\kappa^2}{K_k}, \quad (7.18)$$

and the threshold amplitude is

$$\left|\frac{B_k}{B_0}\right|_{\text{thr}} \approx 3.1 \left(\frac{m_e}{m_i}\right)^{1/2} \frac{\omega}{\Omega_i} \frac{\kappa_1\kappa_2}{\kappa^2} \left(\frac{k_{1z}k_{2z}}{k_z^2}\right)^{1/2} \frac{1}{K} \left(\times \frac{1}{2\kappa} \text{ for } \kappa < 1;\ s = 1\right)$$
$$= \left(\frac{m_e}{m_i}\right)^{1/2} \frac{\omega_k}{\Omega_i} c_s \kappa^{\sigma_s}, \quad (7.19)$$

where $s \equiv s_2$ and $k_{1z}k_{2z}/k_z^2$ is expressed in terms of K_μ (see below) and estimated, resulting in $c_{-1} \approx 0.4$, $\sigma_{-1} = 1$ for $\kappa < 1$ and $c_{-1} \approx 0.5$, $\sigma_{-1} = -1$ for $\kappa > 1$ ($s = -1$ for counterstreaming KAWs); and $c_1 \approx 0.7$, $\sigma_1 = -1$ for $\kappa < 1$, and $c_1 \approx 1.1$, $\sigma_1 = -1$ for $\kappa > 1$ ($s = 1$ for parallel-propagating KAWs). Equation (7.19) shows

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the main tendency for the threshold to decrease with lower frequencies and/or higher perpendicular wavenumbers.

Equations (7.12)–(7.19) clearly demonstrate a unique feature of low-frequency ($\omega_k/\Omega_i \lesssim 0.2$) KAWs with $\kappa \approx 1$ to become nonlinear at very small amplitudes $B_k/B_0 \lesssim 10^{-3}$: even smaller than those predicted by Hasegawa and Chen (1992) on the basis of qualitative arguments.

One should note that, although decay of the $\kappa^2 \ll 1$ pump into counterstreaming KAWs is local in $k_\perp$-space, it is non-local in k_z-space because $|k_{2z}|/k_z \lesssim \kappa^2 \ll 1$. Thus this decay becomes impossible in a non-uniform plasma if the (large) parallel wavelength of counterstreaming (decay) KAWs appears to be comparable to the inhomogeneity scale: $|k_{2z}|^{-1} \approx L_z$. This places a restriction on the pump wavenumbers:

$$k_z \kappa^2 > L_z^{-1}. \tag{7.20}$$

Also, the efficacy of this decay channel in a non-uniform plasma may be reduced by the finite correlation length along field lines, $L_c \lesssim L_z$. Namely, from the condition that the waves should overlap during the growth time, $\gamma_k^{NL} > V_A/L_c$, we estimate the corresponding threshold for decay into counterstreaming KAWs as

$$\left|\frac{B_k}{B_0}\right|_{\rm thr} \approx 2.5\beta^{1/2}(k_z\kappa^2 L_c)^{-1}\frac{\omega_k}{\Omega_i}, \tag{7.21}$$

which is rather high compared with (7.19) near the upper boundary of the condition (7.20) (i.e. for waves of low $k_z\kappa^2$).

8. Discussion

The decay condition (7.7) follows directly from the resonant conditions (7.1). Indeed, using in (7.1) the KAW dispersion relation (5.6), we get

$$|k_{1z}|\,K_1 + |k_{2z}|\,K_2 = |k_{1z}+k_{2z}|\,K_k = \pm\left(s_1\,|k_{1z}| + s_2\,|k_{2z}|\right)K_k,$$

or

$$\frac{\pm s_1 K_k - K_1}{K_2 \mp s_2 K_k} = \frac{|k_{1z}|}{|k_{2z}|} > 0,$$

resulting in (7.7). Therefore the decay condition (7.7) holds automatically for three-wave resonant interaction; this means that the expression (7.5) can be written as a squared quantity, in perfect agreement with general three-wave interaction theory (Stenflo 1994). In more detail, noting that

$$\begin{aligned}(K_2 - s_2K_k)\,(s_1K_k - K_1) &= \frac{\left[(k_{1z}+k_{2z})\,K_2 - s_2k_zK_k\right]\left[s_1k_zK_k - (k_{1z}+k_{2z})\,K_1\right]}{k_z^2}\\ &= \frac{\left[s_1\,|k_{1z}|\,K_2 + s_2\,(\omega_2-\omega_k)/V_A\right]\left[s_1\,(\omega_k-\omega_1)/V_A - s_2\,|k_{2z}|\,K_1\right]}{k_z^2}\\ &= \frac{|k_{1z}|\,|k_{2z}|}{k_z^2}\left(s_1K_2 - s_2K_1\right)^2,\end{aligned}$$

we can rewrite (7.5) in the form

$$\gamma_{NL}^2 = \frac{\omega_1\omega_2\,|V|^2}{W_1W_2},$$

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where $W_k = |B_k|^2/4\pi$ is the KAW energy, and

$$V = \frac{V_A}{V_{Ti}}\frac{\Omega_i}{\omega_k}\frac{K_k(s_1K_2 - s_2K_1)}{\kappa\kappa_1\kappa_2}\left(s_1\frac{\kappa_1^2}{K_1} + s_2\frac{\kappa_2^2}{K_2} + \frac{\kappa^2}{K_k}\right)(\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2)_z\frac{B_k}{B_0}\frac{B_1B_2}{16\pi}.$$

It is interesting to compare the results of kinetic theory in the limit $\kappa^2 \ll 1$ with those obtained in the past by use of the two-fluid MHD approach. To do this, we note that only parallel propagation of interacting KAWs has been implicitly implied previously (Erokhin *et al.* 1978; Voitenko 1989; Volokitin and Dubinin 1989; Yukhimuk and Kucherenko 1993). It appears that the result by Erokhin *et al.* (1978) is less than (7.10) by a factor of 1.75, and the decay instability condition is the same as (7.9). An expression for $|\gamma_{NL}|$ that is close to (7.10) (within a factor of 0.875) have been found by Volokitin and Dubinin (1989) and Yukhimuk and Kucherenko (1993), but, because of minor algebraic errors, an opposite sign of γ_{NL}^2 results in a different instability condition, $k_{1\perp} < k_{2\perp} < k_\perp$. This last condition evidently cannot be satisfied for resonant decay including parallel-propagating KAWs (it is incompatible with the matching conditions $\omega_k = \omega_1 + \omega_2$ and $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$).

The dependence $\gamma_{NL} \propto k_\perp^3$ is the common result for all the above-mentioned papers investigating parallel-propagating ($k_{1z} > 0$, $k_{2z} > 0$) waves for $\kappa_i^2 \ll 1$. However, as we have shown in the previous section, it is much more effective for the $\kappa^2 \ll 1$ pump to excite two counterstreaming KAWs ($k_{1z} > 0$, $k_{2z} < 0$), in which case a κ^{-1} times larger growth rate and $\gamma_{NL} \propto k_\perp^2$ dependence are found. One should keep in mind the non-local character of this decay channel in k_z space, making it more sensitive to the effects introduced by plasma inhomogeneity.

As κ approaches 1, the growth rates of these two decay channels become of the same order of magnitude, and for $\kappa > 1$ the decay into parallel-propagating waves dominates.

9. Applications

(i) Kinetic Alfvén waves of a large amplitude $B_k/B_0 = 10^{-2}$–10^{-3}, propagating along the geomagnetic field lines in the auroral zones of the Earth's magnetosphere, have been observed by satellites at altitudes $h = (0.1\text{–}2)\times10^4$ km on a background of large-scale field-aligned currents (Gurnett *et al.* 1984; Dubinin *et al.* 1988). Current-driven instability of KAWs can develop at altitudes $h = (2\text{–}4)\times10^4$ km and excite the observed wave flux (Voitenko *et al.* 1990). For the magnetospheric conditions at $h = 4\times10^9$ cm, where $\Omega_i \approx 10\ \mathrm{s}^{-1}$, $V_A \approx 10^8$ cm s^{-1}, $\rho_i \approx 2\times10^6$ cm and $T_i = T_e$, the exciting waves have wavenumbers $k_\perp\rho_i \approx 0.5$ and $k_z\rho_i \approx 0.05$, and amplitudes $B_k/B_0 \approx 2.5\times10^{-3}$.

Let us consider the parametric decay of these KAWs, propagating towards the Earth. Using the above parameters in (7.15), we find a growth rate $\gamma_k^{NL} \approx 0.025$ s^{-1}. Since the decay growth time, $\tau_{NL} \approx 1/\gamma_k^{NL} \approx 40$ s, is less than the Alfvén-wave propagation time, $\tau_A = \Delta h/V_A = 50$–$100$ s, we conclude that KAWs excited at the distant magnetosphere undergo a parametric decay instability, and the observed wave spectra should be modified by three-wave couplings among KAWs.

With a field-aligned inhomogeneity $L_z \approx 10^4$ km at $h = (1\text{–}4)\times10^4$ km, we see that the conditions (7.20) and (7.21) are easily satisfied, while the Landau threshold (7.19) places a restriction on the frequencies of unstable waves, $\omega/\Omega_i \lesssim 0.5$, provided their amplitudes are $B_k/B_0 \approx 2.5\times10^{-3}$. The lowest frequency (of the antiparallel-propagating KAWs) is well above the decay growth rate, $\omega_2 \approx 0.5\kappa^2\omega \approx 0.5$ s $\gg$

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$\gamma_k^{NL} \approx 0.025\ \mathrm{s}^{-1}$, and hence the weak-interaction approach is justified. Note that the modified decay instability, including a very low-frequency wave $\omega_2 < \gamma_k^{NL}$ (Dubinin *et al.* 1988; Volokitin and Dubinin 1989), has too long a growth time, about 200 s, and thus cannot develop in our case.

(ii) Although the problem of coronal heating, especially in active regions, has not yet received a definite answer, heating by an Alfvèn-wave flux seems to be the most realistic mechanism. An energy flux $F \approx 10^8$ erg cm^{-2} s^{-1} is required to heat active-region coronal loops of length $L \approx 5 \times 10^9$ cm, where $T_e \approx T_i \approx 5 \times 10^6$ K (Kano and Tsuneta 1996). Then, taking as typical for coronal loops $B_0 = 100$ G and $V_A = 10^8$ cm s^{-1}, we get the wave amplitude $B_k/B_0 = 3.5 \times 10^{-2}$ and corresponding (non-thermal) plasma velocity $V_{\mathrm{pl}} = 35 \times 10^5$ cm s^{-1}. It is interesting to note that observed non-thermal broadening of spectral lines yields the same root mean square velocities $V_{\mathrm{pl}} = (25\text{–}45) \times 10^5$ cm s^{-1} (Saba and Strong 1991).

The common difficulty of all heating mechanisms incorporating Alfvén waves is their weak dissipation in the strong magnetic field $B_0 \approx 100$ G. We suggest that this problem may be eliminated in a plasma with transverse ($\perp \mathbf{B}_0$) inhomogeneity of the Alfvén speed (scale $L_A \approx 10^6$ cm), where the growth of transverse wavenumber $k_\perp \approx (\omega_k/L_A)\,t$ brings about a kinetic effect of Landau damping (7.18) for $\omega_k \gtrsim 1\ \mathrm{s}^{-1}$, or enhanced collisional dissipation for $\omega_k < 1\ \mathrm{s}^{-1}$ (Voitenko 1994). Then the condition of effective wave dissipation over a loop length L ($\gtrsim 10^9$ cm), $\gamma_k^L \gtrsim V_A/L$, gives $\kappa^2\omega_k \gtrsim V_{Te}/L$. Such a wave also becomes nonlinearly unstable if $\gamma_k^{NL} \gtrsim V_A/L$. Taking (7.15) for γ_k^{NL}, we find that the condition for decay instability of KAWs, heating coronal loops, is satisfied in a wide frequency range $\omega_k < 0.3\,(V_{Te}/\rho_i)\,B_k/B_0 \approx 0.5 \times 10^6\ \mathrm{s}^{-1}$, covering the whole MHD spectrum. Therefore the nonlinear spreading and turbulent character of Alfvén-wave spectra should be taken into account when investigating plasma heating and current drive in coronal loops (see e.g. Elfimov *et al.* 1996).

10. Conclusions

A dynamic equation governing the weak interaction between KAWs in a low-β plasma has been derived using a plasma kinetic model. This allowed us to keep kinetic and dispersive finite-ion-Larmor radius corrections in the nonlinear dispersion relation for KAWs. The general expression for the coupling coefficient of the three-wave resonant interaction among KAWs with arbitrary values of the kinetic variable κ has been calculated, and the growth rate and threshold amplitude of the KAW decay instability in a Maxwellian plasma have been obtained. In the limit $k_\perp^2 \rho_i^2 \ll 1$ we found the growth rate scaling $\propto k_\perp^2$, more favourable for three-wave resonant decay than the scaling previously elaborated, $\propto k_\perp^3$. The first study of parametric KAW instability in the domain $k_\perp > \rho_i^{-1}$ has been presented, where the same growth-rate behaviour, $\propto k_\perp^2$, is found. These results show a growing efficiency of the three-wave interaction among Alfvén waves with growing $k_\perp$.

The remarkable tendency of KAWs towards nonlinear destabilization at very low amplitudes may be responsible for the turbulent character of waves observed in different regions of space, such as the Earth's magnetosphere and solar coronal loops. At the same time, magnetospheric turbulence is sometimes observed in a relatively high-β plasma, when waves with a significant compressional magnetic component arise (Anderson *et al.* 1990). One should note that an essential feature of the present paper (and many previous papers) is the restriction to the case of

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a low-β plasma, making it possible to decouple compressible modes, which greatly simplifies calculations, allowing the use of a plasma kinetic model. In the above-mentioned more general cases of high β, one should use different nonlinear models (see e.g. Shukla and Stenflo 1995).

As follows from the decay conditions, the initial tendency of the spectral energy spreading is into both higher and lower wavenumbers, but the dominant direction of the energy cascade is still to be determined. Together with the finding of weak turbulent spectra, this is the aim of the accompanying paper (Voitenko 1998).

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