$$= (p + q)^2 \left(1 + \cot^2 C\right) \frac{h^2}{h^2}$$

$$= (p + q)^2 + \frac{((p + q)\, h \cot C)^2}{h^2}$$

$$= (p + q)^2 + \frac{\left(h^2 - pq\right)^2}{h^2}$$

where we used the theorem in the last equality. From this we get Pandita's formula for the circumradius of an acute triangle

$$R = \frac{1}{2}\sqrt{(p + q)^2 + \left(\frac{h^2 - pq}{h}\right)^2}. \tag{5}$$

Note that formulas (4) and (5) are valid as they stand *only* in acute triangles, but in obtuse triangles one of the segments p or q needs to be taken as negative. In a right triangle, $h^2 = pq$, so (5) simplifies to $R = \frac{1}{2}c$ where c is the hypotenuse.

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MARTIN JOSEFSSON
Västergatan 25d, 285 37 Markaryd, Sweden
e-mail: *martin.markaryd@hotmail.com*

100.31 Heron-like formulas for quadrilaterals

Heron's formula for the area of a triangle and Brahmagupta's formula for the area of a cyclic quadrilateral are famous formulas that have received a lot of attention due to their symmetry. That the former is a special case of the latter is well known, but it is probably not so well known that Brahmagupta's formula can be derived from Heron's formula (see [1]). How about the area of a general convex quadrilateral? There are about a dozen different formulas known, all of which can be found at [2]. Two of these formulas, that we had not seen before, were added without any reference in December 2014. They express the area in terms of the diagonals and the bimedians of the quadrilateral. A *bimedian* is a line segment in a quadrilateral that connects the midpoints of two opposite sides.

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What is quite special about the two new formulas is the number of quantities they involve. Most of the quadrilateral area formulas are expressed in terms of five or six quantities (for example the four sides and the two diagonals), but there can be as few as three quantities (the diagonals and the angle between them). The two formulas we shall study are also expressed in terms of just three quantities, all of which are lengths. This reminds us of Heron's formula. Is it possible that the area of a general convex quadrilateral is closely related to that of a triangle? The answer is, perhaps surprisingly, yes!

In a convex quadrilateral, let the lengths of the diagonals be p and q, and the lengths of the bimedians be m and n. Then we shall prove that the area K of the quadrilateral is given by the formulas

$$K = \tfrac{1}{4}\sqrt{[(p+q)^2 - 4m^2]\,[4m^2 - (p-q)^2]} \tag{1}$$

and

$$K = \tfrac{1}{2}\sqrt{[(m+n)^2 - p^2]\,[p^2 - (m-n)^2]}. \tag{2}$$

In the derivations of these formulas we will use the semi-factorised version of Heron's formula, that the area T of a triangle with sides a, b, c is given by

$$T = \tfrac{1}{4}\sqrt{[(a+b)^2 - c^2]\,[c^2 - (a-b)^2]}.$$

We also need a quite well-known theorem due to the French mathematician Pierre Varignon (1654-1722), which states that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram whose sides are parallel to and half as long as the diagonals of the quadrilateral, and with an area that is half the area of the original quadrilateral.

Applying Varignon's theorem to a convex quadrilateral, where E, F, G, H are the midpoints of the sides, we can prove (1) like this (see Figure 1):

$$\begin{aligned} K &= 2K_{EFGH} = 4T_{EFH} \\ &= 4 \times \frac{1}{4}\sqrt{\left[\left(\frac{p}{2}+\frac{q}{2}\right)^2 - m^2\right]\left[m^2 - \left(\frac{p}{2}-\frac{q}{2}\right)^2\right]} \\ &= \tfrac{1}{4}\sqrt{[(p+q)^2 - 4m^2]\,[4m^2 - (p-q)^2]} \end{aligned}$$

where T_{EFH} stands for the area of triangle EFH. We also used that a diagonal in a parallelogram divides it into two triangles with equal areas.

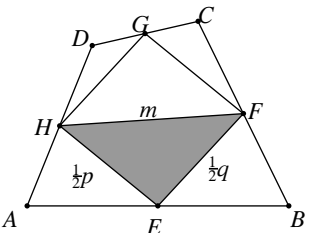


FIGURE 1: Varignon's parallelogram *EFGH*

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Formula (2) is derived in a similar way. If J is the intersection of the bimedians, then (see Figure 2)

$$\begin{aligned} K &= 2K_{EFGH} = 4T_{EFH} = 8T_{EJH} \\ &= 8 \times \frac{1}{4}\sqrt{\left[\left(\frac{m}{2}+\frac{n}{2}\right)^2-\left(\frac{p}{2}\right)^2\right]\left[\left(\frac{p}{2}\right)^2-\left(\frac{m}{2}-\frac{n}{2}\right)^2\right]} \\ &= \tfrac{1}{2}\sqrt{[(m+n)^2-p^2]\,[p^2-(m-n)^2]} \end{aligned}$$

where we used that the diagonals in a parallelogram bisect each other.

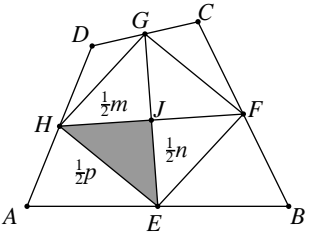


FIGURE 2: Bisecting bimedians EG and FH

To see the resemblance with Heron's formula better, let us introduce $s_1 = \frac{1}{2}(p+q+2m)$. Then it is easy to show that (1) is equivalent to

$$K = \sqrt{s_1(s_1-p)(s_1-q)(s_1-2m)}. \tag{3}$$

In the same way, if we put $s_2 = \frac{1}{2}(m+n+p)$, then (2) is equivalent to

$$K = 2\sqrt{s_2(s_2-m)(s_2-n)(s_2-p)}. \tag{4}$$

Formulas (1) and (2) are fascinating in the following way. If we draw a convex quadrilateral on a piece of paper, then measuring its four sides is not enough to calculate its area. But of the four distances the two diagonals and the two bimedians constitute, only three are needed to complete the same task!

After having studied a proof of Heron's formula in a class, a natural continuation is to study quadrilaterals. In this note we have shown that this does not have to be any more difficult since there is a close connection between triangles and quadrilaterals via Varignon's theorem.

We conclude with an exercise for the reader. First prove that the lengths of the diagonals p and q and the lengths of the bimedians m and n in a quadrilateral are related according to the equality

$$p^2+q^2 = 2\left(m^2+n^2\right),$$

and then use it to prove the following area formula for convex quadrilaterals:

$$K = \tfrac{1}{2}\sqrt{(m+n+p)(m+n-p)(m+n+q)(m+n-q)}.$$

This formula, which can be used if we know the lengths of the two diagonals and the length of the sum of the two bimedians, is due to the

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French mathematician Georges Dostor [3, p. 73]. As a corollary we have that the area of a parallelogram with sides of lengths a and b is given by

$$K = \tfrac{1}{4}\sqrt{(p+q+2a)(p+q-2a)(p+q+2b)(p+q-2b)}$$

where p and q are the lengths of the diagonals [3, p.72].

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MARTIN JOSEFSSON
Västergatan 25d, 285 37 Markaryd, Sweden
e-mail: *martin.markaryd@hotmail.com*

100.32 Linear quadrisection and the Kiepert hyperbola

For a general scalene triangle ABC, the Cartesian coordinate system provided by the asymptotes of the associated Kiepert hyperbola has one obvious advantage, namely, that a point $P\left(x, \frac{1}{x}\right)$ is instantly recognised as lying on this curve ([1]).

In this Note we will locate two significant points lying on different branches of the hyperbola and also encounter three examples of linear quadrisection. However, we opt just to outline some of the routine coordinate geometry.

In our Cartesian framework, the vertices $A\left(a, \frac{1}{a}\right)$, $B\left(b, \frac{1}{b}\right)$, $C\left(c, \frac{1}{c}\right)$, centroid $G\left(g, \frac{1}{g}\right)$ and orthocentre $H\left(h, \frac{1}{h}\right)$ lie on the curve $xy = 1$. Thus $3g = a + b + c$, $\frac{3}{g} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ with $abch = -1$ for rectangularity. The origin $O(0, 0)$ is the Kiepert centre and g, h and two of the parameters a, b, c may be taken as positive.

We will need the following result.

Lemma: We have

$$\begin{aligned}(b+c)(c+a)(a+b) &= (3g-a)(3g-b)(3g-c)\\ &= 27g^3 - 9(a+b+c)g^2 + 3(bc+ca+ab)g - abc\\ &= \left(3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)g - 1\right)abc\\ &= 8abc = -\frac{8}{h}.\end{aligned}$$

The triangle $A'B'C'$

The tangents to any circumscribing rectangular hyperbola at the vertices A, B, C are given by the equations $a^2y + x = 2a$, $b^2y + x = 2b$, $c^2y + x = 2c$, respectively, and these may be shown to meet in pairs at the vertices of the triangle $A'B'C'$ with coordinates

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