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VARIANCE BOUNDS UNDER A HAZARD RATE CONSTRAINT

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Consider an absolutely continuous distribution on $[0, \infty)$ with known mean μ, and hazard rate function, h satisfying, $0 < a \leq h(t) \leq b < \infty$, for almost all $t \geq 0$. We derive the sharp range for σ^2, under these constraints.

1. INTRODUCTION

Consider a non-negative random variable, X, with an absolutely continuous distribution, F, and hazard rate function h, satisfying, $0 < a \leq h(t) \leq b < \infty$, for almost all $t \geq 0$. This paper continues a study of optimization problems under this constraint, began in Brown and Kemperman, (3). The work is based on my conversations with the late Professor J.H.B. Kemperman of Rutgers University. Our motivation is that in biostatistics, reliability, actuarial science and other areas, modeling is often based on hazard functions. It is fundamental to understand the influence of hazard function conditions on the properties of the corresponding probability distributions. There is a large literature on properties of distributions under hazard function constraints, but not for the constraint considered here. These include monotonicity of h (increase failure rate (IFR), decreasing failure rate (DFR)), star-shaped behavior of $H(t) = \int_0^t h\, dx$, (increasing failure rate on the average (IFRA), decreasing failure rate on the average (DFRA)), and sub or superadditivity of H (new worse than used (NWU), new better than used (NBU)). Marshall and Olkin [5], and Barlow and Proschan, [1], are excellent references.

In, (3) we found the sharp range of $\bar{F}$, for a fixed t_0, under $a \leq h \leq b$ and known mean μ. In the present paper, given the same a, b, μ, we study the range of the variance, σ^2.

It is easy to show, and is well known that

$$\beta \stackrel{\text{def}}{=} b^{-1} \leq \mu \leq a^{-1} \stackrel{\text{def}}{=} \alpha, \tag{1}$$

$$\beta^2 = b^{-2} \leq \sigma^2 \leq a^{-2} \leq \alpha^2. \tag{2}$$

Given, $\mu \epsilon (\alpha, \beta)$ we derive sharp lower and upper bounds for σ^2. All values in between are also attainable. The sharp lower bound is given by,

$$\sigma_L^2 = \mu^2 + 2(\mu - \beta)(\alpha - \mu) - 2\alpha(\alpha - \mu) \log\left(\frac{\alpha - \beta}{a - \mu}\right), \tag{3}$$

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and the sharp upper bound by,

$$\sigma_U^2 = \mu^2 + 2(\mu-\beta)(\alpha-\mu) + 2\beta(\mu-\beta)\log\left(\frac{\alpha-\beta}{\mu-\beta}\right). \tag{4}$$

For example if $a=1, b=2$ and $\mu=0.7$, then $\sigma_L^2 \approx 0.3035, \sigma_U^2 \approx 0.7933$. Thus, $\sigma^2 \epsilon [0.3035, 0.7933]$, an improvement over (2) under which $\sigma^2 \epsilon [\frac{1}{4}, 1]$.

The bounds are derived in Section 2. Furthermore, comments and implications are found in Section 3.

Section 2. Consider, $\mu = p\alpha + q\beta$, with $p\epsilon(0,1), q = 1-p$. Define the class V,

$$V = \{F : a \le h \le b, \mu_F = \mu\}. \tag{5}$$

We construct two extreme distributions in V,

$$\bar{F}_1(t) = \begin{cases} e^{-at}, 0 \le t \le S \stackrel{\text{def}}{=} \alpha \log q^{-1} \\ e^{-aS}e^{-b(t-S)}, t > S \end{cases} \tag{6}$$

and,

$$\bar{F}_2(t) = \begin{cases} e^{-bt}, 0 \le t \le T \stackrel{\text{def}}{=} \beta \log p^{-1} \\ e^{-bT}e^{-a(t-T)}, t > T \end{cases} \tag{7}$$

The means of F_1, F_2 are both equal to μ. For example,

$$E_{F_1}X = \int_0^\infty \bar{F}_1(t)dt = \frac{1-e^{-aS}}{a} + \frac{e^{-aS}}{b}$$
$$= \alpha - (\alpha-\beta)q = p\alpha + q\beta = \mu.$$

Similarly, $E_{F_2}X = \mu$.

By F_1, F_2 being extreme we mean that for any $F \epsilon V$, and $t \ge 0$, that

$$\int_t^\infty \bar{F}_1(x)\,dx \le \int_t^\infty \bar{F}(x)\,dx \le \int_t^\infty \bar{F}_2(x)\,dx. \tag{8}$$

Since F_1, F, F_2 all have common mean μ, (8) is equivalent to the stationary renewal distributions being stochastically ordered,

$$\bar{G}_1(t) = \mu^{-1}\int_t^\infty \bar{F}_1(x)\,dx \le \bar{G}(t) \le \bar{G}_2(t) = \mu^{-1}\int_t^\infty \bar{F}_2(x)\,dx.$$

Since, $\mu_G = \frac{\mu_2}{2\mu}$, with $\mu_2 = EX^2$, it follows that,

$$E_{F_1}X^2 \le E_F X^2 \le E_{F_2}X^2,$$

and (because of the common mean), that

$$\sigma_L^2 \stackrel{\text{def}}{=} \text{Var}_{F_1}X \le \sigma^2 = \text{Var}X \le \text{Var}_{F_2}X \stackrel{\text{def}}{=} \sigma_U^2.$$

The condition (8), along with equality of means is equivalent to convex ordering of distributions. This leads to further inequalities as is discussed in Section 3.

To prove, (8), consider, (F_1, F), the case, (F, F_2) follows by the same argument.

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For $0 \le t \le S = \alpha \log q^{-1}$,

$$\frac{\bar{F}_1(t)}{\bar{F}(t)} = e^{\int_0^t (h-a)\,dx},$$

thus for $t \le S, \bar{F}_1(t)/\bar{F}(t)$ is non-decreasing and is ≥ 1. For, $t > S$,

$$\frac{\bar{F}_1(t)}{\bar{F}(t)} = \frac{\bar{F}(S)}{\bar{F}_1(S)} e^{-\int_S^t (b-h)dx},$$

thus $\bar{F}_1(t)|\bar{F}(t)$ is non-increasing in (S, ∞). For $F \ne F_1, \bar{F}(S) > \bar{F}_1(S)$. Since,

$$\int_0^\infty (\bar{F}(x) - \bar{F}_1(x))dx = 0,$$

it follows that, $\lim(\bar{F}_1(t)|\bar{F}(t)) < 1$, and there exists $t^* > S$, with $\bar{F}_1(t) \ge \bar{F}(t)$ for $t \le t^*$, and $\bar{F}_1(t) \le \bar{F}(t)$ for $t \ge t^*$, with $\bar{F}_1|\bar{F}$ ultimately <1. Thus, for $t \ge t^*$,

$$\int_0^t (\bar{F}_1(x) - \bar{F}(x)dx \ge 0, \quad \text{and is strictly}$$

positive for, $S \le t \le \tilde{t}$, for some $\tilde{t} < t^*$. Moreover, for $t > t^*$,

$$\int_0^t (\bar{F}_1(x) - \bar{F}(x))dx = \int_t^\infty (\bar{F}(x) - \bar{F}_1(x)) \ge 0.$$

Consequently, for all t,

$$\int_t^\infty (\bar{F}_1(x) - \bar{F}(x))dx = -\int_0^t (\bar{F}_1(x) - \bar{F}(x))dx \le 0.$$

and (8) holds. $\diamondsuit$

Now we compute σ_L^2, the calculation of σ_U^2 being similar. First (recall that $e^{-aS} = q$) from,

$$\begin{aligned}\frac{1}{2}E_{F_2}X^2 &= \int_0^S te^{-at}dt + e^{-aS}\int_S^\infty [(t-S)+S]e^{-b(t-S)}dt \\ &= \alpha^2 - (\alpha^2 + \alpha S)e^{-aS} + e^{-aS}[\beta^2 + \beta S], \\ &= p\alpha^2 + q\beta^2 - (\alpha - \beta)qS. \end{aligned} \tag{9}$$

From (9), and $\mu = p\alpha + q\beta$, we obtain

$$\begin{aligned}\frac{\mu_2}{2} - \mu^2 &= pq(\alpha-\beta)^2 - q(\alpha-\beta)S \\ &= (\mu-\beta)(\alpha-\mu) - \alpha(\alpha-\mu)\log\left(\frac{\alpha-\beta}{\alpha-\mu}\right). \end{aligned} \tag{10}$$

Finally from (10),

$$\begin{aligned}\sigma^2 &= \mu^2 + 2\left(\frac{\mu_2}{2} - \mu^2\right) \\ &= \mu^2 + 2(\mu-\beta)(\alpha-\mu) - 2\alpha(\alpha-\mu)\log\left(\frac{\alpha-\beta}{\alpha-\mu}\right), \end{aligned} \tag{11}$$

which is σ_L^2, as given in (3).

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2. COMMENTS AND ADDITIONS

(1) Suppose that F is absolutely continuous and satisfies, $h \geq a > 0$, but with no upper bound on the hazard function. If we let $b \to \infty$ in (3) and (4) we find,

$$\lim_{b\to\infty} \sigma_L^2 = \mu(2\alpha - \mu) - 2\alpha(\alpha - \mu)\log\left(\frac{\alpha}{\alpha - \mu}\right), \quad \text{and} \tag{12}$$

$$\lim_{b\to\infty} \sigma_U^2 = \mu(2\alpha - \mu). \tag{13}$$

We can achieve σ_L^2 and σ_U^2 with distributions not in V, but in the closure of V under convergence in distribution.

$$\text{Define, } \bar{F}_1(t) = \begin{cases} e^{-at}, t < \alpha \log(\frac{\alpha}{\alpha-\mu}) \stackrel{\text{def}}{=} S \\ 0, t \geq S. \end{cases}$$

Note that F_1 has support $[0, S]$, and has an atom at S. One can easily compute $E_{F_1} X = \mu$, and, $\text{var}_{T_1} X = \sigma_L^2$, given by (12).
Define, $\bar{F}_2(t) = a\mu\, e^{at}, t \geq 0$. The distribution F_2 has an atom at zero. A straightforward calculation shows that $E_{F_1} X = \mu$ and $\text{Var}_{F_2} X = \sigma_U^2$, given by (13).

(2) When, $h \leq b < \infty$, but with no positive lower bound on h, the sharp range for σ^2 is $[\beta^2, \infty]$.
To achieve infinite variance choose,

$$f(x) = 2\mu^2(x + \mu)^{-3}, \quad 0 \leq x < \infty.$$

Then the mean equals μ and the variance is infinite.
To achieve, $\sigma^2 = \beta^2$, take, $X = (\mu - \beta) + Y$, with Y exponential with mean β. To see that, β^2, is a lower bound, note that if $h \leq b$, then

$$m(t) = E(X|X > t) \geq \beta, \quad \text{and}$$
$$\sigma^2 = Em^2(X) \geq \beta^2, \quad \text{(Bremaud [2])}.$$

(3) The inequality, (8) combined with equality of means, tells us (Marshall and Olkin [5], p. 63), that,

$$E_{F_1}(\varphi(X)) \leq E_F \varphi(X) \leq E_{F_2}(\varphi(X)). \tag{14}$$

For all $F \epsilon V$, and convex φ with the expectations existing. For example, let $\varphi(x) = e^{cx}$, with $c < a$. Then (14) translates to,

$$\frac{a}{a-c} - \frac{bc(1-\mu a)}{(a-c)(b-c)}\left(\frac{\alpha-\beta}{\alpha-\mu}\right)^{\frac{c}{a}} \leq E_F e^{cX} \leq \frac{b}{b-c} - \frac{ac(b\mu - 1)}{(a-c)(b-c)}\left(\frac{\alpha-\beta}{\mu-\beta}\right)^{\frac{c}{b}}.$$

(4) If F is non-lattice with $\mu_2 < \infty$, then (Feller [4], p. 357),

$$M(t) = EN(t) = \frac{t}{\mu} + \left(\frac{\mu_2}{2\mu^2} - 1\right) + o(1)$$

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where, $N(t)$ is the number of renewals in $[0,t]$. Consider a distribution in V. Then it follows from (3) and (4), that,

$$\left(1-\frac{\beta}{\mu}\right)\left(\frac{\alpha}{\mu}-1\right)-\frac{\alpha}{\mu}\left(\frac{\alpha}{\mu}-1\right)\log\left(\frac{\alpha-\beta}{\alpha-\mu}\right)\leq\frac{\mu^2}{2\mu^2}-1\leq\left(1-\frac{\beta}{\mu}\right)\left(\frac{\alpha}{\mu}-1\right)+\frac{\beta}{\mu}\left(1-\frac{\beta}{\mu}\right)\log\left(\frac{\alpha-\beta}{\mu-\beta}\right).$$

We thus have sharp two sided bounds for, $\lim_{t\to\infty}(M(t)-\frac{t}{\mu})$, for distributions in V.

(5) The distribution F_1, (6), is IFR, and thus new better than used in expectation (NBUE). For NBUE distributions, $\sigma^2\leq\mu^2$ (Barlow and Proschan [1], p. 187). Thus for any class of distributions W containing IFR and contained in NBUE, the sharp range for the variance for $F\epsilon W\cap V$, is $[\sigma_L^2,\mu^2]$. The upper bound, μ^2 is attained by an exponential distribution with mean μ. Similarly F_2 is DFR, so that for any class of distributions W', containing DFR and containedin new better than used in expectation (NWUE), the sharp range for σ^2 for $F\epsilon W'\cap V$ is, $[\mu^2,\sigma_U^2]$. In the example previously considered, $(a=1,b=2,\mu=0.7)$, if in addition to belonging to V, F is NBUE, then $\sigma^2\epsilon[0.3035,0.4900]$, while if F is NWUE then, $\sigma^2\epsilon[0.4900,0.7933]$.

(6) Define Z to be a mixture of exponential distributions; Z is exponential with mean α with probability $p=\frac{\mu-\beta}{\alpha-\beta}$, and exponential with mean β with probability, $q=\frac{\alpha-\mu}{\alpha-\beta}$, so that,

$$Pr(Z>t)=p\,e^{-at}+q\,e^{-bt}.$$

Then mean of Z equals μ, while,

$$\text{Var}\,Z=\mu^2+2(\mu-\beta)(\alpha-\mu).$$

Thus, curiously,

$$\sigma_L^2=\text{Var}\,Z-2\alpha(\alpha-\mu)\log\left(\frac{\alpha-\beta}{\alpha-\mu}\right),\quad\text{and}$$

$$\sigma_U^2=\text{Var}\,Z+2\beta(\mu-\beta)\log\left(\frac{\alpha-\beta}{\mu-\beta}\right).$$

I have no intuitive interpretation for the presence of Var Z in the above expressions.

(7) **Example.** Consider the birth and death example given in [3], p. 34–35. In this example, the random variable of interest is DFR. The values of (a,b,μ) are given by $a=6-\sqrt{6},b=30|31$ and $\mu=333|310$. By Remark 5 of Section 3, the sharp two-sided bound for σ^2 is given by, $[\mu^2,\sigma_U^2]$. To five decimal places this gives the interval,

$$\sigma^2\epsilon(1.15389,1.20984),\quad\text{and}$$

$$\sigma\epsilon(1.07419,1.09993).$$

The quantity, $\alpha=a^{-1}$, is known as the mean quasi-stationary hitting time, while $\beta=b^{-1}$ is the mean post recovery exit time, and μ the mean stationary hitting time.

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