Probability in the Engineering and Informational Sciences, **21**, 2007, 315–334. Printed in the U.S.A.
DOI: 10.1017/S0269964807070192

ROLE OF EQUILIBRIUM DISTRIBUTION IN RELIABILITY STUDIES

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The equilibrium distribution arises as the limiting distribution of the forward recurrence time in a renewal process. The purpose of this article is to study the relationships between the equilibrium distributions (including its higher derivates) and the original distributions. Some stochastic order relations and the relations between their aging properties are investigated and some applications in the field of insurance and financial investments are given. In addition, the relation between the equilibrium distribution of a series system and the series system of equilibrium distribution is investigated. Bivariate equilibrium distribution whose reliability properties are consistent with those of the univariate equilibrium distribution is defined.

1. INTRODUCTION

Let X be a continuous random variable with probability density function (p.d.f.) $f(x)$, distribution function $F(x)$, and survival function $\bar{F}(x) = 1 - F(x)$. We define the p.d.f. $f^*(x)$as

$$f^*(x) = \frac{\bar{F}(x)}{\mu}, \quad x \geq 0, \tag{1.1}$$

where $\mu = E(X) < \infty$. Then $f^*(x)$ is called the p.d.f. of an equilibrium distribution or induced distribution.

The above distribution arises as the limiting distribution of the forward recurrence time in a renewal process. It also arises as the marginal distribution of W_1, where the joint p.d.f. of (W_1, W_2) is given by

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$$g(w_1, w_2) = \frac{f(w_2)}{\mu}, \quad 0 < w_1 < w_2 < \infty$$

$$= 0, \quad \text{elsewhere;}$$

see Brown [10].

Note that, in this case, the p.d.f. of W_2 is given by the length-biased version of the original distribution as

$$f_{W_2}(w_2) = \frac{w_2 f(w_2)}{\mu}, \quad 0 < w_2 < \infty.$$

The equilibrium distribution (1.1) is intimately connected to its parent distribution and many of the reliability properties of the original distribution can be easily studied by means of the properties of the equilibrium distribution.

The purpose of this article is to study the relationships between (1.1) (including its higher derivatives) and the original distribution. Some stochastic order relations and the relations between their aging properties are investigated and some applications in the field of insurance and financial investments are given. This is primarily a review article. However, the examples in Section 6 are new.

The organization of this article is as follows. In Section 2 we present some definitions and background material encountered in reliability studies, including some criteria of aging and their relationships. Some definitions of stochastic order relations and their relationships are also provided. Section 3 deals with higher-order equilibrium distributions and stop loss moments. In Section 4 aging properties of equilibrium distributions and their stochastic ordering with the original distribution are explored. In Section 5 the relation between the equilibrium distribution of a series system and a series system of equilibrium distributions, consisting of two components, is investigated. Section 6 contains the bivariate equilibrium distribution along with two examples. Finally, in Section 7 we provide some conclusions and comments.

2. DEFINITIONS AND BACKGROUND

Let X be a continuous positive random variable representing a survival time with an absolutely continuous distribution function $F(t)$, survival function $\bar{F}(t) = 1 - F(t)$, and cumulative hazard rate $\Lambda(t) = -\ln \bar{F}(t)$ with $\bar{F}(0) = 1$. Wherever necessary, it will be assumed that $r_F(t) = \Lambda'(t)$ is the hazard rate corresponding to $F(t)$. A key role in this article will be played by the mean residual life function (MRLF) $\mu_F(t)$ defined as

$$\mu_F(t) = E(X - t | X > t) = \frac{\int_t^\infty \bar{F}(x)\, dx}{\bar{F}(t)}, \quad t \geq 0.$$

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It will be assumed that $\mu_F(0) = E(X) < \infty$. When discussing the variance of the residual lifetime $X - t | X > t$, it will be assumed that $E(X^2) < \infty$. The variance residual life function (VRLF) is defined as

$$\sigma_F^2(t) = \mathrm{Var}(X - t | X > t)$$
$$= \frac{2}{\bar{F}(t)} \int_t^\infty \int_x^\infty \bar{F}(y)\, dy\, dx - \mu_F^2(t), \quad t \geq 0.$$

We refer to Gupta and Kirmani [17,18], Launer [28], and Gupta, Kirmani, and Launer [19] for details about the MRLF and the VRLF.

The above-defined functions highlight different aspects of survival and residual life distributions.The hazard rate and the mean residual life function are related by

$$r_F(t) = \frac{1 + \mu_F'(t)}{\mu_F(t)}. \tag{2.1}$$

Further, the hazard rate, the MRLF, and the VRLF are tied together by the relation

$$\frac{d}{dt}\, \sigma_F^2(t) = r_F(t)\{\sigma_F^2(t) - \mu_F^2(t)\}; \tag{2.2}$$

see Gupta [16].

It is well known that $r_F(t)$ determines the distribution function uniquely and, hence, $\mu_F(t)$ also characterizes the distribution. Additionally, $\bar{F}(t)$ and $\mu_F(t)$ are connected by

$$\bar{F}(t) = \frac{\mu_F(0)}{\mu_F(t)} \exp\left\{ - \int_0^t \frac{dx}{\mu_F(x)} \right\}. \tag{2.3}$$

Thus, $r_F(t)$, $\mu_F(t)$, and $\bar{F}(t)$ are equivalent in the sense that given one of them, the other two can be determined. Hence, in the analysis of survival data, one sometimes estimates $r_F(t)$ or $\mu_F(t)$ instead of $\bar{F}(t)$, according to the convenience of the procedure available.

In addition to the above functions, the residual coefficient of variation is given by $\gamma_F(t) = \sigma_F(t)/\mu_F(t)$. Further, the MRLF, the VRLF, and the residual coefficient of variation are connected by the relation

$$\frac{d}{dt}\, \sigma_F^2(t) = \mu_F(t)(1 + \mu_F'(t))(\gamma_F^2(t) - 1); \tag{2.4}$$

see Gupta [16].

We now describe briefly some aging classes of life distributions and their relationships

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2.1. Some Criteria of Aging for the RLF

In this section we review some of the aging criteria and their relationships. We also describe how the aging properties of the original distribution are transformed into the aging properties of the residual life.

Let X be a continuous positive random variable representing the life of a component. Let F be the cumulative distribution function of X and $\bar{F}(x) = 1 - F(x)$ be the reliability function or the survival function of X. Then $\overline{F}_t(x) = P(X > x + t | X > t)$ is the survival function of a unit of age t. Evidently, any study of the phenomenon of aging should be based on $\overline{F}_t(x)$ and functions related to this. Thus, the following hold:

1. F is said to be PF_2 if $\ln f(x)$ is concave, where $f(\cdot)$ is the density corresponding to $F(\cdot)$.
2. F is said to have increasing (decreasing) failure rate [IFR (DFR)] if $\overline{F}_t(x) = \bar{F}(x + t)/\bar{F}(t)$ is decreasing (increasing) in t. If F is absolutely continuous with density f, then F is in the IFR (DFR) class if $r_F(t) = f(t)/\bar{F}(t)$ is increasing (decreasing) in t.
3. F is said to have increasing (decreasing) failure rate average [IFRA (DFRA)] if $\int_0^t r_F(x)\, dx/t$ is increasing (decreasing).
4. F is said to have new better (worse) than used [NBU (NWU)] if $\bar{F}_t(x) \leq (\geq) \bar{F}(x)$ for $x \geq 0$ and $t \geq 0$.
5. F is said to have decreasing (increasing) mean residual life [DMRL (IMRL)] if the mean residual life $\mu_F(t) = \int_t^\infty \bar{F}(x)\, dx/\bar{F}(t)$ is decreasing (increasing) assuming that the mean $\mu_F(0)$ *exists*.
6. F is said to have new better (worse) than used in expectation [NBUE (NWUE)] if $\mu_F(t) \leq (\geq) \mu_F(0)$ for all $t \geq 0$.
7. F is said to have decreasing variance residual life (increasing variance residual life) [DVRL (IVRL)] if $\sigma_F^2(t)$ is decreasing (increasing).

The chain of implications among these classes of distributions is

$$
\begin{array}{lllll}
PF_2 \Rightarrow \text{IFR} & \Rightarrow \text{IFRA} & \Rightarrow \text{NBU} \\
\Downarrow & & \Downarrow \\
\text{DMRL} & \Rightarrow & \text{NBUE}. \\
\Downarrow & & \\
\text{DVRL} &
\end{array}
$$

The reverse implications are not true; for counterexamples, see Bryson and Siddiqui [11]. Some extensions of these classes of distributions are contained in Klefsjo [26,27], Shaked [32], Singh and Deshpande [35], Deshpande, Kochar, and Singh [12], Basu and Ebrahimi [5,6], Abouammoh [1], Abouammoh and Ahmad [2], and Loh [30].

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2.2. Stochastic Order Relations

Let X and Y be nonnegative absolutely continuous random variables with density functions $f(x)$ and $g(x)$ and survival functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively. Then, we have the following:

1. X is said to be smaller than Y in the likelihood ratio ordering, written as $X \leq^{\mathrm{LR}} Y$, if $f(x)/g(x)$ is nonincreasing in x.
2. X is said to be smaller than Y in the failure (hazard) rate ordering, written as $X \leq^{\mathrm{FR}} Y$, if $r_F(x) \geq r_G(x)$ for all x.
3. X is said to be smaller than Y in the stochastic ordering, written as $X \leq^{\mathrm{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x.
4. X is said to be smaller than Y in the mean residual life ordering, written as $X \leq^{\mathrm{MRL}} Y$, if $\mu_F(x) \leq \mu_G(x)$ for all x. Deshpande, Singh, Bagai, and Jain [13] show that $X \leq^{\mathrm{MRL}} Y$ if and only if $\int_x^\infty \bar{F}(u)\,du / \int_x^\infty \bar{G}(u)\,du$ is decreasing in x.
5. X is said to be smaller than Y in the increasing convex order, written as $X \leq^{\mathrm{icx}} Y$, if $\int_x^\infty \bar{F}(u)\,du \leq \int_x^\infty \bar{G}(u)\,du$ for all x. Note that in the literature the increasing convex order has also been called stop loss ordering (Hesselager [22]) and ST2 ordering; see Belzunce, Candel, and Ruiz [7] and Shaked and Shanthikumar [33]. For some generalized variability ordering, see Zarek [39], Li and Zhu [29], and Bhattacharjee and Sethuraman [8].
6. X is said to be smaller than Y in the variance residual life ordering, written as $X \leq^{\mathrm{VRL}} Y$, if $\sigma_F^2(x) \geq \sigma_G^2(x)$ for all x.
7. X is said to be smaller than Y in the Laplace transform ordering, written as $X \leq^{\mathrm{LT}} Y$, if $E(e^{-sX}) \geq E(e^{-sY})$. Shaked and Shanthikumar [33] showed that this is equivalent to $\int_0^\infty e^{-sx}\bar{F}(x)\,dx \leq \int_0^\infty e^{-sx}\bar{G}(x)\,dx$.
8. X is said to be smaller than Y in the moment generating function ordering, written as $X \leq^{\mathrm{MGF}} Y$, if $E(e^{sX}) \leq E(e^{sY})$.

It is well known that

$$\begin{array}{ccccc} X \leq^{\mathrm{LR}} Y & \Rightarrow & X \leq^{\mathrm{FR}} Y & \Rightarrow & X \leq^{\mathrm{MRL}} Y \Rightarrow X \leq^{\mathrm{VRL}} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq^{\mathrm{st}} Y & & X \leq^{\mathrm{icx}} Y; \end{array}$$

see Shaked and Shanthikumar [33].

3. HIGHER-ORDER EQUILIBRIUM DISTRIBUTIONS AND STOP LOSS MOMENTS

We define the sequence of induced distributions as follows. Let $\bar{F}(x)$ be the survival function of a nonnegative random variable X with MRLF given by $\mu_F(t)$. We now define a sequence $\{\bar{F}_1, \bar{F}_2, \ldots\}$ of survival functions induced by $\bar{F}$ as

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$$\overline{F_n}(t) = \frac{\int_t^\infty \bar{F}_{n-1}(u)\,du}{\mu_{n-1}}, \tag{3.1}$$

where μ_n is the mean of the distribution F_n. Assume that $E(X_n) < \infty$ for the largest n used above so that $\mu_1, \mu_2, \ldots, \mu_n$ will all be finite. In what follows, we will denote by $X^{[0]} = X$, corresponding to the survival function $\bar{F}_0 = \bar{F}$, the original random variable and $X^{[1]}, X^{[2]}, \ldots$ corresponding to $\bar{F}_1, \bar{F}_2, \ldots$.

We now present the following result.

THEOREM 3.1: *Let ϕ be a convex function. Then*

$$E_F[\phi(X-t)|X>t] = \phi(0) + E_F[(X-t)|X>t]E_{F_1}[\phi'(X-t)|X>t]. \tag{3.2}$$

PROOF:

$$\begin{aligned} &E_F[\phi(X-t)|X>t] \\ &\quad= \frac{-\int_t^\infty \phi(x-t)\,d\bar{F}(x)}{\bar{F}(t)} \\ &\quad= \phi(0) + \frac{\int_t^\infty \phi'(x-t)\bar{F}(x)\,dx}{\bar{F}(t)} \\ &\quad= \phi(0) + \int_t^\infty \phi'(x-t)\left[\frac{\bar{F}(x)/\mu}{\int_t^\infty (\bar{F}(x)/\mu)\,dx}\right]\frac{\int_t^\infty \bar{F}(x)\,dx}{\bar{F}(t)}\,dx \\ &\quad= \phi(0) + E_F[(X-t)|X>t]\int_t^\infty \phi'(x-t)\,\frac{f_1(x)}{\overline{F_1}(x)}\,dx, \end{aligned}$$

where $f_1(x) = \bar{F}(x)/\mu$ is the p.d.f. of the first induced distribution and $\overline{F_1}(x)$ is its survival function. Thus,

$$E_F[\phi(X-t)|X>t] = \phi(0) + E_F[(X-t)|X>t]E_{F_1}[\phi'(X-t)|X>t]. \quad \blacksquare$$

Particular Cases:

$$E_F[(X-t)^2|X>t] = 2\mu_F(t)\mu_{F_1}(t) \tag{3.3}$$

and, in general,

$$E_F[(X-t)^k|X>t] = k!\prod_{j=0}^{k-1}\mu_j(t), \tag{3.4}$$

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where $\mu_0(t) = \mu(t) =$ MRLF of F and $\mu_j(t)$ is the MRLF of the jth induced distribution; see also Stein and Dattero [36]. From this, it is clear that

$$E_F(X^k \mid X > t) = k! \prod_{j=0}^{k-1} \mu_j, \tag{3.5}$$

where μ_j is the mean of the jth induced distribution.

3.1. Stop Loss Transform

DEFINITION 3.1: *The function $\pi_X(t) = E[(X - t)_+]$ is called the stop loss transform of X, where $(X - t)_+ = \text{Max}(X - t, 0)$ represents the amount by which X exceeds the threshold t.*

The kth stop loss moment is given by

$$R_k(t) = E[(X - t)_+^k] = \int_t^\infty (x - t)^k \, dF(x); \tag{3.6}$$

see Willmot, Drekic, and Cai [38] and Hesselager, Wang, and Willmot [23]. Note that $R_k(t)$ has also been called the kth partial moment in the literature; see Gupta and Gupta [15]. We now present the following result.

THEOREM 3.2: *The survival function of the kth equilibrium distribution is given by*

$$\bar{F}_k(t) = E[(X - t)_+^k]/E(X^k). \tag{3.7}$$

PROOF:

$$\begin{aligned}
&E[(X - t)_+^k]/E(X^k) \\
&\quad = \int_t^\infty (x - t)^k \, dF(x)/E(X^k) \\
&\quad = \int_t^\infty k(x - t)^{k-1} \bar{F}(x) \, dx/E(X^k) \\
&\quad = k\mu \int_t^\infty (x - t)^{k-1} f_1(x) \, dx/E(X^k) \\
&\quad = \frac{k\mu}{E(X^k)} \int_t^\infty (k - 1)(x - t)^{k-2} \bar{F}_1(x) \, dx \\
&\quad = \frac{k(k-1)\mu\mu_1}{E(X^k)} \int_t^\infty (x - t)^{k-2} f_2(x) \, dx.
\end{aligned}$$

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Proceeding in this way, we get

$$E[(X-t)_+^k]/E(X^k) = \frac{k!\Pi_{j=0}^{k-1}\mu_j}{E(X^k)}\bar{F}_k(t)$$
$$= \bar{F}_k(t). \quad \blacksquare$$

We now show that $R_k(t)$ determines the distribution function uniquely.

THEOREM 3.3: *Under the above-stated conditions,*

$$\bar{F}(t) = \frac{(-1)^k}{k!}R_k^{(k)}(t), \tag{3.8}$$

where $R_k^{(k)}(t)$ denotes the kth derivative of $R_k(t)$.

PROOF: Applying Taylor's theorem to $F(x)$, we get

$$F(x) = 1 - \frac{(-1)^k}{k!}\int_t^\infty (x-t)^k F^{(k)}(x)\,dF(x)$$
$$= 1 - \frac{(-1)^k}{k!}\left(\frac{d}{dt}\right)^k \int_0^\infty u^k f(t+u)\,du$$
$$= 1 - \frac{(-1)^k}{k!}\left(\frac{d}{dt}\right)^k \int_t^\infty (x-t)^k\,dF(x)$$
$$= 1 - \frac{(-1)^k}{k!}R_k^{(k)}(t). \tag{3.9}$$

This gives the desired result. $\blacksquare$

Remark 3.1: For other proofs, see Gupta and Gupta [15] and Navarro, Franco, and Ruiz [31].

4. AGING PROPERTIES OF EQUILIBRIUM DISTRIBUTIONS AND SOME ORDER RELATIONS

It is well known that the failure rate of F_1 is the reciprocal of the MRLF of F and, in general, the failure rate of F_i $(i = 1,2,\ldots)$ is the reciprocal of the MRLF of F_{i-1} $(i = 1,2,\ldots)$. This means that F_i is IFR is equivalent to F_{i-1} is DMRL. In the following, we show that F_{i-1} is DMRL is equivalent to F_{i-2} is DVRL.

THEOREM 4.1: *F_{i-1} is DMRL is equivalent to F_{i-2} is DVRL* $(i = 2,3,\ldots)$.

PROOF: It is enough to prove the result for $i = 2$. We have

$$\frac{E((X-t)^2|X>t)}{E((X-t)|X>t)} = 2\mu_{F_1}(t). \tag{4.1}$$

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Also,

$$\mu_{F_1}(t) = \frac{\sigma_F^2(t) + \mu_F^2(t)}{\mu_F(t)}$$

$$= \frac{1}{2}\,\mu_F(t)(\gamma_F^2(t) + 1).$$

This gives

$$\frac{d}{dt}\,[E((X-t)^2 | X > t)]/E(X - t | X > t)$$

$$= 2\mu'_{F_1}(t)$$

$$= 2(r_{F_1}(t)\mu_{F_1}(t) - 1)$$

$$= 2\left(\frac{\mu_{F_1}(t)}{\mu_F(t)} - 1\right)$$

$$= 2\left[\frac{1}{2}\,(\gamma_F^2(t) + 1 - 1)\right]$$

$$= \gamma_F^2(t) - 1.$$

This proves the result; see also Gupta et al. [19], Singh [34], Fagiuoli and Pellerey [14], and Willmot [37]. ■

COROLLARY 4.1: *Suppose F is a strictly increasing life distribution. Then F is DVRL (IVRL) if and only if the induced distribution corresponding to F is DMRL.*

Remark 4.1: The restriction to strictly increasing F in Theorem 4.1 and Corollary 4.1 cannot be relaxed. This can be seen from Example 1 of Gupta et al. [19], where F is IVRL but the function $E((X-t)^2 | X > t)/E((X-t) | X > t)$ is not increasing in the neighborhood of $t = \frac{1}{2}$.

Remark 4.2: It is also clear from the above derivation that

$$\frac{\mu_{F_1}(t)}{\mu_F(t)} = \frac{1}{2}\,(1 + \gamma_F^2(t)).$$

From the above, we can state the following.

THEOREM 4.2: *Suppose F is a strictly increasing life distribution. Then F is DVRL (IVRL) if and only if* $\mu_{F_1}(t) \leq (\geq)\ \mu_F(t)$.

Remark 4.3: The above result was also noticed by Deshpande et al. [13] and Abouammoh, Kanjo, and Khalique [3].

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It is also clear from the above that $\mu_{F_1}(t) = \mu_F(t)$ if and only if F has an exponential distribution.

The following theorem addresses the stochastic comparisons of $X^{[1]}$ and X.

THEOREM 4.3: *Suppose X is a nonnegative random variable with $E(X) < \infty$. Then X is NBUE (NWUE) if $X^{[1]} \leq^{\text{st}} (\geq^{\text{st}}) X$.*

PROOF: The survival function of $X^{[1]}$ is given by

$$\begin{aligned}\bar{F}_{X^{[1]}}(t) &= \int_t^\infty \bar{F}(x)\,dx/\mu \\ &= \bar{F}_X(t)\bar{F}_X(t)/\mu.\end{aligned}$$

This gives

$$\frac{\bar{F}_{X^{[1]}}(t)}{\bar{F}_X(t)} = \frac{\mu_F(t)}{\mu}.$$

Thus,

$$\bar{F}_{X^{[1]}}(t) \leq \bar{F}_X(t) \quad \text{if and only if } X \text{ is NBUE.}$$

Similarly, $\bar{F}_{X^{[1]}}(t) \geq \bar{F}_X(t)$ if and only if X is NWUE. ■

We now present the following result dealing with the hazard rate comparison of $X^{[1]}$ and X.

THEOREM 4.4: *Suppose X is a nonnegative random variable with $E(X) < \infty$. Then X is IMRL (DMRL) if and only if $X \leq^{\text{FR}} (\geq^{\text{FR}}) X^{[1]}$.*

PROOF:

$$\begin{aligned}X \leq X^{[1]} &\Leftrightarrow \bar{F}_{X^{[1]}}(t)/\bar{F}_X(t) \text{ is increasing} \\ &\Leftrightarrow \frac{\bar{F}_{X^{[1]}}(v)}{\bar{F}_X(v)} \geq \frac{\bar{F}_{X^{[1]}}(u)}{\bar{F}_X(u)} \quad \text{if } u \leq v \\ &\Leftrightarrow \bar{F}_{X^{[1]}}(v)\bar{F}_X(u) \geq \bar{F}_{X^{[1]}}(u)\bar{F}_X(v) \\ &\Leftrightarrow \int_v^\infty \frac{\bar{F}(t)}{\mu}\,dt\,\bar{F}_X(u) \geq \int_u^\infty \frac{\bar{F}(t)}{\mu}\,dt\,\bar{F}_X(v) \\ &\Leftrightarrow \frac{\int_v^\infty \bar{F}(t)\,dt}{\bar{F}_X(v)} \geq \frac{\int_u^\infty \bar{F}(t)\,dt}{\bar{F}_X(u)} \\ &\Leftrightarrow \mu_X(v) \geq \mu_X(u) \\ &\Leftrightarrow X \text{ is IMRL.}\end{aligned}$$

■

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The following result deals with the Laplace ordering of two random variables and their corresponding equilibrium distributions.

THEOREM 4.5: *Let X and Y be two nonnegative random variables with $E(X) = E(Y)$. Then*

$$X \leq^{\mathrm{LT}} Y \Leftrightarrow X^{[1]} \leq^{\mathrm{LT}} Y^{[1]}.$$

PROOF: We have

$$\begin{aligned} L_{X^{[1]}}(t) &= t \int_0^\infty e^{-tx} \left(\int_0^x \frac{\bar{F}_X(y)}{\mu}\, dy \right) dx \\ &= \frac{t}{E(X)} \int_0^\infty \left(\int_y^\infty e^{-tx}\, dx \right) \bar{F}_X(y)\, dy \\ &= \frac{1}{E(X)} \int_0^\infty e^{-ty} \bar{F}_X(y)\, dy. \end{aligned}$$

Now

$$\begin{aligned} X \leq^{\mathrm{LT}} Y &\Leftrightarrow \int_0^\infty e^{-tx} \bar{F}_X(x)\, dx \leq \int_0^\infty e^{-ty} \bar{F}_Y(y)\, dy \\ &\Leftrightarrow L_{X^{[1]}}(t) \leq L_{Y^{[1]}}(t) \quad \text{since } E(X) = E(Y). \end{aligned}$$

■

The following two results deal with the stop loss ordering of two random variables and their corresponding equilibrium variables.

THEOREM 4.6: *Let X and Y be two nonnegative random variables with $E(X) = E(Y)$. Then*

$$X \leq^{\mathrm{SL}} Y \Leftrightarrow X^{[1]} \leq^{\mathrm{st}} Y^{[1]}.$$

PROOF:

$$\begin{aligned} X^{[1]} \leq^{\mathrm{st}} Y^{[1]} &\Leftrightarrow \frac{\pi_X(t)}{E(X)} \leq \frac{\pi_Y(t)}{E(Y)} \quad \text{for all } t \geq 0 \\ &\Leftrightarrow \pi_X(t) \leq \pi_Y(t) \quad \text{since } E(X) = E(Y) \\ &\Leftrightarrow X \leq^{\mathrm{SL}} Y. \end{aligned}$$

■

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THEOREM 4.7: *Let X and Y be two nonnegative random variables with finite means. Then*

$$X \leq^{\mathrm{HR}} Y \Leftrightarrow X^{[1]} \leq^{\mathrm{LR}} Y^{[1]}.$$

PROOF:

$$\begin{aligned} X^{[1]} &\leq^{\mathrm{LR}} Y^{[1]} \\ &\Leftrightarrow \frac{f_{X^{[1]}}(t)}{f_{Y^{[1]}}(t)} \text{ is decreasing} \\ &\Leftrightarrow \frac{f_{X^{[1]}}(u)}{f_{Y^{[1]}}(u)} \geq \frac{f_{X^{[1]}}(v)}{f_{Y^{[1]}}(v)} \quad \text{if } u \leq v \\ &\Leftrightarrow \frac{\bar{F}_X(u)}{\bar{F}_Y(u)} \geq \frac{\bar{F}_X(v)}{\bar{F}_Y(v)} \\ &\Leftrightarrow \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} \text{ is decreasing} \\ &\Leftrightarrow X \leq^{\mathrm{HR}} Y. \end{aligned}$$

■

We now define the convex ordering as follows.

DEFINITION 4.1: $X \leq^{\mathrm{icx}} Y$ *if* $E(f(X)) \leq E(f(Y))$ *for all increasing convex functions. This is equivalent to*

$$X \leq^{\mathrm{icx}} Y \quad \text{if } \bar{F}_X(t) \leq \bar{F}_Y(t) \text{ for all } t \geq 0.$$

Following the above concept, we define a similar ordering between the sth equilibrium as follows.

DEFINITION 4.2: $X \leq^{s-\mathrm{icx}} Y$ *if* $\bar{F}_X^{(s)}(t) \leq \bar{F}_Y^{(s)}(t)$ *for all* $t \geq 0$, *where* $\bar{F}_X^{(s)}(t)$ *and* $\bar{F}_Y^{(s)}(t)$ *are the survival functions of their sth equilibrium distributions; see Klar and Muller [25].*

Comparing the original variable and its equilibrium variable in the above ordering, we have

$$X \geq^{s-\mathrm{icx}} X^{[s]} \quad \text{for } s \geq 1 \text{ if } E(X-t)_+^{s-1} \geq E(X^{[s]}-t)_+^{s-1}.$$

This is equivalent to

$$\frac{E(X-t)_+^s}{E(X-t)_+^{s-1}} \leq \mu s.$$

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Denoting

$$M_n(t) = \frac{E(X-t)_+^n}{nE(X-t)_+^{n-1}},$$

the above inequality can be written as

$$M_s(t) \leq \mu;$$

see Stein and Dattero [36].

5. EQUILIBRIUM DISTRIBUTION OF A SERIES SYSTEM

Let $X_1, X_2, \ldots, X_n$ be independent random variables with survival functions $\bar{F}_{X_1}$, $\bar{F}_{X_2}, \ldots, \bar{F}_{X_n}$, respectively. When the components are arranged in a series system, we observe $X_1 \wedge X_2 \wedge \cdots \wedge X_n = \text{Min}(X_1, X_2, \ldots, X_n)$. In this section, we will compare two systems whose components are the nth equilibrium distributions of X and Y and the nth equilibrium distribution of a system having components X and Y. More precisely, we have the following result due to Bon and Illayk [9].

THEOREM 5.1: *Let X and Y be independent nonnegative random variables. Let $X^{[k]}$ and $Y^{[k]}$ be respectively the independent equilibrium variables of X and Y of order k. If X and Y have the DMRL property and if the nth moments of X and Y exist, then*

$$X^{[n]} \wedge Y^{[n]} \leq^{\text{LR}} (X \wedge y)^{[n]},$$

where $(X \wedge y)^{[n]}$ is the equilibrium variable of $X \wedge Y$ of order n.

PROOF: For $n = 1, X^{[1]}$ and $Y^{[1]}$ are independent and the p.d.f. of $X^{[1]} \wedge Y^{[1]}$ is given by

$$f_{X^{[1]} \wedge Y^{[1]}}(t) = f_{X^{[1]}}(t)\bar{F}_{Y^{[1]}}(t) + f_{Y^{[1]}}(t)\bar{F}_{X^{[1]}}(t).$$

Also, the p.d.f. of $(X \wedge Y)^{[1]}$ is

$$f_{(X \wedge Y)^{[1]}}(t) = \frac{\bar{F}_X(t)\bar{F}_Y(t)}{E(X \wedge Y)}.$$

These give

$$\frac{f_{X^{[1]} \wedge Y^{[1]}}(t)}{f_{(X \wedge Y)^{[1]}}(t)} = \frac{E(X \wedge Y)[\mu_X(t) + \mu_Y(t)]}{E(X)E(Y)}.$$

Since X and Y have the DMRL property, the right-hand side of the above equation is decreasing and, hence, $X^{[1]} \wedge Y^{[1]} \leq^{\text{LR}} (X \wedge Y)^{[1]}$.

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For the general case, we proceed by induction and assume that the result is true of $k = n$. It can be verified that

$$\frac{f_{X^{[n]} \wedge Y^{[n]}}(t)}{f_{(X \wedge Y)^{[n]}}(t)}$$

$$= E(X \wedge Y)^{[n-1]} \left[\frac{\bar{F}_{X^{[n-1]}}(t) \bar{F}_{Y^{[n]}}(t)}{E(X^{[n-1]}) \bar{F}_{(X \wedge Y)^{[n-1]}}(t)} + \frac{\bar{F}_{Y^{[n-1]}}(t) \bar{F}_{X^{[n]}}(t)}{E(Y^{[n-1]}) \bar{F}_{(X \wedge Y)^{[n-1]}}(t)} \right],$$

$$= A(t) + B(t),$$

where

$$A(t) = \frac{\bar{F}_{X^{[n-1]}}(t) \bar{F}_{Y^{[n]}}(t)}{E(X^{[n-1]}) \bar{F}_{(X \wedge Y)^{[n-1]}}(t)},$$

and likewise for $B(t)$. We now show that $A(t)$ and $B(t)$ are decreasing functions of t.

Since Y has the DMRL property, all of its equilibrium variables have the IFR property and also the DMRL property. This means that $Y^{[n]} \leq^{\text{HR}} Y^{[n-1]}$. Thus,

$$X^{[n-1]} \wedge Y^{[n]} \leq^{\text{HR}} X^{[n-1]} \wedge Y^{[n-1]}.$$

Also, the induction hypothesis implies that

$$X^{[n-1]} \wedge Y^{[n-1]} \leq^{\text{HR}} (X \wedge Y)^{[n-1]}.$$

Therefore, $X^{[n-1]} \wedge Y^{[n]} \leq^{\text{HR}} (X \wedge Y)^{[n-1]}$. This implies that $A(t)$ is decreasing. Similary, $B(t)$ is decreasing and the proof is complete. ■

Remark 5.1: For the lower bound on the nth moment of a series system, see the above remark.

6. BIVARIATE EQUILIBRIUM DISTRIBUTION

In order to define bivariate equilibrium distribution, we first review the bivariate hazard rates and bivariate MRLF as follows. Let (X_1, X_2) be a nonnegative bivariate random variable with survival function $\bar{F}(x_1, x_2)$.The hazard components of (X_1, X_2) are given by $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$, where

$$h_i(x_1, x_2) = \frac{\partial}{\partial x_i} [-\ln \bar{F}(x_1, x_2)], \qquad i = 1, 2,$$

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and the MRLF of (X_1, X_2) is given by $(\mu_1(x_1, x_2), \mu_2(x_1, x_2))$, where

$$\mu_1(x_1, x_2) = E(X_1 - x_1 | X_1 > x_1, X_2 > x_2)$$

$$= \frac{1}{\bar{F}(x_1, x_2)} \int_{x_1}^{\infty} \bar{F}(x_1, x_2)\, dx_1,$$

and likewise for $\mu_2(x_1, x_2)$. The hazard components and the MRLF components are related by

$$h_i(x_1, x_2) = \frac{1 + \dfrac{\partial}{\partial x_i} \mu_i(x_1, x_2)}{\mu_i(x_1, x_2)}, \qquad i = 1, 2.$$

Keeping these properties in mind, Gupta and Sankaran [21] defined the bivariate equilibrium random variable (Y_1, Y_2) by means of the conditional distribution as follows.

The p.d.f. of Y_1 given $Y_2 > y_2$ is

$$g(y_1 | Y_2 > y_2) = \frac{P(X_1 > y_1 | X_2 > y_2)}{E(X_1 | X_2 > y_2)}. \tag{6.1}$$

Likewise, the p.d.f. of Y_2 given $Y_1 > y_1$ is

$$g(y_2 | Y_1 > y_1) = \frac{P(X_2 > y_2 | X_1 > y_1)}{E(X_2 | X_1 > y_1)}. \tag{6.2}$$

It can be easily seen that the marginal distributions of Y_1 and Y_2 coincide with the equilibrium distributions of X_1 and X_2, respectively. Following the above scheme, one can easily define equilibrium distributions in higher dimensions.

We now present two examples.

Example 6.1: Gumbel's bivariate exponential distribution with survival function

$$\bar{F}(x_1, x_2) = \exp(-ax_1 - bx_2 - cx_1 x_2), x_1, x_2 > 0, a, b, c > 0.$$

It is well known that for this distribution, the conditional distribution of X_1 given $X_2 > x_2$ and the conditional distribution of X_2 given $X_1 > x_1$ are exponential. Also, the marginal distributions of X_1 and X_2 are exponentials. However, the conditional distribution of X_1 given $X_2 = x_2$ and that of X_2 given $X_1 = x_1$ are not exponential. For other properties of this model, see Arnold [4].

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It can be easily verified that in this case

$$g(y_1|Y_2 > y_2) = (a + cy_2)e^{-ay_1 - cy_1 y_2}, \qquad y_1 > 0, y_2 > 0.$$

This shows that the distribution of Y_1 given $Y_2 > y_2$ is exponential with parameter $a + cy_2$. Similarly, the distribution of Y_2 given $Y_1 > y_1$ is exponential with parameter $b + cy_1$.

We will now obtain the equilibrium distributions in the case of series and parallel systems.

Series System: In this case, we observe $T_1 = \text{Min}(X_1, X_2)$ and the survival function is given by

$$\bar{F}(t) = \exp(-at - bt - ct^2), \qquad t > 0,$$

and

$$E(T_1) = \frac{1}{2}\, c^{-1/2} e^{((a+b)^2/4c)\Gamma}\left(\frac{1}{2}, \frac{(a+b)^2}{4c}\right),$$

where

$$\Gamma(m, x) = \int_x^\infty t^{m-1} e^{-t}\, dt;$$

see Gupta, Tajdari, and Bresinsky [20]. The above two expressions yield the p.d.f. of the equilibrium distribution in the case of a series system.

Parallel System: In this case, we observe $T_2 = \text{Max}(X_1, X_2)$ and the survival function is given by

$$\bar{F}(t) = e^{-at} + e^{-bt} - e^{-at-bt-ct^2}, \qquad t > 0,$$

and

$$E(T_2) = \frac{a+b}{ab} - E(T_1).$$

The above two expressions yield the equilibrium distribution in the case of a parallel system.

Example 6.2: Generalized bivariate Pareto distribution with survival function

$$\bar{F}(x_1, x_2) = \frac{1}{[1 + a(x_1 + x_2) + a^2 b x_1 x_2]^c},$$

$$x_1, x_2 > 0,\ a, c > 0,\ 0 \le b \le 1 + c.$$

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The above survival function was obtained as a mixture of independent exponential random variables; see Hutchinson and Lee [24, p. 118].

It can be verified that, in this case,

$$g(y_1|Y_2 > y_2) = \frac{\bar{F}(y_1, y_2)}{P(X_2 > y_2)E(X_1|X_2 > y_2)}$$

$$= \left[\frac{1 + ay_2}{1 + a(y_1 + y_2) + a^2 b y_1 y_2}\right]^c \frac{1}{E(X_1|X_2 > y_2)},$$

where

$$E(X_1|X_2 > y_2) = \int_0^\infty \left[\frac{1 + ay_2}{1 + a(y_1 + y_2) + a^2 b y_1 y_2}\right]^c dy_1$$

$$= (c-1)(a + a^2 b y_2) \frac{(1 + ay_2)^{c-1}}{[1 + a(y_1 + y_2) + a^2 b y_1 y_2]^c}.$$

We will now obtain the equilibrium distributions in the case of series and parralel systems.

Series System: In this case, we observe $T_1 = \text{Min}(X_1, X_2)$ and the survival function is given by

$$\bar{F}(t) = \frac{1}{(1 + 2at + a^2 b t^2)^c}, \qquad t > 0,$$

and

$$E(T_1) = \frac{2^{c-1/2}}{a\sqrt{b}} \left(\frac{1}{b} - 1\right)^{(1-2c)/4} \Gamma\left(c + \frac{1}{2}\right) \beta(2c-1, 1) P_{c-3/2}^{1/2-c}\left(\frac{1}{\sqrt{b}}\right),$$

$$c > \frac{1}{2},$$

where

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu + 1, 1 - \mu, \frac{1-z}{2}\right)$$

is the associated Legendre function of the first kind; see Gupta et al. [20].

The above two expressions yield the p.d.f. of the equilibrium distribution in the case of a series system.

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Parallel System: In this case, we observe $T_2 = \text{Max}(X_1, X_2)$ with survival function

$$\bar{F}(t) = \frac{2}{(1+ab)^c} - \frac{1}{(1+2at+a^2bt^2)^c}$$

and

$$E(T_2) = \frac{2}{a(c-1)} - E(T_1).$$

The above two expressions yield the distribution of the equilibrium distribution in the case of a parallel system.

7. SOME CONCLUSIONS AND COMMENTS

In this article we have presented the equilibrium distribution including its higher derivates from a reliability point of view. Since the equilibrium distribution is intimately connected with the original distribution, it provides a useful tool in studying several important properties of the original distribution. Aging properties of equilibrium distributions and their stochastic orderings with the original distributions are explored. The relation between the equilibrium distribution of a series system and a series system of equilibrium distributions, consisting of two components, is investigated. Extension to bivariate equilibrium distributions is provided along with some examples. It is hoped that this article will be useful to theoreticians and practitioners in studying the applications of equilibrium distribution in reliability.

Acknowledgment

The author is thankful to the Editor Sheldon Ross for some useful comments that enhanced the presentation.

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