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SEMI-STATIC VARIANCE-OPTIMAL HEDGING IN STOCHASTIC VOLATILITY MODELS WITH FOURIER REPRESENTATION

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Abstract

We introduce variance-optimal semi-static hedging strategies for a given contingent claim. To obtain a tractable formula for the expected squared hedging error and the optimal hedging strategy we use a Fourier approach in a multidimensional factor model. We apply the theory to set up a variance-optimal semi-static hedging strategy for a variance swap in the Heston model, which is affine, in the 3/2 model, which is not, and in a market model including jumps.

Keywords: Variance-optimal and semi-static hedging; volatility model; Fourier representation; square-integrable martingale

2010 Mathematics Subject Classification: Primary 60H05; 60H30; 60G46
Secondary 91G20; 91G80

1. Introduction

Variance-optimal hedging was introduced in [12] as a tractable method of hedging contingent claims in incomplete markets (see [23] for an overview). The main idea of variance-optimal hedging is to find a self-financing trading strategy ϑ^0, for a given claim η^0, which minimizes the (risk-neutral) variance of the residual hedging error at a terminal time $T > 0$. As shown in [12], the solution of this optimization problem is given by the so-called Galtchouk–Kunita–Watanabe (GKW) decomposition of η^0 with respect to the underlying stock S. We assume that we are working under a risk-neutral probability measure P and denote by E the expectation with respect to P: The stock S is a square-integrable martingale with respect to P. Then, the GKW decomposition of η^0 with respect to S takes the form

$$\eta^0 = \mathrm{E}[\eta^0] + \int_0^T \vartheta_s^0 \, \mathrm{d}S_s + L_T^0,$$

where L^0 is a square-integrable martingale orthogonal to S. For applications, it is important to have tractable formulas for the variance-optimal strategy ϑ^0 and for the expected squared hedging error $\varepsilon^2 := \mathrm{E}[(L_T^0)^2]$. To this end, several authors have combined variance-optimal hedging with *Fourier methods*. This idea was first pursued in [15], where in an exponential Lévy market the expected hedging error was explicitly computed for contingent claims with

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an integrable and invertible Laplace transform. This method has been further investigated in [18], where the underlying market model is a stochastic volatility model described by an *affine semimartingale*.

In this paper we extend the results of [15, 18] in two directions: First, we consider a very general setting of semimartingale factor models that is not limited to processes with independent increments or with affine structure. Second, in addition to classic variance-optimal hedging, we also consider the variance-optimal *semi-static* hedging problem, i.e. we combine dynamic trading in the underlying S with static (i.e. buy-and-hold) positions in a finite number of *given* contingent claims $\boldsymbol{\eta} := (\eta^1, \dots, \eta^d)^\top$, e.g. European puts and calls. Semi-static hedging strategies have been considered for the hedging of barrier options (see [6]), in model-free hedging approaches based on martingale optimal transport (see [3, 4]), and for the semi-static replication of variance swaps by Neuberger's formula (see [21]).

As shown in [9] and summarized in Section 2 below, the semi-static hedging problem can be solved under a variance-optimality criterion when the *covariances* $\varepsilon_{ij} := \mathrm{Cov}[L_T^i, L_T^j]$ of the residuals in the GKW decompositions of all supplementary claims $\boldsymbol{\eta}$ are also known. This leads to a natural extension of the questions investigated in [15, 18]: If the model for S allows for explicit calculation of the moment-generating function of $X_T = \log S_T$ (such as Lévy models, the Heston model, or the 3/2 model), and each of the supplementary claims $\eta^1, \dots, \eta^d$ has a Fourier representation (such as European puts and calls), how can we compute the quantities ε_{ij}? In our main results, Theorems 3.2, 4.1, 4.2, and 4.3, we provide answers to this question and show that the covariances ε_{ij} and the optimal strategies can be recovered by Fourier-type integrals, extending the results of [15, 18].

We stress that for classical variance-optimal hedging the relevant quantity is $A := \mathrm{E}[(L_T^0)^2]$. For the variance-optimal semi-static hedging problem, we additionally need all the covariances $B = (B^j)_{j=1,\dots,d}$, $B^j := \mathrm{Cov}[L_T^0, L_T^j]$ and $C = (C^{ij})_{i,j=1,\dots,d}$, $C^{ij} := \varepsilon_{ij}$. Furthermore, we do not assume that η^0 has a Fourier representation. Rather, we require that the supplementary claims $\boldsymbol{\eta}$ do. The computation of the covariances B and C is completely new, even in the well-studied Heston model.

We now give a brief overview of the companion paper [9], where we develop numerical results connected with the variance-optimal semi-static hedging problem in the Heston model: We fix a d-dimensional basket of supplementary claims $\boldsymbol{\eta}$ of European put and call options and set up a variance-optimal semi-static strategy for the variance swap. The quantities A, B, and C, whose analytic expressions are obtained in the present paper, are numerically computed in [9]. Furthermore, as we shall see below, the variance-optimal semi-static problem can be formulated by two nested minimization problems. The outer minimization problem is interesting because it leads to questions related to optimal subset selection: Given $\boldsymbol{\eta}$, which subset of supplementary claims $\boldsymbol{\eta}_k := (\eta^{i_1}, \dots, \eta^{i_k})^\top$ of cardinality $k \leq d$ minimizes the squared hedging error? We call this problem *sparse* variance-optimal semi-static hedging and we study it in [9] also comparing different optimal-subset-selection algorithms (e.g. greedy forward selection, Leaps and Bounds, and Lasso).

This paper has the following structure: In Section 2 we collect some notions about stochastic analysis and discuss semi-static variance-optimal strategies. Section 3 is devoted to the study of the GKW decomposition for some square-integrable martingales that play a fundamental role in our setting. In Section 4, we discuss certain problems related to stochastic analysis of processes depending on a parameter and Fourier methods in a general multidimensional factor model. In Section 5 we apply the results of the previous sections to the Heston model (which is an affine model), to the 3/2 model (which is non-affine), and to an exponential Lévy model with jumps.

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2. Basic tools and motivation

Let $[0, T]$ be a fixed time interval, $T > 0$. By $(\Omega, \mathcal{F}, \mathrm{P})$ we denote a complete probability space and by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration satisfying the usual conditions. We assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F} = \mathcal{F}_T$.

We denote by $\mathcal{H}^2_{\mathbb{C}}$ (resp. $\mathcal{H}^2$) the Hilbert space of càdlàg complex-valued (resp. real-valued) $\mathbb{F}$-adapted square-integrable martingales.

If $X, Y \in \mathcal{H}^2$, then $\langle X, Y\rangle$ denotes the predictable covariation of X and Y. Notice that X and Y are orthogonal if and only if $X_0 Y_0 = 0$ and $\langle X, Y\rangle = 0$.

For $Z^j := X^j + \mathrm{i}Y^j \in \mathcal{H}^2_{\mathbb{C}}$, $j = 1, 2$, we define

$$\langle Z^1, Z^2\rangle := (\langle X^1, X^2\rangle - \langle Y^1, Y^2\rangle) + \mathrm{i}(\langle X^1, Y^2\rangle + \langle Y^1, X^2\rangle).$$

For a real- or complex-valued process A of finite variation, we denote by $\mathrm{Tv}(A)$ the total-variation process of A. If H is a (real- or complex-valued) measurable process and A is of finite variation, we denote by $\int_0^{\cdot} H_s \,\mathrm{d}A_s$ the (Lebesgue–Stieltjes) integral process, if it exists. Recall that $\int_0^{\cdot} H_s \,\mathrm{d}A_s$ is of finite variation if and only if $\int_0^t |H_s(\omega)| \,\mathrm{dTv}(A)_s(\omega) < +\infty$ for every t and ω. Furthermore, for such an H and A real-valued, we have $\mathrm{Tv}(\int_0^{\cdot} H_s \,\mathrm{d}A_s) = \int_0^{\cdot} |H_s| \,\mathrm{dTv}(A)_s$.

Notice that for $Z^j := X^j + \mathrm{i}Y^j \in \mathcal{H}^2_{\mathbb{C}}$, $j = 1, 2$, by the Kunita–Watanabe inequality for real-valued martingales, we immediately get

$$\mathrm{E}[\mathrm{Tv}(\langle Z^1, Z^2\rangle)_t] \le 2\mathrm{E}[\langle Z^1, \overline{Z}^1\rangle_t]^{1/2}\mathrm{E}[\langle Z^2, \overline{Z}^2\rangle_t]^{1/2},$$

where, for $Z \in \mathcal{H}^2_{\mathbb{C}}$, we denote by $\overline{Z} \in \mathcal{H}^2_{\mathbb{C}}$ the complex conjugate of Z.

For $X \in \mathcal{H}^2_{\mathbb{C}}$, we introduce

$$L^2_{\mathbb{C}}(X) := \{\vartheta \text{ predictable and complex-valued} : \mathrm{E}[\textstyle\int_0^T |\vartheta_s|^2 \,\mathrm{d}\langle X, \overline{X}\rangle_s] < +\infty\},$$

the space of complex-valued integrands for X. For $\vartheta \in \mathrm{L}^2_{\mathbb{C}}(X)$, $\int_0^{\cdot} \vartheta_s \,\mathrm{d}X_s \in \mathcal{H}^2_{\mathbb{C}}$ denotes the stochastic integral process of ϑ with respect to X. By $L^2(X)$ we denote the subspace of $L^2_{\mathbb{C}}(X)$ consisting of *real-valued* elements.

We conclude by recalling the GKW decomposition: Let $X, Y \in \mathcal{H}^2_{\mathbb{C}}$. Then there exist a unique $\vartheta \in L^2_{\mathbb{C}}(X)$ and a unique $L \in \mathcal{H}^2_{\mathbb{C}}$, $L_0 = 0$, such that $\langle X, \overline{L}\rangle = 0$ (and hence $\langle L, \overline{X}\rangle = 0$), and the decomposition, called the GKW decomposition of Y with respect to X,

$$Y = Y_0 + \int_0^{\cdot} \vartheta_s \,\mathrm{d}X_s + L,$$

holds. Notice that $\vartheta = \mathrm{d}\langle Y, \overline{X}\rangle / \mathrm{d}\langle X, \overline{X}\rangle$. We call the orthogonal component L *the residual* in the GKW decomposition of Y with respect to X. For brevity we say that (ϑ, L) is the GKW decomposition of Y with respect to X. Identifying a complex-valued square-integrable and $\mathcal{F}_T$-measurable random variable y with the *unique* (up to indistinguishability) càdlàg martingale $Y \in \mathcal{H}^2_{\mathbb{C}}$ satisfying $Y_t = \mathrm{E}[y \mid \mathcal{F}_t]$ a.s., $t \in [0, T]$, we also speak of the GKW decomposition of y with respect to X.

2.1. Semi-static variance-optimal hedging

Let $S \in \mathcal{H}^2$ be strictly positive and assume that S models the price process of some stock. This is a crucial assumption on the probability measure P: We require that P is already

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an equivalent martingale measure and we do not consider the problem under the historical measure, say Q, with respect to which S is, in general, a non-negative semimartingale. This latter case has been considered, e.g., in [22]. See also [23] and the references therein.

A fixed basket of contingent claims $\boldsymbol{\eta} := (\eta^1, \dots, \eta^d)^\top$, $\eta^j \in L^2(\mathrm{P})$, $j = 1, \dots, d$, is given. We stress that we regard $\mathbb{R}^d$ as a space of column vectors.

Definition 2.1.

(i) A *semi-static* strategy is a pair $(\vartheta, \boldsymbol{\nu}) \in L^2(S) \times \mathbb{R}^d$. A semi-static strategy is called *dynamic* if $\boldsymbol{\nu} = 0$; it is called static if $\vartheta = 0$.

(ii) A semi-static variance-optimal hedging strategy for the square-integrable contingent claim η^0 is a solution of the *semi-static variance-optimal hedging problem* given by

$$\varepsilon^2 = \min_{\boldsymbol{\nu} \in \mathbb{R}^d, \vartheta \in \mathrm{L}^2(S), c \in \mathbb{R}} \mathrm{E}\left[\left(c - \mathrm{E}[\boldsymbol{\nu}^\top \boldsymbol{\eta}] + \int_0^T \vartheta_s \, \mathrm{d}S_s - (\eta^0 - \boldsymbol{\nu}^\top \boldsymbol{\eta})\right)^2\right]. \tag{2.1}$$

In other words, a semi-static strategy is a strategy which, beyond the dynamic (i.e. continuously rebalanced) position in S, also allows a static (i.e. buy-and-hold) position in the fixed basket $\boldsymbol{\eta} = (\eta^1, \dots, \eta^d)^\top$ of contingent claims.

Clearly, (2.1) is a generalization of the classic variance-optimal hedging problem (see, e.g., [12] or [23]), and semi-static variance-optimal strategies typically allow for a reduction of the quadratic hedging error in comparison with classic variance-optimal strategies.

Following [9], the semi-static hedging problem can be split into an inner and an outer optimization problem. That is, it can be written as an *inner problem*,

$$\varepsilon^2(\boldsymbol{\nu}) = \min_{\vartheta \in L^2(S), c \in \mathbb{R}} \mathrm{E}\left[\left(c - \mathrm{E}[\boldsymbol{\nu}^\top \boldsymbol{\eta}] + \int_0^T \vartheta_s \, \mathrm{d}S_s - (\eta^0 - \boldsymbol{\nu}^\top \boldsymbol{\eta})\right)^2\right], \tag{2.2}$$

and an *outer problem*,

$$\varepsilon^2 = \min_{\boldsymbol{\nu} \in \mathbb{R}^d} \varepsilon^2(\boldsymbol{\nu}). \tag{2.3}$$

Now let (ϑ^j, L^j) be the GKW decomposition of η^j with respect to S, $j = 0, \dots, d$. Since (2.2) is a classic variance-optimal hedging problem, the optimal solution $(\vartheta^{\boldsymbol{\nu},*}, c^*)$ is given by the integrand $\vartheta^{\boldsymbol{\nu},*} \in L^2(S)$ appearing in the GKW decomposition of $\eta^0 - \boldsymbol{\nu}^\top \boldsymbol{\eta}$ with respect to S, that is, $\vartheta^{\boldsymbol{\nu},*} = \vartheta^0 - \sum_{j=1}^d \nu_j \vartheta^j$, and $c^* = \mathrm{E}[\eta^0]$.

The outer problem (2.3) is a finite-dimensional quadratic optimization problem, since $\varepsilon^2(\boldsymbol{\nu})$ is of the form

$$\varepsilon^2(\boldsymbol{\nu}) = A - 2\boldsymbol{\nu}^\top B + \boldsymbol{\nu}^\top C \boldsymbol{\nu},$$

where the coefficients A, B, and C are given by

$$A := \mathrm{E}[\langle L^0, L^0 \rangle_T], \quad B^j := \mathrm{E}[\langle L^0, L^j \rangle_T], \quad C^{ij} := \mathrm{E}[\langle L^i, L^j \rangle_T], \quad i, j = 1, \dots, d. \tag{2.4}$$

If C is invertible, then the optimal strategy $(\vartheta^*, \boldsymbol{\nu}^*, c^*)$ for the semi-static variance-optimal hedging problem (2.1) is given by

$$\boldsymbol{\nu}^* = C^{-1} B, \quad \vartheta^* = \vartheta^0 - \sum_{j=1}^d \nu_j^* \vartheta^j, \quad c^* = \mathrm{E}[\eta^0].$$

Hence, to solve the semi-static variance-optimal hedging problem, it is necessary to compute the predictable covariations of the residuals in the GKW decomposition of η^j with

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respect to S, $j = 1, \ldots, d$, and then their expectation. This extends beyond the classic variance-optimal hedging problem, where it is enough to determine A in order to compute the squared hedging error and the optimal strategy.

3. The GKW decomposition in factor models

Let S, $Y^i \in \mathcal{H}^2_{\text{loc}}$ and let (ϑ^i, L^i) be the GKW decomposition of Y^i with respect to S, $i = 1, 2$. In this section we compute ϑ^i, $i = 1, 2$, and the predictable covariation $\langle L^1, L^2 \rangle$. This is motivated by Section 2, where we have seen that these predictable covariations are needed to compute the solution of (2.1).

3.1. Martingale representation

We consider a quasi-left continuous locally square-integrable semimartingale $X = (X^1, \ldots, X^n)^\top$ with state space $(E, \mathcal{E}) := (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ (see [16, Definition II.2.27]). Then, for $j = 1, \ldots, n$, the unidimensional semimartingale X^j admits the unique decomposition

$$X^j = X^j_0 + M^j + A^j, \tag{3.1}$$

where A^j is a continuous adapted process of finite variation and $M^j \in \mathcal{H}^2_{\text{loc}}$ is quasi-left continuous, $j = 1, \ldots, n$. We denote by $X^{j,c}$ the continuous martingale part of X^j. Let $Y \in \mathcal{H}^2_{\text{loc}}$ be of the form

$$Y_t = f(t, X_t), \quad f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n). \tag{3.2}$$

We are going to show that Y has the *representation property relative to* X (see [16, Definition III.4.22]) and also obtain an explicit form of the integrand. To tackle the problem, we pursue the following classical idea: We consider $Y \in \mathcal{H}^2_{\text{loc}}$, which is given by a function $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ of time and X. Then, applying Itô's formula, we show that Y has the representation property relative to X.

Such kinds of martingale representation results have already been considered in the literature, especially for pricing of European options: In [8, Section 3] this result has been shown in a Brownian context, and in [11, Section 3] for the homogeneous Poisson case. In [5, Section 2] the case of a continuous semimartingale is considered. We add a new element to this picture by considering locally square-integrable quasi-left continuous semimartingales.

We denote by μ be the jump measure of X, by ν its predictable compensator, and we put $\overline{\mu} := \mu - \nu$. By $\mathcal{G}^2_{\text{loc}}(\mu)$ we denote the space of predictable integrands W on $[0, T] \times \Omega \times E$ such that the stochastic integral $Z = \int_0^\cdot \int_E W(s, x)\overline{\mu}(\mathrm{d}s, \mathrm{d}x)$ is the unique (up to indistinguishability) purely discontinuous element of $\mathcal{H}^2_{\text{loc}}$ such that $Z_0 = 0$ and $\Delta Z = W(\cdot, \Delta X)1_{\{\Delta X \neq 0\}}$. For details, see [16, Section II.1d].

For $Y \in \mathcal{H}^2_{\text{loc}}$, we denote by $Y^d \in \mathcal{H}^2_{\text{loc}}$ the purely discontinuous part of Y (see [16, Theorem I.4.18]).

Theorem 3.1. *Let $X = (X^1, \ldots, X^n)^\top$ be a quasi-left continuous locally square-integrable semimartingale such that X^j has the canonical decomposition as in (3.1), $j = 1, \ldots, n$. Let μ be the jump measure of X, ν the predictable compensator of μ, and $\overline{\mu} := \mu - \nu$. Let $Y = (Y_t)_{t \in [0,T]} \in \mathcal{H}^2_{\text{loc}}$ be as in (3.2). Then, denoting by $\partial_\cdot$ the partial derivative,*

$$Y = Y_0 + \sum_{j=1}^n \int_0^\cdot \partial_{x_j} f(s, X_{s-})\, \mathrm{d}X^{j,c}_s + \int_0^\cdot \int_E [f(s, x + X_{s-}) - f(s, X_{s-})]\, \overline{\mu}(\mathrm{d}s, \mathrm{d}x). \tag{3.3}$$

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Proof. By Itô's formula and (3.1), we deduce that

$$\begin{aligned} Y - Y_0 &= \sum_{j=1}^{n} \int_0^{\cdot} \partial_{x_j} f(s, X_{s-})\, \mathrm{d}M_s^j + \int_0^{\cdot} \partial_t f(s, X_{s-})\, \mathrm{d}s \\ &\quad + \sum_{j=1}^{n} \int_0^{\cdot} \partial_{x_j} f(s, X_{s-})\, \mathrm{d}A_s^j + \frac{1}{2} \sum_{j,k=1}^{n} \int_0^{\cdot} \partial^2_{x_j x_k} f(s, X_{s-})\, \mathrm{d}\langle X^{j,c}, X^{k,c}\rangle_s \\ &\quad + \sum_{s \le \cdot} \left[\Delta Y_s - \sum_{j=1}^{n} \partial_{x_j} f(s, X_{s-}) \Delta X_s^j \right], \end{aligned} \tag{3.4}$$

where $\partial^2_{\cdot\cdot}$ denotes the second partial derivative. We denote by N the first term and by G the sum of the second, the third, and the fourth terms on the right-hand side of (3.4). Finally, we set $U(t, x) := f(t, x + X_{t-}) - f(t, X_{t-}) - \sum_{j=1}^{n} \partial_{x_j} f(t, X_{t-}) x_j$. The process U belongs to $\mathcal{G}^2_{\mathrm{loc}}(\mu)$. Indeed, because of (3.4) and the continuity of G, $\Delta(Y - N) = U(\cdot, \Delta X)$. Hence, $\sum_{s \le \cdot} U_s^2 \le [Y - N, Y - N]$ and $[Y - N, Y - N]$ is locally integrable because $Y - N \in \mathcal{H}^2_{\mathrm{loc}}$. Thus, $Z := \int_0^{\cdot} \int_E U(s, x)\, \overline{\mu}(\mathrm{d}s, \mathrm{d}x) \in \mathcal{H}^2_{\mathrm{loc}}$. Furthermore, because of (3.4), we have $J := \sum_{s \le \cdot} U(s, \Delta X_s) + G \in \mathcal{H}^2_{\mathrm{loc}}$ and J is of finite variation. Hence, J is a purely discontinuous local martingale and $\Delta Z = \Delta J$. From [16, Corollary I.4.19], Z and J are indistinguishable and (3.4) becomes

$$Y = Y_0 + \sum_{j=1}^{n} \int_0^{\cdot} \partial_{x_j} f(s, X_{s-})\, \mathrm{d}M_s^j + \int_0^{\cdot} \int_E U(s, x)\, \overline{\mu}(\mathrm{d}s, \mathrm{d}x).$$

Since, for $j = 1, \ldots, n$, we have $M^j = X^{j,c} + (M^j)^d$ and $(M^j)^d = x_j * \overline{\mu}$, we get the desired result by [16, Proposition II.1.30 (b)] and by linearity. □

We now assume that $S \in \mathcal{H}^2_{\mathrm{loc}}$ is of the form $S = \mathrm{e}^{X^1}$. Denoting by $\check{S}$ the stochastic logarithm of S, we have

$$S = S_0 + \int_0^{\cdot} S_{s-}\, \mathrm{d}\check{S}_s. \tag{3.5}$$

Applying Theorem 3.1 to $f(t, x) = \mathrm{e}^{x_1}$ yields

$$\check{S} := \check{S}_0 + (X^{1,c} + \check{S}^d), \quad \check{S}^d := \int_0^{\cdot} \int_E (\mathrm{e}^{x_1} - 1)\, \overline{\mu}(\mathrm{d}s, \mathrm{d}x) \tag{3.6}$$

and, since $x \mapsto (\mathrm{e}^{x_1} - 1) \in \mathcal{G}^2_{\mathrm{loc}}(\mu)$, we have $\check{S} \in \mathcal{H}^2_{\mathrm{loc}}$. If Y is as in (3.2), then Theorem 3.1, (3.5), (3.6), and [16, Theorem II.1.33(a)] imply that

$$\begin{aligned} \langle S, Y\rangle &= \sum_{j=1}^{n} \int_0^{\cdot} S_{s-} \partial_{x_j} f(s, X_{s-})\, \mathrm{d}\langle X^{j,c}, X^{1,c}\rangle_s \\ &\quad + \int_0^{\cdot} \int_E (\mathrm{e}^{x_1} - 1) S_{s-} W(s, x) \nu(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{3.7}$$

Theorem 3.2. *Let $X = (X^1, \ldots, X^2)^\top$ be a quasi-left continuous locally square-integrable semimartingale such that X^j has the canonical decomposition as in (3.1), $j = 1, \ldots, d$. Let μ be the jump measure of X, ν the predictable compensator of μ, and $\overline{\mu} := \mu - \nu$. Let $S = \mathrm{e}^{X^1} \in$*

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$\mathcal{H}^2_{\mathrm{loc}}$. *Let* $Y^i \in \mathcal{H}^2_{\mathrm{loc}}$ *be such that* $Y^i_t = f^i(t, X_t)$ *as in* (3.2) *and put* $Y^{i,d} := (Y^i)^d$, $i = 1, 2$. *Then the GKW decomposition* (ϑ^i, L^i) *of* Y^i *with respect to* S *is given by* $L^i = Y^i - Y^i_0 - \int_0^\cdot \vartheta^i_s \, \mathrm{d}S_s$ *and, for all* $t \in [0, T]$,

$$\vartheta^i_t = \frac{1}{S_{t-}} \left(\sum_{j=1}^n \partial_{x_j} f^i(t, X_{t-}) \frac{\mathrm{d}\langle X^{1,c}, X^{j,c} \rangle_t}{\mathrm{d}\langle \check{S}, \check{S} \rangle_t} + \frac{\mathrm{d}\langle \check{S}^d, Y^{i,d} \rangle_t}{\mathrm{d}\langle \check{S}, \check{S} \rangle_t} \right), \quad i = 1, 2. \tag{3.8}$$

Moreover, setting

$$Q^{jk} := \langle X^{j,c}, X^{k,c} \rangle - \int_0^\cdot \frac{\mathrm{d}\langle X^{1,c}, X^{j,c} \rangle_s}{\mathrm{d}\langle \check{S}, \check{S} \rangle_s} \, \mathrm{d}\langle X^{1,c}, X^{k,c} \rangle_s,$$

we have

$$\begin{aligned} \mathrm{d}\langle L^1, L^2 \rangle_t = & \sum_{j=1}^n \sum_{k=1}^n \partial_{x_j} f^1(t, X_{t-}) \partial_{x_k} f^2(t, X_{t-}) \, \mathrm{d}Q^{jk}_t \\ & - \sum_{j=1}^n \partial_{x_j} f^1(t, X_{t-}) \frac{\mathrm{d}\langle \check{S}^d, Y^{2,d} \rangle_t}{\mathrm{d}\langle \check{S}, \check{S} \rangle_t} \, \mathrm{d}\langle X^{1,c}, X^{j,c} \rangle_t \\ & - \sum_{j=1}^n \partial_{x_j} f^2(t, X_{t-}) \frac{\mathrm{d}\langle \check{S}^d, Y^{1,d} \rangle_t}{\mathrm{d}\langle \check{S}, \check{S} \rangle_t} \, \mathrm{d}\langle X^{1,c}, X^{j,c} \rangle_t \\ & + \mathrm{d}\langle Y^{1,d}, Y^{2,d} \rangle_t - \frac{\mathrm{d}\langle \check{S}^d, Y^{1,d} \rangle_t}{\mathrm{d}\langle \check{S}, \check{S} \rangle_t} \, \mathrm{d}\langle \check{S}^d, Y^{2,d} \rangle_t, \end{aligned} \tag{3.9}$$

where, denoting $W^i(t, x) := f^i(t, x + X_{t-}) - f^i(t, X_{t-})$, $i = 1, 2$, *we have*

$$\langle \check{S}, \check{S} \rangle = \langle X^{1,c}, X^{1,c} \rangle + \int_0^\cdot \int_E (\mathrm{e}^{x_1} - 1)^2 \nu(\mathrm{d}s, \mathrm{d}x), \tag{3.10}$$

$$\langle \check{S}^d, \check{S}^d \rangle = \int_0^\cdot \int_E (\mathrm{e}^{x_1} - 1)^2 \, \nu(\mathrm{d}s, \mathrm{d}x), \tag{3.11}$$

$$\langle \check{S}^d, Y^{i,d} \rangle = \int_0^\cdot \int_E (\mathrm{e}^{x_1} - 1) W^i(s, x) \, \nu(\mathrm{d}s, \mathrm{d}x), \tag{3.12}$$

$$\langle Y^{1,d}, Y^{2,d} \rangle = \int_0^\cdot \int_E W^1(s, x) W^2(s, x) \, \nu(\mathrm{d}s, \mathrm{d}x). \tag{3.13}$$

Proof. Clearly, (3.10) and (3.11) follow from (3.6); (3.12) follows from (3.7); and (3.13) is a consequence of Theorem 3.1. Formula (3.8) follows by Theorem 3.1 and (3.6), because of $\vartheta^i = \mathrm{d}\langle S, Y^i \rangle / \mathrm{d}\langle S, S \rangle$, $i = 1, 2$, and of $\langle S, S \rangle = S_-^2 \cdot \langle \check{S}, \check{S} \rangle$. To show (3.9) we consider the identity $\langle L^1, L^2 \rangle = \langle Y^1, Y^2 \rangle - \int_0^\cdot \vartheta^1_s \vartheta^2_s \, \mathrm{d}\langle S, S \rangle_s$, compute $\langle Y^1, Y^2 \rangle$ from Theorem 3.1, and use (3.8) and (3.10). The proof is complete. □

Let (B, C, ν) denote the semimartingale characteristics of X (cf. [16, II.§2a]) relative to the truncation function $h = (h^1, \ldots, h^n)^\top$, where $h^j(x) = x_j 1_{\{|x_j| \le 1\}}$, $j = 1, \ldots, n$. Let us assume that (B, C, ν) are absolutely continuous with respect to the Lebesgue measure on $[0, T]$ (which we denote by Leb.): By (b, c, F) we denote the *predictable differential characteristics* of X. We stress that $F_t(\omega, \mathrm{d}x)$ is a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and $C^{jk} = \int_0^\cdot c^{jk}_s \, \mathrm{d}s$, $j, k = 1, \ldots, n$. We get the following corollary to Theorem 3.2.

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Corollary 3.1. *Let (b, c, F) be the predictable differential characteristics of X.*
(i) *Setting $\xi_t := c_t^{11} + \int_E (\mathrm{e}^{x_1} - 1)^2 F_t(\mathrm{d}x)$, $t \in [0, T]$, we get,* Leb.*-a.e.,*

$$\vartheta_t^i = \frac{1}{S_{t-}\xi_t}\left(\sum_{j=1}^n \partial_{x_j} f^i(t, X_{t-}) c_t^{1j} + \int_E (\mathrm{e}^{x_1} - 1) W^i(t, x) F_t(\mathrm{d}x) \right), \quad i = 1, 2, \tag{3.14}$$

and (3.9) *becomes*

$$\begin{aligned} \frac{\mathrm{d}\langle L^1, L^2\rangle_t}{\mathrm{d}t} &= \sum_{j=1}^n \sum_{k=1}^n \partial_{x_j} f^1(t, X_{t-}) \partial_{x_k} f^2(t, X_{t-}) \left(c_t^{jk} - \frac{c_t^{1j}}{\xi_t} c_t^{1k} \right) \\ &\quad - \frac{1}{\xi_t} \sum_{j=1}^n \partial_{x_j} f^1(t, X_{t-}) c_t^{j1} \int_E (\mathrm{e}^{x_1} - 1) W^2(t, x) F_t(\mathrm{d}x) \\ &\quad - \frac{1}{\xi_t} \sum_{j=1}^n \partial_{x_j} f^2(t, X_{t-}) c_t^{j1} \int_E (\mathrm{e}^{x_1} - 1) W^1(t, x) F_t(\mathrm{d}x) \\ &\quad + \int_E W^1(t, x) W^2(t, x) F_t(\mathrm{d}x) \\ &\quad - \frac{1}{\xi_t} \left(\int_E (\mathrm{e}^{x_1} - 1) W^1(t, x) F_t(\mathrm{d}x) \right) \left(\int_E (\mathrm{e}^{x_1} - 1) W^2(t, x) F_t(\mathrm{d}x) \right). \end{aligned} \tag{3.15}$$

(ii) *If, furthermore, X is continuous, then*

$$\vartheta_t^i = \frac{1}{S_t c_t^{11}} \sum_{j=1}^n \partial_{x_j} f^i(t, X_t) c_t^{1j}, \quad i = 1, 2,$$

and

$$\frac{\mathrm{d}\langle L^1, L^2\rangle_t}{\mathrm{d}t} = \sum_{j=2}^n \sum_{k=2}^n \partial_{x_j} f^1(t, X_t) \partial_{x_k} f^2(t, X_t) \left(c_t^{jk} - \frac{c_t^{1j}}{c_t^{11}} c_t^{1k} \right). \tag{3.16}$$

We remark that in the case of absolutely continuous semimartingale characteristics as described in Corollary 3.1, by (3.5) and (3.6) we get $\langle S, S\rangle = \int_0^{\cdot} S_{s-}^2 \xi_s \, \mathrm{d}s$.

4. Models with Fourier representation

In this section we combine variance-optimal semi-static hedging with Fourier methods. We do not assume a special stochastic volatility model (in [18] affine models were considered). This requires some technical considerations about stochastic processes depending on a parameter. A seminal paper about this topic is [24].

4.1. Complex-valued processes depending on a parameter

We denote by $\mathcal{S} \subseteq \mathbb{C}^n$ the space of parameters and consider the Borel σ-algebra $\mathcal{B}(\mathcal{S})$. On $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ a finite complex-valued measure ζ is given. We recall that with ζ we can associate the *positive* measure $|\zeta|$, called the *total variation* of ζ. Moreover, $\mathrm{d}\zeta = h \, \mathrm{d}|\zeta|$, where h is a complex-valued function such that $|h| = 1$. So, $L^1(\zeta) = L^1(|\zeta|)$.

We set $\widetilde{\Omega} := \Omega \times [0, T] \times \mathcal{S}$. If $\mathcal{O}$ (resp. $\mathcal{P}$) is the optional (resp. predictable) σ-algebra, then we set $\widetilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{B}(\mathcal{S})$ (resp. $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathcal{S})$).

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We denote by $\widetilde{U}$ a function on $\widetilde{\Omega}$ and we say that $(t, \omega, z) \mapsto \widetilde{U}(\omega, z)_t$ is an optional (resp. a predictable) function if it is $\widetilde{\mathcal{O}}$-measurable ($\widetilde{\mathcal{P}}$-measurable). Sometimes, to specify the σ-algebra $\mathcal{B}(\mathcal{S})$, we say that $\widetilde{U}$ is an optional (resp. predictable) function with respect to $\mathcal{S}$. We denote by Leb. the Lebesgue measure on $[0, T]$.

Convention. *We assume that all quantities explicitly depending on $z \in \mathcal{S}$ are complex-valued.*

Lemma 4.1. *Let $\widetilde{U}$ be an optional function such that $\widetilde{U}(z)$ is a complex-valued square-integrable martingale (non-necessarily càdlàg) for every $z \in \mathcal{S}$, and furthermore assume that $\sup_{z\in\mathcal{S}} \mathrm{E}[|\widetilde{U}(z)_t|^2] < +\infty$, for every $t \in [0, T]$. Then the process $\widehat{U} := (\widehat{U}_t)_{t\in[0,T]}$, $\widehat{U}_t := \int_{\mathcal{S}} \widetilde{U}(z)_t \zeta(\mathrm{d}z)$, is a (non-necessarily càdlàg) complex-valued square-integrable martingale. If, additionally, $t \mapsto \widetilde{U}(\omega, z)_t$ is càdlàg, then $\widehat{U}$ is a.s. càdlàg.*

Proof. We observe that Tonelli's theorem yields

$$\mathrm{E}\left[\int_{\mathcal{S}} |\widetilde{U}(z)_t|^2 |\zeta|(\mathrm{d}z)\right] \le |\zeta|(\mathcal{S}) \sup_{z\in\mathcal{S}} \mathrm{E}[|\widetilde{U}(z)_t|^2] < +\infty, \quad t \in [0, T].$$

So, $z \mapsto \widetilde{U}(\omega, z)_t \in L^2(\zeta)$ a.s., $t \in [0, T]$, and $\widehat{U}$ is adapted, $(\omega, z) \mapsto \widetilde{U}(\omega, z)_t$ being $\mathcal{F}_t \otimes \mathcal{B}(\mathcal{S})$-measurable, for every $t \in [0, T]$. Fubini's theorem now yields that $\widehat{U}$ is a complex-valued square-integrable martingale (non-necessarily càdlàg). If $t \mapsto \widetilde{U}(\omega, z)_t$ is càdlàg, then $(s, \omega, z) \mapsto \sup_{t\in[0,s]} |\widetilde{U}(\omega, z)_t|$ is optional and Tonelli's theorem and Doob's inequality yield $\mathrm{E}[\int_{\mathcal{S}} \sup_{t\in[0,T]} |\widetilde{U}(z)_t|^2 |\zeta|(\mathrm{d}z)] < +\infty$. Therefore, we have $|\widetilde{U}(z)_t| \le \sup_{t\in[0,T]} |\widetilde{U}(z)_t| \in L^2(\zeta)$ a.s. Hence, by dominated convergence, $\widehat{U}$ is a.s. càdlàg. The proof is complete. □

To compute the point brackets of square-integrable martingales with a representation as in Lemma 4.1, we shall use the next proposition.

Proposition 4.1. *Let $\widetilde{V}$ be a function on $\widetilde{\Omega}$ such that $\widetilde{V}(z) \in \mathcal{H}^2_{\mathbb{C}}$, for all $z \in \mathcal{S}$, and $\sup_{z\in\mathcal{S}} \mathrm{E}[|\widetilde{V}(z)_T|^2] < +\infty$. Let $Z \in \mathcal{H}^2_{\mathbb{C}}$ and let $(t, \omega, z) \mapsto \langle Z, \widetilde{V}(z)\rangle_t(\omega)$ be a predictable function.*

(i) *Then the function $(t, \omega, z) \mapsto \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_t(\omega)$ is predictable and*

$$\mathrm{E}\left[\int_{\mathcal{S}} \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_T |\zeta|(\mathrm{d}z)\right] < +\infty. \tag{4.1}$$

Moreover, the process $D = (D_t)_{t\in[0,T]}$, $D_t := \int_{\mathcal{S}} \langle Z, \widetilde{V}(z)\rangle_t \zeta(\mathrm{d}z)$, is well defined for all $t \in [0, T]$ a.s. Furthermore, D is càdlàg, predictable, of finite variation, and $D_0 = 0$.

(ii) *Let $\widetilde{U}$ be an optional function and $\widetilde{U}(z)$ be a (non-necessarily càdlàg) version of $\widetilde{V}(z)$ for every $z \in \mathcal{S}$. Then $\sup_{z\in\mathcal{S}} \mathrm{E}[|\widetilde{U}(z)_t|^2] < +\infty$, $t \in [0, T]$, and the process $\widehat{U} = (\widehat{U}_t)_{t\in[0,T]}$, $\widehat{U}_t := \int_{\mathcal{S}} \widetilde{U}(z)_t \zeta(\mathrm{d}z)$, is a complex-valued square-integrable martingale. Let U^* be a càdlàg version of $\widehat{U}$. Then*

$$\langle Z, U^*\rangle = \int_{\mathcal{S}} \langle Z, \widetilde{V}(z)\rangle \zeta(\mathrm{d}z). \tag{4.2}$$

(iii) *If $\widetilde{K}$ is a predictable function such that $\int_0^T |\widetilde{K}(\omega, z)_s| \,\mathrm{d}s < +\infty$, $(\omega, z) \in \Omega \times \mathcal{S}$, and*

$$\langle Z, \widetilde{V}(z)\rangle_t(\omega) = \int_0^t \widetilde{K}(\omega, z)_s \,\mathrm{d}s,$$

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then $(t, \omega, z) \mapsto \langle Z, \widetilde{V}(z)\rangle_t(\omega)$ *is a predictable function,* $\widetilde{K} \in L^1(\widetilde{\mathcal{P}}, \mathrm{P} \otimes \mathrm{Leb.} \otimes \zeta)$*, and*

$$\langle Z, U^*\rangle = \int_{\mathcal{S}} \int_0^{\cdot} \widetilde{K}(z)_s \, \mathrm{d}s \, \zeta(\mathrm{d}z) = \int_0^{\cdot} \int_{\mathcal{S}} \widetilde{K}(z)_s \zeta(\mathrm{d}z) \, \mathrm{d}s \tag{4.3}$$

holds. Furthermore, $\mathrm{E}[\langle Z, U^*\rangle] = \int_0^{\cdot} \int_{\mathcal{S}} \mathrm{E}[\widetilde{K}(z)_s]\zeta(\mathrm{d}z) \, \mathrm{d}s$.

Proof. We first prove (i). The predictability of $(t, \omega, z) \mapsto \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_t(\omega)$ follows from the predictability of $(t, \omega, z) \mapsto \langle Z, \widetilde{V}(z)\rangle_t(\omega)$ and from

$$\mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_t = \lim_{n \to +\infty} \sum_{k=1}^{n} |\langle Z, \widetilde{V}(z)\rangle_{tk/n} - \langle Z, \widetilde{V}(z)\rangle_{t(k-1)/n}|.$$

To prove (4.1), we observe that Tonelli's theorem, the Kunita–Watanabe inequality, and the relation $\mathrm{E}[\langle X, \overline{X}\rangle_t] \le \mathrm{E}[|X_t|^2]$, $t \ge 0$, valid for every $X \in \mathcal{H}^2_{\mathbb{C}}$ ($\overline{X}$ denoting the complex conjugate of X), imply that

$$\mathrm{E}\left[\int_{\mathcal{S}} \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_T |\zeta|(\mathrm{d}z)\right] \le 2\mathrm{E}[|Z_T|^2]^{1/2} \Big(\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{V}(z)_T|^2] \Big)^{1/2} |\zeta|(\mathcal{S}) < +\infty.$$

This proves (4.1). From (4.1), since $\sup_{t \in [0,T]} |\langle Z, \widetilde{V}(z)\rangle_t| \le \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_T$, we get $z \mapsto \langle Z, \widetilde{V}(z)\rangle_t(\omega) \in L^1(\zeta)$ for all $t \in [0, T]$ a.s. So, the process D is defined for every $t \in [0, T]$ a.s., and D is predictable. Since $(t, \omega, z) \mapsto \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_t(\omega) \in L^1(\zeta)$ a.s., by dominated convergence we see that D is a.s. càdlàg. From the definition of the total variation for $t \mapsto D_t(\omega)$, by monotone convergence we obtain that D is of finite variation. Since $D_0 = 0$ holds, this concludes the proof of (i). We now verify (ii). Since $\widetilde{U}(z)$ is a version of $\widetilde{V}(z)$ for every z, we have $\mathrm{E}[|\widetilde{U}(z)_t|^2] = \mathrm{E}[|\widetilde{V}(z)_t|^2]$ for every $t \in [0, T]$ and every $z \in \mathcal{S}$. Hence, by the assumptions on $\widetilde{V}$, we get $\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{U}(z)_t|^2] < +\infty$ for every $t \in [0, T]$. By Lemma 4.1, the process $\widehat{U} = \int_{\mathcal{S}} \widetilde{U}(z)\zeta(\mathrm{d}z)$ is a complex-valued square-integrable martingale. We denote by U^* a càdlàg version of $\widehat{U}$. Hence, $U^*_t = \int_{\mathcal{S}} \widetilde{U}(z)_t \zeta(\mathrm{d}z)$ a.s. for every $t \in [0, T]$. Since $\widetilde{U}(z)$ is a version of $\widetilde{V}(z)$, we get that $Z\widetilde{U}(z) - \langle Z, \widetilde{V}(z)\rangle$ is a martingale. Furthermore, by Hölder's inequality, for every $t \in [0, T]$ we have

$$\mathrm{E}\left[\int_{\mathcal{S}} |Z_t \widetilde{U}(z)_t| |\zeta|(\mathrm{d}z)\right] \le \mathrm{E}[|Z_t|^2]^{1/2} \sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{U}(z)_t|^2]^{1/2} |\zeta|(\mathcal{S}) < +\infty.$$

To see (4.2), because of this latter estimate and (4.1), we can apply Fubini's theorem to get that $ZU^* - D$ is a martingale. Indeed, a.s. and for every $s \le t$, we have

$$\mathrm{E}[Z_t U^*_t - D_t | \mathcal{F}_s] = \int_{\mathcal{S}} \mathrm{E}[Z_t \widetilde{U}(z)_t - \langle Z, \widetilde{V}(z)\rangle_t | \mathcal{F}_s] \zeta(\mathrm{d}z) = Z_s U^*_s - D_s.$$

So, (4.2) holds and the proof of (ii) is complete. Finally, we show (iii). The predictability of the function $(t, \omega, z) \mapsto \int_0^t \widetilde{K}(\omega, z)_s \, \mathrm{d}s$ can be verified in a standard way. By (ii) we then deduce that $\langle Z, U^*\rangle = \int_{\mathcal{S}} \int_0^{\cdot} \widetilde{K}(z)_s \, \mathrm{d}s \zeta(\mathrm{d}z)$, which shows the first equality on the right-hand side of (4.3). From Tonelli's theorem, we get

$$\begin{aligned} \int_0^T \int_{\mathcal{S}} |\widetilde{K}(z)_s| |\zeta|(\mathrm{d}z) \, \mathrm{d}s &= \int_{\mathcal{S}} \int_0^T |\widetilde{K}(z)_s| \, \mathrm{d}s |\zeta|(\mathrm{d}z) \\ &= \int_{\mathcal{S}} \mathrm{Tv}(\langle Z, \widetilde{V}(z)\rangle)_T |\zeta|(\mathrm{d}z) < +\infty, \quad \text{a.s.,} \end{aligned} \tag{4.4}$$

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where, in the last estimate, we used the properties of the variation process and (4.1). Hence, taking the expectation in (4.4) and using (4.1) yields $\widetilde{K} \in L^1(\widetilde{\mathcal{P}}, \mathrm{P} \otimes \mathrm{Leb.} \otimes \zeta)$. So, Fubini's theorem implies that $z \mapsto \widetilde{K}(\omega, z)_t$ belongs to $L^1(\zeta)$ $\mathrm{P} \otimes \mathrm{Leb.}$-a.e. and that $(t, \omega) \mapsto \int_{\mathcal{S}} \widetilde{K}(\omega, z)_t \zeta(\mathrm{d}z)$ is predictable, $\mathrm{P} \otimes \mathrm{Leb.}$-integrable, and, finally, for every $t \in [0, T]$, $\mathrm{E}[\int_{\mathcal{S}} \int_0^t \widetilde{K}(z)_s \, \mathrm{d}s \zeta(\mathrm{d}z)] = \int_0^t \int_{\mathcal{S}} \mathrm{E}[\widetilde{K}(z)_s] \zeta(\mathrm{d}z) \, \mathrm{d}s$. From (4.4), we also get $\int_{\mathcal{S}} \int_0^t \widetilde{K}(z)_s \, \mathrm{d}s \zeta(\mathrm{d}z) = \int_0^t \int_{\mathcal{S}} \widetilde{K}(z)_s \zeta(\mathrm{d}z) \, \mathrm{d}s$, $t \in [0, T]$, a.s. The proof is complete. □

4.2. Fourier representation of the GKW decomposition

Let X be a factor process taking values in $\mathbb{R}^n$, and let $R \in \mathbb{R}^n$ be such that $\mathrm{E}[\exp(2R^\top X_T)] < \infty$. We define $\mathcal{S} := \{z \in \mathbb{C}^n : \mathrm{Re}(z) = R\}$. A European option $\eta \in L^2(\mathrm{P})$ is given and its payoff is $\eta = h(X_T)$ for some *real-valued function* h with domain in $\mathbb{R}^n$. We remark that η is a *real-valued* random variable. We assume that the two-sided Laplace transform $\hat{h}$ of h exists in R and that it is integrable on $\mathcal{S}$. Then, h has the following representation:

$$h(x) = \frac{1}{(2\pi \, \mathrm{i})^n} \int_{R - \mathrm{i}\infty}^{R + \mathrm{i}\infty} \exp(z^\top x) \hat{h}(z) \, \mathrm{d}z = \int_{\mathcal{S}} \exp(z^\top x) \zeta(\mathrm{d}z), \tag{4.5}$$

where ζ is the finite complex-valued absolutely continuous measure on $\mathcal{S}$ defined by

$$\zeta(\mathrm{d}z) := \frac{1}{(2\pi \, \mathrm{i})^n} \hat{h}(z) \, \mathrm{d}z. \tag{4.6}$$

Let $H \in \mathcal{H}^2$ be the *unique* (up to indistinguishability) càdlàg *real-valued* martingale satisfying $H_t = \mathrm{E}[\eta \mid \mathcal{F}_t]$ a.s., $t \in [0, T]$.

Assumption 4.1. *There exists an optional function $(t, \omega, z) \mapsto \widetilde{H}(\omega, z)_t$ such that $\widetilde{H}(z)$ has càdlàg paths and $\widetilde{H}(z)_t = \mathrm{E}[\mathrm{e}^{z^\top X_T} | \mathcal{F}_t]$ a.s. for every $t \in [0, T]$, for every $z \in \mathcal{S}$.*

Proposition 4.2. *Let Assumption 4.1 hold. Then $\widetilde{H}(z) \in \mathcal{H}^2_{\mathbb{C}}$, for every $z \in \mathcal{S}$, and $\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{H}(z)_T|^2] < +\infty$. Furthermore, $\widehat{H} \in \mathcal{H}^2_{\mathbb{C}}$, where $\widehat{H}_t := \int_{\mathcal{S}} \widetilde{H}(z)_t \zeta(\mathrm{d}z)$, and $\widehat{H}$ and H are indistinguishable.*

Proof. From Assumption 4.1, $(t, \omega) \mapsto \widetilde{H}(\omega, z)_t$ is càdlàg and hence $\widetilde{H}(z) \in \mathcal{H}^2_{\mathbb{C}}$ for each $z \in \mathcal{S}$, since $\widetilde{H}(z)_t = \mathrm{E}[\mathrm{e}^{z^\top X_T} | \mathcal{F}_t]$ a.s. for every $t \in [0, T]$ and $z \in \mathcal{S}$. Furthermore, $(t, \omega, z) \mapsto \widetilde{H}(\omega, z)_t$ is optional and $|\widetilde{H}(z)_t|^2 \le \widetilde{H}(2R)_t \in L^1(\mathrm{P})$ holds, a.s., for every $t \in [0, T]$, for every $z \in \mathcal{S}$. This yields $\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{H}(z)_t|^2] < +\infty$, $t \in [0, T]$. By Lemma 4.1, we get $\widehat{H} \in \mathcal{H}^2_{\mathbb{C}}$. The identity $H_t = \mathrm{E}[\eta | \mathcal{F}_t] = \widehat{H}_t$ holds a.s. and for every $t \in [0, T]$ by Fubini's theorem. This yields that H and $\widehat{H}$ are modifications and, therefore, $\widehat{H}$ is, in particular, real-valued. Since H and $\widehat{H}$ are a.s. càdlàg and modifications, they are indistinguishable. The proof is complete. □

Let $S \in \mathcal{H}^2$ describe the price process of some traded asset. We stress that S is a *real-valued* process. Theorem 4.1 below shows that the Fourier representation (4.5) of η and the GKW decomposition of η with respect to S can be interchanged.

Theorem 4.1. *Let $S \in \mathcal{H}^2$, let $\eta = h(X_T) \in L^2(\mathrm{P})$ have the Fourier representation given in (4.5) and (4.6), and let (ϑ, L) denote the GKW decomposition of η with respect to S. Let Assumption 4.1 hold and let $(\vartheta(z), L(z))$ be the GKW decomposition of $\widetilde{H}(z)$ with respect to S, for every $z \in \mathcal{S}$, where $\widetilde{H}$ is the optional function in Assumption 4.1. Let, moreover,*

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$(t, \omega, z) \mapsto \widetilde{\vartheta}(\omega, z)_t$ be a predictable function such that $\widetilde{\vartheta}(\omega, z)_t = \vartheta(\omega, z)_t$ $\mathrm{P} \otimes \mathrm{d}\langle S, S\rangle$-a.e. for every $z \in \mathcal{S}$.

(i) *We have $\widetilde{\vartheta} \in L^2(\widetilde{\mathcal{P}}, \mathrm{P} \otimes \langle S, S\rangle \otimes \zeta)$, $\int_{\mathcal{S}} \widetilde{\vartheta}(z)\zeta(\mathrm{d}z) \in L^2(S)$, and there exists an optional function $(t, \omega, z) \mapsto \widetilde{Y}(\omega, z)_t$ such that $\widetilde{Y}(z)$ is a version of $\int_0^{\cdot} \widetilde{\vartheta}(z)_s \,\mathrm{d}S_s$ (and hence of $\int_0^{\cdot} \vartheta(z)_s \,\mathrm{d}S_s$) for every $z \in \mathcal{S}$ and*

$$\int_0^t \int_{\mathcal{S}} \widetilde{\vartheta}(z)_s \zeta(\mathrm{d}z)\,\mathrm{d}S_s = \int_{\mathcal{S}} \widetilde{Y}(z)_t \zeta(\mathrm{d}z), \qquad \text{a.s., } \textit{for all } t \in [0, T]. \tag{4.7}$$

(ii) *There exists an optional function $(t, \omega, z) \mapsto \widetilde{L}(\omega, z)_t$ such that $\widetilde{L}(z)$ is a version of $L(z)$, for every $z \in \mathcal{S}$. Furthermore, $\sup_{z\in\mathcal{S}} \mathrm{E}[|\widetilde{L}(z)_t|^2] < +\infty$, $t \in [0, T]$, and*

$$L_t = \int_{\mathcal{S}} \widetilde{L}(z)_t \zeta(\mathrm{d}z), \qquad \text{a.s., } \textit{for all } t \in [0, T]. \tag{4.8}$$

(iii) *The GKW decomposition of $H \in \mathcal{H}^2$, $H_t = \mathrm{E}[\eta \mid \mathcal{F}_t]$ a.s., $t \in [0, T]$, is*

$$H = H_0 + \int_0^{\cdot} \int_{\mathcal{S}} \widetilde{\vartheta}(z)_s \zeta(\mathrm{d}z)\,\mathrm{d}S_s + L, \tag{4.9}$$

and ϑ is given by

$$\vartheta = \int_{\mathcal{S}} \widetilde{\vartheta}(z)\zeta(\mathrm{d}z). \tag{4.10}$$

Proof. First, we observe that, for every $z \in \mathcal{S}$, the identity

$$\mathrm{E}[|\widetilde{H}(z)_t|^2] = \mathrm{E}\left[\left|\int_0^{\cdot} \vartheta_s(z)\,\mathrm{d}S_s\right|^2 + |L(z)_t|^2 + |\widetilde{H}(z)_0|^2\right]$$

holds. Hence, Proposition 4.2 implies that

$$\sup_{z\in\mathcal{S}} \mathrm{E}\left[\left|\int_0^t \vartheta_s(z)\,\mathrm{d}S_s\right|^2\right] < +\infty\,, \qquad \sup_{z\in\mathcal{S}} \mathrm{E}\left[|L(z)_t|^2\right] < +\infty, \quad t \in [0, T]. \tag{4.11}$$

We now verify (i). The stochastic integrals $\int_0^{\cdot} \vartheta_s(z)\,\mathrm{d}S_s$ and $\int_0^{\cdot} \widetilde{\vartheta}_s(z)\,\mathrm{d}S_s$ are indistinguishable, for every $z \in \mathcal{S}$. So, from (4.11), we get $\sup_{z\in\mathcal{S}} \mathrm{E}[|\int_0^t \widetilde{\vartheta}_s(z)\,\mathrm{d}S_s|^2] < +\infty$, $t \in [0, T]$. From Tonelli's theorem and Itô's isometry, we then have

$$\begin{aligned}\mathrm{E}\left[\int_0^T \int_{\mathcal{S}} |\widetilde{\vartheta}(z)_t|^2 |\zeta|(\mathrm{d}z)\,\mathrm{d}\langle S, S\rangle_t\right] &= \int_{\mathcal{S}} \mathrm{E}\left[\int_0^T |\widetilde{\vartheta}(z)_t|^2\,\mathrm{d}\langle S, S\rangle_t\right] |\zeta|(\mathrm{d}z) \\ &\le |\zeta|(\mathcal{S}) \sup_{z\in\mathcal{S}} \mathrm{E}\left[\left|\int_0^T \widetilde{\vartheta}(z)\,\mathrm{d}S_s\right|^2\right] < +\infty,\end{aligned}$$

which shows that $\widetilde{\vartheta} \in L^2(\widetilde{\mathcal{P}}, \mathrm{P} \otimes \langle S, S\rangle \otimes \zeta)$. Therefore, applying Fubini's theorem yields that $z \mapsto \widetilde{\vartheta}(\omega, z)_t \in L^2(\zeta)$ $\mathrm{P} \otimes \langle S, S\rangle$-a.e., and that $(t, \omega) \mapsto \int_{\mathcal{S}} |\widetilde{\vartheta}(\omega, z)_t|^2 |\zeta|(\mathrm{d}z)$ is predictable and belongs to $L^1(\mathcal{P}, \mathrm{P} \otimes \langle S, S\rangle)$. So, $\int_{\mathcal{S}} \widetilde{\vartheta}(z)\zeta(\mathrm{d}z) \in L^2(S)$. To see (4.7), we apply the *stochastic Fubini theorem* as in [20, Theorem 3(a) and Corollary 2]: There exists an optional function $(t, \omega, z) \mapsto \widetilde{Y}(\omega, z)_t$ such that $\widetilde{Y}(z)$ is a version of $\int_0^{\cdot} \widetilde{\vartheta}(z)_s \,\mathrm{d}S_s$, for every $z \in \mathcal{S}$, and we have the identity (4.7). The proof of (i) is complete. We now verify (ii) and (iii). We define $\widetilde{L}(\omega, z)_t := \widetilde{H}(\omega, z)_t - \widetilde{H}(z)_0 - \widetilde{Y}(\omega, z)_t$, for every $(t, \omega, z) \in \widetilde{\Omega}$. Since $\widetilde{L}(z)_t = \widetilde{H}(z)_t - \widetilde{H}(z)_0 -$

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$\int_0^{\cdot} \widetilde{\vartheta}(z)_s \, \mathrm{d}S_s$, $\widetilde{Y}(z)$ being a version of $\int_0^{\cdot} \widetilde{\vartheta}(z)_s \, \mathrm{d}S_s$ for every $z \in \mathcal{S}$, we obtain that $\widetilde{L}(z)$ is a version of $L(z)$ for every $z \in \mathcal{S}$. Hence, (4.11) yields the estimate $\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{L}(z)_t|] < +\infty$ for every $t \in [0, T]$. Therefore, since $\widetilde{L}$ is optional, by Lemma 4.1, $\widehat{L}_t = \int_{\mathcal{S}} \widetilde{L}(z)_t \zeta(\mathrm{d}z)$ is a square-integrable martingale. We denote by L^* a càdlàg version of $\widehat{L}$. We can now compute a.s., and for every $t \in [0, T]$,

$$\begin{aligned} H_t = \int_{\mathcal{S}} \widetilde{H}(z)_t \zeta(\mathrm{d}z) &= H_0 + \int_{\mathcal{S}} \widetilde{Y}(z)_t \zeta(\mathrm{d}z) + \widehat{L}_t \\ &= H_0 + \int_0^t \int_{\mathcal{S}} \widetilde{\vartheta}(z)_s \zeta(\mathrm{d}z) \, \mathrm{d}S_s + L_t^*, \end{aligned} \tag{4.12}$$

where in the first equality we used Proposition 4.2. But the left-hand side and the right-hand side of the last equality in (4.12) are càdlàg. Hence, they are indistinguishable. To prove (4.9), it remains to show that L^* is orthogonal to S. This yields $L = L^*$. By Tonelli's theorem and Hölder's inequality we obtain, for every $t \in [0, T]$, $\mathrm{E}[\int_{\mathcal{S}} |S_t \widetilde{L}(z)_t| |\zeta|(\mathrm{d}z)] < +\infty$. Since $SL(z)$ is a martingale for every $z \in \mathcal{S}$, $S\widetilde{L}(z)$ is a martingale for every $z \in \mathcal{S}$ too, $\widetilde{L}(z)$ being a version of $L(z)$ for every $z \in \mathcal{S}$. By Fubini's theorem, we get, a.s., for every $0 \le s \le t \le T$,

$$\mathrm{E}[S_t L_t^* \mid \mathcal{F}_s] = \mathrm{E}[S_t \widehat{L}_t \mid \mathcal{F}_s] = \int_{\mathcal{S}} \mathrm{E}[S_t \widetilde{L}(z)_t \mid \mathcal{F}_s] \zeta(\mathrm{d}z) = S_s L_s^*.$$

So, L^* is orthogonal to S, and (4.9) and (4.8) hold. From (4.9) we get (4.10) because of $\vartheta = \mathrm{d}\langle S, H\rangle / \mathrm{d}\langle S, S\rangle = \int_{\mathcal{S}} \widetilde{\vartheta}(z) \zeta(\mathrm{d}z)$. The proof is complete. □

Let η^j be a *real-valued* square-integrable European option with GKW decomposition (ϑ^j, L^j) with respect to $S, j = 1, 2$. In the following theorem we compute $\langle L^1, L^2 \rangle$ assuming that $\eta^j = h^j(X_T)$ and h^j has the representation given in (4.5) and (4.6) on $\mathcal{S}^j := \{z \in \mathbb{C}^n : \mathrm{Re}(z) = R^j\}$ with $\mathrm{E}[\exp(2(R^j)^\top X_T)] < +\infty, j = 1, 2$. We use the notation $\mathcal{S} := \mathcal{S}^1 \cup \mathcal{S}^2$. We also introduce $\overline{\Omega} := [0, T] \times \Omega \times \mathcal{S}^1 \times \mathcal{S}^2$ and $\overline{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathcal{S}^1) \otimes \mathcal{B}(\mathcal{S}^2)$.

Theorem 4.2. *Let $\eta^j \in L^2(\mathrm{P})$ have the representation (4.5) on $\mathcal{S}^j$ with respect to the measure ζ^j and let (ϑ^j, L^j) be the GKW decomposition of η^j with respect to S, $j = 1, 2$. Let $\widetilde{H}$ be the optional mapping in Assumption 4.1 with $\mathcal{S} := \mathcal{S}^1 \cup \mathcal{S}^2$ and with the following measurability assumptions: The restriction of $\widetilde{H}$ to $[0, T] \times \Omega \times \mathcal{S}^j$ is optional with respect to $\mathcal{S}^j$, $j = 1, 2$. Let $(\vartheta(z), L(z))$ be the GKW decomposition of $\widetilde{H}(z)$ with respect to S for $z \in \mathcal{S}$ and assume that there exists a mapping $\widetilde{\vartheta}$ on $[0, T] \times \Omega \times \mathcal{S}$ such that $\widetilde{\vartheta}(\omega, z)_t = \vartheta(\omega, z)_t$ $\mathrm{P} \otimes \langle S, S\rangle$-a.e. for every $z \in \mathcal{S}$ and the restriction of $\widetilde{\vartheta}$ to $[0, T] \times \Omega \times \mathcal{S}^j$ is predictable with respect to $\mathcal{S}^j$, $j = 1, 2$.*

(i) *Then there exists an optional mapping $\widetilde{L}$ on $[0, T] \times \Omega \times \mathcal{S}$ such that the restriction of $\widetilde{L}$ to $[0, T] \times \Omega \times \mathcal{S}^j$ is optional with respect to $\mathcal{S}^j$, $\widetilde{L}(z)$ is a version of $L(z)$, for every $z \in \mathcal{S}$, and $L_t^j = \int_{\mathcal{S}^j} \widetilde{L}(z)_t \zeta^j(\mathrm{d}z)$ a.s. for every $t \in [0, T]$, $j = 1, 2$.*

(ii) *If the function $(t, \omega, z_1, z_2) \mapsto \langle L(z_1), L(z_2)\rangle_t(\omega)$ on $\overline{\Omega}$ is $\overline{\mathcal{P}}$-measurable then so is $(t, \omega, z_1, z_2) \mapsto \mathrm{Tv}(\langle L(z_1), L(z_2)\rangle)_t(\omega)$, and*

$$\mathrm{E}\left[\int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \mathrm{Tv}(\langle L(z_1), L(z_2)\rangle)_T |\zeta^2|(\mathrm{d}z_2) |\zeta^1|(\mathrm{d}z_1)\right] < +\infty. \tag{4.13}$$

Moreover, $\int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \langle L(z_1), L(z_2)\rangle \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1)$ is a predictable process and

$$\langle L^1, L^2 \rangle = \int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \langle L(z_1), L(z_2)\rangle \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1). \tag{4.14}$$

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(iii) *If* $\widetilde{K}$ *is a* $\overline{\mathcal{P}}$*-measurable function on* $\overline{\Omega}$ *such that* $\int_0^T |\widetilde{K}(\omega, z_1, z_2)_s| \, \mathrm{d}s < +\infty$*, for* $(\omega, z_1, z_2) \in \Omega \times \mathcal{S}^1 \times \mathcal{S}^2$*, and*

$$\langle L(z_1), L(z_2) \rangle_t(\omega) = \int_0^t \widetilde{K}(\omega, z_1, z_2)_s \, \mathrm{d}s,$$

then $\widetilde{K}$ *is* $\overline{\mathcal{P}}$*-measurable,* $\widetilde{K} \in L^1(\overline{\mathcal{P}}, \mathrm{P} \otimes \mathrm{Leb.} \otimes \zeta^1 \otimes \zeta^2)$*,*

$$\langle L^1, L^2 \rangle = \int_0^{\cdot} \int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \widetilde{K}(z_1, z_2)_s \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1) \, \mathrm{d}s$$

and $\mathrm{E}[\langle L^1, L^2 \rangle_T] = \int_0^T \int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \mathrm{E}[\widetilde{K}(z_1, z_2)_s] \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1) \, \mathrm{d}s$*.*

Notice that the process $\langle L^1, L^2 \rangle$ in Theorem 4.2 is real valued, while $\langle L(z_1), L(z_2) \rangle$ is complex valued, $z_j \in \mathcal{S}^j$, $j = 1, 2$.

Proof of Theorem 4.2. We have $\sup_{z \in \mathcal{S}^j} \mathrm{E}[|\widetilde{H}(z)_T|^2] < +\infty$, $j = 1, 2$; this means that $\sup_{z \in \mathcal{S}} \mathrm{E}[|\widetilde{H}(z)_T|^2] < +\infty$. To see (i), it is enough to apply Theorem 4.1 to $\mathcal{S}^1$ and $\mathcal{S}^2$ to find a functional $\widetilde{L}^j$ which is optional with respect to $\mathcal{S}^j$, such that $\widetilde{L}^j(z)$ is a version of $L(z)$ for $z \in \mathcal{S}^j$ and $L_t^j = \int_{\mathcal{S}^j} \widetilde{L}^j(z) \zeta^j(\mathrm{d}z)$ a.s., $t \in [0, T]$, $j = 1, 2$. Since $\mathcal{S}^1$ and $\mathcal{S}^2$ are either disjoint or identical, we define $\widetilde{L}$ on $\mathcal{S}$ setting $\widetilde{L}(\omega, t, z) := \widetilde{L}^j(\omega, z)_t$, for (t, ω, z) in $[0, T] \times \Omega \times \mathcal{S}^j$, $j = 1, 2$, and $\widetilde{L}$ has the desired properties. We now come to (ii). By (4.11), we get $\sup_{z \in \mathcal{S}^j} \mathrm{E}[|L(z)_t|^2] \le \sup_{z \in \mathcal{S}} \mathrm{E}[|L(z)_t|^2] < +\infty$, $t \in [0, T]$, $j = 1, 2$, and (4.13) can be proven as (4.1). From (4.13) we see that $\int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \langle L(z_1), L(z_2) \rangle \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1)$ is defined for every $t \in [0, T]$ a.s. and it is a predictable process of finite variation starting at zero. By the properties of $\widetilde{L}$, the mapping $(t, \omega, z_1, z_2) \mapsto \widetilde{L}(\omega, z_1)_t \widetilde{L}(\omega, z_2)_t$ on $\overline{\Omega}$ is $\mathrm{P} \otimes |\zeta^1| \otimes |\zeta^2|$-integrable. Hence, since $\widetilde{L}(z_1)\widetilde{L}(z_2) - \langle L(z_1), L(z_2) \rangle$ is a martingale, applying Fubini's theorem, we see that $L^1 L^2 - \int_{\mathcal{S}^1} \int_{\mathcal{S}^2} \langle L(z_1), L(z_2) \rangle \zeta^2(\mathrm{d}z_2) \zeta^1(\mathrm{d}z_1)$ is a martingale. Thus, we get (4.14). The proof of (iii) is similar to the proof of Proposition 4.1(iii), and is therefore omitted. The proof is complete. □

We now combine the results of this section with those of Section 3 for semimartingales with absolutely continuous characteristics. Let us consider the factor process $X = (X^1, \ldots, X^n)^\top$ and let $R^j \in \mathbb{R}$ be such that $\mathrm{E}[\exp(2R^j X_T^1)] < +\infty$. We define $\mathcal{S}^j := \{z \in \mathbb{C} : \mathrm{Re}(z) = R^j\}$, $j = 1, \ldots, d$, and set $\mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$.

Theorem 4.3. *Let* X *be a locally square-integrable semimartingale with differential characteristics* (b, c, F)*, and let* $S = \mathrm{e}^{X^1} \in \mathcal{H}^2$*. Let* $\eta^j = h^j(S_T) = (h^j \circ \exp)(X_T^1)$ *be a real-valued square-integrable European option, let* $h^j \circ \exp$ *have the representation given in* (4.5) *and* (4.6) *on* $\mathcal{S}^j$*, and let* (ϑ^j, L^j) *be the GKW decomposition of* η^j *with respect to* S*,* $j = 1, \ldots, d$*. Additionally:*

(i) *Let* $f : [0, T] \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{C}$ *be such that the restriction of* f *to* $[0, T] \times \mathbb{R}^n \times \mathcal{S}^j$ *is* $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathcal{S}^j)$*-measurable,* $j = 1, \ldots, d$*,* $f(\cdot, \cdot, z) \in C^{1,2}([0, T] \times \mathbb{R}^n)$*, for* $z \in \mathcal{S}$*, and* $\widetilde{H}(\omega, z)_t = f(t, X_t(\omega), z)$*,* $\widetilde{H}(z)_t = \mathrm{E}[\mathrm{e}^{z X_T^1} \mid \mathcal{F}_t]$*, a.s.,* $t \in [0, T]$*,* $z \in \mathcal{S}$*.*

(ii) *We define the complex-valued function* $W(t, \omega, x, z) := f(t, x + X_{t-}(\omega), z) - f(t, X_{t-}(\omega), z)$*, for* $z \in \mathcal{S}$*, and assume that* $\int_E |W(t, \omega, x, z)|^2 F_t(\omega, \mathrm{d}x) < +\infty$*, for every* $(t, \omega, z) \in [0, T] \times \Omega \times \mathcal{S}$*, and that* $\int_E (\mathrm{e}^{x_1} - 1)^2 F_t(\omega, \mathrm{d}x) < +\infty$*,* $\xi_t(\omega) > 0$ *for every* $(t, \omega) \in [0, T] \times \Omega$ *(see Corollary* 3.1*(i) for the definition of the process* ξ*).*

(iii) *Let* $(\vartheta(z), L(z))$ *denote the GKW decomposition of* $\widetilde{H}(z)$ *with respect to* S*, for* $z \in \mathcal{S}$*, and let* $(t, \omega, z) \mapsto \widetilde{\vartheta}(\omega, z)_t$ *be the function defined on the right-hand side of* (3.14) *in terms of*

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$f(t, X_{t-}(\omega), z)$ and $W(t, \omega, x, z)$ (see (ii) above), that is,

$$\widetilde{\vartheta}(z)_t = \frac{1}{S_{t-}\xi_t}\left(\sum_{k=1}^{n} \partial_{x_k} f(t, X_{t-}, z) c_t^{1k} + \int_E (\mathrm{e}^{x_1} - 1) W(t, x, z) F_t(\mathrm{d}x)\right), \quad z \in \mathcal{S}.$$

Claim 1. *The mapping $(t, \omega) \mapsto \widetilde{H}(\omega, z)_t$ is càdlàg, for $z \in \mathcal{S}$, and the restriction of $\widetilde{H}$ to $\mathcal{S}^j$ is optional with respect to $\mathcal{S}^j$, $j = 1, \ldots, d$. Furthermore, $\widetilde{\vartheta}(z) = \vartheta(z)$ $\mathrm{P} \otimes \mathrm{d}\langle S, S\rangle$-a.e., for $z \in \mathcal{S}$, holds, and the restriction of $\widetilde{\vartheta}$ to $\mathcal{S}^j$ is a predictable function with respect to $\mathcal{S}^j$, $j = 1, \ldots, d$. In particular, the assumptions of Theorem 4.1 are satisfied with respect to $\mathcal{S}^j$, $j = 1, \ldots, d$.*

(iv) *We denote by f^j and W^j the complex-valued functions obtained taking the restriction of f and W to $\mathcal{S}^j$, respectively, so that $\widetilde{H}(\omega, z)_t = f^j(t, X_t(\omega), z)$, $z \in \mathcal{S}^j$, $j = 1, \ldots, d$. Let $\widetilde{K}(\omega, z_i, z_j)$ denote the density of $\langle L(z_i), L(z_j)\rangle$ with respect to the Lebesgue measure given in (3.15) (now defined with respect to f^i, f^j, W^i, and W^j, $i, j = 1, \ldots, d$) and let $\int_0^T |\widetilde{K}(\omega, z_i, z_j)_t| \,\mathrm{d}t < +\infty$ for every $(t, \omega, z_i, z_j) \in [0, T] \times \Omega \times \mathcal{S}^i \times \mathcal{S}^j$, for every $i, j = 1, \ldots, d$.*

Claim 2. *The function $(t, \omega, z_i, z_j) \mapsto \int_0^t \widetilde{K}(\omega, z_i, z_j)_s \,\mathrm{d}s$ is predictable with respect to $\mathcal{S}^i \times \mathcal{S}^j$ and we have $\mathrm{E}[\langle L^i, L^j\rangle_T] = \int_0^T \int_{\mathcal{S}^i} \int_{\mathcal{S}^j} \mathrm{E}[\widetilde{K}(\omega, z_i, z_j)_t] \zeta^j(\mathrm{d}z_j) \zeta^i(\mathrm{d}z_i) \,\mathrm{d}t$, for $i, j = 1, \ldots, d$.*

Proof. We first show claim 1. By assumption (i) of the theorem, $(t, \omega, z) \mapsto \widetilde{H}(\omega, z)_t$ is optional with respect to $\mathcal{S}^j$, $j = 1, \ldots, d$, and $(t, \omega) \mapsto \widetilde{H}(\omega, z)_t$ is càdlàg, $z \in \mathcal{S}$. By assumptions (i) and (ii), the function $(t, \omega, z) \mapsto \int_E (\mathrm{e}^{x_1} - 1) W(t, \omega, x, z) F_t(\omega, \mathrm{d}x)$ is predictable with respect to $\mathcal{S}^j$, for every $j = 1, \ldots, d$, the process ξ is predictable, and hence the restriction of $\widetilde{\vartheta}$ to $\mathcal{S}^j$ is predictable with respect to $\mathcal{S}^j$, $j = 1, \ldots, d$. Now let $(\vartheta(z), L(z))$ be the GKW decomposition of $\widetilde{H}(z)$ with respect to S, where $z \in \mathcal{S}$. From Corollary 3.1, we have $\vartheta(z) = \widetilde{\vartheta}(z)$ $\mathrm{P} \otimes \mathrm{d}\langle S, S\rangle$-a.e. for every $z \in \mathcal{S}$. This shows claim 1. To see claim 2, since $\langle L(z_i), L(z_j)\rangle_t = \int_0^t \widetilde{K}(\omega, z_i, z_j)_s \,\mathrm{d}s$ by Corollary 3.1, we use claim 1 and Theorem 4.2(iii) applied to $\mathcal{S}^i$ and $\mathcal{S}^j$. □

5. Applications

We apply the results of Sections 3 and 4 to two continuous stochastic volatility models, the Heston model, which is affine, and the 3/2 model, which is not, and to a simple model with jumps.

We set up a variance-optimal semi-static hedging strategy for a variance swap η^0: Let X be a square-integrable semimartingale and let $S = \mathrm{e}^X \in \mathcal{H}^2$; then

$$\eta^0 = [X, X]_T - k,$$

where $k = \mathrm{E}[[X, X]_T]$ so that $\mathrm{E}[\eta^0] = 0$.

A basket $(\eta^1, \ldots, \eta^d)$ of European options written on S is fixed. We assume that $\eta^j \in L^2(\mathrm{P})$ and $\eta^j = h^j(S_T)$, where $h^j \circ \exp$ has the representation given in (4.5) and (4.6), with respect to the complex-valued measure ζ^j on $\mathcal{S}^j = \{z \in \mathbb{C} : \mathrm{Re}(z) = R^j\}$, $\mathrm{E}[\exp(2R^j X_T)] < +\infty$, $j = 1, \ldots, d$.

5.1. The Heston model

The Heston model for $(X, V)^\top$ (see [14]) is given by

$$\mathrm{d}X_t = -\frac{1}{2} V_t \,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^1, \quad \mathrm{d}V_t = -\lambda(V_t - \kappa)\,\mathrm{d}t + \sigma\sqrt{V_t}\,\mathrm{d}W_t^2,$$

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where W^1 and W^2 are correlated Brownian motions with $\langle W^1, W^2\rangle_t = \rho t$, $\rho \in [-1, 1]$ and $\lambda, \sigma, \kappa > 0$, $2\lambda\kappa \geq \sigma^2$ so that $V_t > 0$. Notice that $(X, V)^\top$ is affine and

$$\mathrm{E}[\exp(u_1 X_T + u_2 V_T) \mid \mathcal{F}_t] = \exp(\phi_{T-t}(u_1, u_2) + u_1 X_t + \psi_{T-t}(u_1, u_2) V_t) =: \widetilde{H}(z)_t, \quad (5.1)$$

where $(t, u_1, u_2) \mapsto \phi_t(u_1, u_2)$ and $(t, u_1, u_2) \mapsto \psi_t(u_1, u_2)$ are complex-valued functions that can be expressed in terms of elementary functions. See [14], [2, Section 4.2.1], [19], [1] etc. for concrete expressions, domain of validity, and numerical issues.

For $z \in \mathcal{S}^j$, we have $\widetilde{H}(z) \in \mathcal{H}^2_{\mathbb{C}}$, $j = 1, \ldots, d$. Defining $r^0_{z_1,z_2} := \phi_t(z_1, 0) + \phi_t(z_2, 0)$ and $r^1_{z_1,z_2} := \psi_t(z_1, 0) + \psi_t(z_2, 0)$ for $z_1 \in \mathcal{S}^i$, $z_2 \in \mathcal{S}^j$, from [18, Equation (5.18)], for $t \in [0, T]$, the random variable $V_t \widetilde{H}(z_1)_t \widetilde{H}(z_2)_t$ is integrable and

$$\begin{aligned} \mathrm{E}[V_t \widetilde{H}(z_1)_t \widetilde{H}(z_2)_t] = S_0^{z_1+z_2} \mathrm{e}^{r^0_{z_1,z_2}} \times \exp(&\phi_t(z_1 + z_2, r^1_{z_1,z_2}) \\ &+ \psi_t(z_1 + z_2, r^1_{z_1,z_2}) V_0) \times \partial_{u_2}[\phi_t(z_1 + z_2, u_2) \\ &+ \psi_t(z_1 + z_2, u_2) V_0]|_{u_2 = r^1_{z_1,z_2}}. \end{aligned} \quad (5.2)$$

5.1.1. *Semi-static variance-optimal hedging.* We now compute the quantities A, B, and C defined in (2.4) for the variance swap $\eta^0 = \int_0^T V_s \, \mathrm{d}s - k$ in the Heston model. We start by computing A.

Proposition 5.1. *Setting* $\alpha(t) := \lambda^{-1}(1 - \mathrm{e}^{-\lambda(T-t)})$, *we have*

$$A = \sigma^2(1 - \varrho^2) \int_0^T \alpha^2(t)[(V_0 - \kappa)\mathrm{e}^{-\lambda t} + \kappa] \, \mathrm{d}t. \quad (5.3)$$

Proof. The process $(X, V, [X, X])^\top$ is a locally square-integrable semimartingale whose characteristics are absolutely continuous with respect to the Lebesgue measure. The second differential characteristic c is given by $c^{11} = V$, $c^{12} = c^{21} = \rho\sigma V$, $c^{22} = \sigma^2 V$, and $c^{i3}_t = 0$, $i = 1, 2, 3$. Since V is a Cox–Ingersoll–Ross process, we have

$$\mathrm{E}[V_s \mid \mathcal{F}_t] = \kappa(1 - \mathrm{e}^{-\lambda(s-t)}) + \mathrm{e}^{-\lambda(s-t)} V_t, \quad s \geq t. \quad (5.4)$$

We consider $H^0 \in \mathcal{H}^2$ such that $H^0_t = \mathrm{E}[\int_0^T V_s \, \mathrm{d}s | \mathcal{F}_t]$ a.s., $t \in [0, T]$. By Fubini's theorem we obtain $H^0_t = f^0(t, X_t, V_t, [X, X]_t)$, where $f^0(t, x, v, w) = \tilde{\alpha}(t) + \alpha(t)v + w$, with $\tilde{\alpha}(t) := \kappa\lambda^{-1}(\lambda(T - t) - 1 + \mathrm{e}^{-\lambda(T-t)})$ and $\alpha(t)$ as above, $t \in [0, T]$. We can now apply Corollary 3.1(ii) with $Y^i = H^0$, $f^i = f^0$, $i = 1, 2$, and $(X^1, X^2, X^3)^\top = (X, V, [X, X])^\top$. The only relevant derivative is $\partial_v f^0 = \alpha(t)$. Taking the expectation in (3.16), and applying Fubini's theorem and (5.4) with $t = 0$, we get (5.1). The proof is complete. □

We stress that according to Corollary 3.1(ii) we have $\vartheta^0_t = \rho\sigma\alpha(t)S_t^{-1}$. As a next step, we compute the vector B in (2.4).

Proposition 5.2. *For* $B = (B^1, \ldots, B^d)^\top$ *we have*

$$B^j = \sigma^2(1 - \varrho^2) \int_0^T \int_{\mathcal{S}^j} \alpha(t) E(t, z) \psi_{T-t}(z, 0) \zeta^j(\mathrm{d}z) \, \mathrm{d}t,$$

where $\alpha(t) = \lambda^{-1}(1 - \mathrm{e}^{-\lambda(T-t)})$ *and the explicit expression of* $E(t, z) := \mathrm{E}[\widetilde{H}(z)_t V_t]$ *is given by* (5.2) *with* $z_1 = z \in \mathcal{S}^j$, $j = 1, \ldots, d$, *and* $z_2 = 0$.

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Proof. Let (ϑ^j, L^j) and $(\vartheta(z), L(z))$ be the GKW decomposition with respect to S of η^j and $\widetilde{H}(z)$, respectively, $z \in \mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$, $j = 0, \dots, d$. We now compute $\langle L^0, L(z)\rangle$, $z \in \mathcal{S}$. Let H^0 and f^0 be as in the proof of Proposition 5.1. According to Corollary 3.1 (applied now to the complex-valued square-integrable martingale $\widetilde{H}(z)$ and to H^0) with $f^1 = f^0$ and $f^2 = f$, $f(t, x, v, w, z) = \exp(\phi_t(z, 0) + \psi_t(z, 0)v + zx)$, we deduce that $\vartheta(z)_t = \frac{\widetilde{H}(z)_t}{S_t}(z + \sigma\rho\psi_{T-t}(z, 0)) =: \widetilde{\vartheta}(z)_t$ $\mathrm{P} \otimes \mathrm{d}\langle S, S\rangle$-a.e., $z \in \mathcal{S}$. Thus, the restriction of $\widetilde{\vartheta}$ to $\mathcal{S}^j$ is a predictable function with respect to $\mathcal{S}^j$, $j = 1, \dots, d$. Moreover,

$$\mathrm{d}\langle L^0, L(z)\rangle_t = \sigma^2(1-\rho^2)\alpha(t)\psi_{T-t}(z, 0)\widetilde{H}(z)_t V_t \,\mathrm{d}t.$$

Because of (5.1), $\widetilde{K}(z)_t := \sigma^2(1-\rho^2)\alpha(t)\psi_{T-t}(z, 0)\widetilde{H}(z)_t V_t$ is a predictable function with respect to each $\mathcal{S}^j$, $j = 1, \dots, d$. Furthermore, it is continuous in t. Because of (5.1) and Theorem 4.3, the assumptions of Theorem 4.1 are satisfied with respect to $\mathcal{S}^j$, for each $j = 1, \dots, d$, so we have $L^j_t = \int_{\mathcal{S}^j} \widetilde{L}(z)\zeta^j(\mathrm{d}z)$, a.s., $t \in [0, T]$. Since $\sup_{z\in\mathcal{S}^j} \mathrm{E}[|\widetilde{L}(z)_t|^2] < +\infty$, $t \in [0, T]$, $j = 1, \dots, d$, from Proposition 4.1(iii) we get

$$B^j = \mathrm{E}[\langle L^0, L^j\rangle_T] = \sigma^2(1-\rho^2)\int_0^T \int_{\mathcal{S}^j} \alpha(t)\psi_{T-t}(z, 0)\mathrm{E}[\widetilde{H}(z)_t V_t]\zeta^j(\mathrm{d}z)\,\mathrm{d}t.$$

Finally, from (5.2) we get the explicit expression of $\mathrm{E}[\widetilde{H}(z)_t V_t]$. □

Proposition 5.3. *For $C = (C^{ij})_{i,j=1,\dots,d}$, we have*

$$C^{ij} = \sigma^2(1-\varrho^2)\int_0^T \int_{\mathcal{S}^i}\int_{\mathcal{S}^j} \psi_{T-t}(z_1, 0)\psi_{T-t}(z_2, 0)J(t, z_1, z_2)\zeta^j(\mathrm{d}z_2)\zeta^i(\mathrm{d}z_1)\,\mathrm{d}t,$$

where $J(t, z_1, z_2) := \mathrm{E}[\widetilde{H}(z_1)_t\widetilde{H}(z_2)_t V_t]$ is given by (5.2), $i, j = 1, \dots, d$.

Proof. Let (ϑ^j, L^j) and $(\vartheta(z), L(z))$ be the GKW decomposition with respect to S of η^j and of $\widetilde{H}(z)$, respectively, $j = 1, \dots, d$, $z \in \mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$. By Corollary 3.1(ii) with $f^1(t, x, v, w, z) = f(t, x, v, w, z_1)$, $z \in \mathcal{S}^i$, and $f^2(t, x, v, w, z) = f(t, x, v, w, z_2)$, $z \in \mathcal{S}^j$, $f(t, x, v, w, z) = \exp(\phi_t(z, 0) + \psi_t(z, 0)v + zx)$, we get

$$\mathrm{d}\langle L(z_1), L(z_2)\rangle_t = \sigma^2(1-\varrho^2)\psi_{T-t}(z_1, 0)\psi_{T-t}(z_2, 0)\widetilde{H}(z_1)_t\widetilde{H}(z_2)_t V_t\,\mathrm{d}t =: \widetilde{K}(z_1, z_2)_t\,\mathrm{d}t,$$

where, $(t, \omega, z_1, z_2) \mapsto \widetilde{K}(\omega, z_1, z_2)_t$ is $\mathcal{P} \otimes \mathcal{B}(\mathcal{S}^i) \otimes \mathcal{B}(\mathcal{S}^j)$-measurable and continuous in t. Since we have $V_t > 0$, the claim follows from Theorem 4.3. □

5.2. The 3/2 model

The 3/2 model for $(X, V)^\top$ is given by

$$\mathrm{d}X_t = -\frac{V_t}{2}\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W^1_t, \qquad \mathrm{d}V_t = V_t(\lambda - \kappa V_t)\,\mathrm{d}t + \sigma V_t^{3/2}\,\mathrm{d}W^2_t, \quad V_0 > 0, \tag{5.5}$$

where W^1 and W^2 are Brownian motions with $\langle W^1, W^2\rangle_t = \rho t$, $\rho \in [-1, 1]$. We assume $\kappa > 0$, which ensures both non-explosion of V (see [10, Equation (3)]) and $V_t > 0$, $t \in [0, T]$. The two-dimensional semimartingale $(X, V)^\top$ given by (5.5) is *not an affine process*. For the computation of the conditional moment-generating function we mainly refer to [13], that is, we regard the 3/2 model as a special case of the so-called 4/2 model. More precisely, setting

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$a=0$, $b=1$, and $r=0$ in [13, Equation (2.1)], we get the dynamics of the price process $S=\mathrm{e}^X$ (and hence of $X=\log S$) under the local martingale measure in the 3/2 model. We stress that, denoting by κ_G, θ_G, σ_G the parameters in [13, Equation (2.2)], the relation between λ, κ, and σ in (5.5) and κ_G, θ_G, σ_G is

$$\kappa=\kappa_G\theta_G-\sigma_G^2,\quad \lambda=\kappa_G,\quad \sigma=-\sigma_G. \tag{5.6}$$

In [17, Proposition 6.3.2.1] the transition density of the process $R=(R_t)_{t\in[0,T]}$, $R_t:=1/V_t$ is given. From this, we obtain the density function $f_{V_t}(v)$ of V_t: $f_{V_t}(v)=0$ for $v\le 0$, and for $v>0$

$$\begin{aligned} f_{V_t}(v)=&\frac{1}{2v^2}\,\frac{4\lambda}{\sigma^2(\mathrm{e}^{\lambda t}-1)}\left(\frac{V_0}{v}\right)^{q/2}\\ &\times\exp\left(\lambda\left(1+\frac{q}{2}\right)t-\frac{2\lambda}{\sigma^2(\mathrm{e}^{\lambda t}-1)}\left(\frac{1}{V_0}+\frac{\mathrm{e}^{\lambda t}}{v}\right)\right)I_q\left(\frac{4\lambda\mathrm{e}^{\frac{\lambda}{2}t}}{\sigma^2(\mathrm{e}^{\lambda t}-1)\sqrt{V_0v}}\right),\end{aligned} \tag{5.7}$$

where $q:=2(\kappa+\sigma^2)/\sigma^2-1$ and $I_p(\,\cdot\,)$ denotes the modified Bessel function of the first kind. From [13, p. 17], denoting by $M(a,b,\cdot)$ the confluent hypergeometric function, we have

$$\begin{aligned}\mathrm{E}[V_t^\eta]=&\left(\frac{2\lambda\mathrm{e}^{\lambda t}}{\sigma^2(\mathrm{e}^{\lambda t}-1)}\right)^\eta\exp\left(-\frac{2\lambda}{\sigma^2(\mathrm{e}^{\lambda t}-1)V_0}\right)\frac{\Gamma(q+1-\eta)}{\Gamma(q+1)}\\ &\times M\left(q+1-\eta,\,q+1,\,\frac{2\lambda}{\sigma^2(\mathrm{e}^{\lambda t}-1)V_0}\right)\end{aligned}$$

for $\eta\in\mathbb{R}$ such that $q+1-\eta>0$, which always holds for $\eta=2+\delta$, for some fixed $\delta>0$: In the 3/2 model, contrary to the Heston model, V_t does not have finite moments of every order. According to [13, Proposition 3.1], for $z\in\mathbb{C}$ we have $\exp(zX_T)\in L^1(\mathrm{P})$ if $\mathrm{Re}(z)\in\mathcal{A}_{0,+\infty}$, where the set $\mathcal{A}_{0,+\infty}$ is defined in [13, Equation (3.7)].

Let $\exp(zX_T)$ be integrable, $z\in\mathbb{C}$. By the relation $M(x,y,z)=\mathrm{e}^z M(y-x,y,-z)$, $z\in\mathbb{C}$, for the confluent hypergeometric function, from [13, Equation (3.3)] with $a=0$ and $b=1$, we obtain $\widetilde{H}(z)_t=\mathrm{E}[\exp(zX_T)\mid\mathcal{F}_t]$, a.s., $t\ge 0$, where

$$\widetilde{H}(z)_t=\mathrm{e}^{zX_t}g(t,V_t,z), \tag{5.8}$$

and the function g is given by

$$\begin{aligned} g(t,v,z):=&\frac{\Gamma(\beta_z-\alpha_z)}{\Gamma(\beta_z)}\left(\frac{2\lambda}{v\sigma^2(\mathrm{e}^{\lambda(T-t)}-1)}\right)^{\alpha_z}\\ &\times M\left(\alpha_z,\,\beta_z,\,-\frac{2\lambda}{v\sigma^2(\mathrm{e}^{\lambda(T-t)}-1)}\right)\end{aligned} \tag{5.9}$$

and $\alpha_z=-\frac{1}{2}-\frac{\tilde\kappa_z}{\sigma^2}+c_z$, $\beta_z=1+2c_z$, $\tilde\kappa_z=k-z\varrho\sigma$, $c_z:=\{(\frac{1}{2}+\frac{\tilde\kappa_z}{\sigma^2})^2+\frac{z-z^2}{\sigma^2}\}^{1/2}$.

In the proof of [7, Theorem 3, Section 8], it was shown that $v\mapsto g(t,v,z)$ (therein ℓ) solves an ordinary differential equation of the second order and $g(T,V_T,z)=1$. So, $t\mapsto\widetilde{H}(\omega,z)_t$ is continuous. Taking m_z as in [13, Equation (3.5)] with $a=0$ and $b=1$, then $c_z=\frac{m_z}{2}$, by the properties of the confluent hypergeometric function $M(x,y,z)$, and we get:

$$\begin{aligned}\partial_v g(t,v,z)=&\frac{\alpha_z}{v}(\delta(t,v))^{\alpha_z}\frac{\Gamma(\beta_z-\alpha_z)}{\Gamma(\beta_z)}\\ &\times\left[\frac{\delta(t,v)}{\beta_z}M(\alpha_z+1,\beta_z+1,-\delta(t,v))-M(\alpha_z,\beta_z,-\delta(t,v))\right],\end{aligned} \tag{5.10}$$

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where $\delta(t,v) := \frac{2\lambda}{\sigma^2(\mathrm{e}^{\lambda(T-t)}-1)v}$. We require $\kappa - \varrho\sigma \geq -\frac{\sigma^2}{2}$ (i.e., *Feller's condition*), which ensures that $S = \mathrm{e}^X$ is a true martingale (see [13, Remark 3.3] with the identification of parameters in (5.6)). To get $S \in \mathcal{H}^2$, we require $2 \in \mathcal{A}_{0,+\infty}$.

We observe that the variance swap η^0 is square integrable in this model. This follows because, analyzing the explicit expression of the Laplace transform of η^0, which was calculated in [7, Theorem 3] (or [13, Appendix A]), we recognize that it is defined in an open neighborhood of zero.

5.2.1. *Variance-optimal semi-static hedging.* We now compute the quantities A, B, and C defined in (2.4) for the variance swap $\eta^0 = \int_0^T V_s\,\mathrm{d}s - k$ in the 3/2 model.

We start by computing A. First, we introduce the function

$$h'(y) = \frac{\sigma^2}{2}\Gamma(q)\mathrm{e}^{-2/\sigma^2 y}y^{-1}\gamma^*\Big(q, -\frac{2}{\sigma^2 y}\Big), \tag{5.11}$$

where $\gamma^*(a,z) := \frac{1}{\Gamma(a)}\int_0^1 u^{a-1}\mathrm{e}^{-zu}\,\mathrm{d}u$, such that $\gamma^*(a,z) = \frac{z^{-a}}{\Gamma(a)}\,\gamma(a,z)$, γ denoting the lower incomplete Gamma function. We then put $h(y) := \int_0^y h'(x)\,\mathrm{d}x$. The function h was introduced in [7, Theorem 4] and this is just a more compact way to write h.

Proposition 5.4. *Setting $\beta(t) := \lambda^{-1}(\mathrm{e}^{\lambda(T-t)} - 1)$, we have*

$$A = \sigma^2(1-\rho^2)\int_0^T\int_0^\infty \beta(t)^2[h'(\alpha(t)v)]^2 v^3 f_{V_t}(v)\,\mathrm{d}v\;\mathrm{d}t,$$

where $f_{V_t}(v)$ is as in (5.7) and h' is from (5.11).

Proof. The process $(X, V, [X,X])^\top$ is a locally square-integrable semimartingale with absolutely continuous characteristics. The second differential characteristic c is given by $c_t^{11} = V_t$, $c_t^{12} = \rho\sigma V_t^2$, $c_t^{22} = \sigma^2 V_t^3$, and $c_t^{i3} = 0$, $i = 1,2,3$. Since $[X,X] = \int_0^\cdot V_s\,\mathrm{d}s$, we consider $H^0 \in \mathcal{H}^2$ such that $H_t^0 = \mathrm{E}[\int_0^T V_s\,\mathrm{d}s \mid \mathcal{F}_t] - k$ a.s., $t \in [0,T]$. By the Markov property of V and [7, Theorem 4], we get $H_t^0 + k = f^0(t, X_t, V_t, [X,X]_t)$, with $f^0(t,x,v,w) := h(\beta(t)v) + w$, and moreover $h(y) := \int_0^y h'(x)\,\mathrm{d}x$. The function f^0 is twice continuously differentiable (see [7, Equation (81)]) and $\partial_v f^0(t,v,w) = \beta(t)h'(\beta(t)v)$. Therefore, by Corollary 3.1(ii), we obtain

$$\langle L^0, L^0\rangle_T = \sigma^2(1-\rho^2)\int_0^T \beta(t)^2[h'(\beta(t)V_t)]^2 V_t^3\,\mathrm{d}t.$$

Taking the expectation and using Fubini's theorem yields the result. □

We stress that according to Corollary 3.1(ii) we have $\vartheta_t^0 = \rho\sigma\beta(t)h'(\beta(t)V_t)V_t S_t^{-1}$.

As the next step, we compute the vector B. The explicit expression of $\mathrm{E}[\mathrm{e}^{zX_T} \mid V_t]$ can be obtained from [13, Proposition 4.1], with $a = 0$ and $b = 1$ (observe that $R_t = V_t^{-1}$, where $(R_t)_{t\in[0,T]}$ denotes the volatility process in [13, Equation (2.2)]), and reads

$$\mathrm{E}[\mathrm{e}^{zX_t} \mid V_t] = \exp\Big(zX_0 - z\frac{\lambda\rho}{\sigma}t\Big)\Big(\frac{V_t}{V_0}\Big)^{\frac{z\rho}{\sigma}}\frac{I_{\frac{2}{\sigma^2}\sqrt{B_z}}\Big(\frac{4\lambda\mathrm{e}^{\frac{\lambda}{2}t}}{\sigma^2(\mathrm{e}^{\lambda t}-1)\sqrt{V_0V_t}}\Big)}{I_q\Big(\frac{4\lambda\mathrm{e}^{\frac{\lambda}{2}t}}{\sigma^2(\mathrm{e}^{\lambda t}-1)\sqrt{V_0V_t}}\Big)}, \tag{5.12}$$

where $B_z := (\kappa + \frac{\sigma^2}{2})^2 + 2\sigma^2[z(\frac{\rho}{\sigma}(\kappa+\frac{\sigma^2}{2}) - \frac{1}{2}) + \frac{z^2}{2}(1-\rho^2)]$.

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Proposition 5.5. *For* $B = (B^1, \dots, B^d)^\top$, *setting* $\beta(t) = \lambda^{-1}(\mathrm{e}^{\lambda(T-t)} - 1)$,

$$B^j = \sigma^2(1-\rho^2)\int_0^T \int_{\mathcal{S}^j} \int_0^\infty \beta(t)h'(\beta(t)v)\partial_v g(t,v,z)G(t,v,z)v^3 f_{V_t}(v)\,\mathrm{d}v\,\zeta^j(\mathrm{d}z)\,\mathrm{d}t,$$

where $G(t,v,z) := \mathrm{E}[\mathrm{e}^{zX_T} \mid V_t = v]$ *is given by* (5.12) *with* $V_t \to v$, $\partial_v g(t,v,z)$ *by* (5.10), $f_{V_t}(v)$ *by* (5.7), *and* $z \in \mathcal{S}^j$.

Proof. Let (ϑ^j, L^j) and $(\vartheta(z), L(z))$ be the GKW decompositions with respect to S of η^j and $\widetilde{H}(z)$, respectively, $j = 0, \dots, d$, $z \in \mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$. We now compute $\langle L^0, L(z)\rangle$, $z \in \mathcal{S}$. Let H^0 and f^0 be as in the proof of Proposition 5.4. According to Corollary 3.1(ii) with $f^1 = f^0$ and $f^2 = f$, $f(t,x,v,w,z) = \mathrm{e}^{zx}g(t,v,z)$, and g from (5.9), $\vartheta(z) = (zH(z)_t + \sigma\rho\mathrm{e}^{zX_t}\partial_v g(t,V_t)V_t)S_t^{-1} =: \widetilde{\vartheta}(z)$ $\mathrm{P}\otimes\,\mathrm{d}\langle S,S\rangle$-a.e., $z \in \mathcal{S}$. Thus, the restriction of $\widetilde{\vartheta}$ to $\mathcal{S}^j$ is a predictable function with respect to $\mathcal{S}^j$, $j = 1, \dots, d$. Furthermore,

$$\mathrm{d}\langle L^0, L(z)\rangle_t = \sigma^2(1-\rho^2)h'(\beta(t)V_t)\beta(t)\partial_v g(t,V_t,z)V_t^3\mathrm{e}^{zX_t}\,\mathrm{d}t =: \widetilde{K}(z)_t\,\mathrm{d}t.$$

Since h is twice continuously differentiable and g is twice continuously differentiable in the second variable, $t \mapsto \widetilde{K}(\omega,z)_t$ is a continuous function. Because of (5.8) and $V_t > 0$, by Theorem 4.3, the assumptions of Theorem 4.1 are satisfied with respect to $\mathcal{S}^j$, for each $j = 1, \dots, d$, so we have $L_t^j = \int_{\mathcal{S}^j} \widetilde{L}(z)\zeta^j(\mathrm{d}z)$, a.s., $t \in [0,T]$. Since $\sup_{z\in\mathcal{S}^j}\mathrm{E}[|\widetilde{L}(z)_t|^2] < +\infty$, $t \in [0,T]$, $j = 1, \dots, d$, from Proposition 4.1(iii), the identity $\mathrm{E}[\,\cdot\,] = \mathrm{E}[\mathrm{E}[\,\cdot \mid V_t]]$, and (5.12), we obtain the claim. □

Proposition 5.6. *For* $C = (C^{ij})_{i,j=1,\dots,d}$ *we have*

$$C^{ij} = \sigma^2(1-\rho^2)\int_0^T \int_{\mathcal{S}^i}\int_{\mathcal{S}^j}\int_0^\infty \Big\{\partial_v g(t,v,z_1)\partial_v g(t,v,z_2) \times G(t,v,z_1+z_2)v^3 f_{V_t}(v)\Big\} \times \mathrm{d}v\,\zeta^j(\mathrm{d}z_2)\,\zeta^i(\mathrm{d}z_1)\,\mathrm{d}t,$$

where $\partial_v g$ *is as in* (5.10), $G(t,v,z)$ *as in Proposition* 5.5, *and* $f_{V_t}(v)$ *is from* (5.7).

Proof. Let (ϑ^j, L^j) and $(\vartheta(z), L(z))$ be the GKW decompositions with respect to S of η^j and of $H(z)$, $j = 0, \dots, d$, $z \in \mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$. By Corollary 3.1(ii) with $f^1(t,x,v,w,z) = f(t,x,v,w,z_1)$, $z \in \mathcal{S}^i$, and $f^2(t,x,v,z) = f(t,x,v,w,z_2)$, $z \in \mathcal{S}^j$, $i,j = 1, \dots, d$, $f(t,x,v,z) = \mathrm{e}^{zx}g(t,v,z)$, we get

$$\mathrm{d}\langle L(z_1), L(z_2)\rangle_t = \sigma^2(1-\varrho^2)\partial_v g(t,V_t,z_1)\partial_v g(t,V_t,z_2)\mathrm{e}^{(z_1+z_2)X_t}V_t^3\,\mathrm{d}t =: \widetilde{K}(z_1,z_2)\,\mathrm{d}t.$$

Since $\widetilde{K}$ is $\mathcal{P}\otimes\mathcal{B}(\mathcal{S}^i)\otimes\mathcal{B}(\mathcal{S}^j)$-measurable, $t \mapsto \widetilde{K}(\omega,z_1,z_2)_t$ is continuous, and $V_t > 0$, the claim follows from Theorem 4.3 and (5.12). □

5.3. An example with jumps

We now consider a special exponential Lévy model with unit jump sizes and compute the quantities A, B, and C for the variance swap.

Let N^1 and N^2 be independent Poisson processes. Let λ_i be the parameter of N^i, that is, $(\overline{N}^i)_{t\in[0,T]}$ is a martingale, $\overline{N}_t^i := N_t^i - \lambda_i t$, $i = 1, 2$. We now introduce the processes $M_t^1 := N_t^1 - \lambda_1 t(\mathrm{e}-1)$ and $M_t^2 := N_t^2 + \lambda_2 t(\mathrm{e}^{-1}-1)$ and consider the semimartingale $X =$

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$(X^1, X^2) := (X_0^1 + M^1 - M^2, N^2)$. Since $\Delta N^1 \Delta N^2 = 0$, the jump measure μ of X is given on $E = \{(-1, 1), (1, 0)\}$ by

$$\mu(\omega, \mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2) = \mathrm{d}N_t^1(\omega)\delta_{(1,0)}(\mathrm{d}x_1, \mathrm{d}x_2) + \mathrm{d}N_t^2(\omega)\delta_{(-1,1)}(\mathrm{d}x_1, \mathrm{d}x_2),$$

δ_a denoting the Dirac measure concentrated at the point a. Hence, the predictable compensator ν of μ is

$$\nu(\omega, \mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2) = \lambda_1\delta_{(1,0)}(\mathrm{d}x_1, \mathrm{d}x_2)\,\mathrm{d}t + \lambda_2\delta_{(-1,1)}(\mathrm{d}x_1, \mathrm{d}x_2)\,\mathrm{d}t.$$

The third differential semimartingale characteristic of X is

$$F_t(\omega, \mathrm{d}x) = F(\mathrm{d}x) = \lambda_1\delta_{(1,0)}(\mathrm{d}x_1, \mathrm{d}x_2) + \lambda_2\delta_{(-1,1)}(\mathrm{d}x_1, \mathrm{d}x_2).$$

We consider the market model $S = \mathrm{e}^{M^1 - M^2}$. Then, S is a square-integrable martingale. We now compute the quantities A, B, and C combining Corollary 3.1(i) and Theorem 4.3. The quantity ξ_t in Corollary 3.1(i) fulfills $\xi_t = \xi = (\mathrm{e} - 1)^2(\lambda_1 + \frac{\lambda_2}{\mathrm{e}^2}) > 0$. For the variance swap we now have $\eta^0 + k = [X^1, X^1]_T = N_T^1 + N_T^2$ and $k = (\lambda_1 + \lambda_2)T$. We now define $f^0(t, x) := x^1 + 2x^2 + [\lambda_1(\mathrm{e} - 2) + \lambda_2(\mathrm{e}^{-1} - 2)]t$. Then $H_t^0 = \mathrm{E}[\eta^0 \mid \mathcal{F}_t] = f^0(t, X)$ holds. Therefore, Corollary 3.1(i) with $L^i = L^0$ and $W^i = W^0$, $i = 1, 2$, where we define $W^0 := f^0(t, x + X_{t-}) - f^0(t, X_{t-}) = x_1 + 2x_2$, yields

$$A = \langle L^0, L^0\rangle_T = \left[\lambda_1 + \lambda_2 - \frac{\left(\lambda_1 - \frac{\lambda_2}{\mathrm{e}}\right)^2}{\lambda^1 + \frac{\lambda^2}{\mathrm{e}^2}}\right] T.$$

We now compute B. Let $\mathcal{S} := \bigcup_{j=1}^d \mathcal{S}^j$. We observe that X^1 has exponential moments of every order. Furthermore, since X^1 has independent increments with respect to the filtration $\mathbb{F}$, setting $q(z) := \lambda_1(\mathrm{e}^z - 1 - z(\mathrm{e} - 1)) + \lambda_2(\mathrm{e}^{-z} - 1 - z(\mathrm{e}^{-1} - 1))$, we get $\mathrm{E}[\mathrm{e}^{zX_T^1} \mid \mathcal{F}_t] = f(t, X_t, z) =: \widetilde{H}(z)_t$, where $f(t, x, z) := \mathrm{e}^{zx_1 + q(z)(T-t)}$, $z \in \mathbb{C}$. Let us consider the GKW decomposition $(\vartheta(z), L(z))$ of $\widetilde{H}(z)$ with respect to S, $z \in \mathcal{S}$. We define $W(t, x, z) := H(z)_{t-}(\mathrm{e}^{zx_1} - 1)$. Then, by Corollary 3.1(i) applied to $L^1 = L^0$ and $L^2 = L(z)$, with $W^1 = W^0$ and $W^2 = W$, we get $\mathrm{d}\langle L^0, L(z)\rangle_t = Q(z)\widetilde{H}(z)_{t-}\,\mathrm{d}t$, where we define

$$Q(z) := (\mathrm{e}^z - 1)\left(\lambda_1 - \frac{\lambda_2}{\mathrm{e}^z} - \frac{(\lambda_1 - \frac{\lambda_2}{\mathrm{e}})(\lambda_1 + \frac{\lambda_2}{\mathrm{e}^{z+1}})}{\lambda_1 + \frac{\lambda_2}{\mathrm{e}^2}}\right).$$

Assumptions (ii) of Theorem 4.3 are satisfied, and hence the assumptions of Theorem 4.1 are satisfied. Thus, $L_t^j = \int_{\mathcal{S}^j} \widetilde{L}(z)_t\zeta^j(\mathrm{d}z)$, a.s., $t \in [0, T]$, $j = 1, \ldots, d$. Since $\widetilde{H}(z)_-$ is locally bounded, it is integrable with respect to Leb. on $[0, T]$. So, Proposition 4.1(iii) yields $B^j = T\int_{\mathcal{S}^j} \exp(zX_0^1 + q(z)T)Q(z)\zeta^j(\mathrm{d}z)$.

We now conclude by computing the covariance matrix C. We consider $\mathcal{S}^i$ and $\mathcal{S}^j$ and define $f^1(t, x, z) := f(t, x, z_1)$, $z_1 \in \mathcal{S}^i$, $f^2(t, x, z) := f(t, x, z_2)$, $z_2 \in \mathcal{S}^j$, and

$$P(z_1, z_2) := (\mathrm{e}^{z_1} - 1)(\mathrm{e}^{z_2} - 1)\left(\lambda_1 + \frac{\lambda_2}{\exp(z_1 + z_2)} - \frac{\left(\lambda_1 + \frac{\lambda_2}{\exp(z_1+1)}\right)\left(\lambda_1 + \frac{\lambda_2}{\exp(z_2+1)}\right)}{\lambda_1 + \frac{\lambda_2}{\mathrm{e}^2}}\right).$$

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Then Corollary 3.1(i) yields $\mathrm{d}\langle L(z_1), L(z_2)\rangle_t = P(z_1, z_2)H(z_1)_{t-}H(z_2)_{t-}\,\mathrm{d}t,\ z_1 \in \mathcal{S}^i,\ z_2 \in \mathcal{S}^j$, and the density is locally bounded. From Theorem 4.3, we get

$$C^{ij} = \int_0^T \int_{\mathcal{S}^i} \int_{\mathcal{S}^j} \Big\{ \exp\left((z_1 + z_2)X_0^1 + q(z_1 + z_2)t + (q(z_1) + q(z_2))(T - t)\right) \times P(z_1, z_2) \Big\} \zeta^j(\mathrm{d}z_2)\zeta^i(\mathrm{d}z_1)\,\mathrm{d}t.$$

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