Ergod. Th. & Dynam. Sys. (1997), **17**, 663–666
Printed in the United Kingdom © *1997 Cambridge University Press*

Conformal isomorphism of inner functions

N. F. G. MARTIN
Department of Mathematics, Math-Astronomy Building, University of Virginia, Charlottesville, VA 22903, USA

(*Received* 30 *January* 1995)

Abstract. If a non-Möbius inner function on the disk with a fixed point in the open disk has a boundary function which is measure theoretically conjugate to the boundary function of another inner function with fixed point in the open disk, then the inner functions are conformally conjugate.

The open unit disk will be denoted by **D** and its boundary by **T**. An inner function is a bounded analytic function f on **D** whose boundary function f^*, defined a.e. on **T** by $f^*(x) = \lim_{r\to 1^-} f(rx)$, takes its values in **T**.

If an inner function has a fixed point p in **D**, then the harmonic measure with respect to p, λ_p, is a f^*-invariant measure [**4**] and the system $(\mathbf{T}, \lambda_p, f^*)$ is an abstract dynamical system. The study of these systems was started by Aaronson [**1**] and Neuwirth [**8**]. Further work can be found in [**2**, **3**, **5**–**7**].

In this paper we show that if two such systems are measure theoretically conjugate and one of the defining inner functions is not Möbius, then the defining inner functions are conformally conjugate. The idea of the proof is due to Shub and Sullivan [**10**]. Also see [**11**]. The principle tool used is a Markov partition for $(\mathbf{T}, \lambda_p, f^*)$ obtained by Craizer [**2**] with a cannonical family of measures which have real analytic density functions.

THEOREM. *Let f and g be inner functions with fixed points p_1 and p_2 in* **D**. *If either f or g is not Möbius and $(\mathbf{T}, \lambda_{p_1}, f^*)$ and $(\mathbf{T}, \lambda_{p_2}, g^*)$ are measure theoretically conjugate, then f and g are conformally conjugate.*

Proof. Take

$$m_1(z) = \frac{z + p_1}{1 + \overline{p}_1 z} \quad \text{and} \quad m_2(z) = \frac{z + p_2}{1 + \overline{p}_2 z}.$$

Then $m_1^{-1} \circ f \circ m_1$ and $m_2^{-1} \circ g \circ m_2$ have fixed points at zero and if these functions are conformally isomorphic so are f and g. Thus we will assume that f and g have fixed points at zero. Since harmonic measure on **T** with respect to zero is normalized Lebesgue measure, this assumption implies that f^* and g^* preserve Lebesgue measure which will be denoted by λ.

Suppose that ϕ is a metric automorphism of $(\mathbf{T}, \lambda)$ such that $f^* \circ \phi = \phi \circ g^*$ and f is not Möbius. Suppose that g is a Möbius transformation. Then g being a one-to-one

https://doi.org/10.1017/S0143385797084964 Published online by Cambridge University Press

map of **D** to itself which sends **T** to itself it has the form

$$g(z) = c\frac{z-a}{1-\bar{a}z}$$

for $|c| = 1$, $|a| < 1$. However, by assumption $g(0) = 0$ and g is a rotation so $(\mathbf{T}, \lambda, g^*)$ has zero entropy. This is impossible since Aaronson [**1**] shows that non-Möbius inner functions have exact boundary functions which must have positive entropy. Thus g is not Möbius.

Craizer has shown that there exists a finite partition $\mathcal{P} = \{I_i : 1 \leq i \leq p\}$ of **T** into intervals such that for $1 \leq i \leq p$ given that there exists an open disk U_i such that $U_i \cap \mathbf{T} = I_i$ and a normal family of holomorphic functions $F_k^{(n)}$ on U_i with $n \in \mathbf{N}$ and $1 \leq k \leq \#(\mathcal{P} \vee (f^*)^{-1}(\mathcal{P}) \vee \cdots \vee (f^*)^{-n}(\mathcal{P})) = \#\mathcal{P}^{(n)}$, such that $F^{(n)}(I_i) \subset \mathbf{R}$ and

$$\int_S F_k^{(n)}\, d\lambda = \lambda(f^{*-n}(S) \cap B_k)$$

for all measurable $S \subset I_i$ and $B_k \in (\mathcal{P} \vee (f^*)^{-1}(\mathcal{P}) \vee \cdots \vee (f^*)^{-n}(\mathcal{P}))$ and $n \in \mathbf{N}$. (See p. 132 and Theorem 2.2 of [**2**].)

Now take $n = 1$. Then $\mathcal{P}^{(1)} = \{I_i \cap f^{*-1}(I_j) : 1 \leq i, j \leq p\}$. Now for each $1 \leq i \leq p$ there is an open set U_i such that $U_i \cap \mathbf{T} = I_i$ and holomorphic functions $F_{m,j}$ defined on U_i such that $F_{m,j}(I_i) \subset \mathbf{R}$ and

$$\begin{aligned}\int_S F_{m,j}\, d\lambda &= \lambda(f^{*-1}(S) \cap f^{*-1}(I_m) \cap I_j) \\ &= \lambda(f^{*-1}(S \cap I_m) \cap I_j)\end{aligned}$$

for $S \subset I_i$. Since $S \cap I_m = \emptyset$ unless $m = i$, $F_{m,j} = 0$ a.e. unless $m = i$. Thus we have partitions $\{I_i : 1 \leq i \leq p\}$ of **T** into intervals and holomorphic functions $F_{i,j}$ on open sets U_i such that $U_i \cap \mathbf{T} = I_i$ and

$$\int_S F_{i,j}\, d\lambda = \lambda(f^{*-1}(S) \cap I_j),$$

where $S \subset I_i$.

Similarly, there is a partition $\{L_l : 1 \leq l \leq q\}$ of **T** into intervals and holomorphic functions $G_{l,m}$ on open sets V_l such that $V_l \cap \mathbf{T} = L_l$ for $1 \leq l \leq q$ with $G_{l,m}(L_l) \subset \mathbf{R}$ and for any measurable $S' \subset L_l$

$$\int_{S'} G_{l,m}\, d\lambda = \lambda(g^{*-1}(S') \cap L_m). \tag{1}$$

Consider the measure spaces $(\phi^{-1}(I_i) \cap L_l, \nu_{i,l}{}^{j,m})$ and $(I_i \cap \phi(L_l), \mu_{i,l}{}^{j,m})$ with measures defined by

$$\mu_{i,l}{}^{j,m}(S) = \lambda(f^{*-1}(S) \cap I_j \cap \phi(L_m))$$

and

$$\nu_{i,l}{}^{j,m}(S') = \lambda(g^{*-1}(S') \cap \phi^{-1}(I_j) \cap L_m),$$

where $S \subset I_i \cap \phi(L_l)$ and $S' \subset \phi^{-1}(I_i) \cap L_l$. Since λ is f^* and g^* invariant one can easily check that both $\mu_{i,l}{}^{j,m}$ and $\nu_{i,l}{}^{j,m}$ are absolutely continuous with respect to λ. Let

https://doi.org/10.1017/S0143385797084964 Published online by Cambridge University Press

$D\mu_{i,l}{}^{j,m}$ and $D\nu_{i,l}{}^{j,m}$ denote the Radon–Nikodym derivatives of $\mu_{i,l}{}^{j,m}$ and $\nu_{i,l}{}^{j,m}$ with respect to λ.

Since ϕ is a metric automorphism it is an a.e. bijection of $\phi^{-1}(I_i) \cap L_l$ onto $I_i \cap \phi(L_l)$. Since $f^* \circ \phi = \phi \circ g^*$, and λ is ϕ invariant

$$\nu_{i,l}{}^{j,m}(\phi^{-1}(S)) = \mu_{i,l}{}^{j,m}(S)$$

and

$$\mu_{i,l}{}^{j,m}(\phi(S')) = \nu_{i,l}{}^{j,m}(S')$$

for $S \subset I_i \cap \phi(L_l)$ and $S' \subset \phi^{-1}(I_i) \cap L_l$. Thus ϕ is a metric isomorphism of $(\phi^{-1}(I_i) \cap L_l,\ \nu_{i,l}{}^{j,m})$ onto $(I_i \cap \phi(L_l),\ \mu_{i,l}{}^{j,m})$ and since λ is ϕ invariant

$$D\mu_{i,l}{}^{j,m} \circ \phi = D\nu_{i,l}{}^{j,m} \tag{2}$$

a.e. on $\phi^{-1}(I_i) \cap L_l$ for all $1 \leq i, j \leq p$ and $1 \leq l, m \leq q$.

Define

$$D\mu(x) = \sum_{m=1}^{q}\sum_{j=1}^{p}\sum_{l=1}^{q}\sum_{i=1}^{p} D\mu_{i,l}{}^{j,m}(x)\chi_{I_i \cap \phi(L_l)}(x)$$

$$D\nu(x) = \sum_{m=1}^{q}\sum_{j=1}^{p}\sum_{l=1}^{q}\sum_{i=1}^{p} D\nu_{i,l}{}^{j,m}(x)\chi_{\phi^{-1}(I_i)\cap L_l}(x).$$

It follows from equation (2) that $D\mu \circ \phi = D\nu$ a.e. on $\mathbf{T}$.

Let l be fixed and $S' \subset L_l$. Then using equation (1),

$$\begin{aligned}
\int_{S'} D\nu\, d\lambda &= \int_{S'} \sum_{m=1}^{q}\sum_{j=1}^{p}\sum_{i=1}^{p} D\nu_{i,l}{}^{j,m}(x)\chi_{\phi^{-1}(I_i)\cap L_l}(x)\, d\lambda \\
&= \sum_{m=1}^{q}\sum_{j=1}^{p}\sum_{i=1}^{p} \int_{S'} D\nu_{i,l}{}^{j,m}(x)\chi_{\phi^{-1}(I_i)\cap L_l}(x)\, d\lambda \\
&= \sum_{m=1}^{q}\sum_{j=1}^{p}\sum_{i=1}^{p} \lambda(g^{*-1}(S' \cap \phi^{-1}(I_i)) \cap \phi^{-1}(I_j) \cap L_m) \\
&= \sum_{m=1}^{q} \lambda(g^{*-1}(S') \cap L_m) \\
&= \sum_{m=1}^{q} \int_{S'} G_{l,m}\, d\lambda \\
&= \int_{S'} \sum_{m=1}^{q} G_{l,m}\, d\lambda.
\end{aligned}$$

Thus $D\nu = \sum_{m=1}^{q} G_{l,m}$ a.e. on L_l. Similarly, $D\mu = \sum_{j=1}^{p} F_{i,j}$ a.e. on I_i. Let $x \in \phi^{-1}(I_i) \cap L_l$. Then $D\nu(x) = \sum_{m=1}^{q} G_{l,m}(x)$ and $D\mu \circ \phi(x) = \sum_{j=1}^{p} F_{i,j} \circ \phi(x)$ and since $D\mu \circ \phi = D\nu$ a.e. on $\mathbf{T}$,

$$\sum_{j=1}^{p} F_{i,j} \circ \phi = \sum_{m=1}^{q} G_{l,m} \tag{3}$$

https://doi.org/10.1017/S0143385797084964 Published online by Cambridge University Press

a.e. on $\phi^{-1}(I_i) \cap L_l$ for $1 \leq i \leq p$ and $1 \leq l \leq q$.

Select integers i and l such that $\lambda(\phi^{-1}(I_i) \cap L_l) > 0$. Define $F_i = \sum_{j=1}^{p} F_{i,j}$ and $G_l = \sum_{m=1}^{q} G_{l,m}$. By Craizer's construction [**2, 3**], $F_{i,j}$ and $G_{l,m}$ are not identically zero and since F_i and G_l are analytic there is an interval $J \subset I_i$ with $\lambda(\phi^{-1}(J) \cap L_l) > 0$ and such that F_i is bijective on J. Let h be real analytic on $F_i(J)$ to J such that $h(F_i(y)) = y$ for $y \in J$.

By equation (3), $F_i \circ \phi(x) = G_l(x)$ for almost all $x \in \phi^{-1}(I_i) \cap L_l$ so that if $x \in \phi^{-1}(J) \cap L_l$ is a point for which equality holds, $\phi(x) = h(F_i(\phi(x))) = h(G_l(x))$ and

$$\phi = h \circ G_l$$

a.e. on $\phi^{-1}(J) \cap L_l$. Define ψ on $\phi^{-1}(J) \cap L_l$ by $\psi = h \circ G_l$. Since h and G_l are real analytic so is ψ and since $\phi = \psi$ a.e. on $\phi^{-1}(J) \cap L_l$ and the Radon–Nikodym derivative of $\lambda\phi$ with respect to λ is one, $|\psi'| = 1$ and ψ is an isometry. Extend this isometry to $\overline{\mathbf{D}}$ as a rotation.

Since $f \circ \psi - \psi \circ g$ is a bounded analytic function on $\mathbf{D}$, $\log|f^* \circ \psi - \psi \circ g^*|$ is integrable on $\mathbf{T}$ unless $f \circ \psi = \psi \circ g$ [**9**, Theorem 15.19]. However, $f^* \circ \phi = \phi \circ g^*$ a.e. on $\mathbf{T}$ and $\psi = \phi$ a.e. on $\phi^{-1}(J) \cap L_l$, so $f^* \circ \psi - \psi \circ g^* = 0$ on a set of positive measure and $\log|f^* \circ \psi - \psi \circ g^*|$ is not integrable. Thus $f \circ \psi = \psi \circ g$. □

Acknowledgements. This work was started while the author was visiting University College London and he would like to thank Milne Anderson for making the visit possible and the Department of Mathematics for their hospitality.

REFERENCES

[1] J. Aaronson. Ergodic theory of inner functions of the upper half plane. *Ann. Inst. Henri Poincaré* **14** (1978), 233–253.
[2] M. Craizer. Entropy of inner functions. *Israel J. Math.* **74** (1991), 129–168.
[3] M. Craizer. The Bernoulli property of inner functions. *Ergod. Th. & Dynam. Sys.* **12** (1992), 209–215.
[4] G. Letac. Which functions preserve Cauchy laws? *Proc. Amer. Math. Soc.* **67** (1977), 277–286.
[5] N. F. G. Martin. On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms. *Bull. London Math. Soc.* **15** (1983), 343–348.
[6] N. F. G. Martin. Classification of some finite Blaschke products as metric endomorphisms. *J. Math. Anal. Appl.* **114** (1986), 205–209.
[7] N. F. G. Martin. On ergodic properties of restrictions of inner functions. *Ergod. Th. & Dynam. Sys.* **9** (1989), 137–152.
[8] J. H. Neuwirth. Ergodicity of some mappings of the circle and line. *Israel J. Math.* **31** (1978), 359–367.
[9] W. Rudin. *Real and Complex Analysis*, 3rd edn. McGraw Hill, 1987.
[10] M. Shub and D. Sullivan. Analytic maps of the circle with many theoretical moduli and the distortion random variable of endomorphisms. *Preprint.*
[11] M. Shub and D. Sullivan. Expanding endomorphisms of the circle revisted. *Ergod. Th. & Dynam. Sys.* **5** (1985), 285–289.

https://doi.org/10.1017/S0143385797084964 Published online by Cambridge University Press