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A Frattini-like subgroup

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1. *Introduction.* The Frattini subgroup $\phi(G)$ of a group G is the intersection of G and all its maximal subgroups. The following results for finite groups are well known:

THEOREM A0. *If G is a finite group, then the following three conditions are equivalent:*

(i) *G is nilpotent,*

(ii) *$G/\phi(G)$ is nilpotent,*

(iii) *$\phi(G) \geqslant G'$.*

THEOREM B0. *If G is a finite group, then $\phi(G)$ is nilpotent.*

The situation is far less straightforward in infinite groups, largely due to the possibility of G having insufficient maximal subgroups or even none at all. Most of the results similar to A0 and B0 which have been obtained for infinite groups have therefore been for groups with some condition which ensures a large number of maximal subgroups, e.g. finitely generated soluble groups ((10), p. 197) or residually finite soluble minimax groups ((8), § 5).

Our aim in this paper is to introduce a class of proper subgroups of a group G, the *major subgroups*, which are, in some sense, as close as possible to G. If G satisfies the maximal condition, then the major subgroups coincide with the maximal subgroups. We consider the subgroup $\mu(G)$, the intersection of the major subgroups of G, and obtain results similar to A0 and B0 in various classes of groups.

Without restricting ourselves to one of these classes, we are able to show that a certain nilpotency condition on G implies that $\mu(G) \geqslant G'$, thus extending the (i) $\Rightarrow$ (iii) part of A0. Let $\mathfrak{Z}$ be the class of groups G in which each proper subgroup has a proper normal closure. $\mathfrak{Z}$ is a class of generalized nilpotent groups in the sense of Robinson ((10), p. 1) and is Q-closed. Clearly $\mathfrak{Z}$ contains the class of groups in which each subgroup is descendant.

THEOREM A1. *If G is a hypercentral $\mathfrak{Z}$-group, then $\mu(G) \geqslant G'$.*

The more important part of Theorem A0 is the effect of the structure of $G/\mu(G)$ on that of G. For these results, we do have to consider different classes and prove:

THEOREM A2. (*a*) *Let $G \in \mathfrak{U}$; then $G/\mu(G)$ is locally nilpotent if and only if G is locally nilpotent.*

(*b*) *Let G be nilpotent-by-finite; then* (i) *$G/\mu(G)$ is hypercentral if and only if G is hypercentral;* (ii) *$\mu(G) \geqslant G'$ if and only if G is a hypercentral $\mathfrak{Z}$-group.*

(*c*) *Let $G \in \mathfrak{S}_0$; then $G/\mu(G)$ is hypercentral if and only if G is hypercentral.*

(*d*) *Let $G \in \mathfrak{S}_1$; then $\mu(G) \geqslant G'$ if and only if G is nilpotent.*

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(*e*) *Let G be a soluble group satisfying the minimal condition on normal subgroups*; *then* (i) *$G/\mu(G)$ is hypercentral if and only if G is hypercentral*; (ii) *$\mu(G) \geqslant G'$ if and only if G is nilpotent.*

The second type of result concerns the nilpotency of $\mu(G)$ and we prove

THEOREM B. (*a*) *Let $G \in \mathfrak{U}$; then $\mu(G)$ is locally nilpotent.*

(*b*) *Let G be nilpotent-by-finite; then $\mu(G)$ is hypercentral.*

(*c*) *Let $G \in \mathfrak{S}_0$; then $\mu(G)$ is hypercentral.*

The class $\mathfrak{U}$ in part (*a*) of Theorems A 2 and B is the class of locally finite groups G such that, for each $H \leqslant G$ and for each set of primes Π, the Sylow Π-subgroups of H are conjugate in H. This class was introduced in (1) and a large number of results on finite soluble groups have since been extended to this class. The results given in part (*a*) of Theorems A 2 and B can, of course, be stated in a much more general form but at this point we just give the most important case of the general results.

The layout of the paper is as follows. In section 2, we define the major subgroups and prove some basic results about them, including the important fact that every proper subgroup of a group G is contained in a major subgroup of G (2·3). We go on to consider the μ-subgroup of Abelian groups and note some simple counterexamples to possible conjectures. For example, it is not always true that $\mu(G_1 \times G_2) = \mu(G_1) \times \mu(G_2)$. We make use of the very strong restrictions on major subgroups of Abelian groups in proving Theorem A 1 in section 4.

The remaining sections are concerned with the proof of Theorems A 2 and B in the various classes. Whenever we have a suitable Sylow theory, Lemma 2·3 shows that we can apply the usual Frattini argument. This is the method used to obtain the results for $\mathfrak{U}$-groups in section 5. A Frattini argument is also used in section 7 where it is applied to a certain class of conjugate complements in an $\mathfrak{S}_0$-group.

The results for nilpotent-by-finite groups given in section 6 are really obtained for Abelian-by-finite groups and a result of D. Robinson is used to extend them.

Except where otherwise stated, our notation and terminology is that used in Robinson's book ((9) and (10)).

2. *Major subgroups.* Let U be a proper subgroup of G. If

$$U = U_0 < U_1 < \ldots < U_\alpha = G$$

is a properly ascending chain of subgroups from U to G, then we shall say that the ordinal α is the *type* of the chain. We define $m(U)$ (or, more precisely, $m_G(U)$) to be the least upper bound of the types of all properly ascending chains from U to G. Then $m(U)$ can be considered as a measure of how far U is from G. Clearly, $m(U) = 1$ if and only if U is a maximal subgroup of G. If G is either infinite cyclic or a quasicyclic p-group, then $m(1) = \omega$, although, in the first case, there are no properly ascending chains of type ω from 1 to G and, in the second case, there is such a chain. [Although it is not strictly necessary in our definition of $m(U)$, we may clearly restrict our considerations to complete chains from U to G, that is chains in which $U_\lambda = \bigcup_{\mu<\lambda} U_\mu$ whenever λ is a limit ordinal.]

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A proper subgroup M of G is called a *major subgroup* if $m(U) = m(M)$, whenever $M \leqslant U < G$. Roughly speaking, this means that every proper subgroup containing M is as far from G as M is. The major subgroups of a group satisfying the maximal condition (in particular, an infinite cyclic group) are just the maximal subgroups, whereas every proper subgroup of a quasi-cyclic p-group is a major subgroup.

It is clear that if M is a major subgroup of G and $M \leqslant U < G$, then U is also a major subgroup of G.

We begin by considering the possible values of $m(M)$, where M is a major subgroup.

THEOREM 2·1. *Let M be a major subgroup of G and $m(M) = \rho$. Then ρ cannot be expressed as a sum of two smaller ordinals.*

Conversely, if ρ is an ordinal which cannot be expressed as a sum of two smaller ordinals, then there is a group G with a major subgroup M such that $m(M) = \rho$.

Proof. Suppose that $\rho = \sigma + \tau$ with $1 \leqslant \sigma, \tau < \rho$; then there is a chain of type α with $\alpha > \sigma$:

$$M = M_0 < \ldots < M_\sigma < \ldots < M_\alpha = G.$$

But $m(M_\sigma) = m(M) = \rho$ and so there is a chain from M_σ to G having type β with $\beta > \tau$:

$$M_\sigma < M_{\sigma+1} < \ldots < M_{\sigma+\beta} = G.$$

But this gives a properly ascending chain from M to G of type $\sigma + \beta > \sigma + \tau = \rho$ contrary to $m(M) = \rho$.

For the example in the second part of the theorem, let $X = \operatorname{Dr}_{\alpha<\rho} \langle x_\alpha | x_\alpha^2 = 1 \rangle$. For each pair of ordinals α, β with $\alpha < \beta < \rho$, X has an automorphism $y_{\alpha\beta}$ of order 2 which maps x_β to $x_\alpha x_\beta$ and fixes each x_γ with $\gamma \neq \beta$. Let $Y = \langle y_{\alpha\beta} | \alpha < \beta < \rho \rangle$ and let G be the split extension of X by Y. If H is any subgroup of G containing Y, then $H \cap X \lhd G$ and so, to find the chains from Y to G, we first determine the normal subgroups N of G contained in X. Let $x = x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_k} \in N$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_k$. Then N contains $[x, y_{\alpha_i \alpha_k}] = x_{\alpha_i}$, for $i = 1, \ldots, k-1$. Hence $x_{\alpha_k} \in N$ and so N contains $x_\alpha = [x_{\alpha_k}, y_{\alpha\alpha_k}]$, for each $\alpha < \alpha_k$, and thus $N \geqslant \operatorname{Dr}_{\alpha \leqslant \alpha_k} \langle x_\alpha \rangle$. It follows that N is of the form $X_\beta = \operatorname{Dr}_{\alpha<\beta} \langle x_\alpha \rangle$ and that the only proper subgroups of G containing Y are YX_β with $\beta < \rho$. If ρ cannot be written as the sum of two smaller ordinals, then $m(YX_\beta) = \rho = m(Y)$ and Y is a major subgroup.

It is sometimes possible to show that $m(M) < \omega^2$ and it then follows from (2·1) that $m(M) = 1$ or ω. If M is a maximal subgroup and $N \lhd G$, then either $N \leqslant M$ or $NM = G$. If M is a major subgroup we may have $M < NM < G$. The following elementary result is useful in discussing this situation in the case $m(M) = \omega$.

LEMMA 2·2. *Let M be a major subgroup of G. If $m(M) = \rho$ and $M \leqslant H < G$ then every properly ascending chain from M to H has type less than ρ. If $m(M) = \omega$, then there exists a properly ascending chain of type ω from M to G.*

Proof. If there is a chain of type ρ from M to H then adding G to this gives a chain of type $\rho + 1$ from M to G.

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For the second part of the lemma, we construct a chain

$$M = M_0 < M_1 < \ldots < M_n < \ldots < M_\omega = G,$$

by defining the M_n inductively. So suppose that we can define the proper subgroups $M_0, M_1, \ldots, M_{n-1}$. Since $M \leqslant M_{n-1} < G$, we have $m(M_{n-1}) = m(M) = \omega$. Therefore M_{n-1} is not maximal in G and there is a subgroup M_n with $M_{n-1} < M_n < G$.

It is clear that the second part of the lemma can be extended to $m(M) = \rho$, where ρ is any ordinal of the form

$$\rho = \alpha_1 + \alpha_2 + \ldots$$

with $\alpha_i < \rho$, for each i. We leave open the general question.

It is perhaps worth noting that although every chain from M to H has type less than $m(M)$, we may still have $m_H(M) = m_G(M)$. For example, let X be the group generated by $x_1, x_2, \ldots$ such that $x_{i+1}^2 = x_i$ $(i = 1, 2, \ldots)$ and let G be the split extension of X by the automorphism y which maps each element to its inverse. Then $\langle y \rangle$ is a major subgroup of G and $m_G(\langle y \rangle) = \omega$. If $H = \langle x_1, y \rangle$, then H is an infinite dihedral group and $m_H(\langle y \rangle) = \omega$, although every ascending chain from $\langle y \rangle$ to H is finite.

The following result, although elementary, will be of basic importance in what follows as it enables us to use a Frattini argument.

LEMMA 2·3. *Every proper subgroup U of a group G is contained in a major subgroup of G.*

Proof. Consider the set of ordinals $\{m(V) | U \leqslant V < G\}$. This has a smallest element m_0, say. Choose $M \geqslant U$ such that $m(M) = m_0$; then M is a major subgroup of G. For if $M \leqslant W < G$, then $m(W) \leqslant m(M)$; but $m(W) \geqslant m_0 = m(M)$ and so $m(W) = m(M)$.

3. *Abelian groups.*

THEOREM 3·1. *Let M be a major subgroup of the Abelian group G. Then either $m(M) = 1$ and G/M is cyclic of prime order or $m(M) = \omega$ and G/M is a quasi-cyclic group.*

Proof. Let $x \in G - M$ and let L be a subgroup of G maximal with respect to $M \leqslant L$ but $x \notin L$. Writing $\bar{x}$ for the image of x in G/L, $\bar{x}$ is contained in every subgroup of G/L. By considering $\langle \bar{x}^r \rangle$, we see that $\bar{x}$ has prime order p, say. If $\bar{g}_1, \ldots, \bar{g}_n \in G/L$, then $\langle \bar{x} \rangle$ is the unique minimal subgroup of the finitely generated Abelian group $\langle \bar{g}_1, \ldots, \bar{g}_n \rangle$, which is therefore cyclic of order a power of p.

Thus G/L is a locally cyclic p-group.

If G/L is finite, then G/L is cyclic of prime order and $M = L$. If G/L is quasi-cyclic, then $m(M) = m(L) = \omega$ and every properly ascending chain from M to L is finite (2·2). Thus L/M is a finitely generated Abelian group. If L/M contains a subgroup K/M of index q, where q is a prime not equal to p, then G/K is the direct product of L/K and a quasi-cyclic p-group Q/K. But then Q is a maximal subgroup of G and $m(M) = m(Q) = 1$. This shows that L/M is a finite p-group and G/M is the direct product of a quasi-cyclic p-group and a finite p-group. If the finite factor is non-trivial then again M is contained in a maximal subgroup and so $m(M) = 1$. Therefore if $m(M) = \omega$, G/M must be quasi-cyclic.

THEOREM 3·2. *If the Abelian group G has a factor group $G/H \cong C_{p^\infty}$, then $\mu(G)$ is a periodic p'-group.*

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Proof. If x is not a p'-element, then $\langle x^p \rangle < \langle x \rangle$ and so we can choose a subgroup M maximal subject to $x^p \in M$ but $x \notin M$. As in the proof of (3·1), G/M is a locally cyclic p-group. Therefore $G/(H \cap M)$ is isomorphic to C_{p^∞}, $C_{p^\infty} \times C_{p^\infty}$, or $C_{p^\infty} \times C_{p^n}$, for some integer n. In the first two cases, it is obvious that $\mu(G/(H \cap M))$ is trivial and hence $\mu(G) \leqslant M$. In the third case, let the quasicyclic group be

$$\langle x_1, x_2, \ldots | x_1^p = 1, x_{i+1}^p = x_i, i = 1, 2, \ldots \rangle$$

and the finite cyclic group be generated by y so that $G/(H \cap M) \cong \langle y \rangle \times \langle x_1, x_2, \ldots \rangle$. Then $\langle y \rangle$ is a major subgroup and so also is $\langle x_n y \rangle$ and therefore

$$\mu(G/(H \cap M)) \leqslant \langle y \rangle \cap \langle x_n y \rangle = 1.$$

Therefore in all three cases, we have $\mu(G) \leqslant M$ and so $x \notin \mu(G)$.

Since a quasi-cyclic subgroup of an Abelian group is a direct factor, the hypothesis on G in (3·2) is equivalent to G having a section isomorphic to C_{p^∞}. The following result therefore follows immediately.

COROLLARY 3·3. *If G is an Abelian group, then $\mu(G)$ has no quasi-cyclic factor group. In particular, if $\mu(G)$ satisfies the minimal condition, then it is finite.*

If $G_1 \cong C_{p^\infty}$ and $G_2 \cong C_{p^n}$ $(n > 1)$ then $\mu(G_1) = 1$ and $\mu(G_2) \cong C_{p^{n-1}}$. But it follows from (3·2) that $\mu(G_1 \times G_2) = 1$ and so in general $\mu(G_1 \times G_2) \neq \mu(G_1) \times \mu(G_2)$.

If N is a normal subgroup of the finite group G, then $\phi(N) \leqslant \phi(G)$. It is easily seen that this result does not extend to $\mu(G)$ by considering the subgroup $N \cong C_{p^n}$ $(n > 1)$ of $G \cong C_{p^\infty}$.

4. *Proof of Theorem* A 1. If $\mu(G) \geqslant G'$, then by (2·3), it is clear that no proper subgroup of G covers G/G' and this in turn implies that every proper subgroup of G is contained in a proper normal subgroup of G. This is the reason for considering the class $\mathfrak{Z}$ defined in the Introduction.

We begin our proof by showing that the major subgroups M that we consider satisfy $m(M) = 1$ or ω and so (2·2) can be applied. We saw that this was the case in Abelian groups (3·1) and the following result may be considered as an extension of that result.

LEMMA 4·1. *Let M be a major subgroup of the $(SD^{\mathfrak{Q}} \cap \mathfrak{Z})$-group G. Then $MG' < G$ and $m(M) = 1$ or ω.*

(Here $SD^{\mathfrak{Q}}$ denotes the class of groups in which each factor group is hypoabelian.)

Proof. M^G is a proper subgroup of G and so $m(M^G) = m(M)$. Since G/M^G is an SD-group, $M^G G' < G$ and so $MG' < G$ and $m(M) = m(MG')$. It follows from (3·1) that $m(MG') = 1$ or ω.

LEMMA 4·2. *Let M be a major subgroup of the $SD^{\mathfrak{Q}}$-group G. If $M \lhd G$, then $M \geqslant G'$.*

Proof. Since G/M is an SD-group, $MG' < G$ and so $m(M) = m(MG')$. By (3·1), $m(MG') = 1$ or ω. If $m(M) = 1$, then M is maximal in G and so $M = MG' \geqslant G'$. So we may assume that $m(M) = \omega$, G/MG' is a quasi-cyclic p-group and every properly ascending chain from M to MG' is finite. Putting $M = 1$, we have a group G with $G/G' \cong C_{p^\infty}$ and G' satisfying the maximal condition. If $G' \neq 1$, then G'/G'' is a finitely generated Abelian group and so $G'/(G')^q G''$, where q is prime, is a finite elementary Abelian q-group. Thus $G/(G')^q G''$ is a metabelian group satisfying the minimal condi-

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tion and is a finite extension of a quasi-cyclic p-group. Since G has no maximal subgroups, $G/(G')^q G'' \cong C_{p^\infty}$ contrary to $(G')^q G'' < G'$. Thus $G' = 1$ and so $M \geqslant G'$.

THEOREM 4·3. *If G is a hypercentral $\mathfrak{Z}$-group, then $\mu(G) \geqslant G'$.*

Proof. Let M be a major subgroup of G. Since G is hypercentral, it is an SD^q-group ((9), p. 48) and so $MG' < G$ and $m(M) = 1$ or ω (4·1). If $m(M) = 1$, then $M = MG' \geqslant G'$ and so we may assume that $m(M) = \omega$.

Therefore G/MG' is a quasi-cyclic p-group (3·1) and every properly ascending chain from M to MG' is finite (2·2). If $M \not\geqslant G'$, then there is a maximal subgroup X of MG' containing M. X is a major subgroup of G with $m(X) = \omega$ and $XG' = MG' > X$.

Let $K = \mathrm{core}_G(X)$ and $Y/K = \zeta(G/K) \cap XG'/K$. Then Y/K is non-trivial and so $YX = XG'$. It follows that $XG'/X \cong Y/K$ is cyclic of prime order q, say. $K = \bigcap_{g \in G} X^g$ and XG'/X^g is cyclic of order q. Thus XG'/K is an elementary Abelian q-group and G/K is locally finite. If $q \neq p$, then G/K is the direct product of a quasi-cyclic p-group P/K and the elementary Abelian q-group XG'/K ((10), Corollary to Theorem 6·14). Thus X is contained in the maximal subgroup PX of G, contrary to $m(X) = \omega$. It follows that G/K is the extension of an elementary Abelian p-group XG'/K by a quasi-cyclic p-group.

Let $Z_1/K = \zeta(G/K)$. If $Z_1 X = G$, then $X \lhd G$ and it follows from (4·2) that $X \geqslant G'$, contrary to our construction of X. So we may assume that $XG' \leqslant Z_1 X < G$. Let $Z_2/Z_1 = \zeta(G/Z_1)$; then $Z_2 \cap Z_1 X > Z_1$ and we can choose an element $a \in (Z_2 \cap Z_1 X) - Z_1$. The mapping $\phi: G \to Z_1$, defined by $\phi(g) = [g, a]$, is a homomorphism. Z_1 has finite exponent and so $G/\mathrm{Ker}\,\phi$ has finite exponent. But $\mathrm{Ker}\,\phi = KC_G(a) \geqslant Z_1 X$ and $G/Z_1 X \cong C_{p^\infty}$. This final contradiction completes the proof of the theorem.

We observed when defining $\mathfrak{Z}$ that it was clearly necessary for G to be a $\mathfrak{Z}$-group for $\mu(G)$ to contain G'. This is illustrated by the group G which is the split extension of a quasi-cyclic 2-group by its involution α. For then $\langle\alpha\rangle$ is a major subgroup of G and so $\mu(G) \leqslant \mathrm{core}_G(\langle\alpha\rangle) = 1$ although G is a non-Abelian hypercentral group.

On the other hand, the Heineken–Mohamed group (6) is a group without centre, in which each subgroup is subnormal and so is certainly a non-hypercentral $\mathfrak{Z}$-group. This group is an extension of a countably infinite elementary Abelian p-group H by a quasi-cyclic p-group. It is easily seen that each subgroup of finite index in H is a major subgroup of G and so $\mu(G) = 1$.

5. *Locally finite groups.* In order to state the basic result of this section more concisely, we define the class $\mathfrak{D}$ of locally finite groups G such that, for each $H \leqslant G$ and for each prime p, the Sylow p-subgroups of H are conjugate in H. It is clear that $\mathfrak{D}$ contains all $\mathfrak{U}$-groups; it also contains the class of groups satisfying the minimal condition on p-subgroups for each prime p (11) and the class of periodic (locally nilpotent)-by-finite groups.

THEOREM 5·1. *Let N be a normal $\mathfrak{D}$-subgroup of the group G such that $N/(N \cap \mu(G))$ is locally nilpotent. Then N is locally nilpotent.*

Proof. Let P be a Sylow p-subgroup of N and suppose $N_G(P) < G$. Then $N_G(P)$ is contained in a major subgroup M of G. But then $N_G(P(N \cap \mu(G))) = N_G(P) \cdot (N \cap \mu(G)) \leqslant M < G$, contrary to $P(N \cap \mu(G))/(N \cap \mu(G))$ being the unique Sylow p-subgroup

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of $N/(N \cap \mu(G))$ ((3), Lemma 2·2). Therefore $N_G(P) = G$ and N has a unique Sylow p-subgroup for each prime p, i.e. N is locally nilpotent.

By considering the two cases $N = G$ and $N = \mu(G)$, we immediately obtain the following corollaries:

COROLLARY 5·2. *Let G be a $\mathfrak{D}$-group. Then G is locally nilpotent if and only if $G/\mu(G)$ is locally nilpotent.*

COROLLARY 5·3. *Let G be any group such that $\mu(G)$ is a $\mathfrak{D}$-group. Then $\mu(G)$ is locally nilpotent.*

For $\mathfrak{U}$-groups, a more general result concerning the influence of the structure of $G/\mu(G)$ on that of G can be obtained by considering formations. The necessary definitions will be found in (1).

THEOREM 5·4. *Let $\mathfrak{R}$ be a QS-closed subclass of $\mathfrak{U}$ and let $\mathfrak{F}$ be a saturated $\mathfrak{R}$-formation. If G is a $\mathfrak{R}$-group then $G \in \mathfrak{F}$ if and only if $G/\mu(G) \in \mathfrak{F}$.*

Proof. It is sufficient to prove that if $G/\mu(G) \in \mathfrak{F}$ then $G \in \mathfrak{F}$. Let $\Pi = \tilde{\omega}(\mathfrak{F})$ and let S_Π be a Sylow Π-subgroup of G. If $S_\Pi < G$, then S_Π is contained in a major subgroup M of G. But then $S_\Pi \mu(G) \leqslant M < G$ contrary to $G/\mu(G)$ being a Π-group and $S_\Pi \mu(G)/\mu(G)$ being a Sylow Π-subgroup of $G/\mu(G)$ ((1), Lemma 2·1 (ii)).

For each prime $p \in \Pi$, there is a normal subgroup C of G such that $G/C \in \mathfrak{f}(p)$ and $C/\mu(G) \in \mathfrak{S}_{p'}\mathfrak{S}_p$. Let $S_{p'}$ be a Sylow p'-subgroup of C so that $N_G(S_{p'}\mu(G)) = G$. If $N_G(S_{p'}) \neq G$, then $N_G(S_{p'})$ is contained in a major subgroup M of G and so

$$N_G(S_{p'}\mu(G)) = N_G(S_{p'}) \cdot \mu(G) \leqslant M < G.$$

This contradiction shows that $S_{p'} \lhd G$ and so $C \in \mathfrak{S}_{p'}\mathfrak{S}_p$. Thus

$$G \in \mathfrak{S}_\pi \cap \bigcap_{p \in \Pi} \mathfrak{S}_{p'}\mathfrak{S}_p \mathfrak{f}(p) = \mathfrak{F}.$$

Hartley (2) has considered $\Gamma(G)$, the intersection of the Galois subgroups of the $\mathfrak{U}$-group G, and has obtained similar results. (The extra hypothesis '$G \in \mathfrak{S}_\Pi$' in his Lemma 3·4 (iii) is unnecessary as the same argument as in (5·4) above can be applied.) A Galois subgroup of G is a proper subgroup H such that $G = PH$, where

$$P/\mathrm{core}_G(H) = O_p(G/\mathrm{core}_G(H)),$$

for some prime p. This definition is based on the well-known property of maximal subgroups of finite soluble groups and, in some ways, they behave rather better than the major subgroups; for example, $\Gamma(G_1 \times G_2) = \Gamma(G_1) \times \Gamma(G_2)$. On the other hand, a finite soluble group may possess Galois subgroups that are not maximal and so $\Gamma(G)$ may be smaller than $\phi(G)$.

If U is a proper subgroup of G, then U is contained in a major subgroup M of G. Since $M < G$, M is contained in a Galois subgroup T of G. T is also a major subgroup of G and so every proper subgroup of G is contained in a major Galois subgroup of G. Therefore all the above results could be obtained for the intersection of the major Galois subgroups and it seems possible that these are the 'right' analogues of maximal subgroups for $\mathfrak{U}$-groups.

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In the class of FC-groups, the Frattini subgroup works fairly well. It is certainly true that $\phi(G)$ is locally nilpotent and that G is locally nilpotent if and only if $G/\phi(G)$ is locally nilpotent. It is also a simple matter to prove:

THEOREM 5·5. *Let G be a $\mathfrak{G}$-group and let $\mathfrak{F}$ be a local formation of $\mathfrak{G}$-groups with $\tilde{\omega}(\mathfrak{F}) = \Pi$. Then $G \in \mathfrak{F}$ if and only if G is a Π-group and $G/\phi(G) \in \mathfrak{F}$.* (see (12) for the necessary definitions.)

However, one may have $G/\phi(G)$ being a Π-group without G being a Π-group simply by including a quasi-cyclic Π'-group in the centre of G. This slight difficulty can be avoided by using $\mu(G)$ and we get a similar result to (5·4).

6. *Nilpotent-by-finite groups.* We first consider Abelian-by-finite groups and then use a result of D. Robinson to extend the results obtained to nilpotent-by-finite groups. The following result enables us to make use of the results of the preceding section.

LEMMA 6·1. *Let H/K be a chief factor of the Abelian-by-finite group G. Then there is a chief factor U/V of G such that $H/K \overset{G}{\cong} U/V$, G/V is periodic and U/V is the unique minimal normal subgroup of G/V.*

Proof. Let V be a normal subgroup of G maximal with respect to $V \cap H = K$. Let $U = HV$ so that $U/V \overset{G}{\cong} H/K$. U/V is the unique minimal normal subgroup of G/V. Putting $V = 1$, it remains to show that if the Abelian-by-finite group G has a unique minimal normal subgroup U, then G is periodic. G has a normal Abelian subgroup A such that G/A is finite. We may assume that $A \neq 1$ and so $A \geqslant U$. Let $a \in A$; then $B = \langle a^G \rangle$ is a finitely generated Abelian normal subgroup containing U. There is an integer n such that B^n is torsion-free and so $B^n \cap U = 1$, since U is finite. But $B^n \lhd G$ and so $B^n = 1$ and B is finite. Therefore a has finite order and G is periodic. (In fact, G is a finite extension of an Abelian p-group.)

THEOREM 6·2. *Let N be a normal subgroup of the Abelian-by-finite group G. Then N is hypercentral if and only if $N/(N \cap \mu(G))$ is hypercentral.*

Proof. We show first that if $N/(N \cap \mu(G))$ is hypercentral then every chief factor of G below N is central in N.

Let H/K be such a chief factor; then $H/K \overset{G}{\cong} U/V$, where U/V is the unique minimal normal subgroup of the periodic group G/V (6·1). N covers U/V and so

$$H/K \overset{G}{\cong} (N \cap U)/(N \cap V)$$

and $N/(N \cap V)$ is periodic. $N/(N \cap V)$ is therefore a $\mathfrak{D}$-group and so is locally nilpotent (5·2). It follows that H/K is a finite central factor in N.

Let A be a normal Abelian subgroup of finite index in G. Let T be the torsion subgroup of $A \cap N$ and let p be a prime not dividing $|N/(N \cap A)|$. Let $x \in (A \cap N) - T$ and $B = \langle x^G \rangle T/T$. B/T is free Abelian of finite rank and B/B^{p^n} is central in N for all n. Therefore B/T is central in N and so x is a central element modulo T. Hence $(A \cap N)/T$ is central and N/T is nilpotent.

Each element of T is contained in a finite normal subgroup of G and so there is an ascending N-normal series from 1 to T with finite factors. Thus there is an ascending N-central series from 1 to T and so N is hypercentral.

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COROLLARY 6·3. *Let N be a normal subgroup of the nilpotent-by-finite group G. Then N is hypercentral if and only if $N/(N \cap \mu(G))$ is hypercentral.*

Proof. Let R be a nilpotent normal subgroup of finite index in G and let $N/(N \cap \mu(G))$ be hypercentral.

Since $\mu(G/R') \geqslant \mu(G)\,R'/R'$, (6·2) shows that NR'/R' is hypercentral. Therefore $NR/R' = (NR'/R') \cdot (R/R')$ is hypercentral. It follows that NR is hypercentral ((9), p. 57) and so N is hypercentral.

Taking the two cases $N = G$ and $N = \mu(G)$ gives Theorems A2 (b) and B(b). We noted in section 4 that if $\mu(G) \geqslant G'$ then G must be a $\mathfrak{Z}$-group. This gives the following partial converse to Theorem A 1.

COROLLARY 6·4. *Let G be nilpotent-by-finite. Then $\mu(G) \geqslant G'$ if and only if G is a hypercentral $\mathfrak{Z}$-group.*

7. *Soluble groups with finite Abelian section rank.* The class $\mathfrak{S}_0$ of soluble groups with finite Abelian section rank consists of those groups G with a finite normal series in which the factors are Abelian groups with finite 0-rank and finite p-rank for all primes p. Our results in this class will depend heavily on the following result which enables us to restrict our attention to chief factors of G.

THEOREM 7·1. *Let N be a normal subgroup of the $\mathfrak{S}_0$-group G. If each chief factor of G below N is central in N, then N is a hypercentral group.*

Proof. There is a normal series of G

$$1 = A_0 < A_1 < \ldots < A_k = N \leqslant G$$

such that the factors A_i/A_{i-1} are one of the following types:

(I) periodic Abelian with each Sylow subgroup being a Černikov group,

(II) torsion-free Abelian and rationally irreducible (as a G-module).

By induction on such a series, we may assume that G has a normal subgroup A of type I or II such that N/A is a hypercentral group and each chief factor of G below A is central in N. We must show that A is contained in the hypercentre of N.

Case I. Let $\Omega_n(A_p)$ denote the subgroup of A generated by all elements of order dividing p^n. Then $\Omega_n(A_p)$ is a finite normal subgroup of G and so is contained in $\zeta_{f(n)}(N)$ for some integer $f(n)$. Thus $A_p = \bigcup_{n=1}^{\infty} \Omega_n(A_p) \leqslant \zeta_\omega(N)$ and hence $A \leqslant \zeta_\omega(N)$, as required.

Case II. Let $C = C_G(A)$. Then $\bar{G} = G/C$ is a soluble irreducible $\mathbf{Q}$-linear group and so $\bar{G}$ has a free Abelian normal subgroup H of finite index ((9), Corollary 1 to Lemma 5·29·1). Since $\bar{G}$ is an $\mathfrak{S}_0$-group, H has finite rank and in particular $\bar{G}$ is polycyclic.

Let $a \in A$ and let $M = \langle a^G \rangle$, so that M may be considered as a finitely generated $\mathbf{Z}\bar{G}$-module. By Corollary 1 to Lemma 9·53 of (10), M contains a free Abelian subgroup X such that M/X is a Π-group for some finite set of primes Π. If p is a prime not in Π, then $M^p \cap X = X^p$ and so $\bigcap_{p \notin \Pi} M^p \cap X = 1$. Thus $\bigcap_{p \notin \Pi} M^p$ is a Π-group and, since M is torsion-free, we have $\bigcap_{p \notin \Pi} M^p = 1$.

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$M^p \lhd G$ and since A has finite rank r, say, we have $|M/M^p| \leqslant p^r$. Therefore $[M, {}_rN] \leqslant \bigcap_{p \notin \Pi} M^p = 1$ and so $M \leqslant \zeta_r(N)$. Thus $a \in \zeta_r(N)$ for each $a \in A$ and so $A \leqslant \zeta_r(N)$, as required.

Putting $N = G$, we have immediately

COROLLARY 7·2. *If the $\mathfrak{S}_0$-group G is a $\bar{Z}$-group, then G is hypercentral.*

This improves the known result that locally nilpotent $\mathfrak{S}_0$-groups are hypercentral ((8), Lemma 3·11). In view of the example of Kargapolov (7) it seems very unlikely that similar results could be obtained in any larger class than $\mathfrak{S}_0$.

THEOREM 7·3. *Let N be a normal subgroup of the $\mathfrak{S}_0$-group G. Then N is hypercentral if and only if $N/(N \cap \mu(G))$ is hypercentral.*

Proof. It is sufficient, by (7·1), to show that if $N/(N \cap \mu(G))$ is hypercentral then each chief factor of G below N is central in N.

As in the proof of (7·1), we may assume that G has a normal subgroup A of type I or II such that N/A is hypercentral and also $A \leqslant N \cap \mu(G)$. If H/K is a chief factor of G below A then H/K is periodic and so in both cases A/K is periodic and we may clearly assume that A/K is a p-group and so is a Černikov group.

Let B/K be the hypercentral residual of N/K. If H/K is not central, then $B > K$, N/B is hypercentral and $[N, B]K = B$. By Theorem A of (5), N/K splits conjugately over B/K. If X is a complement to B/K in N, then by the usual Frattini argument $G = B \cdot N_G(X)$. $N_G(X)$ is a proper subgroup of G and so there is a major subgroup of G not containing B. This is contrary to $\mu(G) \geqslant B$ and so H/K must be central in N. This completes the proof.

Again the two cases $N = G$ and $N = \mu(G)$ give Theorem A 2 (c) and Theorem B (c). Another part of Theorem A 2 is given in the following.

COROLLARY 7·4. *Let G be a soluble group satisfying the minimal condition on normal subgroups. Then G is hypercentral if and only if $G/\mu(G)$ is hypercentral.*

Proof. Let $G/\mu(G)$ be hypercentral; we prove by induction on its derived length n that G is hypercentral. This is clearly so if $n = 1$ and so we may assume that $n > 1$ and $G/G^{(n-1)}$ is hypercentral. But then G is a $\mathfrak{U}$-group ((4), Lemma 4·2) and so, by (5·2), G is locally nilpotent and therefore hypercentral ((9), Corollary 2 to Theorem 5·27).

An $\mathfrak{S}_1$-group G is one having a finite normal series in which the factors are Abelian groups of finite rank, whose torsion subgroups are Černikov groups. We consider $\mathfrak{S}_1$-groups in which $\mu(G) \geqslant G'$, showing that the structure of $G/\mu(G)$ will distinguish between nilpotent groups and hypercentral groups.

THEOREM 7·5. *Let G be an $\mathfrak{S}_1$-group. Then $\mu(G) \geqslant G'$ if and only if G is nilpotent.*

Proof. We may assume that G has a normal Abelian subgroup $A \leqslant \mu(G)$ such that G/A is nilpotent and A is one of the following types:

(I) A is a Černikov group,

(II) A is torsion-free and rationally irreducible.

If A is of type I, then let B be the nilpotent residual of G. G/B is nilpotent and $[B, G] = B$. If $B \neq 1$, then by Theorem C of (5), B has a proper supplement in G. But

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then G has a major subgroup not containing B contrary to $B \leqslant \mu(G)$. Therefore $B = 1$ and G is nilpotent.

If A is of type II, then as in the proof of (7·1) we can show that $[A, {}_rG] \leqslant \bigcap_p A^p = 1$ and so $A \leqslant \zeta_r(G)$, where r is the rank of A. It again follows that G is nilpotent.

COROLLARY 7·6. *Let G be a soluble group satisfying the minimal condition on normal subgroups. Then $\mu(G) \geqslant G'$ if and only if G is nilpotent.*

Proof. Let $\mu(G) \geqslant G'$; we prove that G is nilpotent by induction on the derived length n of G. This is clearly so if $n = 1$ and so we may assume that $n > 1$ and $G/G^{(n-1)}$ is nilpotent. Therefore G is a $\mathfrak{U}$-group ((4), Lemma 4·2) and so is locally nilpotent (5·2). But a locally nilpotent group satisfying the minimal condition on normal subgroups satisfies the minimal condition on all subgroups ((9) Corollary 2 to Theorem 5·27). Therefore we can apply the previous theorem to see that G is nilpotent.

(7·5) and (7·6) show that for $\mathfrak{S}_1$-groups and for soluble groups with the minimal condition on normal subgroups, hypercentral $\mathfrak{Z}$-groups are necessarily nilpotent. The following example shows that there exist hypercentral $\mathfrak{Z}$-groups in the class $\mathfrak{S}_0$ which are not nilpotent and so Theorem 7·5 cannot be extended to the class $\mathfrak{S}_0$.

Let G_i be the split extension of the cyclic group $\langle x_i \rangle$ of order p_i^i by its automorphism y_i of order p_i which maps x_i to $x_i^{p_i+1}$. Then G_i is nilpotent of class i. Let $G = \operatorname{Dr}_{i=1}^{\infty} G_i$, then G is an $\mathfrak{S}_0$-group since $X = \operatorname{Dr}_{i=1}^{\infty} \langle x_i \rangle$ is Abelian of rank one and so is G/X. If U is a proper subgroup of G then there is some G_i not covered by U. Since G_i is finite, it follows that $U^G < G$ and G is a $\mathfrak{Z}$-group. Also G is clearly hypercentral with central height ω.

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