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A Mathematical Analysis of Scale Similarity

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Abstract. Scale similarity is found in many natural phenomena in the universe, from fluid dynamics to astrophysics. In large eddy simulations of turbulent flows, some sub-grid scale (SGS) models are based on scale similarity. The earliest scale similarity SGS model was developed by Bardina et al., which produced SGS stresses with good correlation to the true stresses. In the present study, we perform a mathematical analysis of scale similarity. The analysis has revealed that the ratio of the resolved stress to the SGS stress is γ^2, where γ is the ratio of the second filter width to the first filter width, under the assumption of small filter width. The implications of this analysis are discussed in the context of large eddy simulation.

AMS subject classifications: 35L65, 76F02

Key words: Scale similarity, large eddy simulation, sub-grid scale models.

1 Introduction

After decades of development, large eddy simulations (LES) [5, 14, 19] are being used in the design process to predict vortex dominated, highly separated turbulent flows found in many important applications in aerospace, mechanical and chemical engineering. One important challenge in LES is the determination of the sub-grid scale (SGS) stress resulted from the filtering of the nonlinear governing Navier-Stokes equations. Various models have been developed to compute the SGS stress from the resolved field variables. Popular models comprise the Smagorinsky model (SM) and its many variants including the static [19] and dynamic SM [8], the scale similarity model (SSM) [1], the mixed model of SSM and SM [1], and monotone integrated LES [2] or implicit LES [9], in which no explicit SGS model is used. There is an extensive literature on each model with many applications.

Since the pioneering work by Bardina et al. on the SSM [1] for incompressible flow, there has been an extensive effort in evaluating its performance by comparing with other SGS models [7, 12, 16]. Furthermore the SSM has been extended to other flow problems

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including compressible flow [6] and combustion [11]. Direct numerical simulations and experimental measurements [15, 16, 18] have demonstrated scale similarity in turbulent flows. Many a priori tests using experimental or DNS data have shown a high correlation between the true stress and the modeled SGS stress based on the SSM [15]. In Bardina's original SSM, the second filter or test filter has the same width as the first one. It was proven by Speziale [20] that the Bardina constant must be 1 to satisfy Galilean invariance. The present analysis to be shown later confirms this result. Other researchers suggested using a different filter width for the second filter, e.g., in [15], and many approaches were suggested to determine the Bardina constant [4, 18].

In a recent effort to understand the performance of these SGS models in the context of discontinuous high order methods such as the discontinuous Galerkin method [3] or the flux reconstruction (FR)/correction procedure via reconstruction method (CPR) [10, 21], we performed a priori and a posteriori studies using the 1D Burgers' equation [13]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\qquad x\in[-1,1], \tag{1.1}$$

where u is the state variable, ν is a constant viscosity, chosen to be 8×10^{-5} in the present study to mimic high Reynolds number flow problems. The LES governing equation is then obtained by filtering (1.1) with a low pass filter $G_\Delta(x,\xi)$ satisfying the following conservative property

$$\int_{-\infty}^{+\infty} G_\Delta(x,\xi)d\xi=1, \tag{1.2}$$

where Δ denotes the filter width. Typical filters include the top hat filter defined as

$$G_\Delta(x,\xi)=\begin{cases} 1/\Delta, & |x-\xi|\leqslant\Delta/2,\\ 0, & \text{otherwise,}\end{cases} \tag{1.3}$$

and the Gaussian filter

$$G_\Delta(x,\xi)=\sqrt{\frac{6}{\pi\Delta^2}}e^{-\frac{6(x-\xi)^2}{\Delta^2}}. \tag{1.4}$$

The filtering process is defined mathematically in the physical space as a convolution operator. The filtered variable $\hat{\phi}(x,t)$ of a space-time variable $\phi(x,t)$ in 1D is defined as

$$\hat{\phi}(x,t)=\int_{-\infty}^{+\infty} G_\Delta(x,\xi)\phi(\xi,t)d\xi. \tag{1.5}$$

The filtering process is linear, i.e, $\widehat{\phi+\varphi}=\hat{\phi}+\hat{\varphi}$. If the filter width is constant, the differential and the filter operators commute, i.e., $\frac{\widehat{\partial\phi}}{\partial x}=\frac{\partial\hat{\phi}}{\partial x}$. Applying a low-pass spatial filter to Eq. (1.1), we obtain the following filtered equation

$$\frac{\partial\hat{u}}{\partial t}+\hat{u}\frac{\partial\hat{u}}{\partial x}=\nu\frac{\partial^2\hat{u}}{\partial x^2}-\frac{1}{2}\frac{\partial\tau}{\partial x}, \tag{1.6}$$

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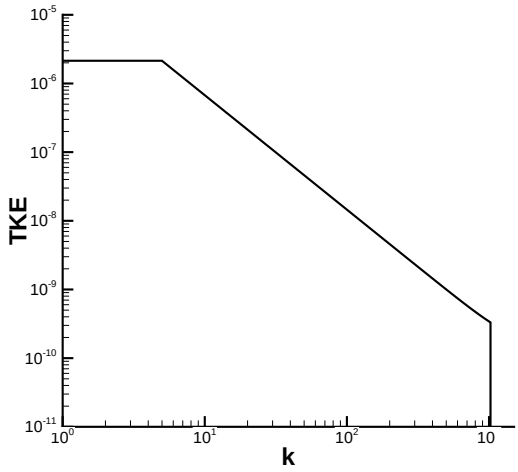


Figure 1: The initial energy spectrum.

where τ is the SGS stress

$$\tau = \widehat{uu} - \hat{u}\hat{u}. \tag{1.7}$$

To evaluate the performance of various models, a DNS was performed first, assuming an initial turbulent energy spectrum of $E_0(k) \sim k^{-5/3}$, as shown in Fig. 1. One realization of the initial condition is displayed in Fig. 2a, with randomly generated phase angles and mean velocity of 1. A filtered solution of is shown in Fig. 2b. Note that high frequency modes are removed by the filter.

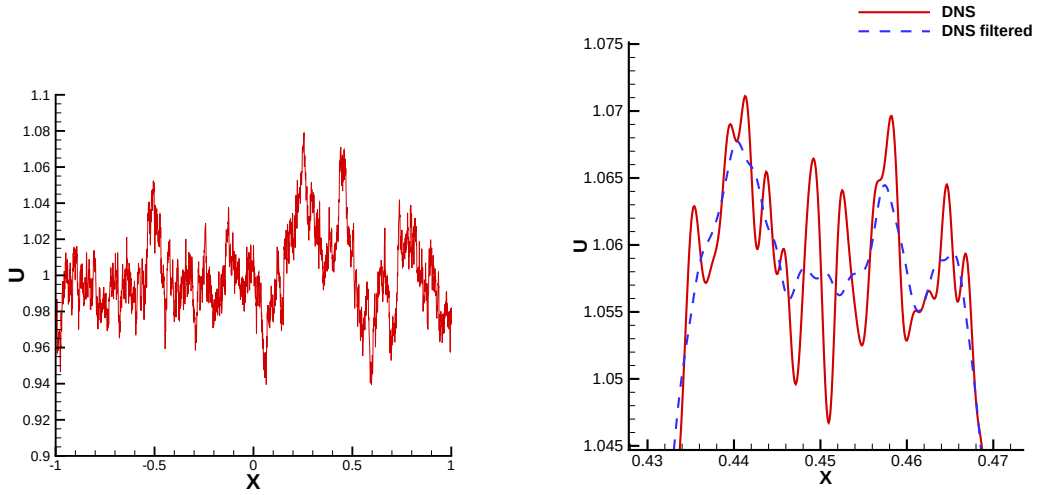
A DNS simulation is performed until $t=0.5$ on a fine mesh with mesh size $h=1/4{,}096$ using a third order CPR method, and the solution and the spectrum at the final time is shown in Fig. 3. It was verified that the DNS data is mesh independent. Note that the solution is heavily dissipated near the high frequency regime. The top hat filter with a filter width of 16h was then used to produce the filtered solution. In addition, the "true" SGS stress (or τ_{true}) with the same filter width is computed from the DNS data. The filtered solution was also used in various models to produce the "modeled" stress. One of the SGS models we tested was the SSM of Bardina [1]. In our implementation, the second filter with filter width of $\Delta_2 = 2\Delta$ was used to produce the SGS stress at the resolved scale, i.e.,

$$\mathrm{L} = \left(\widetilde{\hat{u}\hat{u}} - \widetilde{\hat{u}}\widetilde{\hat{u}}\right), \tag{1.8}$$

where $\sim$ indicates filtering with width Δ_2. In the a priori test, the SGS stress computed with the SSM showed nearly perfect correlation (correlation coefficient >0.99) with the true stress, as shown in Fig. 4. Furthermore, we discovered that they are related by a factor of 4, i.e.,

$$\frac{\mathrm{L}}{\tau_{true}} = 4. \tag{1.9}$$

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(a) (b)

Figure 2: One realization of the initial condition and the filtered solution.

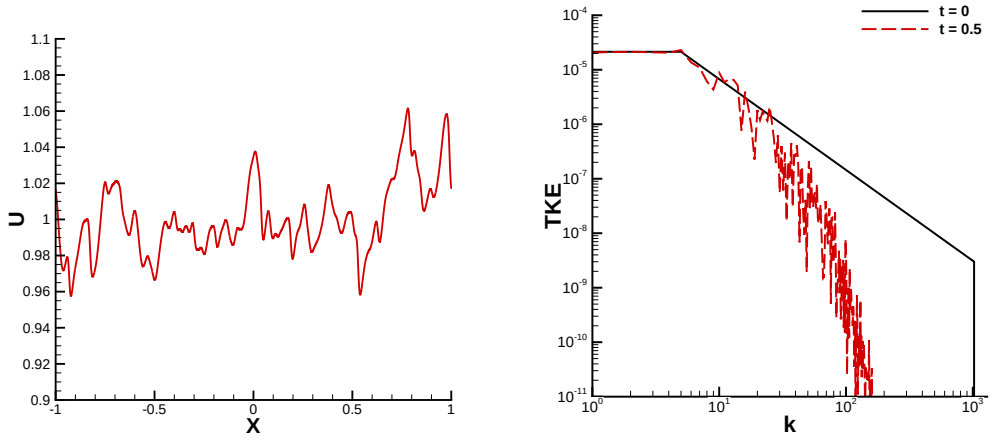


Figure 3: The solution and its spectrum of the DNS solution at $t=0.5$.

2 Analysis of scale similarity with a single Fourier mode

For the sake of simplicity without loss of generality, we consider periodic data $u(x)$ at a given time on domain $[-\pi,\pi]$. The solution can be decomposed into the following Fourier modes

$$u(x)=\sum_{n=0}^{\infty} a_n e^{inx}, \tag{2.1}$$

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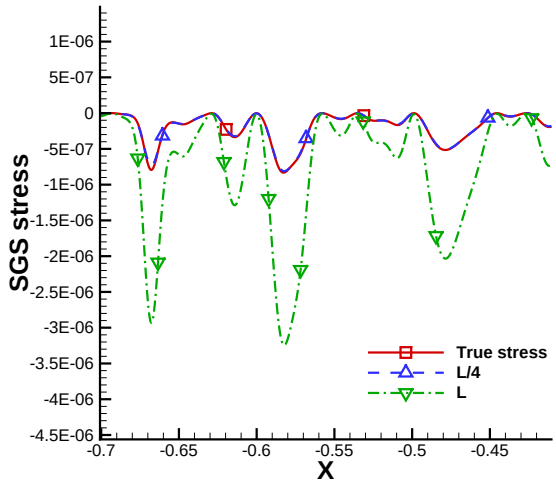


Figure 4: Comparison of the true stress and the modeled stress based on the SSM of Bardina When we first saw the relation (1.9), we suspected that this was a bug. After we tested different initial conditions, we were convinced that it was not a bug. The fact that (1.9) is true for arbitrarily-generated random initial conditions prompted us to look for a deeper reason resulting in the following analysis.

where $i=\sqrt{-1}$, and n is the wave number. To illustrate the basic idea, we first consider a single Fourier mode, i.e., $u(x)=e^{inx}$ and the top hat filter. The filtered solution is then

$$\hat{u}(x)=\frac{1}{\Delta}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} e^{in\xi}d\xi=sinc\left(\frac{n\Delta}{2}\right)\cdot e^{inx}, \tag{2.2}$$

where $sinc(n\Delta/2)=\frac{\sin(n\Delta/2)}{n\Delta/2}$. Obviously the filter only changes the magnitude of the solution, but not the phase. In addition, we have

$$\widehat{u\hat{u}}(x)=\frac{1}{\Delta}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} e^{i2n\xi}d\xi=sinc(n\Delta)\cdot e^{i2nx}. \tag{2.3}$$

The SGS stress is then

$$\begin{aligned}\tau&=\widehat{u\hat{u}}-\hat{u}\hat{u}=sinc(n\Delta)\cdot e^{i2nx}-sinc^2\left(\frac{n\Delta}{2}\right)\cdot e^{i2nx}\\&=\left[sinc(n\Delta)-sinc^2\left(\frac{n\Delta}{2}\right)\right]e^{i2nx}.\end{aligned} \tag{2.4}$$

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Next we apply a second filter with a width of $\Delta_2=\gamma\Delta$ to the resolved variable to obtain

$$\widetilde{\hat{u}}(x)=\frac{1}{\gamma\Delta}\int_{x-\frac{\gamma\Delta}{2}}^{x+\frac{\gamma\Delta}{2}}\hat{u}(\xi)d\xi=sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{\gamma n\Delta}{2}\right)e^{inx}, \tag{2.5}$$

and

$$\widetilde{\hat{u}\hat{u}}(x)=\frac{1}{\gamma\Delta}\int_{x-\frac{\gamma\Delta}{2}}^{x+\frac{\gamma\Delta}{2}}\hat{u}(\xi)\hat{u}(\xi)d\xi=sinc^2\left(\frac{n\Delta}{2}\right)sinc(\gamma n\Delta)e^{i2nx}. \tag{2.6}$$

The SGS stress of the resolved scale is then

$$L=\widetilde{\hat{u}\hat{u}}-\widetilde{\hat{u}}\widetilde{\hat{u}}=sinc^2\left(\frac{n\Delta}{2}\right)\left(sinc(\gamma n\Delta)-sinc^2\left(\frac{\gamma n\Delta}{2}\right)\right)e^{i2nx}. \tag{2.7}$$

From (2.4) and (2.7), we obtain

$$\frac{L}{\tau}=\frac{sinc^2\left(\frac{n\Delta}{2}\right)\left[sinc(\gamma n\Delta)-sinc^2\left(\frac{\gamma n\Delta}{2}\right)\right]}{sinc(n\Delta)-sinc^2\left(\frac{n\Delta}{2}\right)}. \tag{2.8}$$

In the limit of small $n\Delta$, we have

$$sinc(n\Delta)=1-\frac{(n\Delta)^2}{6}+\mathcal{O}(n\Delta)^4. \tag{2.9}$$

Therefore, we obtain

$$\frac{L}{\tau}=\frac{\left[1-\frac{(n\Delta)^2}{12}+\mathcal{O}(n\Delta)^4\right]\left[-\frac{(\gamma n\Delta)^2}{6}+\frac{(\gamma n\Delta)^2}{12}+\mathcal{O}(n\Delta)^4\right]}{-\frac{(n\Delta)^2}{6}+\frac{(n\Delta)^2}{12}+\mathcal{O}(n\Delta)^4}=\gamma^2+\mathcal{O}(n\Delta)^2. \tag{2.10}$$

Note that the error term is quadratic. In the special case of $\gamma=2$, $L=4\tau$. As it turns out this result is also true for the Gaussian filter. The filtered solution with a Gaussian filter is

$$\hat{u}(x)=\int_{-\infty}^{\infty}\sqrt{\frac{6}{\pi\Delta^2}}e^{-\frac{6(x-\xi)^2}{\Delta^2}}e^{in\xi}d\xi=\sqrt{\frac{6}{\pi\Delta^2}}\int_{-\infty}^{\infty}e^{-\frac{6(x-\xi)^2}{\Delta^2}}e^{in\xi}d\xi. \tag{2.11}$$

Set $X=\sqrt{6}(x-\xi)/\Delta$, so that $\xi=x-(\Delta/\sqrt{6})X$, $d\xi=-(\Delta/\sqrt{6})dX$. Thus, we have

$$\hat{u}(x)=\sqrt{\frac{6}{\pi\Delta^2}}\cdot\frac{\Delta}{\sqrt{6}}\int_{-\infty}^{\infty}e^{-X^2+in\left(x-\frac{\Delta}{\sqrt{6}}X\right)}dX=\sqrt{\frac{1}{\pi}}e^{inx}\int_{-\infty}^{\infty}e^{-X^2-in\frac{\Delta}{\sqrt{6}}X}dX=e^{inx}e^{-\frac{(n\Delta)^2}{24}}. \tag{2.12}$$

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Similarly we can derive the following result

$$\widehat{uu}(x)=\int_{-\infty}^{\infty}\sqrt{\frac{6}{\pi\Delta^2}}e^{-\frac{6(x-\xi)^2}{\Delta^2}}e^{i2nx}d\xi=\sqrt{\frac{1}{\pi}}\cdot e^{i2nx}\int_{-\infty}^{\infty}e^{-X^2-i2n\frac{\Delta}{\sqrt{6}}X}dX=e^{i2nx}e^{-\frac{(n\Delta)^2}{6}}. \tag{2.13}$$

The SGS stress is then

$$\tau=\widehat{uu}-\hat{u}\hat{u}=e^{i2nx}e^{-\frac{(n\Delta)^2}{6}}-e^{i2nx}e^{-\frac{(n\Delta)^2}{12}}=e^{i2nx}e^{-\frac{(n\Delta)^2}{6}}\left[1-e^{\frac{(n\Delta)^2}{12}}\right]. \tag{2.14}$$

Again we apply a second filter with a width of $\Delta_2=\gamma\Delta$ to the resolved field to obtain

$$\widetilde{\hat{u}}(x)=e^{-\frac{(n\Delta)^2}{24}}\tilde{u}(x)=e^{-\frac{(n\Delta)^2}{24}}e^{-\frac{(\gamma n\Delta)^2}{24}}e^{inx} \tag{2.15}$$

and

$$\widetilde{\hat{u}\hat{u}}(x)=e^{-\frac{(n\Delta)^2}{12}}\widehat{uu}(x)=e^{-\frac{(n\Delta)^2}{12}}e^{-\frac{(\gamma n\Delta)^2}{6}}e^{i2nx}. \tag{2.16}$$

The SGS stress of the resolved scale is

$$L=\widetilde{\hat{u}\hat{u}}-\widetilde{\hat{u}}\widetilde{\hat{u}}=e^{i2nx}e^{-\frac{\left(1+2\gamma^2\right)(n\Delta)^2}{12}}\left[1-e^{\frac{(\gamma n\Delta)^2}{12}}\right]. \tag{2.17}$$

From (2.14) and (2.17), we obtain

$$\frac{L}{\tau}=e^{\frac{\left(1-2\gamma^2\right)(n\Delta)^2}{12}}\frac{1-e^{\frac{(\gamma n\Delta)^2}{12}}}{1-e^{\frac{(n\Delta)^2}{12}}}. \tag{2.18}$$

In the limit of small $n\Delta$, we have

$$\frac{L}{\tau}=\left(1+\mathcal{O}(n\Delta)^2\right)\frac{1-1-\frac{(\gamma n\Delta)^2}{12}+\mathcal{O}(n\Delta)^4}{1-1-\frac{(n\Delta)^2}{12}+\mathcal{O}(n\Delta)^4}=\gamma^2+\mathcal{O}(n\Delta)^2. \tag{2.19}$$

3 Analysis of scale similarity with all Fourier modes

Next we consider a solution with all the Fourier modes, i.e.,

$$u(x)=\sum_{n=0}^{\infty}a_n e^{inx}. \tag{3.1}$$

With the top hat filter, we obtain the following filtered solution

$$\hat{u}(x)=\sum_{n=0}^{\infty}a_n sinc\left(\frac{n\Delta}{2}\right)\cdot e^{inx}. \tag{3.2}$$

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In addition, we have

$$\widehat{uu}(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n a_m sinc\frac{(n+m)\Delta}{2}e^{i(n+m)x}. \tag{3.3}$$

The SGS stress is then

$$\tau=\widehat{uu}-\hat{u}\hat{u}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n a_m\left[sinc\frac{(n+m)\Delta}{2}-sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)\right]e^{i(n+m)x}. \tag{3.4}$$

Next we apply a second filter with a width of to the resolved variable

$$\widetilde{\hat{u}}(x)=\sum_{n=0}^{\infty}a_n sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{\gamma n\Delta}{2}\right)e^{inx}, \tag{3.5}$$

and

$$\widetilde{\hat{u}\hat{u}}(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n a_m sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)sinc\frac{\gamma(n+m)\Delta}{2}e^{i(n+m)x} \tag{3.6}$$

The SGS stress of the resolved scale is then

$$\begin{aligned}L=&\widetilde{\hat{u}\hat{u}}-\widetilde{\hat{u}}\widetilde{\hat{u}}\\=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n a_m sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)\\&\times\left(sinc\frac{\gamma(n+m)\Delta}{2}-sinc\left(\frac{\gamma n\Delta}{2}\right)sinc\left(\frac{\gamma m\Delta}{2}\right)\right)e^{i(n+m)x}.\end{aligned} \tag{3.7}$$

Now let's consider each term in (3.4) and (3.7). It is obvious that

$$\frac{L_{nm}}{\tau_{nm}}=\frac{sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)\left(sinc\frac{\gamma(n+m)\Delta}{2}-sinc\left(\frac{\gamma n\Delta}{2}\right)sinc\left(\frac{\gamma m\Delta}{2}\right)\right)}{sinc\frac{(n+m)\Delta}{2}-sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)}. \tag{3.8}$$

In the limit of small $(n+m)\Delta$, we have

$$\frac{L_{nm}}{\tau_{nm}}=\frac{-\frac{nm}{12}\gamma^2\Delta^2+\mathcal{O}[(n+m)\Delta]^4}{-\frac{nm}{12}\Delta^2+\mathcal{O}[(n+m)\Delta]^4}=\gamma^2+\mathcal{O}[(n+m)\Delta]^2. \tag{3.9}$$

Therefore, we have

$$\frac{L}{\tau}=\gamma^2+\mathcal{O}[(n+m)\Delta]^2 \tag{3.10}$$

in the same limit. The analysis with the Gaussian filter is similar and is not repeated here.

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4 Analysis of scale similarity in 2D

In two dimensions, we only perform a single mode analysis with the top hat filter. Consider the following two dimensional velocity field

$$u(x,y)=e^{inx}e^{imy}, \qquad v(x,y)=e^{ipx}e^{iqy}. \tag{4.1}$$

The filtered solution is then

$$\hat{u}(x,y)=\frac{1}{\Delta^2}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}}\int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}} e^{in\xi}e^{im\eta}d\eta d\xi = e^{inx}e^{imy}\cdot sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right), \tag{4.2}$$

$$\hat{v}(x,y)=\frac{1}{\Delta^2}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}}\int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}} e^{ip\xi}e^{iq\eta}d\eta d\xi = e^{ipx}e^{iqy}\cdot sinc\left(\frac{p\Delta}{2}\right)sinc\left(\frac{q\Delta}{2}\right). \tag{4.3}$$

In addition, we have

$$\begin{aligned}\widehat{uv}&=\frac{1}{\Delta^2}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}}\int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}} e^{i(n+p)\xi}e^{i(m+q)\eta}d\eta d\xi\\ &=e^{i(n+p)x}e^{i(m+q)y}\cdot sinc\left(\frac{(n+p)\Delta}{2}\right)sinc\left(\frac{(m+q)\Delta}{2}\right).\end{aligned} \tag{4.4}$$

The SGS stress is then

$$\begin{aligned}\tau&=\widehat{uv}-\hat{u}\hat{v}\\ &=e^{i(n+p)x}e^{i(m+q)y}\left[sinc\left(\frac{(n+p)\Delta}{2}\right)sinc\left(\frac{(m+q)\Delta}{2}\right)\right.\\ &\quad\left.-sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)sinc\left(\frac{p\Delta}{2}\right)sinc\left(\frac{q\Delta}{2}\right)\right].\end{aligned} \tag{4.5}$$

Applying a second filter $\sim$ with a width $\gamma\Delta$ to the resolved variable, we obtain

$$\widetilde{\hat{u}}=e^{inx}e^{imy}sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)sinc\left(\frac{\gamma n\Delta}{2}\right)sinc\left(\frac{\gamma m\Delta}{2}\right), \tag{4.6}$$

and

$$\widetilde{\hat{v}}=e^{ipx}e^{iqy}sinc\left(\frac{p\Delta}{2}\right)sinc\left(\frac{q\Delta}{2}\right)sinc\left(\frac{\gamma p\Delta}{2}\right)sinc\left(\frac{\gamma q\Delta}{2}\right). \tag{4.7}$$

Denote $\alpha=sinc(n\Delta/2)sinc(m\Delta/2)sinc(p\Delta/2)sinc(q\Delta/2)$. Then we have

$$\widetilde{\hat{u}\hat{v}}=\alpha\widetilde{uv}, \qquad \widetilde{\hat{u}\hat{v}}=\alpha\tilde{u}\tilde{v}. \tag{4.8}$$

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The SGS stress of the resolved scale is then

$$\begin{aligned}L&=\widehat{\tilde{u}\tilde{v}}-\widehat{\tilde{u}}\widehat{\tilde{v}}=\alpha\left(\widehat{\tilde{u}\tilde{v}}-\hat{u}\hat{v}\right)\\&=\alpha e^{i(n+p)x}e^{i(m+q)y}\left[sinc\left(\frac{\gamma(n+p)\Delta}{2}\right)sinc\left(\frac{\gamma(m+q)\Delta}{2}\right)\right.\\&\left.-sinc\left(\frac{\gamma n\Delta}{2}\right)sinc\left(\frac{\gamma m\Delta}{2}\right)sinc\left(\frac{\gamma p\Delta}{2}\right)sinc\left(\frac{\gamma q\Delta}{2}\right)\right].\end{aligned}\tag{4.9}$$

From (4.5) and (4.9), we obtain

$$\frac{L}{\tau}=\alpha\frac{sinc\left(\frac{\gamma(n+p)\Delta}{2}\right)sinc\left(\frac{\gamma(m+q)\Delta}{2}\right)-sinc\left(\frac{\gamma n\Delta}{2}\right)sinc\left(\frac{\gamma m\Delta}{2}\right)sinc\left(\frac{\gamma p\Delta}{2}\right)sinc\left(\frac{\gamma q\Delta}{2}\right)}{sinc\left(\frac{(n+p)\Delta}{2}\right)sinc\left(\frac{(m+q)\Delta}{2}\right)-sinc\left(\frac{n\Delta}{2}\right)sinc\left(\frac{m\Delta}{2}\right)sinc\left(\frac{p\Delta}{2}\right)sinc\left(\frac{q\Delta}{2}\right)}.$$

In the limit of small $(n+m+p+q)\Delta$, we have $\alpha\approx 1$, and

$$sinc(A+B)\cdot sinc(C+D)-sincA\cdot sincB\cdot sincC\cdot sincD=-\frac{AB+CD}{3}+HOT.\tag{4.10}$$

Finally, we derive the following result using (4.10)

$$\frac{L}{\tau}\approx\gamma^2.\tag{4.11}$$

5 Implications for Large Eddy Simulation

The present analysis shows that perfect scale similarity exists for arbitrary (periodic) data including turbulence *under the assumption that the spectrum contains relatively low frequency contents with respect to the filter width*, regardless of amplitude and phase angle of each mode. Obviously for an arbitrary spectrum including both high and low frequency contents, the present analysis is not valid. This is easily seen in Fig. 5, which displays the modeled and true SGS stress based on the full spectrum shown in Fig. 1, using the same filter width employed in Fig. 4. The correlation between the modeled and true stresses is quite low.

Next let's examine whether (4.11) is true in an actual LES. The promise of the SSM is that the SGS stress is highly correlated with the stress computed based on the resolved scale, taken to be $\hat{u}$. Take the top hat filter for example. Modes of smaller wavelength than D corresponding to the cutoff wavenumber k_Δ are filtered out. In LES, it is believed that the SGS stress from higher modes close to the cutoff wave number k_Δ plays an important role. In the next test, we therefore include modes between k_Δ and $2k_\Delta$ using a filter width $D/2$ to filter the spectrum shown in Fig. 1. *The filtered solution is then treated as DNS data, which is used to obtain the true stress*. This true stress is also compared with the stresses computed using the SSM based on the resolved scale, i.e., $\hat{u}$. Two test filter widths are

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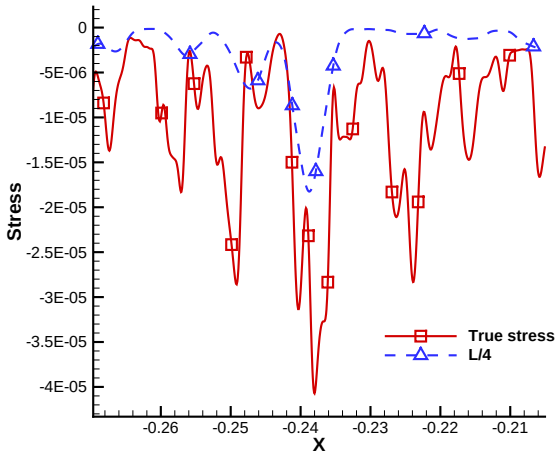


Figure 5: The true stress and the modeled stress for the full spectrum given in Fig. 1.

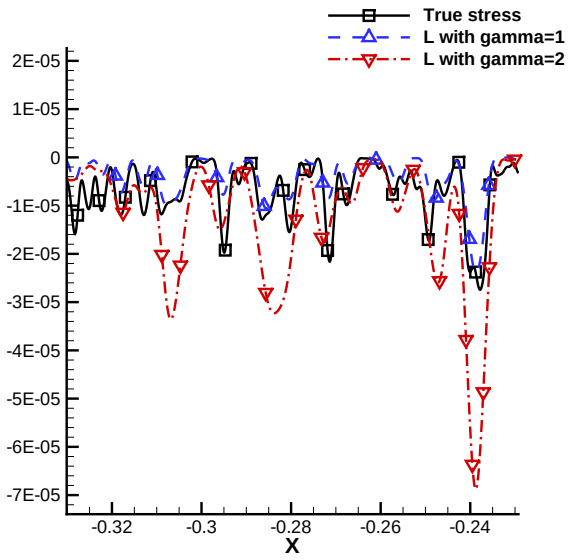


Figure 6: Comparison between the true stress computed with the sub-grid-scale between k_Δ and $2k_\Delta$ and the modelled stresses computed using $\hat{u}$.

used corresponding to $\gamma=1$ and 2. The results are displayed in Fig. 6. Note that there is a reasonably high level of correlation between the stresses.

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Table 1: Correlation coefficients and ratio of averaged stresses from 10 realizations.

		R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
Correlation coefficient	$\gamma=1$	0.85	0.88	0.87	0.90	0.85	0.87	0.85	0.86	0.88	0.88
	$\gamma=2$	0.70	0.71	0.69	0.66	0.68	0.72	0.71	0.69	0.70	0.68
$\frac{L_1(L)}{L_1(\tau)}$	$\gamma=1$	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51
	$\gamma=2$	1.48	1.48	1.48	1.48	1.48	1.48	1.48	1.48	1.48	1.48

The ratio between the true and modeled stresses are computed using simple averages

$$\frac{L}{\tau} \approx \frac{\langle L \rangle}{\langle \tau \rangle}. \tag{5.1}$$

The correlation coefficients and the average stress ratios from 10 realizations are summarized in Table 1. The table confirms that the true stress shows a quite high correlation with the modeled stress, with an average correlation coefficients of 0.88 and 0.69 for $\gamma=1$ and $\gamma=2$, respectively. In addition, $\gamma=1$ demonstrates consistently higher correlation coefficients than $\gamma=2$. This may indicate that one should use the same filter width for the second filter in an SSM implementation. Furthermore, the ratio of the averaged stresses remains a constant with different realizations, indicating that this ratio is only dependent on the spectrum. However, the ratio is much smaller than γ^2. This result appears to agree well with others in the literature [4, 15].

6 Conclusions

The present analysis shows that perfect scale similarity exists for arbitrary (periodic) data including turbulence under the assumption that the spectrum contains relatively low frequency contents with respect to the filter width, regardless of amplitude and phase angle of each mode. In an actual large eddy simulation, in which both large and subgrid scales exist, the present result on the ratio of the resolved scale stress and the SGS stress may be the upper limit. Test results with data including higher modes near the grid cutoff demonstrate that there is a high level of correlation between the modelled and SGS stresses. Furthermore, $\gamma=1$ demonstrates consistently higher correlation coefficients than $\gamma=2$. This may indicate that $\gamma=1$ is preferred in a SSM implementation.

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