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Integrals with a large parameter: paths of descent and conformal mapping

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Abstract. The method of steepest descents is widely used to find the asymptotic expansion of contour integrals of the form

$$\int_a^b g(z)\exp\{Nf(z)\}dz,$$

where N is a large positive parameter, and $f(z)$ and $g(z)$ are analytic functions of z. (For the sake of simplicity it will here be supposed that $f(z)$ and $g(z)$ are regular except at ∞, but our arguments can be extended to include singularities at other points.) In the method of steepest descents the path of integration is deformed into a new equivalent path consisting of the paths of steepest descent through a and b, together with paths of steepest descent through certain saddle points, the *relevant* saddle points. (Paths of descent, etc., are defined in §1 below.) Watson's Lemma then shows that the complete asymptotic expansion depends on the local behaviour of $f(z)$ and $g(z)$ at the end-points and at the relevant saddle-points. On the other hand to determine which saddle-points are relevant is a global (non-local) problem and requires either the tracing of paths of steepest descent all the way from saddle-points and end-points to ∞, or some other global process. It has often been noted that in applications of Watson's Lemma the paths of steepest descent can be replaced by a wider class of paths of descent. In this way we are led to global problems like the following: *Given 2 points $z = \alpha$ and $z = \beta$ where $\Re f(\alpha) > \Re f(\beta)$, can a path of descent be drawn from α to β? If also $\Im f(\alpha) = \Im f(\beta)$, can a path of steepest descent be drawn from α to β?*

In the present paper the global process of curve tracing is replaced by a global process of conformal mapping. The transformation $w = f(z)$ maps a domain D_z in the z-plane on a domain D_w in the w-plane. There are well-known theorems which give useful conditions under which the mapping is (1, 1), and it is then often a simple matter to infer the topography of (for example) the curves of steepest descent within D_z from the topography of the straight lines $\Im w =$ constant within D_w. Three examples are worked out to illustrate various aspects of the method which is seen to remain applicable in certain circumstances where the mapping is not (1, 1). The most fruitful idea appears to be the method of *univalent dissection*. Let the original domain D_z be dissected into non-overlapping subdomains $D_z^{(n)}$, in each of which $f(z)$ is univalent. Then paths of steepest descent can be constructed by pursuing straight lines $\Im w =$ constant from image domain to image domain in the w-plane, and then taking inverse images

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in the z-plane. In a similar manner the existence of other paths of descent can be demonstrated.

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1. *Introduction.* The method of steepest descents is widely used to find the asymptotic expansion of contour integrals of the form

$$\int_a^b g(z)\exp\{Nf(z)\}dz, \tag{1·1}$$

where N is a large positive parameter, and $f(z)$ and $g(z)$ are analytic functions of z. We begin by describing this method briefly. Detailed accounts are given in (1) and (2). To fix ideas it will henceforth be supposed that $f(z)$ and $g(z)$ are regular except at ∞, but many of the following arguments remain valid under wider conditions. Let us write $f(z) = u(x,y) + iv(x,y)$, where u and v are real. When u is treated as a third real coordinate, orthogonal to x and y, the equation $u = u(x,y)$ defines a real surface in (x,y,u) space on which the curves $u =$ constant are level curves, and the curves $v =$ constant are paths of steepest descent or ascent which must end at ∞, the only singularity of $f(z)$. A path of steepest descent from a point z_0 thus satisfies the equation $f(z) = f(z_0) - \tau$ (τ real, $0 < \tau < \infty$). Consider the integral (1·1), in which the path ab will now be suitably deformed. If paths of steepest descent $S(a)$, $S(b)$, from the end-points a and b respectively, terminate in the same valley at ∞ then we have

$$\int_a^b \ldots dz = \int_{S(a)} \ldots dz - \int_{S(b)} \ldots dz;$$

otherwise there is an additional integral on the right-hand side, along a path beginning and ending in two different valleys at ∞. This path is then deformed to lie along paths of steepest descent through saddle-points. (Saddle-points are zeros of the derivative $f'(z)$.) In other words, the original path of integration is deformed into two paths of steepest descent through the end-points, together with the sum of paths of steepest descent through certain of the saddle-points. (These will be called the *relevant* saddle-points.) Each of the integrals along a component path of steepest descent can now be evaluated by Watson's Lemma which shows that the asymptotic expansion of the integral along a component path of the deformed contour depends only on the local behaviour of f and g and their derivatives at the highest point of the path (necessarily an end-point or saddle-point). This is the classical method of steepest descents. We observe that the final result involves only local properties of $f(z)$ and $g(z)$ at end-points and saddle-points, but that the intermediate construction also involves a global (i.e. non-local) property of $f(z)$:

We must decide how to combine paths of steepest descent through end-points and saddle-points so as to obtain a path equivalent to the original path. To decide this is usually the most difficult part of the construction, see e.g. Watson (7), §8·6, and this is the aspect of the problem with which we shall be concerned in the present paper.

It has often been noted that little use is made of the fact that the paths are paths of *steepest* descent: Watson's Lemma is applicable under much wider conditions (Jeffreys (2), ch. 2, Copson (1), chs. 6, 8). The discussion by Jeffreys (2), section 2·3 is

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particularly thorough. The general case is readily reduced to the following special case: Suppose that $\Re f(\alpha) > \Re f(\beta)$. Then the value of

$$\int_{\alpha}^{\beta} g(z) \exp\{Nf(z)\} dz$$

is asymptotically dominated by contributions from the neighbourhood of the point α, if some path $\alpha\beta$ can be found on which $\Re f(z) < \Re f(\alpha)$ except when $z = \alpha$. This condition is clearly satisfied when $\alpha\beta$ is any path of descent (not necessarily of steepest descent), i.e. when $\Re f(z)$ decreases monotonically as z moves from α to β along the path. In the present paper we shall concentrate on paths of descent but it can be seen that most of the arguments remain applicable to any paths on which $\Re f(z) < \Re f(\alpha)$.

Use will be made of the change of variable $w = u + iv = f(z)$, where $f(z)$ is the same function as in (1·1). This changes the integral (1·1) to the form

$$\int g(z) \frac{dz}{dw} \exp(Nw) dw = \int G(w) \exp(Nw) dw,$$

say, where $G(w)$ is a branch of a multivalued function. The variable w was introduced by van der Waerden (6) who noted that paths of steepest descent map into paths $\Im w = v =$ constant in the w-plane. Since, moreover, in the z-plane the path of integration and the end-points and the saddle-points are all known (and therefore also their images in the w-plane are known) it follows that the paths of steepest descent in the w-plane are completely determined. The path of integration in the w-plane being known, it only remains to deform this into loops round branch points by moving the contour to the left. The tracing of paths of steepest descent in the z-plane is therefore unnecessary. It might appear that global arguments are thus avoided, but van der Waerden observed that analytic continuation is needed. In effect, the deformation of the contour takes place not simply in the w-plane but on the Riemann surface of the function $G(w)$. A big enough portion of this Riemann surface must be constructed by analytic continuation to contain (on their appropriate sheets) the loops through the relevant branch points as well as the contour of integration. (Van der Waerden's argument was expressed differently.) To determine the relevant branch points by analytic continuation requires a global process which in any given case may be either more or less convenient than the global process of tracing the paths of steepest descent in the z-plane. (The reader may find it instructive to treat the simple example $g(z) = 1, f(z) = \frac{1}{3}z^3 + z$, by van der Waerden's method.)

To introduce global considerations in yet another way, we shall in the present work treat the transformation $w = f(z)$ as a conformal mapping of a domain D_z in the z-plane on a domain D_w in the w-plane. We then have the global theorem (stated more precisely as Theorem B below): If the mapping of the boundaries is (1, 1), then the mapping of the domains is (1, 1). The mapping of the boundaries is frequently a simple matter, and we can infer almost immediately the topography of (for example) the curves of steepest descent in the z-plane from the known topography of the curves $v =$ constant in the w-plane. (We shall never consider integrals in the w-plane.)

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This idea will be more fully described in section 2 and will be illustrated by means of examples in section 3. We shall need to make use of the following two theorems which are classical and are quoted without proof.

THEOREM A. *Suppose that $F(z)$ is regular in the domain D bounded by the simple closed contour C, and that $F(z) \neq 0$ on C. Then the number of zeros of $F(z)$ inside C is*

$$\frac{1}{2\pi}[\arg F(z)]_C,$$

the change in $(2\pi)^{-1} \arg F(z)$ along the closed contour C.

Proof in Titchmarsh (4), section 3·41.

THEOREM B. *Suppose that C_z is a simple closed contour, and that $F(z)$ is regular inside and on C_z. As z describes C_z in the positive direction, suppose that $w = F(z)$ describes a simple closed contour C_w once. Then $w = F(z)$ gives a conformal representation of the interior of C_z on the interior of C_w.*

The proof, in Titchmarsh (4), section 6·45, and Littlewood (3), Theorem 117, makes use of Theorem A.

2. *The method of conformal mapping.* We have seen that our global problem can be expressed in the following form.

Given a domain D_z in the z-plane, and two points α and β in D_z (in practice the point β is always in a given valley at ∞), *can we in D_z draw a path of descent going from α to β?* (On a path of descent, $\Re f(z)$ decreases monotonically.) The necessary condition $\Re f(\alpha) > \Re f(\beta)$ is assumed to be satisfied. It will be shown that this question can in suitable cases be answered by use of the conformal mapping $w = f(z)$.

To take the simplest case, suppose that $w = f(z)$ gives a (1, 1) mapping of the boundary of D_z onto the boundary of a domain D_w. Then, by Theorem B, the mapping of the interior of D_z onto the interior of D_w is also (1, 1). In particular, we can find in D_w the image points $w = \alpha' = f(\alpha)$ and $w = \beta' = f(\beta)$ of $z = \alpha$ and $z = \beta$. It is often easy to decide, by tracing the boundary of D_w, whether a path $\alpha'\beta'$ of decreasing $\Re w$ can be drawn to lie in D_w. If this is possible then the inverse image of this path lies in D_z and is a path of descent. Moreover, we know that neighbourhoods of $z = \beta$ and $w = \beta'$ correspond, in other words, we know that the path will descend into the given valley at $z = \beta$. These general considerations will now be illustrated by means of examples.

3. *Examples.*

Example 1. $w = f(z) = \frac{1}{3}z^3 + z$.

This simple case will be worked out in some detail. The saddle-points, given by $f'(z) = 0$, are at $z = \pm i$; the valleys at ∞, given by $\Re(z^3) \to -\infty$, $\Im(z^3) = 0$, as $|z| \to \infty$, are at $\arg z = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$. Let us investigate paths of steepest descent from points $z = iy_0$ (where $y_0 > 1$). We have

$$u = \Re f(z) = \tfrac{1}{3}x^3 - xy^2 + x,$$

whence $\partial u/\partial x = 1 - y^2$ when $x = 0$. This is negative when $y > 1$, and it follows that

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paths of descent from points iy_0 (where $y_0 > 1$) initially go into the first quadrant. On the path of steepest descent we have

$$\Im f(z) = \text{constant} = \Im f(iy_0) \quad \text{and} \quad \Re f(z) < \Re f(iy_0),$$

i.e.
$$x^2y - \tfrac{1}{3}y^3 + y = -\tfrac{1}{3}y_0^3 + y_0 \quad \text{and} \quad \tfrac{1}{3}x^3 - xy^2 + x < 0. \tag{3·1}$$

These equations describe the paths of steepest descent not only from the point iy_0 but clearly also from the other two points Z satisfying $f(Z) = f(iy_0)$. Even though the solutions of (3·1) are quite easily computed it is not very easy to disentangle the properties of the path through iy_0 from the properties of the other two paths.

In this simple case it is not difficult to show that the path of steepest descent from $iy_0 (y_0 > 1)$ remains in the first quadrant, and therefore necessarily ends at $\infty \exp(\frac{1}{3}\pi i)$. For the level curves are $\Re f(z) = \frac{1}{3}x^3 - xy^2 + x = \text{constant}$, in particular the line $x = 0$ is a level curve. Since our path starts on $x = 0$, it cannot cross this line again and must remain in $x > 0$. Suppose our path could cross $y = 0$ at $z = x_0 > 0$. The level at $z = x_0$ is

$$\Re f(x_0) = \tfrac{1}{3}x_0^3 + x_0 > 0,$$

the starting level at iy_0 was 0, and the path is a path of descent; this is a contradiction. Thus the path remains in the first quadrant and therefore ends at $\infty \exp(\frac{1}{3}\pi i)$. (An elaborate argument of this kind is given by Watson (7), section 8·61, for the function $f(z) = z\cosh(\alpha + i\beta) - \sinh z$.) The same result, that the path of steepest descent from iy_0 $(y_0 > 1)$ ends at $\infty \exp(\frac{1}{3}\pi i)$, will now be obtained more simply by using the method of conformal mapping. Let us try to map the first quadrant in the z-plane on part of the w-plane; the saddle-point $z = i$ is excluded by a small indentation and the quadrant is closed by a large circular arc at ∞. When z describes the indented boundary $pqrstp$ once, w evidently describes the closed curve $p'q'r's't'p'$ once, the straight lines $p't'$ and $t's'$ being treated as distinct parts of the boundary. (They can be separated by the further transformation $W = (w - \frac{2}{3}i)^{\frac{1}{2}}$.)

Theorem B shows that the interior D_z of $pqrstp$ is mapped onto the interior D_w of $p'q'r's't'p'$, and evidently we can take the radius of qrs arbitrarily large. The unbounded domain D_z is thus mapped onto the unbounded domain D_w. The point $a = iy_0$ lying on ts is mapped on the point $a' = -\frac{1}{3}iy_0^3 + iy_0$ lying on $t's'$. The path of steepest descent from a' in the w-plane is the horizontal straight line

$$w = -\tfrac{1}{3}iy_0^3 + iy_0 - \tau \ (\tau \text{ real}, \ 0 < \tau < \infty)$$

starting at a' and going to $-\frac{1}{3}iy_0^3 + iy_0 - \infty$. This obviously lies in D_w, and therefore its inverse image lies in D_z. The point

$$w = -\tfrac{1}{3}iy_0^3 + iy_0 - \infty$$

corresponds to
$$z = \infty \exp(\tfrac{1}{3}\pi i),$$

as is evident by examining the correspondence between distant points of the two domains D_z and D_w.

If it were desired to show the existence of a path of descent from a to $\infty \exp(\frac{1}{4}\pi i)$, say, this would correspond in the w-plane to a path of decreasing u from a' to $\infty \exp(\frac{3}{4}\pi i)$, and such paths clearly exist in D_w, e.g. the straight line

$$w = a' + \tau \exp(\tfrac{3}{4}\pi i) \quad 0 < \tau < \infty.$$

The inverse image of this path in D_z is a path of descent from a to $\infty \exp(\frac{1}{4}\pi i)$.

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A variant of our method is applicable in some cases where the conformal mapping of the domains may fail. This uses merely the (1, 1) mapping of the boundaries and will now be applied to the present example to show once again that the path of steepest descent from a ends at $\infty \exp(\frac{1}{3}\pi i)$. Consider the closed quadrant $pqrstp$, of large but finite radius. The path of steepest descent from a enters the quadrant at a as

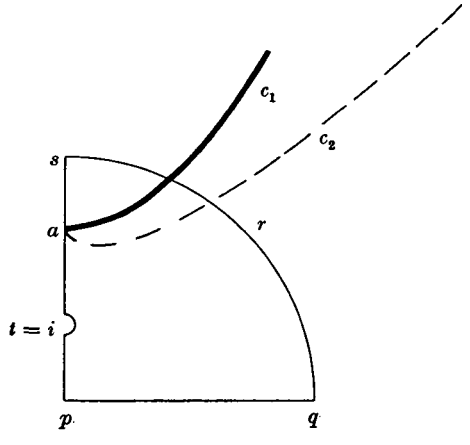


Fig. 1. Example 1. The z-plane.

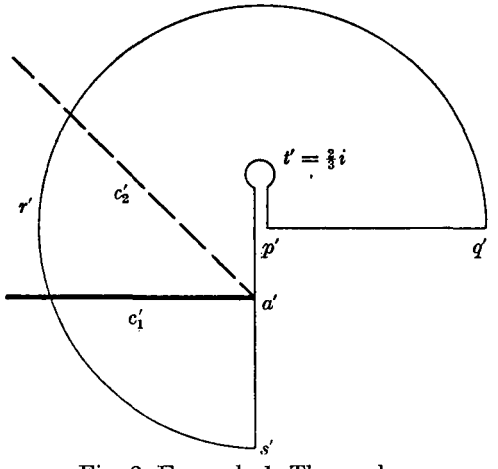


Fig. 2. Example 1. The w-plane.

The path of steepest descent c_1 is mapped on the horizontal line c_1'.

we have seen; where does it leave the quadrant again? This point (l, say) must lie on the boundary of the quadrant, therefore its image l' lies on the boundary $p'q'r's't'p'$, and moreover $\Im(l') = \Im(a')$, and $\Re(l') < \Re(a')$ since the path is a path of steepest descent. It is obvious from Figure 2 that there is only one such point l' on the boundary, and that its inverse image in Figure 1 tends to $\infty \exp(\frac{1}{3}\pi i)$ as the radius of qrs tends to infinity. This is the required result. This variant evidently requires that straight lines $\Im(w) =$ constant should intersect the boundary in the w-plane in exactly one point to the left of the starting point of the path of steepest descent, or that some similar geometrical condition be satisfied.

Example 2. As before, consider $w = f(z) = \frac{1}{3}z^3 + z$. This has a saddle-point at $z = i$, and the valleys at ∞ are in the directions $\arg z = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$, as we have seen. It seems plausible that the 2 paths of steepest descent from $z = i$ should end at $\infty \exp(\frac{1}{3}\pi i)$ and $\infty \exp(\pi i)$ and this can indeed be shown by the method of example 1, if we take for D_z successively the 2 regions $0 < \arg z < \frac{1}{2}\pi$, and $\frac{1}{2}\pi < \arg z < \pi$, both indented at $z = i$. The saddle-point is excluded because near it the mapping is not (1, 1), see Littlewood (3), theorem 115; if the mapping is applied to the complete sector $\frac{1}{3}\pi < \arg z < \pi$, this gives rise to a multiply covered Riemann domain D_w. We shall now see how Theorem B can be generalized to be applicable to this Riemann domain; and how this generalization can be used to demonstrate the existence of paths of descent. Consider Figure 3.

$$\left.\begin{array}{lll} \text{On } pq, & z = r\exp(\frac{1}{3}\pi i) \quad \text{and} & w = -\frac{1}{3}r^3 + r\cos\frac{1}{3}\pi + ir\sin\frac{1}{3}\pi; \\ \text{on } ps, & z = r\exp(\pi i) \quad \text{and} & w = -\frac{1}{3}r^3 - r; \\ \text{on } qrs, & w \sim \frac{1}{3}r^3 e^{3i\theta}, & \pi < 3\theta < 3\pi. \end{array}\right\} \tag{3·2}$$

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We observe that pq and ps are mapped on 2 non-intersecting curves $p'q'$ and $p's'$, and that the complete boundary $pqrsp$ is mapped on $p'q'r's'p'$ which is not a simple closed curve. It will be shown that D_z ($\frac{1}{3}\pi < \arg z < \pi$) is mapped by $w = f(z)$ on a Riemann surface D_w, such that the part of the w-plane bounded by $s'p'q'n's'$ (see Figure 4) is covered twice; there is a cut from t' to ∞.

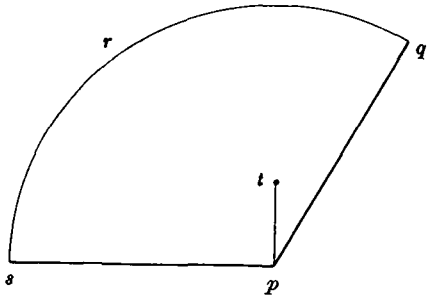


Fig. 3. Example 2. The z-plane.

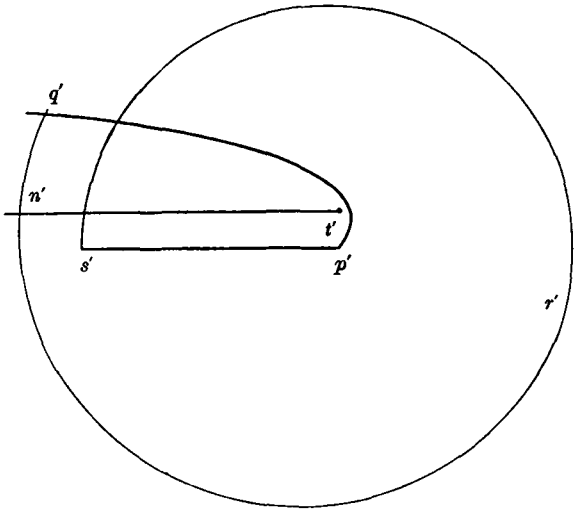


Fig. 4. Example 2. The w-plane.

It will first be shown that $t'(w = \frac{2}{3}i)$, the image point of $t(z = i)$, lies inside the open loop $s'p'q'$, as shown in Figure 4. If not, it lies either on the loop or outside the loop. The former possibility can be excluded by indenting spq suitably at one point. To examine the latter possibility consider

$$\frac{1}{2\pi}[\arg(f(z)-f(i))] = \frac{1}{2\pi}[\arg(w-\tfrac{2}{3}i)]$$

taken round the boundary of D_z. By Theorem A, this is the number of zeros of $f(z)-f(i)$ in D_z, at least 2, since $z = i$ is a double zero. But if t' lies outside $s'p'q'$ then, from Figure 4, $(2\pi)^{-1}[\arg(w-\frac{2}{3}i)]$ is evidently 1. It follows from this contradiction that t' does indeed lie inside the loop $s'p'q'$.

Take a cut from t' to ∞. (For convenience we shall take it inside the loop $s'p'q'$ but this is not necessary.) The curve $p'q'r's'p'$ may now be regarded as the boundary of a Riemann domain, of which the boundary is described once when $pqrsp$ is described once in the z-plane; thus there is a (1, 1) correspondence between points in D_z and points in D_w. This may be proved by introducing the new variable $W = (w-\frac{2}{3}i)^{\frac{1}{2}}$. It is not difficult to see that the boundary curve $p''q''r''s''p''$ is described once in the W-plane when $pqrsp$ is described once in the z-plane, thus by Theorem B there is a (1, 1) correspondence between D_z and D_W and since $w = \frac{2}{3}i + W^2$ there is a (1, 1) correspondence between D_W and the Riemann domain D_w, i.e. there is a (1, 1) correspondence between D_z and the Riemann domain D_w.

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Now a path of descent from t in the z-plane is mapped on a path of decreasing $\Re w$ in the w-plane, and it is readily seen from Figure 4 that many such paths can be drawn, lying in D_w and joining t' to q' or joining t' to s'. In particular, if the cut from t' is taken to be parallel to the negative real axis, let us take the 2 paths lying respectively along the upper and the lower edge of the cut; since $\Im w$ = constant on these paths they are paths of steepest descent, and their inverse images in the z-plane are the

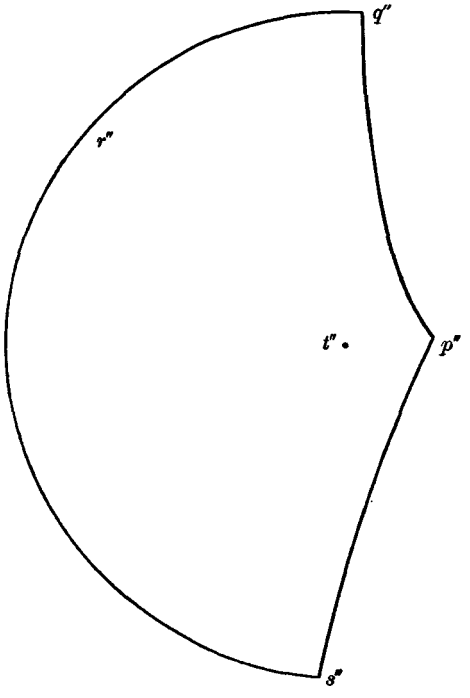


Fig. 5. Example 2. The W-plane.

2 paths of steepest descent from t. (Note. The upper and lower edges of the cut in the w-plane are mapped on distinct curves in the W-plane and the z-plane.) It is readily seen that they terminate at $\infty \exp(\frac{1}{3}\pi i)$ and $\infty \exp(\pi i)$, and this completes the proof.

An argument similar to the above is applicable to all cases in which the image in the w-plane of the boundary of D_z consists of a simple open loop $s'p'q'$ extending to $-\infty$, together with a large circular arc $q'r's'$.

Example 3. $w = f(z) = \frac{1}{9}z^9 + z$. This has saddle-points at the zeros of $f'(z)$, i.e. at $z = \exp\{\frac{1}{8}\pi i(2m+1)\}$, $m = 0, 1, 2, \ldots, 7$; and valleys at $\infty \exp\{\frac{1}{9}\pi i(2s+1)\}$, $s = 0, 1, 2, \ldots, 8$. It will be shown that paths of descent can be drawn, starting at the saddle-point $z = t \equiv \exp \frac{1}{8}\pi i$ and ending at ∞ in either of the 2 'nearest' valleys $\infty \exp \frac{1}{9}\pi i$ and $\infty \exp \frac{1}{3}\pi i$. (The paths of steepest descent from t will also be briefly considered, although, as we have seen in section 1, they are not really needed.)

This example will be used to illustrate the *method of univalent dissection*. The original domain under consideration (D_z, say) is dissected into non-overlapping domains ($D_z^{(n)}$ say), i.e. the closure of the open set $\Sigma D_z^{(n)}$ is equal to the closure of D_z.

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The domains $D^{(n)}$ are chosen so that in each domain $D_z^{(n)}$ the function $f(z)$ is univalent, i.e. $f(z_1) \neq f(z_2)$ when z_1, z_2 are distinct points in $D_z^{(n)}$. (It can be shown that in principle such univalent dissections can always be made by taking the domains $D_z^{(n)}$ sufficiently small.) Since $f(z)$ is not univalent near saddle-points ((3), theorem 115) it is evident that saddle-points cannot lie inside any $D_z^{(n)}$ but must lie on boundaries. When a univalent dissection has been found explicitly it is then not difficult to answer questions relating to paths of descent or paths of steepest descent by pursuing them from domain to domain, as will be seen.

It is convenient to take for our domain D_z the sector $0 < \arg z < \frac{3}{8}\pi$, containing t and the 2 valleys $\infty \exp \frac{1}{9}\pi i$ and $\infty \exp \frac{1}{3}\pi i$. This is dissected into $D_z^{(1)}(0 < \arg z < \frac{1}{8}\pi)$, $D_z^{(2)}(\frac{1}{8}\pi < \arg z < \frac{1}{4}\pi)$ and $D_z^{(3)}(\frac{1}{4}\pi < \arg z < \frac{3}{8}\pi)$, and in each of these smaller sectors the function $f(z)$ is univalent, as will now be shown by means of Theorem B.

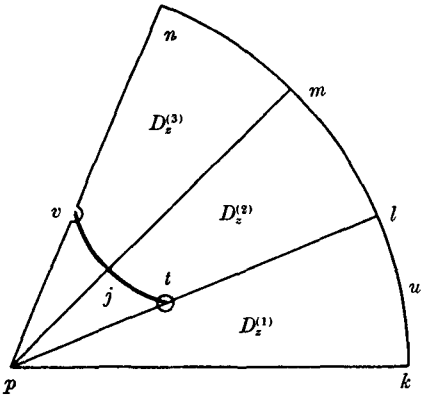


Fig. 6. Example 3. The z-plane. The domains $D_z^{(1)}$, $D_z^{(2)}$ and $D_z^{(3)}$.

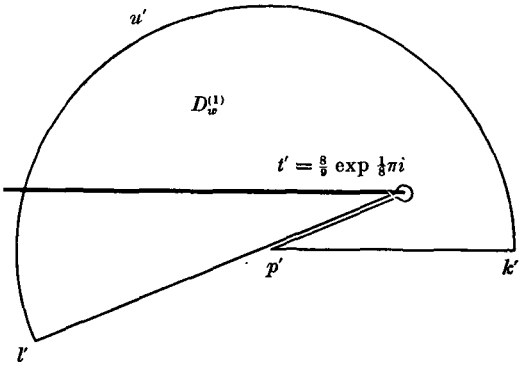


Fig. 7. Example 3. The w-plane. The domain $D_w^{(1)}$.

Consider e.g. the sector $D_z^{(1)}$ bounded by the contour $pkultp$ indented at the saddle-point t (Figure 6). On pk, $z = r > 0$ and $w = \frac{1}{9}r^9 + r > 0$; on ptl, $z = r \exp \frac{1}{8}\pi i$ and $w = (-\frac{1}{9}r^9 + r) \exp \frac{1}{8}\pi i$, a straight line in the w-plane; on kul, $z = \mathrm{Re}^{i\theta}$ $(0 < \theta < \frac{1}{8}\pi)$ and $w \sim \frac{1}{9}R^9 e^{9i\theta}$ $(0 < 9\theta < \frac{9}{8}\pi)$. Thus, when z describes the boundary $pkultp$ once, the image point w describes the boundary $p'k'u'l't'p'$ once (see Figure 7). Since the radius R of kul can be taken to be arbitrarily large it follows from Theorem B that the unbounded domain $D_z^{(1)}$ is mapped conformally on the unbounded domain $D_w^{(1)}$ shown in Figure 7; in other words, $f(z)$ is univalent in $D_z^{(1)}$. Similarly the indented sectors $D_z^{(2)}$ (bounded by $ptlmp$) and $D_z^{(3)}$ (bounded by $pmnvp$ where $v = \exp \frac{3}{8}\pi i$) are mapped on the domains $D_w^{(2)}$ (Figure 8) and $D_w^{(3)}$ (Figure 9).

We can now consider paths of descent from t. In Figure 7, the horizontal line from t' towards the left lies in $D_w^{(1)}$; in other words the path of steepest descent from t into $D_z^{(1)}$ lies entirely in $D_z^{(1)}$. By examining the correspondence between distant points of the domains $D_z^{(1)}$ and $D_w^{(1)}$ it is seen that this path ends at $\infty \exp \frac{1}{9}\pi i$.

Next, let us consider the domain $D_z^{(2)}$. In Figure 8, the horizontal line from t' towards the left intersects $p'm'$; thus the path of steepest descent from t into $D_z^{(2)}$

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ultimately passes into $D_z^{(3)}$. It will now be shown that a path of descent can be drawn from t to $v = \exp \frac{3}{8}\pi i$, and thence to $\infty \exp \frac{1}{3}\pi i$. In Figures 8 and 9, the straight line $t'v'$ intersects $p'm'$, at the point j', say. Evidently $\Re w$ decreases from t' to v', therefore $t'j'$ in Figure 8 and $j'v'$ in Figure 9 are paths of descent, and so are their inverse images in the z-plane (Figure 6); in other words, tjv is a path of descent from t to v. Furthermore, in Figure 9 the horizontal line from v' towards the left lies in $D_w^{(3)}$, in other words, the path of steepest descent from v into $D_z^{(3)}$ lies entirely in $D_z^{(3)}$, and therefore can be seen to end at $\infty \exp \frac{1}{3}\pi i$. On combining this path of steepest descent with the path tjv we obtain a path of descent from t to $\infty \exp \frac{1}{3}\pi i$. (It is evident that there are many such paths.)

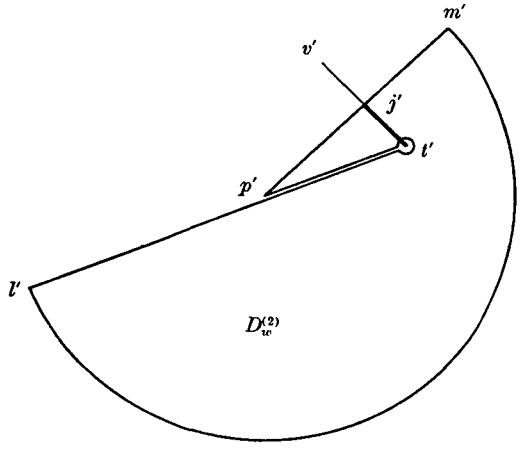


Fig. 8. Example 3. The w-plane. The domain $D_w^{(2)}$.

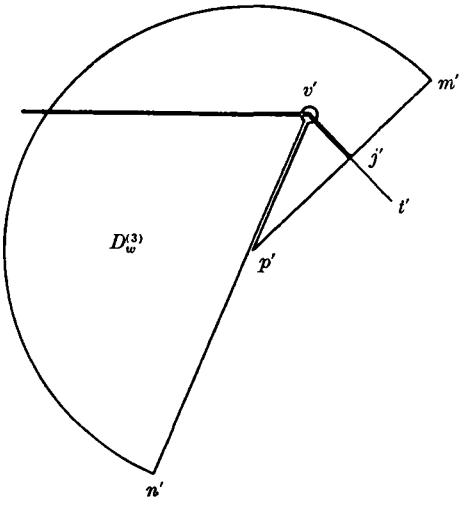


Fig. 9. Example 3. The w-plane. The domain $D_w^{(3)}$.

This example shows how a path of descent can be pursued from one sector to a neighbouring sector. In the same way it can be shown that the path of steepest descent from $\exp \frac{1}{8}\pi i$ into $D_z^{(2)}$ passes through the saddle-point $\exp \frac{7}{8}\pi i$ and thence to $\infty \exp \frac{7}{9}\pi i$ or $\infty \exp \pi i$. The same idea can also be used to find, e.g. the complete topography of the paths of steepest descent $(n+1)^{-1}z^{n+1} + e^{ni\alpha}z$, a function which has a univalent dissection into sectorial domains similar to the dissection of Example 3.

4. *Conclusion.* Given a domain D_z in the z-plane, and 2 points α and β in D_z, is it possible to draw a path $\alpha\beta$ in D_z such that $\Re f(z)$ decreases steadily on this path? (Or alternatively, such that $\Re f(z) < \Re f(\alpha)$ on the path, except at $z = \alpha$?) Three examples have been given above to show how the conformal mapping $w = f(z)$ can be used to answer this question (or the alternative question) in suitable circumstances. In these examples the mapping of the domain D_z could be inferred from the mapping of the boundary, and the existence of the paths of descent was then obvious from geometrical properties of the image domain D_w.

Example 1 illustrated a straightforward case, Example 2 a mapping onto a Riemann surface. Example 3 introduced the idea of a univalent dissection of the original domain

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into smaller domains in each of which the mapping function is univalent; such a dissection allows a path of descent to be pursued from domain to domain and was found in practice to be simpler than the method of Example 2. Another example using conformal mapping was given in an earlier paper (5) where paths of descent for three different functions $f_1(z)$, $f_2(z)$, $f_3(z)$ had to be fitted together. An alternative argument was also given there ((5), p. 183) in which Theorem B was replaced by a weaker local result. A variant of our argument, which uses only the mapping of the boundaries, was treated at the end of Example 1 above.

It is believed that the methods described in the present paper may often be simpler than the customary tracing of paths of steepest descent, or than the direct method of assuming a plausible path $\alpha\beta$ and testing the variation of $\Re f(z)$ on the path, but these earlier methods have been applied to comparatively few cases. For the purpose of comparison the present method was applied to Watson's example $f(z) = z\cosh(\alpha + i\beta) - \sinh z$ ((7), section 8·61). Theorems A and B were used to show that the straight lines $\Re z = \pm\alpha$ and $\Im z = \pm\beta \pm 2m\pi$ $(m = 0, 1, 2, \ldots)$ dissect the z-plane into rectangular domains in each of which $f(z)$ is univalent. (The details are not given here.) The same univalent dissection may be used when the large parameter N in (1·1) is complex, for evidently a dissection univalent for $f(z)$ is also univalent for $cf(z)$, where c is any complex constant. The available evidence from this and other examples suggests that the method of conformal mapping (and particularly the method of univalent dissection) is more practical than earlier methods of tracing paths of descents, but further experience is needed to establish its scope.

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