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Stokes flow between two intersecting circular arcs

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1. *Introduction.* This paper considers a family of viscous flows closely related to the exact Jeffery–Hamel solution ((1), (2)) of the two-dimensional Navier–Stokes equations, for diverging or converging flow in a channel. It is known that if the walls of the channel intersect at an angle less than π then there is a unique solution of the Navier–Stokes equations in which the streamlines are straight lines issuing from the point of intersection of the walls and the flow is everywhere diverging or everywhere converging. The flow parameters depend on the total fluid mass M emitted at the point of intersection and the angle 2α between the walls. By taking the Reynolds number $R = M/\nu$, where ν is the kinematic viscosity, the stream function can be expanded in a power series in R in which the leading term is a Stokes flow. Alternatively the solution can be developed by perturbing the Stokes flow and is one of very few examples known in which a Stokes flow can be regarded as a uniformly valid first approximation everywhere in an infinite fluid region. The class of flows to be considered is a generalization of the Jeffery–Hamel flow by taking the flow region to be finite and bounded by two circular arcs which intersect at an angle less than π. At one point of intersection fluid is forced into the region and an equal amount is absorbed out at the other point. It is found to the first order that the flow at the two points of intersection corresponds to the zero Reynolds number limit for diverging and converging flow, respectively. Now since the flow at these points can be developed by perturbing the Stokes flow solution it is reasonable to assume that the zero Reynolds number flow in the entire finite region bounded by the arcs is a Stokes flow since the most likely region in which this approximation becomes invalid is locally at the points of intersection but here the validity of the approximation is ensured. A comparison of the convection terms with the viscous terms verifies that this conclusion is borne out.

2. *Inversion.* Consider inversion with respect to the unit circle centred at the point $z = -c$. If $\zeta = \xi + i\eta$ is the point in the inverse plane (ξ, η) corresponding to z, then

$$\zeta = \frac{1}{(\bar{z}+c)}. \tag{1}$$

This transformation represents geometrical inversion in which the region

$$-\beta \leqslant \Im M z \leqslant \alpha \quad (0 < \beta < \tfrac{1}{2}\pi), \quad (0 < \alpha < \tfrac{1}{2}\pi)$$

inverts into the region interior to the circular arcs represented by the equations

$$\left(\xi - \frac{1}{2c}\right)^2 + \left(\eta - \frac{1}{2c}\cot\alpha\right)^2 = \frac{1}{4c^2}\operatorname{cosec}^2\alpha, \tag{2a}$$

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and
$$\left(\xi-\frac{1}{2c}\right)^2+\left(\eta+\frac{1}{2c}\cot\beta\right)^2=\frac{1}{4c^2}\operatorname{cosec}^2\beta. \tag{2b}$$

The origin in the z-plane corresponds to $\zeta = 1/2c$ and the point at infinity to $\zeta = 0$.

3. *Inversion of biharmonic functions.* Let ψ_0 be a biharmonic function in the z-plane; then
$$\nabla^4\psi_0 = 16\frac{\partial^4}{\partial z^2\partial\bar{z}^2}\psi_0 = 0, \tag{3}$$

and a solution of the biharmonic equation in the ζ-plane is given by
$$\psi_0' = \frac{\psi_0}{(z+c)(\bar{z}+c)}, \tag{4}$$

satisfying
$$\nabla'^4\psi_0' = 16\frac{\partial^4\psi_0'}{\partial\zeta^2\partial\bar{\zeta}^2} = 0. \tag{5}$$

In the present problem the fluid velocity vanishes on each of the arcs and in addition the total mass flow output at $\zeta = 1/c$ is M and the total mass flow absorbed at $\zeta = 0$ is $-M$. The boundary conditions stated in the z-plane are then of the form
$$\psi(r,\alpha) = (r^2+c^2+2rc\cos\alpha)\,k_1 \quad (\theta=\alpha), \tag{6a}$$
$$\psi(r,\beta) = (r^2+c^2+2rc\cos\beta)\,k_2 \quad (\theta=-\beta), \tag{6b}$$

where k_1 and k_2 are constants, and
$$\partial\psi/\partial\theta\,(r^2+c^2+2rc\cos\alpha)+2rc\sin\alpha\psi = 0 \quad (\theta=\alpha), \tag{7a}$$
$$\partial\psi/\partial\theta\,(r^2+c^2+2rc\cos\beta)-2rc\sin\beta\psi = 0 \quad (\theta=-\beta). \tag{7b}$$

An appropriate solution of (3) satisfying the above boundary conditions is
$$\begin{aligned}\psi(r,\theta) = {} & r^2(A_1\theta+B_1\sin 2\theta+C_1\cos 2\theta)\\ & +rc(A_2\theta\cos\theta+B_2\theta\sin\theta+C_2\sin\theta+D_2\cos\theta)\\ & +(A_3\theta+B_3\sin 2\theta+C_3\cos 2\theta),\end{aligned} \tag{8}$$

where the constants A_1, B_1, C_1, etc., are arbitrary. Two particular configurations will be considered: (a) when the arcs are symmetrical and $\beta = +\alpha$; and (b) one arc is a straight line, i.e. $\beta = 0$, and the other a semi-circle $\alpha = \frac{1}{2}\pi$. For the symmetrical case (a), the constants are determined from the equations
$$C_1 = B_2 = C_2 = C_3 = D_2 = 0,$$
$$A_1 = A_3, \quad B_1 = B_3,$$
$$A_1+2B_1\cos\alpha = C_2+A_2\cos^2\alpha = 0,$$

and the stream function is given by
$$\begin{aligned}\psi(r,\theta) = {} & -k(r^2+c^2)\,(2\theta\cos 2\alpha-\sin 2\theta)\\ & +4rck\frac{(\sin 2\alpha-2\alpha\cos 2\alpha)}{(2\alpha-\sin 2\alpha)}(\theta\cos\theta-\cos^2\alpha\sin\theta),\end{aligned} \tag{9}$$

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the constant k depending on the flux of mass leaving A and being absorbed at B. Now as $r \to 0$, or ∞, then from (4) and (9)

$$\lim \psi' = k(2\theta \cos 2\alpha - \sin 2\theta). \tag{10}$$

Thus
$$M = [\lim \psi']_{-\alpha}^{+\alpha} = -2k(2\alpha \cos 2\alpha - \sin 2\alpha), \tag{11}$$

or
$$k = \tfrac{1}{2}M(\sin 2\alpha - 2\alpha \cos 2\alpha)^{-1}.$$

The flow locally at A, $\zeta = 1/c$, is then

$$\psi' = \frac{M(2\theta \cos 2\alpha - \sin 2\theta)}{2(2\alpha \cos 2\alpha - \sin \alpha\, 2\alpha)} + O\left(\frac{1}{r}\right), \tag{12}$$

or, in terms of the local radius vector ρ_1 and the angular coordinate ϕ_1 at A,

$$\psi' = \frac{M(2\phi_1 \cos 2\alpha - \sin 2\phi)}{2(2\alpha \cos 2\alpha - \sin 2\alpha)} + O(\rho_1). \tag{13}$$

This expression corresponds to the limit of the stream function for the Jeffery–Hamel solution for flow between inclined walls ($0 < \alpha < \frac{1}{2}\pi$) as the Reynolds number tends to zero. A similar situation occurs at B where the fluid mass is absorbed out of the region enclosed by the arcs. In the case for which the arcs intersect at an angle $\frac{1}{2}\pi$, the stream function is

$$\psi'(r, \theta) = \frac{M}{2}\left\{\frac{(r^2 + c^2)\sin 2\theta + 4rc\,(2\theta \cos\theta - \sin\theta)/(\pi - 2)}{(r^2 + c^2 + \sqrt{2}rc)}\right\}. \tag{14}$$

The vorticity $\omega' = 4(\partial^2\psi'/\partial\zeta\partial\bar{\zeta})$ calculated from (4) and (9) is given by

$$\omega' = -\frac{16kc^2\theta \sin 2\alpha(1 - \cos 2\alpha)}{(2\alpha - \sin 2\alpha)} - \frac{4k(c^4 + r^4)}{r^2}\sin 2\theta - 4ck\frac{(r^2 + c^2)}{r}(4\alpha - \sin 4\alpha)\sin\theta, \tag{15}$$

which is clearly harmonic in both planes and takes equal and opposite non-zero values on the boundaries. Hence there is no possibility of laminar separation for this flow and the fluid velocity is everywhere directed in the sense of the source at A to the sink at B. The vorticity derivatives computed from (15) in the z-plane are

$$\frac{\partial\omega'}{\partial r} = -8k\left(r - \frac{c^4}{r^3}\right)\sin 2\theta - 4ck\left(r - \frac{c^2}{r}\right)(4\alpha - \sin 4\alpha)\sin\theta,$$

$$\frac{1}{r}\frac{\partial\omega'}{\partial\theta} = -\frac{16kc^2 \sin 2\alpha(1 - \cos 2\alpha)}{r(2\alpha - \sin 2\alpha)} - \frac{8k(c^4 + r^4)}{r^3}\cos 2\theta - 4ck\frac{(r^2 + c^2)}{r^2}(4\alpha - \sin 4\alpha)\cos\theta.$$

Thus it is readily verified that grad ω' evaluated in the physical ζ-plane is non-zero in the region $-\alpha < \theta < \alpha$, which ensures that

$$-\omega'\frac{\partial\psi'}{\partial\bar{\zeta}}\Big/\left|\nu i\,\frac{\partial\omega'}{\partial\bar{\zeta}}\right|$$

tends to zero with R. The perturbations of the Stokes flow are not given here because of the unwieldy nature of the perturbed stream function due to interaction terms between the source and sink arising from the non-linearity of the equations. However,

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this remains an interesting problem, since it is known from the Jeffery–Hamel solution that as the Reynolds number is increased a critical value is attained in which purely radial divergent flow from the point of intersection of the walls becomes impossible and separation occurs with fluid flowing back, symmetrically along each wall into the point of intersection. In contrast, for converging flow in a channel no such phenomenon occurs, and assuming the flow remains laminar it is purely radially convergent for all Reynolds numbers. From a physical point of view it is reasonable to suppose that for the flow between the arcs separation will commence at the point of intersection where the adverse pressure gradient is infinite and the flow is diverging, with bubbles symmetric about the common chord. With increasing Reynolds numbers the region of separation will spread out along each wall but will never reach the point where the fluid is absorbed out of the region. To conclude, the stream function for case (*b*) will be given. The coefficients A_i, B_i, are determined by

$$A_3 = B_3 = A_1 = B_1 = 0,$$

$$\frac{2}{\pi} A_2 = B_2 = \frac{8c_1}{(4-\pi^2)}, \quad D_2 = \frac{2c_3}{c^2} = 2c_1.$$

Replacing c_1 by k, a constant, the stream function in the inverse plane is given by

$$\psi(r,\theta) = k(r^2+c^2)\cos 2\theta + krc\left\{\frac{4\pi}{4-\pi^2}\left(\theta\cos\theta - \sin\theta + \frac{2}{\pi}\theta\sin\theta\right) + 2\cos\theta\right\}.$$

As $r \to 0$, or ∞, $\qquad \psi' \to k\cos 2\theta.$

The constant k is again determined by the flux leaving or entering the points of intersection, and is for $\zeta = 0$

$$M = [k\cos 2\theta]_{\theta=0}^{\theta=\frac{1}{2}\pi} = -k.$$

The preceding analysis may also recover Rayleigh's solution ((3), page 78) for the case $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, and is also applicable for flows exterior to two intersecting circles with Jeffery–Hamel flow locally at the points of intersection.

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