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On complex vector bundles on rational threefolds

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Introduction

It is known [**14**] that every topological complex vector bundle on a smooth rational surface admits an algebraic structure. In [**10**] one constructs algebraic vector bundles of rank 2 on $\mathbb{P}_3$ with arbitrary Chern classes c_1, c_2 subject to the necessary tcpological condition $c_1 c_2 \equiv 0 \pmod 2$. However, in dimensions greater than 2 the Chern classes don't determine the topological type of a vector bundle. In [**2**] one classifies the topological complex vector bundles of rank 2 on $\mathbb{P}_3$ and one proves that any such bundle admits an algebraic structure.

Inspired by these papers we study these two questions, namely, the topological classification and the existence of algebraic structures for vector bundles on (complex, smooth, projective) rational threefolds. To be more precise, while the topological classification is pushed up to general rational threefolds, and this is a by-product of [**2**], the existence of algebraic structures is investigated here only for rank 2 vector bundles on homogeneous rational threefolds.

The topological classification is given by the theorem below (the only essential cases being rank 2 and rank 3-bundles).

THEOREM 1. *Let X be a compact connected complex manifold of dimension* 3.

(1) *The cohomology classes (c_1, c_2, c_3) are the Chern classes of a topological complex vector bundle of rank* 3 *if and only if* $(c_1 + c_1(X))\, c_2 \equiv c_3 \pmod 2$. *Moreover, (c_1, c_2, c_3) determines the topological type of the bundle.*

(2) *The pair (c_1, c_2) corresponds to the Chern classes of a rank* 2 *topological vector bundle if and only if* $(c_1 + c_1(X))\, c_2 = 0 \pmod 2$.

Moreover, assuming that $H_1(X, \mathbb{Z}) = 0$, one can assert:

(i) *If $c_1 \neq c_1(X)$ in $H^2(X, \mathbb{Z}_2)$, then the topological type of the bundle is completely determined by (c_1, c_2).*

(ii) *If $c_1 = c_1(X)$ in $H^2(X, \mathbb{Z}_2)$, then there exist exactly two topological types with given (c_1, c_2), which are distinguished by an invariant $\alpha \in \mathbb{Z}_2$; if the bundle E admits in addition a holomorphic structure and if it is normalized such that $c_1(E) = c_1(K_X)$, then*

$$\alpha(E) = h^0(E) + h^2(E) \pmod 2.$$

We have denoted by K_X the canonical class, by $c_1(X)$ the first Chern class of the holomorphic tangent bundle and we have as usual identified $H^6(X, \mathbb{Z})$ with $\mathbb{Z}$.

The theorem applies to simply connected threefolds, in particular to rational threefolds (or more generally, by a result of Serre, to unirational ones).

The proof involves K-theory (together with a few homotopical facts), the essential point being the use of [**2**].

The main result of the paper is the following.

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THEOREM 2. *Every topological complex vector bundle of rank 2 on a rational homogeneous threefold admits an algebraic structure.*

We construct as in [2], [10] (see also [12], chapter I, §5) bundles on a threefold X as extensions

$$0 \to L_1 \to E \to L_2 \otimes \mathscr{I}_Y \to 0,$$

where L_1, L_2 are algebraic line bundles and Y is a l.c.i. (local complete intersection) curve. L_1 will be in fact a sufficiently positive line bundle and L_2 will be determined by L_1 and $c_1(E)$. The difficulty lies in finding the appropriate l.c.i. curves Y: the cohomology class $[Y]$ must coincide with $c_2(E) - c_1(L_1)c_1(L_2)$, $\det(N_Y)$ must be isomorphic with $L_1^{-1} \otimes L_2$ and, when $c_1(E) = c_1(K_X)$, the equality

$$h^0(L_1) + h^1(L_2 | Y) = \alpha(E) \pmod 2$$

must be valid.

In [2] disjoint unions of global complete intersections in $\mathbb{P}_3$ are taken into account; afterwards number-theoretical considerations are involved in the proof. Here the key point is a construction due to O. Forster and the first author [4]: starting with a l.c.i. curve in a threefold one constructs new l.c.i. curves with the same support and having triple multiplicity. Once one handles well with these triple structures (we recall the details needed here) and with the double ones [5], then it is not difficult to construct the desired curves. One can understand this as follows. As every natural integer is a combination of 2 and 3, one considers disjoint unions of double and triple structures on some simple rational curves of X in order to obtain l.c.i. curves Y with given (componentwise) positive cohomology class $[Y]$ in $H^4(X, \mathbb{Z})$ and, in the K_X-symplectic case, satisfying in addition the congruence relation concerning the Atiyah–Rees invariant α. On the other hand, when doubling or tripling a given curve one has a degree of freedom which is used then in order to fulfil the necessary condition concerning the normal bundle N_Y. Of course, in the whole proof we will exploit the very concrete nature of the threefold: $\mathbb{P}_3$, the quadric $\mathbb{Q}_3$, the flag $\mathbb{F}_{1,2}$, $\mathbb{P}_2 \times \mathbb{P}_1$, $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

Let us mention that the double structures have already been used in ([8], theorem 2·5) for the construction of large families of bundles on $\mathbb{P}_3$ with given (c_1, c_2), respectively (c_1, c_2, α) and that (very special) triple structures were used in ([3], prop. 2) to construct holomorphic bundles of rank 2 on $\mathbb{P}_3$, topologically trivial with prescribed degree of instability.

The paper is divided into 3 paragraphs: the first contains the topological classification, the second treats preliminaries for the existence Theorem 2 while the third paragraph contains its proof.

1. *The topological classification*

In this section we shall prove Theorem 1.

(1) The fact that in this case the bundle is completely determined by its Chern classes follows by two general results [13], [11]: the first ensures that the stable class is well-determined and the second asserts the equivalence between the stable and the non-stable classifications.

The necessity of the condition is a particular case of the integrality theorem 26·1·1 in [9]. Let c_1, c_2, c_3 be such that $(c_1 + c_1(X))c_2 \equiv c_3 \pmod 2$. We shall find a rank 3 complex vector bundle with Chern classes (c_1, c_2, c_3). In order to do that we will construct an appropriate system of vector bundles on X. Let $e_1, \ldots, e_p$ be a system of generators

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for $H^2(X,\mathbb{Z})$ and $f_1,\dots,f_q$, respectively, for $H^4(X,\mathbb{Z})$. $H^6(X,\mathbb{Z})$ will be identified with $\mathbb{Z}$ via the fundamental cohomology class $[X]$. We shall use the group $K(X)$. Any ξ of $K(X)$ can be written in our case as $\xi=[E]+l\cdot 1$, where E is a vector bundle of rank 3, $l\in\mathbb{Z}$ and 1 is the class of the trivial rank 1 vector bundle. We choose a simplicial decomposition of X. We apply the following result, which is derived from analysis of the Atiyah–Hirzebruch spectral sequence [**1**] (see also [**6**], ch. VII, th. I): if α from $H^{2r}(X,\mathbb{Z})$ satisfies $d_{2k+1}\alpha=0$ for every $k\geqslant 1$, denoting by d_q the differential in the spectral sequence, then there is ξ in $K(X)$, trivial on the skeleton X^{2r-1}, such that $c_r(\xi)=(r-1)!\,\alpha$.

By this argument, used for $r=3$, there is a vector bundle G, such that $c_1(G)=0$, $c_2(G)=0$ and $c_3(G)=2$. Analogously, for $r=2$ and for each f_j there is a vector bundle F_j, such that $c_1(F_j)=0$, $c_2(F_j)=f_j$, $c_3=(F_j)=\eta_j[X]$, where $\eta_j\in\mathbb{Z}$ (in fact, by modifying F_j in $K(X)$ with a multiple of G and by taking a new representative of rank 3, we may assume in addition that $\eta_j=0$ or 1). By the integrality condition,

$$\eta_j=c_1(X)f_j\,(\mathrm{mod}\,2).$$

We can also find line bundles E_i with $c_1(E_i)=e_i$. We will choose in $K(X)$ an element of the form

$$\sum_i\alpha_i[E_i]+\sum_j\beta_j[F_j]+\gamma[G]$$

whose total Chern class is $1+c_1+c_2+c_3$. By writing this element as $[E]+l\cdot 1$, with E a vector bundle of rank 3, the proof will be over. Hence we must find α_i, β_j, $\gamma\in\mathbb{Z}$ so that

$$(*)\quad\prod_i(1+e_i)^{\alpha_i}\prod_j(1+f_j+\eta_j[X])^{\beta_j}(1+2[X])^{\gamma}=1+c_1+c_2+c_3.$$

If $c_1=\sum_i a_ie_i$, $a_i\in\mathbb{Z}$, then putting $\alpha_i=a_i$ for $i=1,\dots,p$ the relation $(*)$ holds good for the terms in $H^2(X,\mathbb{Z})$. Let us denote by $\xi=\sum_i a_i[E_i]$, so that $c_1(\xi)=c_1$. By identifying the terms in $H^4(X,\mathbb{Z})$ from $(*)$, one gets

$$c_2(\xi)+\sum_j\beta_jf_j=c_2.$$

There are (not necessarily unique) $\beta_1,\dots,\beta_q$ which do that. The terms in $H^6(X,\mathbb{Z})$ of $(*)$ give rise to the equation

$$2\gamma+\sum_j\beta_j\eta_j+c_3(\xi)+c_1(\xi)\Big(\sum_j\beta_jf_j\Big)=c_3.$$

The compatibility condition for the unknown γ is that the remaining part should be even. By reducing (mod 2) and by using $\eta_j\equiv c_1(X)f_j$, the defining relation of β_j and the assumption $c_3\equiv(c_1+c_1(X))c_2$, the condition becomes

$$c_3(\xi)\equiv(c_1(\xi)+c_1(X))\,c_2(\xi).$$

But this is true in virtue of the integrality theorem cited above.

(2) The 'only if' part follows from (1). Let (c_1,c_2) such that $(c_1+c_1(X))c_2=0$ (mod 2). There is by (1) a vector bundle F of rank 3, with Chern classes $(c_1,c_2,0)$. Since $c_3=0$, F contains a trivial line sub-bundle and the quotient is the desired bundle.

In the proof of (i) and (ii) the fact that the cup product map

$$H^2(X,\mathbb{Z}_2)\times H^4(X,\mathbb{Z}_2)\to\mathbb{Z}_2$$

is non-degenerate will be used implicitly. Let us suppose that $H_1(X,\mathbb{Z})=0$. Then $H^1(X,\mathbb{Z})=H^5(X,\mathbb{Z})=0$. Consider (c_1,c_2) provided that $(c_1+c_1(X))c_2\equiv 0$ (mod 2).

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By analysing the first stages of the Postnikov tower associated to the map $BU(2) \to BU$, one gets that there are at most two isomorphism classes of rank 2-vector bundles with Chern classes (c_1, c_2) and that the corresponding obstruction space is

$$H^6(X, \mathbb{Z}_2)/(\mathrm{Sq}^2 + c_1)H^4(X, \mathbb{Z}_2),$$

(see [**15**], theorem 2, where the analysis is pushed up to polyhedra X of dimension $\leqslant 12$). In our case Sq^2 coincides with multiplication by $c_1(X)$ (mod 2).

(i) If $c_1 \neq c_1(X)$, then the obstruction space is trivial.

(ii) Let us assume that $c_1 = c_1(X)$ in $H^2(X, \mathbb{Z}_2)$. The obstruction space is in this case $\mathbb{Z}_2$ and we shall see that there are indeed two isomorphism classes. This is derived from the twisted symplectic K-theory method in [**2**], as follows. Let us consider, for a fixed line bundle L on X, the complex vector bundles together with non-degenerate skew-symmetric L-valued bilinear forms and the associated Grothendieck group $\mathrm{KSP}_L(X)$. As $\dim_{\mathbb{R}} X = 6$ and $H^1(X, \mathbb{Z}) = 0$, the corresponding reduced group $\widetilde{\mathrm{KSP}}_L(X)$ classifies rank 2 vector bundles with $c_1 = c_1(L)$. We can assume for our purposes, after a normalization, that $c_1 = -c_1(X)$, therefore the reduced K_X-symplectic group classifies our bundles. There is an exact sequence ($K = K_X$)

$$0 \to \widetilde{\mathrm{KSP}}_K(X) \to \mathrm{KSP}_K(X) \to \mathbb{Z} \to 0$$

and the direct image morphisms [**2**]

$$\alpha_K : \mathrm{KSP}_K(X) \to \mathrm{KO}^{-2}(\text{point}) = \mathbb{Z}_2,$$

$$\rho_K : \mathrm{KO}^{-2}(\text{point}) \to \mathrm{KSP}_K(X),$$

associated to the maps $X \to$ point and point $\to X$, which identify $\mathbb{Z}_2$ with a direct summand of $\mathrm{KSP}_K(X)$. As $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0$, $\mathbb{Z}_2$ will be in fact a direct summand of $\widetilde{\mathrm{KSP}}_K(X)$, the induced maps being denoted also by α_K and ρ_K. On the other hand let

$$\widetilde{\mathrm{KSP}}_K(X) \xrightarrow{c_2} H^4(X, \mathbb{Z})$$

be the morphism corresponding to the second Chern class evaluation. It is onto and its kernel has at most two elements. But $c_2 \circ \rho_K = 0$, because the direct image morphism associated to $\{x_0\} \to X$ factorizes through

$$\mathrm{Ker}\,(\mathrm{KSP}_K(X) \to \mathrm{KSP}_K(X \setminus U))$$

denoting by U a neighbourhood of x_0 (we take x_0 outside the skeleton X^5 and recall that the zero element in KSP_K is the class of the bundle $K_X \oplus 1$). Therefore $\mathbb{Z}_2$ is mapped one to one, hence also onto, in $\mathrm{Ker}\, c_2$. Consequently

$$\widetilde{\mathrm{KSP}}_K(X) \cong H^4(X, \mathbb{Z}) \times \mathbb{Z}_2,$$

the first projection map being c_2 and the second one α_K.

The last assertion of the Theorem 1 is a particular case of theorem 4·2 in [**2**].

2. *Preliminaries to the proof of Theorem* 2

2·1. Let X be a complex projective manifold of dimension 3, L_1 and L_2 algebraic line bundles on X, and Y a local complete intersection curve in X. Let us assume

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further that $H^2(X, L_1 \otimes L_2^{-1}) = 0$ and $\det(N_Y) \cong L_1^{-1} \otimes L_2$. Then there exists an algebraic bundle E of rank 2, realized as an extension

$$0 \to L_1 \to E \to L_2 \otimes \mathscr{I}_Y \to 0.$$

The condition concerning the normal bundle implies that $\mathrm{Ext}^1(L_2 \otimes \mathscr{I}_Y, L_1)$ is a monogenic $\mathscr{O}_Y$-module and the condition $H^2(X, L_1 \otimes L_2^{-1}) = 0$ ensures the lifting of every generator of this $\mathscr{O}_Y$-module to an element in $\mathrm{Ext}^1(L_2 \otimes \mathscr{I}_Y, L_1)$. This gives the desired extension (see [**10**] or [**12**], ch. I, § 5 for details). Moreover,

$$c_1(E) = c_1(L_1) + c_1(L_2),$$
$$c_2(E) = c_1(L_1)\, c_1(L_2) + [Y].$$

When X is a rational threefold, then the algebraic line bundles are well-determined by their Chern classes, hence necessarily $L_2 \cong L_1^{-1} \otimes \mathscr{O}(c_1(E))$, denoting by $\mathscr{O}(c_1(E))$ the algebraic line bundle with first Chern class $c_1(E)$. One obtains in this case that, if $c_1(E)$ is given and if one takes L_1 sufficiently positive, then the condition

$$H^2(X, L_1 \otimes L_2^{-1}) = 0$$

is satisfied. By using the extension, the exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O} \to \mathscr{O}_Y \to 0$ and duality, one has also, for L_1 sufficiently positive,

$$h^0(E) + h^2(E) = h^0(L_1) + h^1(L_2 | Y),$$

a relation which will be needed in computing the Atiyah–Rees invariant.

The difficulty in performing this construction lies in finding the right curves.

2·2. Let us recall Ferrand's procedure of 'doubling' a curve along a direction in the normal bundle [**5**], in the setting of [**4**]. Let X be again a projective manifold of dimension 3 and let Y be a l.c.i. curve in X. Start with an algebraic line bundle L on Y and with an epimorphism

$$N_Y^* = \mathscr{I}_Y / \mathscr{I}_Y^2 \xrightarrow{\epsilon} L \to 0.$$

Its kernel has the form $\mathrm{Ker}\,\epsilon = \mathscr{J}/\mathscr{I}_Y^2$. Then $\mathscr{J}$ is the ideal sheaf of a new curve Y' and $L \cong \mathscr{I}_Y/\mathscr{I}_{Y'}$. It turns out that Y' will be again l.c.i. and $Y \subset Y' \subset Y^{(1)}$, where $Y^{(1)} = (Y, \mathscr{O}/\mathscr{I}_Y^2 | Y)$ is the first infinitesimal neighbourhood of Y (one can find locally f, g such that $\mathscr{I}_Y = (f, g)$ and $\mathscr{I}_{Y'} = (f^2, g)$). At the level of cohomology classes one has the relation $[Y'] = 2[Y]$ (see [**7**], appendix A, for the definition of the cohomology classes of non-reduced subspaces).

One has the exact sequence derived from the inclusion of the ideals,

$$0 \to \mathscr{I}_Y^2/\mathscr{I}_Y\mathscr{I}_{Y'} \to \mathscr{I}_{Y'}/\mathscr{I}_Y\mathscr{I}_{Y'} \to \mathscr{I}_Y/\mathscr{I}_Y^2 \to \mathscr{I}_Y/\mathscr{I}_{Y'} \to 0,$$

and the isomorphisms

$$\mathscr{I}_Y/\mathscr{I}_{Y'} \cong L, \quad \mathscr{I}_Y^2/\mathscr{I}_Y\mathscr{I}_{Y'} \cong L^2, \quad \mathscr{I}_{Y'}/\mathscr{I}_Y\mathscr{I}_{Y'} \cong N_{Y'}^* | Y.$$

By taking det one gets $\det(N_{Y'}) | Y \cong \det(N_Y) \otimes L^{-1}$.

On the other hand, the exact sequence $0 \to L \to \mathscr{O}_{Y'} \to \mathscr{O}_Y \to 0$ gives the exact sequence

$$H^1(Y, L) \to \mathrm{Pic}\, Y' \to \mathrm{Pic}\, Y \to 1.$$

Notice also that, if L is sufficiently positive, then there are always epimorphisms. One has also in this case $\mathrm{Pic}\, Y' \cong \mathrm{Pic}\, Y$.

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2·3. In [4] non-reduced curves of small multiplicity on a threefold are treated systematically. In particular one describes there a method of 'tripling' a curve. One starts with a doubling as above and one asks when there exists a lifting to the next infinitesimal neighbourhood $Y^{(2)}$, i.e. one demands l.c.i. curves Y'' contained in $Y^{(2)}$ and such that $Y'' \cap Y^{(1)} = Y'$. As in other lifting questions, there is a cohomological obstruction, which can be described in this case as follows. From the above section one gets an exact sequence of bundles on Y:

$$0 \longrightarrow L^2 \xrightarrow{i} N^*_{Y'}|Y \longrightarrow \det(N^*_Y) \otimes L^{-1} \longrightarrow 0,$$

where $L^2 \cong \mathscr{I}_Y^2/\mathscr{I}_Y\mathscr{I}_{Y'} \overset{i}{\hookrightarrow} \mathscr{I}_{Y'}/\mathscr{I}_Y\mathscr{I}_{Y'} \cong N^*_{Y'}|Y$ is given by inclusions of ideals. It turns out that the obstruction is the splitting of this.

Let us assume that the previous sequence splits and let τ be a retraction for i. Write $\operatorname{Ker}\tau = \mathscr{J}/\mathscr{I}_Y\mathscr{I}_{Y'}$ and let Y'' be the subspace associated to $\mathscr{J}$. Then Y'' is l.c.i., $Y'' \subset Y^{(2)}$ and $Y'' \cap Y^{(1)} = Y'$. Indeed, the statements are local, so that we can assume $\mathscr{I}_Y = (f, g)$, $\mathscr{I}_{Y'} = (f^2, g)$. Then L^2 is generated by $f^2 \bmod (\mathscr{I}_Y\mathscr{I}_{Y'})$ and $N^*_{Y'}|Y$ is generated by the classes $\dot{f}^2, \dot{g}$. We write $\tau(\dot{g}) = \dot{a}f^2$, where $a \in \mathscr{O}_X$. Then denoting $g' = g - af^2$ one gets easily: $\mathscr{I}_Y = (f, g')$, $\mathscr{I}_{Y'} = (f^2, g')$ and $\mathscr{I}_{Y''} = (f^3, g')$. The cohomology classes are related by $[Y''] = 3Y$.

From the exact sequence

$$0 \to \mathscr{I}_Y^2 \cap \mathscr{I}_{Y''}/\mathscr{I}_Y\mathscr{I}_{Y''} \to \mathscr{I}_{Y'}/\mathscr{I}_Y\mathscr{I}_{Y''} \to \mathscr{I}_Y/\mathscr{I}_Y^2 \to \mathscr{I}_Y/\mathscr{I}_{Y''} + \mathscr{I}_Y^2 \to 0$$

and the isomorphisms

$$N^*_{Y''}|Y \cong \mathscr{I}_{Y''}/\mathscr{I}_Y\mathscr{I}_{Y''}, \quad \mathscr{I}_Y/\mathscr{I}_{Y''} + \mathscr{I}_Y^2 = \mathscr{I}_Y/\mathscr{I}_{Y'} \cong L,$$

$$L^3 \cong \mathscr{I}_Y^2 \cap \mathscr{I}_{Y''}/\mathscr{I}_Y\mathscr{I}_{Y''}$$

(for the last, one defines a map $\to$, then one verifies locally that it is an isomorphism) one gets

$$\det(N_{Y''})|Y \cong \det(N_Y) \otimes L^{-2}.$$

One obtains also the exact sequences

$$0 \to L \to \mathscr{O}_{Y'} \to \mathscr{O}_Y \to 0, \quad 0 \to L^2 \to \mathscr{O}_{Y''} \to \mathscr{O}_{Y'} \to 0,$$

which allow us to connect Pic Y'', Pic Y', Pic Y and which will be needed also to compute the Atiyah–Rees invariant.

The obstruction to the above splitting lies in $H^2(Y, N_Y \otimes L^3)$. Therefore, if L is sufficiently positive, we can always perform the above construction. Moreover, in this case Pic $Y' \cong$ Pic Y.

2·4. The (complex, projective) homogeneous, rational manifolds are (up to isomorphism) exactly the quotients G/H, where G is a semi-simple complex Lie group and H a parabolic subgroup of G. The manifold is called irreducible if G is simple and any rational manifold is isomorphic to a product of irreducible ones. It turns out then that the rational irreducible threefolds are $\mathbb{P}_3$, $\mathbb{Q}_3$, $\mathbb{F}_{1,2}$ and the decomposable ones $\mathbb{P}_2 \times \mathbb{P}_1$, $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

From now on X will be one of these models and all statements below can be checked easily case by case. There are smooth rational disjoint curves $Y_1, \ldots, Y_q$ and smooth surfaces $Z_1, \ldots, Z_q$ (in fact $q \leqslant 3$), so that, denoting $e_i = [Z_i]$ and $f_j = [Y_j]$ one obtains dual bases $(e_1, \ldots, e_q)$, $(f_1, \ldots, f_q)$ for the cohomology groups $H^2(X, \mathbb{Z})$, respectively

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$H^4(X,\mathbb{Z})$. Moreover, in every cohomology class $[Y_i]$ there is a countable set of smooth rational curves $Y_{i,n}$ such that $Y_{i,n}\cap Y_{j,m}=\varnothing$ whenever $(i,n)\neq(j,m)$.

For $a=\Sigma a_i e_i\in H^2(X,\mathbb{Z})$, $a_i\in\mathbb{Z}$, $\mathcal{O}(a)$ will be the unique algebraic line bundle with first Chern class a. $\mathcal{O}(a)$ corresponds to the divisor $\Sigma a_i Z_i$ and

$$\mathcal{O}(a)\cong\mathcal{O}(e_i)^{\otimes a_i}\otimes\ldots\otimes\mathcal{O}(e_q)^{\otimes a_q}.$$

By identifying each Y_i with $\mathbb{P}_1$, one has $\mathcal{O}(e_j)|Y_i\cong\mathcal{O}_{\mathbb{P}_1}(\delta_{ij})$. Fix a curve Y_i and study double and triple structures on it. We identify Y_i with a $\mathbb{P}_1$. For $r\gg 0$ there are epimorphisms $\mathcal{I}_{Y_i}/\mathcal{I}^2_{Y_i}\to\mathcal{O}_{Y_i}(r)\to 0$ which give l.c.i. curves Y_i' with the support Y_i, such that $[Y_i']=2f_i$ and

$$\det(N_{Y_i'})|Y_i\cong(\det N_{Y_i})\otimes\mathcal{O}_{Y_i}(-r).$$

We can then thicken the curves Y_i' in order to obtain triple l.c.i. curves Y_i'' supported on Y_i, with $[Y_i'']=3f_i$ and such that

$$\det(N_{Y_i''})|Y_i\cong\det(N_{Y_i})\otimes\mathcal{O}_{Y_i}(-2r).$$

3. *The proof of Theorem 2*

3·1. *The K_X-symplectic case.* Let $c_1\in H^2(X,\mathbb{Z})$ and $c_2\in H^4(X,\mathbb{Z})$, let us assume that c_1 and $c_1(X)$ have the same image in $H^2(X,\mathbb{Z}_2)$ and let F be a topological vector bundle of rank 2 with $c_i(F)=c_i$. We can tensor F with an algebraic line bundle in such a way that the new c_1 is now $c_1(K_X)$. To prove Theorem 2 in this case we must find algebraic vector bundles E of rank 2 such that the Chern classes are $c_1(E)=c_1=c_1(K_X)$, $c_2(E)=c_2$, and $h^0(E)+h^2(E)$ (mod 2) should take both values 0 and 1. Let $a=\sum_i a_i e_i$ be an element of $H^2(X,\mathbb{Z})$, $a_i\in\mathbb{Z}$. If all a_i are sufficiently large, in short $a\gg 0$, then the line bundle $\mathcal{O}(a)$ will be sufficiently positive so that all the desired vanishing theorems hold (this is plain by general arguments, or by inspecting each case separately).

We look for fibre bundles E as extensions

$$0\to\mathcal{O}(a)\to E\to\mathcal{I}_Y(c_1-a)\to 0$$

with $a\gg 0$. By 2·1 we have to find $a\gg 0$ and l.c.i. curves Y subject to the restrictions: $[Y]=c_2+a(a-c_1)$, $\det(N_Y)\cong\mathcal{O}_Y(c_1-2a)$ and $h^0\mathcal{O}(a)+h^1\mathcal{O}_Y(a-c_1)$ (mod 2) is the given $\alpha\in\mathbb{Z}_2$. One has $c_2+a(a-c_1)=\sum_i b_i f_i$, $b_i\in\mathbb{Z}$. If $a\gg 0$, then $b_i\gg 0$ (because $e_i\cdot e_j=\sum_k\epsilon_{ij}^k f_k$ with $\epsilon_{ij}^k\geqslant 0$ and the intersection matrix $(e_i\cdot e_j)$ is positive definite).

Let us write $b_i=2\alpha_i+3\beta_i$, where α_i and β_i are sufficiently large integers (in fact we need only $\alpha_i\geqslant 0$, $\beta_i\geqslant 2$).

Let $c_1(K_X)=\sum_i\tau_i e_i$, $\tau_i\in\mathbb{Z}$. If $a_i\gg 0$ we can double by 2·4 the curve Y_i (or any other representative $Y_{i,n}$ of f_i) with an epimorphism

$$\mathcal{I}_{Y_i}/\mathcal{I}^2_{Y_i}\to\mathcal{O}_{Y_i}(r_i)$$

where $r_i=2(a_i-\tau_i-1)$. The new curves Y_i' are l.c.i., $[Y_i']=2f_i$ and

$$\det N_{Y_i'}\cong\mathcal{O}_{Y_i'}(\tau_i-2a_i)\cong\mathcal{O}_{Y_i'}(c_1(K)-2a).$$

We derive in addition from the exact sequence

$$0\to\mathcal{O}_{Y_i}(r_i)\to\mathcal{O}_{Y_i'}\to\mathcal{O}_{Y_i}\to 0,$$

$$h^1\mathcal{O}_{Y_i'}(c_1(K)-a)=h^1\mathcal{O}_{Y_i'}(\tau_i-a_i)=a_i-\tau_i-1.$$

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Starting with Y_i (or with any other representative $Y_{i,n}$) we can similarly double with an epimorphism $\mathscr{I}_{Y_i}/\mathscr{I}^2_{Y_i} \to \mathcal{O}_{Y_i}(l_i)$ and then triple, where $l_i = a_i - \tau_i - 1$. One obtains in this way new l.c.i. curves Y_i'', $[Y_i''] = 3f_i$, such that $\det N_{Y_i''} \cong \mathcal{O}_{Y_i''}(c_1(K) - 2a)$ and $h^1\mathcal{O}_{Y_i''}(c_1(K) - a) = a_i - \tau_i - 1$.

Now we find our desired curves! One takes Y to be the disjoint union of curves Y'_{i,s_i} and Y''_{i,t_i} with $i = 1, \ldots, q$, $s_i = 1, \ldots, \alpha_i$ and $t_i = 1, \ldots, \beta_i$. Then Y is a l.c.i., $[Y] = c_2 + a(a - c_1)$ and $\det(N_Y) \cong \mathcal{O}_Y(c_1 - 2a)$. The algebraic bundles E obtained as extensions as above have $c_1(E) = c_1 = c_1(K_X)$, $c_2(E) = c_2$ and

$$\alpha(E) \equiv h^0(E) + h^2(E) \equiv h^0\mathcal{O}(a) + h^1\mathcal{O}_Y(c_1 - a) \equiv h^0\mathcal{O}(a) + \textstyle\sum_i (\alpha_i + \beta_i)(a_i - \tau_i - 1) \pmod 2.$$

Let us further write $b_1 = 2(\alpha_1 + 3) + 3(\beta_1 - 2)$ and keep unchanged the other values for b_j, $2 \leqslant j \leqslant q$. Then one obtains new curves $\tilde{Y}$ and new bundles $\tilde{E}$, so that

$$\alpha(\tilde{E}) \equiv \alpha(E) + a_1 - \tau_1 - 1 \pmod 2.$$

Finally one can choose an appropriate a_1, such that $\alpha(\tilde{E}) \neq \alpha(E)$. The proof in the K_X-symplectic case is over.

3·2. *The case when* $c_1 + c_1(X)$ *is nonvanishing in* $H^2(X, \mathbb{Z}_2)$. Then the topological type is completely described by c_1, c_2. The topological constraint

$$(c_1 + c_1(X))\, c_2 \equiv 0 \pmod 2$$

can be written $\sum_i (c_1^i + \tau_i) c_2^i = 0 \pmod 2$, where $c_1 = \sum_i c_1^i e_i$, $c_2 = \sum_i c_2^i f_i$ (recall that $c_1(K_X) = \sum_i \tau_i e_i$). Moreover, we are under the assumption that at least one of $c_1^i + \tau_i$ is odd.

We shall look again for rank 2-fibre bundles E with $c_1(E) = c_1$, $c_2(E) = c_2$ as extensions

$$0 \to \mathcal{O}(a) \to E \to \mathscr{I}_Y(c_1 - a) \to 0.$$

We need $a \gg 0$ and l.c.i. curves Y satisfying

$$[Y] = c_2 + a(a - c_1) \quad \text{and} \quad \det(N_Y) \cong \mathcal{O}_Y(c_1 - 2a).$$

In this case there don't always exist triple structures Y_i'' on the curves Y_i, subject to the condition $\det(N_{Y_i''}) \cong \mathcal{O}_{Y_i''}(c_1 - 2a)$.

The idea of the proof is the following: we pick out the indices i which permit the tripling procedure and prove that a suitable choice of a allows us to solve the problem only by doubling for the other indices. The proof consists of an inspection of each case separately, the intersection matrix $(e_i \cdot e_j)$ being involved explicitly in the computations.

We remark firstly that for $a_i \gg 0$ we can double Y_i (or any $Y_{i,n}$) by epimorphisms $\mathscr{I}_{Y_i}/\mathscr{I}^2_{Y_i} \to \mathcal{O}_{Y_i}(r) \to 0$, with $r = 2a_i - 2 - \tau_i - c_1^i$, and obtain l.c.i. curves Y_i' with $[Y_i'] = 2f_i$ and $\det(N_{Y_i'}) \cong \mathcal{O}_{Y_i'}(c_1 - 2a)$.

Secondly, if $c_1^i + \tau_1^i = 0 \pmod 2$, then by taking $l = a_i - 1 - \frac{1}{2}(\tau_i + c_1^i)$ we can triple Y_i (or any $Y_{i,n}$), starting with an epimorphism $\mathscr{I}_{Y_i}/\mathscr{I}^2_{Y_i} \to \mathcal{O}_{Y_i}(l) \to 0$ and then performing the construction of 2·3 and 2·4. The new curves Y_i'' are l.c.i., $[Y_i''] = 3f_i$ and $\det(N_{Y_i''}) \cong \mathcal{O}_{Y_i''}(c_1 - 2a)$.

Consider $X = \mathbb{P}_3$. Then the assumption is c_1 odd and c_2 even. It is enough to choose a sufficiently large even and double conveniently $\frac{1}{2}(c_2 + a(a - c_1))$ disjoint lines.

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Consider $X = \mathbb{Q}_3$. Then the class of a hyperplane section gives the generator e of $H^2(\mathbb{Q}_3, \mathbb{Z}) = \mathbb{Z}$, and the class of a line contained in $\mathbb{Q}_3$ gives the generator f of

$$H^4(\mathbb{Q}, \mathbb{Z}) = \mathbb{Z}.$$

One has $e^2 = 2f$, $e^3 = 2ef$ and ef is the positive generator of $H^6(X, \mathbb{Z}) = \mathbb{Z}$. K_X has Chern class $-3e$, thus $\tau_1 = -3$. We are in the non-K_X-symplectic case: $c_1 = c_1^1 e$ with even c_1^1 and $c_2 = c_2^1 f$ with even c_2^1. We look for $a = a_1 e$ with $a_1 \gg 0$ and l.c.i. Y with $[Y] = c_2^1 f + a_1 e(a_1 e - c_1^1 e) = (c_2^1 + 2a_1(a_1 - c_1^1))f$ and $\det(N_Y) \cong \mathcal{O}_Y(c_1 - 2a)$. We can again conveniently double $\frac{1}{2}(c_2^1 + 2a_1(a_1 - c_1^1))$ disjoint lines.

In the above two cases $X = \mathbb{P}_3, \mathbb{Q}_3$, the condition $c_1^1 + \tau_1^1 = 0 \pmod 2$, necessary for the existence of triple structures, is not satisfied, but the doubling procedure was sufficient.

In *the next case*, $X = \mathbb{F}_{1,2}$, the argument is a little bit more complicated. Let us consider $\mathbb{F}_{1,2}$ as the incidence manifold in $\mathbb{P}_2 \times \mathbb{P}_2^*$ and let us denote by π_1, π_2 the projection maps onto $\mathbb{P}_2$, respectively $\mathbb{P}_2^*$. $H^2(\mathbb{F}_{1,2}, \mathbb{Z}) = \mathbb{Z}^2$ is generated by the pull-backs e_1, e_2 by π_1, π_2 of the generators of $H^2(\mathbb{P}_2, \mathbb{Z})$, respectively $H^2(\mathbb{P}_2^*, \mathbb{Z})$. They are the cohomology classes associated to π_1^{-1} (line), respectively π_2^{-1} (line). The generators of $H^4(\mathbb{F}_{1,2}, \mathbb{Z}) = \mathbb{Z}^2$ are $f_1 = e_2^2$ and $f_2 = e_1^2$ and they are the corresponding cohomology classes to the curves $Y_1 = \pi_2^{-1}(\text{point})$, $Y_2 = \pi_1^{-1}(\text{point})$. One has the relations

$$e_1^2 + e_2^2 - e_1 e_2 = 0, \quad e_1^3 = e_2^3 = 0 \quad \text{and} \quad e_1 f_1 = e_2 f_2$$

is the positive generator of $H^6(X, \mathbb{Z})$. For $\tau = c_1(K_X)$ one has $\tau = -2e_1 - 2e_2$.

We assumed that at least one entry of c_1, say c_1^1, is odd, so that the relation

$$(c_1 + c_1(X))\, c_2 = 0 \pmod 2$$

becomes $c_2^1 + c_1^2 c_2^2 = 0 \pmod 2$. We need pairs $(a_1, a_2) \gg 0$ and l.c.i. curves Y such that

$$[Y] = [c_2^1 - c_1^2(a_1 + a_2) + a_2^2 + 2a_1 a_2 - a_2 c_1^1] f_1 + [c_2^2 - c_1^1(a_1 + a_2) + a_1^2 + 2a_1 a_2 - a_1 c_1^2] f_2$$

and $\det(N_Y) \cong \mathcal{O}_Y(c_1 - 2a)$. Denoting by A_1 and A_2 the above coefficients, we can write more simply $[Y] = A_1 f_1 + A_2 f_2$. Because c_1^1 is an odd number, we can't consider the needed triple structures on Y_1. We look then to c_1^2. There are two possibilities:

(1) c_1^2 odd. Then we can't triple the curves Y_2 either. But in this case

$$A_1 = c_2^1 + a_1 + a_2 (\text{mod } 2), \quad A_2 = c_2^2 + a_1 + a_2 (\text{mod } 2).$$

As $c_2^1 = c_2^2 \pmod 2$, we can choose $a_1 \gg 0$, $a_2 \gg 0$ so that A_1 and A_2 should be even, and can find the required curves by doubling.

(2) c_1^2 even. Then $A_1 = c_2^1 \pmod 2$, $A_2 = c_2^2 + a_2 \pmod 2$. In this case c_1^2 is even, hence A_1 is even. Therefore we can double curves in the class f_1 and double and triple (it suffices, by an appropriate choice of a_2, only to double) curves in the class f_2, in order to obtain the required curves.

The remaining cases $\mathbb{P}_2 \times \mathbb{P}_1$, $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ are completely analogous and we omit the details.

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A last remark: the paper corresponds to a preprint of the authors (December 1983). More recently, I. Coandă and the first author (Preprint Increst-Bucharest, nr. 27, May 1984) have proved Theorem 2 also for rank 3, in accordance with the theorem of Vogelaar on $\mathbb{P}_3$.

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